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SFT-stability and Krull dimension in power series rings over an almost pseudo-valuation domain

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Abstract Let R be an APVD with maximal ideal M. We show that the power series ring $R[[x_1, \ldots, x_n]]$ is an SFT-ring if and only if the integral closure of R is an SFT-ring if and only if (R is an SFT-ring and M is a Noether strongly primary ideal of $(M : M)$). We deduce that if R is an m-dimensional APVD that is a residually *-domain, then dim $R[[x_1, \ldots, x_n]] = nm + 1$ or $nm + n$.

Keywords Power series ring · Krull dimension · APVD · SFT-ring

Mathematics Subject Classification 13C15 · 13F25 · 13F30

1 Introduction

In this paper, all rings are commutative with identity and the dimension of a ring means its Krull dimension. Throughout this paper, if D denotes an integral domain with quotient field K, then D' denotes its integral closure, $\mathrm{Min}(D)$ denotes the set of height-one prime ideals of D and if $X_n = \{x_1, \ldots, x_n\}$ is a set of indeterminates over K, then we write $D[[X_n]]$ rather than $D[[x_1, \ldots, x_n]]$. An ideal I is called an SFT-ideal if there exist a natural number k and a finitely generated ideal $J \subseteq I$ such that $a^k \in J$ for each $a \in I$. An SFT-ring is a ring in which every ideal is an SFT-ideal. SFT-rings are similar to Noetherian rings, and they have many nice properties. For properties about SFT-rings, the readers are referred to [1,2,4].

In [2], Arnold studied the Krull dimension of power series ring and showed that if a ring R fails to have the SFT-property, then $R[[X]]$ has infinite dimension. Arnold's result forces us to consider only SFT-rings when we study finite-dimensional power series extensions. Also, it forces us to raise the question: does $R[[X]]$ is an SFT-ring when R is an SFT-ring?. In fact, this is an old question which was raised by R. Gilmer. In Coykendall [8], answered

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this question in the negative by constructing a one-dimensional SFT-domain R such that $R[[X_n]]$ is not an SFT-ring. At the same time (2002), in Badawi and Houston [5], introduced the concept of an almost pseudo-valuation domain. An integral domain R is said to be an almost pseudo-valuation domain (or, for short, APVD) if R is a quasi-local domain with maximal ideal M, and there is a valuation overring in which M is a primary ideal. One very remarkable thing in [8] is that Coykendall's example is a one-dimensional APVD (see [13, Example 2.4]). So it is natural to study the SFT-instability via power series extension over an APVD regardless of its dimension and ask which things cause the SFT-instability in such extensions, which is the first purpose of this paper.

In this paper, we give a necessary and sufficient condition to win the SFT-stability via power series extension over an APVD. Recall from [7] that a P-primary ideal Q of an integral domain R is called a Noether strongly primary ideal of R if $P^k \subseteq Q$ for some positive integer k. We prove that if R is an APVD with maximal ideal M, then $R[[x_1]]$ is an SFT-ring if and only if $R[[x_1, x_2]]$ is an SFT-ring if and only if the integral closure of R is an SFT-ring if and only if R is an SFT-ring and M is a Noether strongly primary ideal of $(M : M)$ (Theorem 2.3).

We show first that if R is a one-dimensional APVD with maximal ideal M, then $R[[X]]$ is an SFT-ring if and only if R is an accp-ring (i.e., satisfies the ascending chain condition on principal ideals) and M is a Noether strongly primary ideal of $(M : M)$.

We use rings of the form $D + M$ to construct several examples which prove the SFT-instability via power series extension over an APVD regardless of its dimension (Proposition 2.7 and Example 2.8).

Recall from [13], an integral domain R is called a $*$-domain if the set $\mathrm{Min}(R)$ is nonempty and if for each $P \in \mathrm{Min}(R)$, R satisfies the ascending chain condition on P-principal ideals (ideals of the form aP, $a \in R$). Also from [13], R is said to be a residually $*$-domain if for each nonmaximal prime ideal P of R, the quotient ring R/P is a $*$-domain. In [13], we proved that if R is an APVD with nonzero finite dimension, then $\dim R[[X]] < \infty$ if and only if $\dim R[[X_n]] < \infty$ if and only if R is a residually $*$-domain if and only if the ring $R[[X]]$ is catenary ([13, Theorem 3.5]). And, in this case, we have $\dim R[[X]] = 1+\dim R$, $1+n \dim R \leq \dim R[[X_n]] \leq n+(n+1) \dim R$ and $\mathrm{ht} P[[X_n]] = n\, \mathrm{ht} P$ for each nonmaximal prime ideal P of R ([13, Theorem 2.9 and Corollary 2.17]). As an application of Theorem 2.3 (instability theorem on an APVD), we show that $\dim R[[X_n]] = n \dim R+1$ or $n \dim R + n$ and $\mathrm{ht} M[[X_n]] = n(\mathrm{ht} M - 1) + 1$ or $n\, \mathrm{ht} M$ and for each value we give sufficient and necessary conditions on R (Theorem 3.1, Corollary 3.4 and Corollary 3.5). Also we give examples for each value (Example 3.8 and Example 3.9). Finally, we show that the ring $R[[X_n]]$ is not catenary if $n > 1$ and $\dim R > 1$.

