# All solutions of the CMC-equation in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motion

**Maria Fernanda Elbert · Ricardo Sa Earp**

**Abstract** In this paper, we give all solutions of the constant mean curvature equation in $\mathbb{H}^n \times \mathbb{R}$ that are invariant by parabolic screw motion and we give the full description of their geometric behaviors. Some of these solutions give examples of non-trivial entire stable horizontal graphs that are not vertical graphs.

**Keywords** Constant mean curvature · Parabolic screw motions · Complete graph

**Mathematics Subject Classification** 53C42 · 53A10 · 35J15

## Introduction

In the last years, the ancient theory of minimal and constant mean curvature surfaces has been revisited for "new" ambient spaces, one of them being $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic space (see for instance [4,7–10] and the references therein). More generally, dealing with greater dimensions, the study of hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ has also been pursued (see for instance [1,2,5]). The construction of examples is an important tool that contributes to the understanding of what happens in this new ambient spaces. In this paper, we search for constant mean curvature (CMC) hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ that are invariant by parabolic screw motions, i.e., invariant by a $(n-1)$-parameter group of isometries such that each element is given by a composition of a parabolic translation in $\mathbb{H}^n$ followed by a vertical translation. In fact, we give all solutions of the constant mean curvature equation in $\mathbb{H}^n \times \mathbb{R}$ that are

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invariant by parabolic screw motion and we give the full description of their geometric behaviors.

We use the half-space model of the hyperbolic space $\mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, x_n = y) \in \mathbb{R}^n | y > 0\}$. We begin with a curve $t = \lambda(y)$ that after the action of such a group gives rise to a vertical graph.

After imposing the mean curvature equation, we are able to characterize the generating curves $\lambda(y)$ depending on the value of $H$. From the results about $\lambda(y)$, we construct many families of interesting examples of complete CMC hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, including embedded and stable ones.

Besides giving explicit examples, we prove the existence of families of hypersurfaces. These existence results are briefly stated below.

**Theorem A** *There exists a family of non-entire horizontal minimal complete graphs in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motions. The asymptotic boundary of each graph of this family is formed by two parallel $(n-1)$-planes.*

**Theorem B** (i) *For each $H$, $0 < |H| < \frac{(n-1)}{n}$, there exists a family of complete horizontal H-CMC graphs in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motions.*

(ii) *For each $H$, $0 < |H| \leq \frac{(n-1)}{n}$, there exists a family of complete non-embedded H-CMC hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motions.*

**Theorem C** *For each $H$, $|H| > \frac{n-1}{n}$, there exists a family of complete non-embedded and t-periodic H-CMC hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motions.*

Theorems A, B, and C are restated with more details in Sects. 3 and 4, where we give good descriptions of the behavior of the hypersurfaces. We point out that the parabolic translations hypersurfaces ($l = 0$) given in Theorems A and B (i) are stable (see Proposition 2.2). Some of them are examples of complete stable CMC hypersurfaces that are neither entire vertical graphs nor the trivial examples, namely, cylinders over totally umbilic $(n-1)$-submanifolds of $\mathbb{H}^n \times \{0\}$.

In [11], the second author studies the case $n = 2$. In the following papers, and in the references therein, the interested reader will find related topics [12,13].

## 1 Preliminaries

We use the half-space model of the hyperbolic space $\mathbb{H}^n$ ($n \geq 2$), i.e., we consider

$$\mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, x_n = y) \in \mathbb{R}^n | y > 0\}$$

endowed with the metric

$$g_{\mathbb{H}} := \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dy^2}{y^2}.$$

On $\mathbb{H}^n \times \mathbb{R}$, with coordinates $(x_1, \ldots, x_{n-1}, y, t)$, we consider the product metric

$$g = g_{\mathbb{H}} + dt^2.$$

A *vertical graph* of a real function $u$ defined over $\Omega \subset \mathbb{H}^n$ is the set $G = \{(x, u(x)) \in \mathbb{H}^n \times \mathbb{R} | x \in \Omega\}$. A computation shows that when $u$ is $C^2$ and we choose the orientation given by the upper unit normal, the mean curvature $H$ of the graph $G$ is given by

$$div\left(\frac{\nabla u}{\sqrt{1 + y^2 ||\nabla u||^2}}\right) + \frac{(2-n)u_y}{y\sqrt{1 + y^2||\nabla u||^2}} = \frac{nH}{y^2}. \tag{1.1}$$

Here, $div$, $\nabla$, and $||.||$ denote quantities in the Euclidean metric of $\mathbb{R}^n$. For a proof see, for instance, [12], Proposition (3.1).

We notice that in $\mathbb{H}^n \times \mathbb{R}$, we can also define the *horizontal graph*, $y = g(x_1, \ldots, x_{n-1}, t)$, of a real and positive function $g$ (see [11]).