2 When is $R[[X_n]]$ an SFT-ring?

Throughout this section, R will denote an APVD with quotient filed K and maximal ideal M and V will denote the overring $(M : M) = \{x \in K; xM \subseteq M\}$ of R. By [5, Theorem 3.4], V is a valuation domain and M is primary to the maximal ideal N of V.

Lemma 2.1 1. *V and R have the same set of nonmaximal prime ideals. In particular, $\dim V = \dim R$ and every prime ideal of R is a proper ideal of V.*
2. *Let P be a nonmaximal prime ideal of R. Then $V/P = (\widetilde{M} : \widetilde{M})$ where $\widetilde{M} = M/P$.*

Proof (1) By [13, Lemma 2.13], every nonmaximal prime ideal P of R is also a nonmaximal (because $P \subset M \subseteq N$) prime ideal of V. Since $R \subseteq V$, it is easy to show that every nonmaximal prime ideal P of V (thus $P \subset M$ because $\sqrt{MV} = N$) is also a nonmaximal prime ideal of R.

(2) Since $\widetilde{M}$ is an ideal of V/P, $(\widetilde{M} : \widetilde{M})$ is an overring of the valuation domain V/P. Thus $(\widetilde{M} : \widetilde{M}) = (V/P)_{Q/P}$ for some prime ideal Q of V which contains P. Then $(\widetilde{M} : \widetilde{M}) = V_Q/PV_Q = V_Q/P$. So Q/P is the maximal ideal of $(\widetilde{M} : \widetilde{M})$. Since $\widetilde{M}$ is a proper ideal of $(\widetilde{M} : \widetilde{M})$, $\widetilde{M} \subseteq Q/P$ and thus $M \subseteq Q$. So $N = \sqrt{MV} \subseteq Q$, and then $Q = N$. Hence $(\widetilde{M} : \widetilde{M}) = V/P$. □

The following lemma gives new characterizations of a one-dimensional APVD R so that $R[[X]]$ is an SFT-ring.

Lemma 2.2 *If* $\dim R = 1$*, then the following assertions are equivalent:*

1. $R[[X]]$ *is an SFT-ring.*
2. R *is a* $*$*-domain.*
3. R *is an SFT-ring and* M *is a Noether strongly primary ideal of* $(M : M)$.
4. R *is an accp-ring and* M *is a Noether strongly primary ideal of* $(M : M)$.
5. $(M : M)$ *is an SFT-ring*

Proof For (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5), see [13, Theorem 2.2 and Theorem 2.9].
(5) $\Rightarrow$ (4) Note that a rank-one valuation domain is SFT if and only if it is Noetherian if and only if it is discrete. If $(a_n R)_{n\geq 0}$ is an ascending chain of nonzero proper principal ideals of R, then $(a_n V)_{n\geq 0}$ is an ascending chain of principal ideals of V. Thus, there exists a positive integer k such that $a_n V = a_k V$ for every integer $n \geq k$. Let $c \in V$ and $d \in R$ such that $a_n = ca_k$ and $a_k = da_n$. We have $dc = 1$, and thus, d is invertible in V. So $d \notin M$, and then d is invertible in R because R is quasi-local. Hence R is an accp-ring. Suppose that $N^n \nsubseteq M$ for each integer $n > 0$. Then $M \subseteq \bigcap_{n=1}^{\infty} N^n$ which is a prime ideal of V by [9, Theorem 17.1]. Thus $N = \sqrt{M} \subseteq N^2$, a contradiction.
(4) $\Rightarrow$ (3) Note that for a nonzero proper ideal I of a valuation domain, I is an SFT-ideal if and only if $I^2 \neq I$. Let s be a positive integer such that $N^s \subseteq M$ and suppose that $M^2 = M$. Then V is not SFT and so not Noetherian. So there exists a strictly increasing chain $(a_n V)_{n\geq 0}$ of principal ideals of V. Thus $\frac{a_n}{a_{n+1}} \in N$ and so $a_n^s \in a_{n+1}^s M$. Then $(a_n^s R)_{n\geq 0}$ is a strictly increasing chain of principal ideals of R, a contradiction. Necessarily $M^2 \neq M$, and hence R is SFT by [1, Proposition 2.2] and [13, Proposition 1.4].
(3) $\Rightarrow$ (5) Let s be a positive integer such that $N^s \subseteq M$. By [13, Proposition 1.4], there exists an $a \in M$ such that $M^6 \subseteq aR$. So $N^{6s} \subseteq aV$. Thus, N is an SFT-ideal of V. Hence, V is an SFT-ring. □

In the following result, we give a necessary and sufficient condition on an APVD R so that $R[[X_n]]$ is an SFT-ring. Also many characterizations of nonzero finite-dimensional APVDs which are residually $*$-domains are given.

Theorem 2.3 *Let R be an APVD with maximal ideal M. The following statements are equivalent:*

1. $R[[X_n]]$ *is an SFT-ring.*
2. $R[[X]]$ *is an SFT-ring.*
3. *The integral closure of* R *is an SFT-ring.*
4. R *is an SFT-ring and* M *is a Noether strongly primary ideal of* $(M : M)$.
5. $(M : M)$ *is an SFT-ring.*

Moreover, if $0 < \dim R < \infty$*, then each of the previous statements is equivalent to the following:*

6. *R is a residually ∗-domain.*

Proof (1) ⇒ (2) Note that the homomorphic image of an SFT-ring is also an SFT-ring [1, Proposition 2.3]. The result follows from the fact that $R[[X]] \cong R[[X_n]]/ < X_{n-1} >$.
(2) ⇒ (4) Since $R \cong R[[X]]/ < X >$, R is an SFT-ring. We can assume that $M \neq 0$ (because in this case $N = 0$). Denote Q to be the union of the nonmaximal prime ideals of R. Since R is a quasi-local domain with linearly ordered prime ideals, Q is a prime ideal of R by [12, Theorem 9]. Also Q is a prime ideal of V by Lemma 2.1. If $Q = M$, then $N = M$. So, we can assume that $Q \neq M$. Then Q is the prime ideal just below M in R, and so dim $\widetilde{R} = 1$ where $\widetilde{R} = R/Q$. Note that the homomorphic image (not be field) of a finite-dimensional APVD with the property residually ∗ is also a finite-dimensional APVD with the property residually ∗ [13, Remark 1.2-(4) and Proposition 1.9]. Since $\widetilde{R}[[X]] \cong R[[X]]/Q[[X]]$ is an SFT-domain, $\widetilde{M}$ is a Noether strongly primary ideal of $(\widetilde{M} : \widetilde{M})$ by Lemma 2.2, where $\widetilde{M} = M/Q$. By Lemma 2.1, $(\widetilde{M} : \widetilde{M}) = V/Q$. Thus $(N/Q)^s \subseteq M/Q$ for some integer $s > 0$. Hence $N^s \subseteq M + Q = M$.
(4) ⇒ (1) Suppose that $R[[X_n]]$ is not an SFT-domain. Then there exists an infinite chain $Q_1 \subset Q_2 \subset \cdots$ of prime ideals in $R[[X_n]][[x_{n+1}]]$ by [2, Proof of Theorem 1]. Denote $P = (\bigcup_{i=1}^{\infty} Q_i) \cap R$. Since P is an SFT-ideal of R, $Q_k \supseteq P$ for some integer $k > 0$. Then $Q_k \supseteq P[[X_{n+1}]]$ by [4, Proposition 2.1]. We have an infinite chain of prime ideals $\widetilde{Q_t} \subset \widetilde{Q_{t+1}} \subset \cdots$ of $\widetilde{R}[[X_{n+1}]]$ where $\widetilde{R} = R/P$, $\widetilde{Q_i} = Q_i/P[[X_{n+1}]]$. Since $\widetilde{Q_i} \cap \widetilde{R} = (0)$ for all integer $i \geq t$, dim $\widetilde{R}[[X_{n+1}]]_{\widetilde{R}\setminus(0)} = \infty$ (in particular $P \neq M$ and so dim $\widetilde{R} \geq 1$), and so Min$(\widetilde{R})$ is nonempty [13, Remark 2.1]. Let $s > 0$ be an integer such that $N^s \subseteq M$. Thus $(N/P)^s \subseteq \widetilde{M}$, and then $\widetilde{M}$ is a Noether strongly primary ideal of $(\widetilde{M} : \widetilde{M})$ by Lemma 2.1. Hence $\widetilde{R}$ is a ∗-domain by Lemma 2.2 and [13, Lemma 1.11], a contradiction by [13, Theorem 2.2].
(4) ⇒ (5) Let P be a nonzero prime ideal of V. If P is not maximal in V, then P is a prime ideal of R by Lemma 2.1. Thus $P^2 \neq P$. So P is an SFT-ideal of V. Now, we suppose that $P = N$. Let $s > 0$ be an integer such that $N^s \subseteq M$, and let $a \in M\setminus M^2$. Then $M^6 \subseteq aR$ by [13, Proposition 1.4]. Thus $N^{6s} \subseteq aV \subseteq N$. Then N is an SFT-ideal of V.
(5) ⇒ (4) Every nonzero prime ideal of R is a proper ideal of V, and so it is not idempotent. Hence R is an SFT-ring. Note that a nonidempotent maximal ideal of a valuation domain is a principal ideal. Let $a \in N$ and $s > 0$ an integer such that $N = aV$ and $a^s \in M$ for some integer $s > 0$. Hence $N^s = a^s V \subseteq MV = M$.
(5) ⇔ (3) By [5, Proposition 3.7], R' is a PVD with maximal ideal $Q := \sqrt{MR'}$ (Recall that a domain R with quotient field K is called a pseudo-valuation domain (for short, PVD) if each prime ideal P of R is strongly prime, in the sense that $x, y \in K$ and $xy \in P$ implies that $x \in P$ or $y \in P$). Let $W = (Q : Q)$ be the valuation domain associated with R'. Since $R' \subseteq V$ [9, Theorem 19.8], $Q \subseteq N$ and so $V \subseteq W$. Let P be a prime ideal of V such that $W = V_P$. Thus, $Q = PV_P = P$ is a prime ideal of V, and then $W = V_Q$. Since $R \subseteq R'$ is an integral extension and Q is a maximal ideal of R', $Q \cap R = M$. Thus $Q = N$ by Lemma 2.1. Hence $V = W$. The result follows from the fact that a PVD is SFT if and only its associated valuation overring is SFT [6, Chapitre 11-Proposition 8.10].
In case $0 < \dim R < \infty$, the equivalence (6) ⇔ (2) follows from [13, Theorem 2.9]. □

The proof of the following result is straightforward because the maximal ideal M of a PVD is always a Noether strongly primary ideal of $(M : M)$ (in fact, M is the maximal ideal of $(M : M)$, see [10, Corollary 6]).

Corollary 2.4 *Let R be a PVD with maximal ideal M. The following statements are equivalent:*

1. *$R[[X_n]]$ is an SFT-ring.*
2. *$R[[X]]$ is an SFT-ring.*
3. *The integral closure of R is an SFT-ring.*
4. *R is an SFT-ring.*
5. *$(M : M)$ is an SFT-ring.*
   Moreover, if $0 < \dim R < \infty$, then each of the previous statements is equivalent to the following:
6. *R is a residually $*$-domain.*

In the following example, we show that for each $m \in \mathbb{N}^* \cup \{\infty\}$, there exists an m-dimensional APVD R (not PVD) such that R and $R[[X_n]]$ are SFT-rings.

Example 2.5 Let F be a field. It is known that there exists an SFT-valuation domain V of the form $F + N$ with maximal ideal N and Krull dimension m. By [4, Lemma 3.1], there exists $a \in V$ such that $N = aV$. Denote $R = F + N^2$. The ideal N^2 is maximal in R because $R/N^2 \cong F$. Since the conductor $(R : V) = N^2 \neq (0)$, V is a valuation overring of R. Since $a^2 \in N^2$ and $a \notin N^2$, N^2 is not strongly prime and then R is not a PVD. Since N^2 is a primary ideal of V, N^2 is a strongly primary ideal of R by [5, Theorem 2.11] and then R is an APVD by [5, Theorem 3.4]. Since $(N^2 : N^2) = (a^2V : a^2V) = V$, $\dim R = m$ (by Lemma 2.1) and $R[[X_n]]$ is an SFT-ring by Theorem 2.3.

Example 2.6 [8, Theorem 4.1](Coykendall's example). We follow [13, Example 2.4]. Take $R = V_1 = \mathbb{F}_2 + XV$ where $V = \mathbb{F}_2[X^\alpha]_M$, with $\mathbb{F}_2[X^\alpha] = \{\sum_{i=0}^n \epsilon_i X^{\alpha_i} | \epsilon_i \in \mathbb{F}_2, \alpha_i \in \mathbb{Q}^+\}$ and M is the maximal ideal of $\mathbb{F}_2[X^\alpha]$ generated by the monomials. The ring R is a one-dimensional SFT–APVD which is not a residually $*$-domain. Hence, by Theorem 2.3, $R[[X_n]]$ is not an SFT-domain and so $(MV)^n \nsubseteq XV$ for each integer $n > 0$. Thus, XV is not a Noether strongly primary ideal of V. This last fact is the cause of the SFT-instability in Coykendall's example.

Coykendall [8] constructed a one-dimensional APVD R which is SFT but $R[[X_n]]$ is not an SFT-ring. We will show that for each $m \in \mathbb{N}^* \cup \{\infty\}$, there exists an m-dimensional APVD R which is SFT but $R[[X_n]]$ is not an SFT-ring. The following result will help us to construct the desired examples.

Proposition 2.7 *Let $m \in \mathbb{N}^* \cup \{\infty\}$ and V be an m-dimensional valuation domain of the form $F + N$, where F is a field and N is the maximal ideal of V. Let D be a domain with quotient field F and set $R = D + N$. If V is an SFT-ring and D is a one-dimensional SFT–APVD with maximal ideal M, then R is an $(m + 1)$-dimensional SFT–APVD with maximal ideal $M + N$.*

Proof Since $R/(M + N) \cong D/M$, $M + N$ is a maximal ideal of R. The domains R and V have the same quotient field K because the conductor $(R : V) = N$ is nonzero. Let $x \in K$ such that $x^n \notin M + N$ for each integer $n > 0$. Since V is a valuation overring of R and $x \notin R$, either $x^{-1} \in N$ or x is a unit of V. If $x^{-1} \in N$, then $x^{-1}(M + N) \subseteq N \subseteq (M + N)$. If $x = f + z$ is a unit of V, where f is a nonzero element of F and $z \in N$. Thus $f^n \notin M$ for each integer $n > 0$ and there exists $z' \in N$ such that $x^{-1} = f^{-1} + z'$. So $f^{-1}M \subseteq M$ by [5, Lemma 2.3]. Thus $x^{-1}(M + N) \subseteq (M + N)$. Then R is an APVD by [5, Lemma 2.3 and Theorem 3.4]. Easily one can show that N is a strongly prime ideal of R and $\mathrm{ht}_R N = m$. Since N is a divided prime ideal of R, $\dim R = \mathrm{ht}N + \dim(R/N) = m + \dim D = m + 1$. Let P be a nonzero nonmaximal prime ideal of R. So P is a prime ideal of V. Thus, $P^2 \neq P$ and then P is an SFT-ideal of R. By [13, Proposition 1.4], $M^2 \neq M$ and $M^6 \subseteq aD$ for each

$a \in M \backslash M^2$. Since N is a divided prime ideal of R, $N \subset aR$. So $(M+N)^6 \subseteq M^6+N \subseteq aR$. Hence $(M+N)$ is an SFT-ideal of R. □

In the following example, we show that for every integer $m \in \mathbb{N}^* \cup \{\infty\}$, there exists an m-dimensional APVD R (not PVD) such that R is SFT but $R[[X_n]]$ is not SFT.

Example 2.8 Let W be a rank-one nondiscrete valuation domain (so $Q^2 = Q$ and W is not SFT) of the form $H + Q$, where H is a field and Q is the maximal ideal of W. Let $0 \neq a \in Q$ and set $D = H + a^2W$. By an argument similar to the one just given in Example 2.5, one can show that D is an APVD (not PVD) with maximal ideal a^2W, that $\dim D = 1$ and that $D[[X_n]]$ is not an SFT-ring. The ideal a^2W is the unique nonzero prime ideal of D and $(a^2W)^2 = a^4W \subseteq a^2D$. Hence D is an SFT-ring. Let $m \in (\mathbb{N} \cup \{\infty\}) - \{0, 1\}$ and F be the quotient field of D. Let V be an $(m-1)$-dimensional SFT-valuation domain of the form $F + N$, where N is the maximal ideal of V. Set $R = D + N$. By Proposition 2.7, R is an m-dimensional SFT–APVD. Since $R[[X_n]]/N[[X_n]] \cong D[[X_n]]$, $R[[X_n]]$ is not an SFT-ring.

3 Application: Krull dimension of $R[[X_n]]$

Let $F \subset L$ be fields, let K_0 be the maximal separable extension of F in L, and let p denote the characteristic of F if F has nonzero characteristic but set $p := 1$ if F has characteristic zero. We say that L has finite exponent over K_0 if $L^{p^n} \subseteq K_0$ for some positive integer n.

Park [14] proved that if R is a finite-dimensional SFT-PVD with maximal ideal M, then

$$\dim R[[X_n]] = \begin{cases} n \dim R + 1 & \text{if } L \text{ has finite exponent over } K_0 \text{ and } [K_0 : F] < \infty \\ n \dim R + n & \text{otherwise} \end{cases}$$

where $F = R/M$, L is the residue field of $(M : M)$ and K_0 is the maximal separable extension of F in L.

We will show that Park's result also remains true if R is a nonzero finite-dimensional APVD such that it is a residually $*$-domain. In fact, Park proved that if R is a finite-dimensional SFT GPVD with maximal ideal M, then $\dim R[[X_n]] = n \dim R + 1$ or $n \dim R + n$, where GPVDs is a generalization of the concept of PVD in the nonquasi-local case (Recall that an integral domain R is called a globalized pseudo-valuation domain (or, for short, GPVD) if there exists a Prüfer overring T satisfying the following two conditions: (1) $R \subseteq T$ is a unibranched extension; (2) there exists a nonzero radical ideal I common to T and R such that each prime ideal of T (respectively, R) that contains I is a maximal ideal of T (respectively, R).). It is easy to see that an APVD is a GPVD if and only if it is a PVD. As our goal was proved by Park in the case of a PVD, so we can suppose that our APVD is not a PVD and consequently necessarily is not a GPVD. However, Park's proof in [14, Theorem 2.4] is also valid for an APVD which is not PVD.

An immediate consequence of Theorem 2.3 and Park's technique is that we can calculate $\dim R[[X_n]]$ as the following result shows.

Theorem 3.1 *Let R be an APVD with maximal ideal M, residue field F and nonzero finite Krull dimension such that R is a residually $*$-domain. Let L be the residue field of $(M : M)$ and K_0 be the maximal separable extension of F in L. Then*

$$\dim R[[X_n]] = \begin{cases} n \dim R + 1 & \textit{if } L \textit{ has finite exponent over } K_0 \textit{ and } [K_0 : F] < \infty \\ n \dim R + n & \textit{otherwise} \end{cases}$$

Proof Assume that L has finite exponent over K_0 and $[K_0 : F] < \infty$. By [14, Lemma 2.1] (also [3, (3.8) p. 107 and Theorem 3.9]), $L[[X_n]]$ is integral over $F[[X_n]]$. By Theorem 2.3,

$N^s \subseteq M$ for some integer $s > 0$ and so $N[[X_n]]^s \subseteq M[[X_n]]$. It follows that $V[[X_n]]$ is integral over $R[[X_n]]$ and hence $\dim R[[X_n]] = \dim V[[X_n]] = n \dim R + 1$ by [4, Theorem 3.6] and Lemma 2.1.

For the case "L has infinite exponent over K_0 or $[K_0 : F] = \infty$", we follow the same way in [14, Theorem 2.4] given by Park (the proof needs [14, Lemma 2.3] which is also true if we replace an SFT-PVD by an APVD which is a residually $*$-domain, the proof is similar to Park's proof). □

Let R be an APVD with maximal ideal M, residue field F and nonzero finite Krull dimension such that R is a residually $*$-domain. Let L be the residue field of $(M : M)$ and K_0 be the maximal separable extension of F in L. In [13], we proved that $\text{ht}P[[X_n]] = n\text{ht}P$ for each prime ideal $P \neq M$ of R [13, Corollary 2.17]. We will show that $\text{ht}M[[X_n]] = n\text{ht}M$ or $n(\text{ht}M - 1) + 1$, but first we need to recall some background material.

Let D and J be two integral domains such that $D \subseteq J$. Let $Z = \{z_i\}_{i=1}^{\infty}$ be a countable set of indeterminates over J. Let $\mathcal{F} = \{f_i\}_{i=2}^{\infty}$ be a subset of $z_1 J[[z_1]]$. By Arnold ([4, p. 899]), $\mathcal{F}$ is said to be a suitable subset of $z_1 J[[[z_1]]$ if $\{i;\ f_i \notin z_1^k J[[z_1]]\}$ is finite for each positive integer k. If $\mathcal{F}$ is a suitable subset of $z_1 J[[z_1]]$, then we can define a unique $D[[z_1]]$-homomorphism $\Phi_{\mathcal{F}} : D[[Z]] \longrightarrow J[[z_1]]$ by $\Phi(z_i) = f_i$, $i \geq 2$ (where $D[[Z]]$ denotes (in the notation of [9, p. 10]) the full power series ring $D[[\{z_i\}_{i=1}^{\infty}]]_3$). By Arnold, J is called a special algebraic extension of D if for each suitable subset $\mathcal{E} = \{g_i\}_{i=2}^{\infty}$ of $z_1 J[[z_1]]$, the $D[[z_1]]$-homomorphism $\Phi_{\mathcal{E}}$ is not an isomorphism.

In other words, J is a special algebraic extension of D if and only if for each integral domain T such that $D[[X]] \subseteq T \subseteq J[[X]]$, $T \ncong D[[X]][[\{Y_i\}_{i=1}^{\infty}]]$ via a $D[[X]]$-isomorphism.

Lemma 3.2 *Suppose that* $\dim R = 1$. *If L has infinite exponent over K_0 or $[K_0 : F] = \infty$, then $\text{ht}M[[X_n]] = n$.*

Proof $R \subseteq V$ is a special algebraic extension because the conductor $(R : V) = M$ is nonzero. By [14, Lemma 2.1], $F \subseteq L$ is not a special algebraic extension and hence $\text{ht}M[[X_n]] \geq n$ by [4, Proposition 2.3]. Since $\text{ht}M[[X_n]] + n \leq \text{ht}(M + < X_n >)$, $\text{ht}M[[X_n]] \leq n$ by Theorem 3.1. □

Remark 3.3 In fact, another proof for the last lemma is similar to the one given by Park in [14, Theorem 2.4], but we choose to give a new proof.

Corollary 3.4 *If L has infinite exponent over K_0 or $[K_0 : F] = \infty$, then $\text{ht}M[[X_n]] = n\,\text{ht}M$.*

Proof Since $\text{ht}M[[X_n]] + n \leq \text{ht}(M + (X_n))$, $\text{ht}M[[X_n]] \leq n\,\text{ht}M$ by Theorem 3.1. Let P be the prime ideal just below M in R. Denote $\widetilde{R} = R/P$ and $\widetilde{M} = M/P$. Note that $\widetilde{R}$ is a one-dimensional APVD with maximal ideal $\widetilde{M}$ and is a residually $*$-domain [13, Remark 1.2-(4) and Proposition 1.10]. Also note that the valuation domain associated with $\widetilde{R}$ is V/P by Lemma 2.1. Since residue field $(\widetilde{R}) = F$ and residue field $(V/P) = L$, $\text{ht}\widetilde{M}[[X_n]] = n$ by Lemma 3.2. Hence $\text{ht}M[[X_n]] \geq \text{ht}\widetilde{M}[[X_n]] + \text{ht}Q[[X_n]] = n + n\,\text{ht}Q = n\,\text{ht}M$ by [13, Corollary 2.17]. □

Corollary 3.5 *If L has finite exponent over K_0 and $[K_0 : F] < \infty$, then $\text{ht}M[[X_n]] = n(\text{ht}M - 1) + 1$.*

Proof It is easy to see that $\text{ht}M[[X_n]] \leq \text{ht}(M + (X_n)) - n = \dim R[[X_n]] - n$. Thus $\text{ht}M[[X_n]] \leq n(\text{ht}M - 1) + 1$ by Theorem 3.1. Since $N[[X_n]] \cap R[[X_n]] = M[[X_n]]$ and $R[[X_n]] \subseteq V[[X_n]]$ is an integral extension (so it satisfies incomparability), $\text{ht}M[[X_n]] \geq \text{ht}N[[X_n]] = n(\text{ht}M - 1) + 1$ by [11, Theorem 13]. □

It is well known that if Q is a prime ideal of $D[X_n]$ with $Q \cap D = P$, then ht$Q$ = ht$P[X_n]$ + ht$(Q/P[X_n])$ [9, Theorem 30.18]. Kang and Park [11] asked if the power series analogue of this last result is true. They were especially interested in Prüfer domains and they answered their question in some cases of Prüfer domains [11, Corollary 15]. The following result gives an answer for their question in some cases of an APVD.

Corollary 3.6 *If L has infinite exponent over K_0 or $[K_0 : F] = \infty$, then ht$Q$ = ht$P[[X_n]]$+ ht$(Q/P[[X_n]])$ for all prime ideal Q of $R[[X_n]]$ with $Q \cap R = P$.*

Proof "$\geq$" is obvious. "$\leq$": Use induction on $m :=$ ht$P$. If $m = 0$, then it is obvious. Suppose $m > 0$ and the statement holds when ht$P < m$. Let Q be a prime ideal of $R[[X_n]]$ such that ht$P = m$ where $P = Q \cap R$. Let $(0) = Q_0 \subset Q_1 \subset \ldots \subset Q_s = Q$ be a chain of prime ideals of $R[[X_n]]$ arriving at Q. Let k be the smallest integer $i \in \{1, \ldots, s\}$ such that $Q_i \cap R \neq (0)$. Thus $Q_0 \cap R = \ldots = Q_{k-1} \cap R = (0)$. So $k - 1 \leq \dim R[[X_n]]_{R\setminus(0)} = n$ by [13, Theorem 2.2]. Denote $P_0 = Q_k \cap R$. Hence $P_0[[X_n]] \subseteq Q_k$.

Case 1 $P_0 = M$ (and so $P = M$). If $M[[X_n]] = Q_k$, then $s = k + (s - k) \leq$ ht$M[[X_n]]$+ht$(Q/M[[X_n]])$. If $M[[X_n]] \subset Q_k$, then $s = (k - 1) + (s - k + 1) \leq$ n+ht$(Q/M[[X_n]]) \leq$ ht$M[[X_n]]$+ht$(Q/M[[X_n]])$ by Corollary 3.4.
Case 2 $P_0 \neq M$. Denote $\widetilde{R} = R/P_0$ and $\widetilde{P} = P/P_0$. So, by considering $R[[X_n]]/P_0[[X_n]] \cong \widetilde{R}[[X_n]]$, since $ht\widetilde{P} < m$ and $(Q/P_0[[X_n]]) \cap \widetilde{R} = \widetilde{P}$, then ht$(Q/P_0[[X_n]])$ = ht$\widetilde{P}[[X_n]]$+ ht$(Q/P[[X_n]])$ by induction hypothesis.
Case 2.1 $Q_k = P_0[[X_n]]$. Then $s = k + (s - k) \leq$ ht$P_0[[X_n]]$+ht$(Q/P_0[[X_n]]$ = ht$P_0[[X_n]]$+ht$\widetilde{P}[[X_n]]$+ht$(Q/P[[X_n]]) \leq$ ht$P[[X_n]]$+ht$(Q/P[[X_n]])$.
Case 2.2 $P_0[[X_n]] \subset Q_k$. Then $s = (k - 1) + (s - k + 1) \leq n$+ht$(Q/P_0[[X_n]])$ = n+ht$\widetilde{P}[[X_n]]$+ht$(Q/P[[X_n]]) \leq$ ht$P_0[[X_n]]$+ht$\widetilde{P}[[X_n]]$+ht$(Q/P[[X_n]])$ because $n \leq$ n htP_0 =ht$P_0[[X_n]]$. Thus $s \leq$ ht$P[[X_n]]$+ht$(Q/P[[X_n]])$. □

Of course, the last result forces us to ask if the ring $R[[X_n]]$ is catenarian (Recall that a ring D is said to be catenary if for each pair $P \subset Q$ of prime ideals of D, all saturated chains of prime ideals of D between P and Q have a common finite length). In [13], we proved that if D is an APVD with nonzero finite dimension, then $D[[X]]$ is catenarian if and only if D is a residually $*$-domain. The question is still open for $R[[X_n]]$ with $n > 1$.

Corollary 3.7 *If $n > 1$ and $\dim R > 1$, then $R[[X_n]]$ is not catenarian.*

Proof Let $n > 1$ be an integer and suppose that $R[[X_n]]$ is catenarian. Then ht(x_n) + ht$(M + (X_n))/(x_n)$ = ht$(M + (X_n))$. Thus $1 +$ ht$(M + (X_{n-1}))$ = ht$(M + (X_n))$. If L has finite exponent over K_0 and $[K_0 : F] < \infty$, then $1 + (n - 1)\dim R + 1 = n \dim R + 1$ by Theorem 3.1. Thus $\dim R = 1$, a contradiction. If L has infinite exponent over K_0 or $[K_0 : F] = \infty$, then $1 + (n - 1)(\dim R + 1) = n(\dim R + 1)$ again by Theorem 3.1. Thus $\dim R = 0$, a contradiction. □

In the following example, we show that for every integer $m \geq 1$, there exists an m-dimensional APVD R such that R is a residually $*$-domain and (L has infinite exponent over K_0 or $[K_0 : F] = \infty$).

Example 3.8 Let $F \subset L$ be an extension of fields such that L is not algebraic over F. Let K_0 be the maximal separable extension of F in L. It is known that there exists an SFT-valuation domain V of the form $L + N$ with maximal ideal N and Krull dimension m. By [4, Lemma 3.1], there exists $a \in V$ such that $N = aV$. Denote $R = F + N^2$. By an argument similar to the one just given in Example 2.5, we show that R is an APVD with maximal ideal

N^2, $\dim R = m$ and R is a residually $*$-domain. It is clear that F and L are the residue fields, respectively, of R and $(N^2 : N^2) = V$. Since $F \subset L$ is not an algebraic extension, $F \subset L$ is not a special algebraic extension by [3, Theorem 2.1]. Then L has infinite exponent over K_0 or $[K_0 : F] = \infty$ by [14, Lemma 2.1].

In the following example, we show that for every integer $m \geq 1$, there exists an m-dimensional APVD R such that R is a residually $*$-domain and (L has finite exponent over K_0 and $[K_0 : F] < \infty$).

Example 3.9 Let F, m, V, N and R be as in Example 2.5. The result follows from the fact that $R/N^2 \cong F \cong V/N$.

SFT-stability and Krull dimension questions in power series rings over an APVD are solved, but catenarity question is not yet. It is still open if $R[[X_n]]$ is catenarian where $n > 1$ and R is a one-dimensional APVD.

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