We search for constant mean curvature (CMC) hypersurfaces that are invariant by parabolic screw motions, that is, a parabolic translation in $\mathbb{H}^n$ followed by a vertical translation. We recall that a parabolic translation can be identified with a horizontal Euclidean translation in this model for $\mathbb{H}^n$. We begin with a *generating curve* $t = \lambda(y)$ in the $yt$-vertical 2-plane of $\mathbb{H}^n \times \mathbb{R}$. The induced metric in this vertical 2-plane is Euclidean. We fix $(l_1, \ldots, l_{n-1}) \in \mathbb{R}^{n-1}$. After a parabolic translation composed with a vertical translation gives rise to a vertical graph

$$t = u(x_1, \ldots, x_{n-1}, y) = \lambda(y) + l_1 x_1 + \cdots + l_{n-1} x_{n-1}. \tag{1.2}$$

Imposing the mean curvature Eq. (1.1) to this graph, we obtain that $\lambda$ and $(l_1, \ldots, l_{n-1})$ satisfy

$$\left( \frac{\lambda'(y)}{\sqrt{1 + y^2 l^2 + y^2 (\lambda')^2(y)}} \right)' + \frac{(2-n)\lambda'(y)}{y\sqrt{1 + y^2 l^2 + y^2 (\lambda')^2(y)}} = \frac{nH}{y^2} \tag{1.3}$$

where $l^2 = l_1^2 + \cdots + l_n^2$. Equivalently, we may write

$$y^{2-n} \left( \frac{\lambda'(y)}{\sqrt{1 + y^2 l^2 + y^2 (\lambda')^2(y)}} \right)' + \frac{(2-n)y^{1-n}\lambda'(y)}{\sqrt{1 + y^2 l^2 + y^2 (\lambda')^2(y)}} = nH y^{-n}.$$

If we suppose that $H$ is constant, integrate and conveniently rearrange the terms, we obtain

$$\frac{\lambda'(y)}{\sqrt{1 + y^2 l^2 + y^2 (\lambda')^2(y)}} = \frac{d(n-1)y^{n-1} - nH}{y(n-1)}, \tag{1.4}$$

where $d$ is the constant that comes from integrating. After another rearrangement and integration, we get

$$\lambda(y) = \int_*^y \frac{\sqrt{1 + \xi^2 l^2} \left( d(n-1)\xi^{n-1} - nH \right)}{\xi(n-1)\sqrt{1 - \left( \frac{d(n-1)\xi^{n-1} - nH}{n-1} \right)^2}} d\xi. \tag{1.5}$$

Since $\lambda$ depends also on $d$ and $l$, we write $\lambda(y, d, l)$ instead of $\lambda(y)$ whenever convenient.

## 2 Some general facts

The function $\lambda(y, d, l)$ is given by (1.5) and we can suppose that $d \geq 0$. The case $d < 0$ is obtained from this by vertical translations or symmetries. In order to simplify notation, we set $g = \left( \frac{d(n-1)y^{n-1} - nH}{n-1} \right)$ and write

$$\lambda(y) = \int_*^y \frac{\sqrt{1 + \xi^2 l^2} g}{\xi\sqrt{1 - g^2}} d\xi. \tag{2.1}$$

Then

$$\lambda'(y) = \frac{\sqrt{1 + y^2 l^2} g}{y\sqrt{1 - g^2}}, \tag{2.2}$$

where we are assuming that $1 - g^2 > 0$. We notice that the sign of $\lambda'(y)$ is that of $g$ and that, when $d \neq 0$, both vanish at $y = \tau_0^{1/n-1}$, where $\tau_0 = \frac{nH}{d(n-1)}$.

A computation shows that

$$\lambda''(y) = \frac{(1 + y^2 l^2)g'y - g(1 - g^2)}{(1 + y^2 l^2)^{1/2}y^2(1 - g^2)^{3/2}}$$

which helps studying the behavior of the curve $t = \lambda(y, d, l)$. For notation purposes, we set $\eta(y) = (1 + y^2 l^2)g'y - g(1 - g^2)$, i.e., the numerator of $\lambda''(y)$. We point out that the sign of $\eta(y)$ is that of $\lambda''(y)$. The following lemma will be useful.

**Lemma 2.1** *If $H \geq 0$ then $\lambda'' > 0$.*

*Proof* We need to study the sign of $\eta(y)$. Since $g' = d(n - 1)y^{n-2} > 0$, we easily see that $\eta(y) > 0$ when $g \leq 0$. Now we see what happens when $g > 0$. Since, in this case

$$\eta(y) = (1 + y^2 l^2)g'y - g(1 - g^2) > (1 + y^2 l^2)g'y - g$$
$$= (1 + y^2 l^2)d(n - 1)y^{n-1} - dy^{n-1} + \frac{nH}{n-1} \geq (n - 2)dy^{n-1} + \frac{nH}{n-1},$$

the lemma follows. □

We now analyze the behavior of $\lambda(y)$ near $y = 0$. If we write

$$\lambda(y) = \int\limits_*^y \frac{\sqrt{1 + \xi^2 l^2}\,g}{\xi\sqrt{1 - g^2}}\,d\xi = \int\limits_*^y \frac{f(\xi)}{\xi}\,d\xi,$$

we can see that $f(y)$ is differentiable in $y = 0$. By observing the expansion

$$f(y) = -\frac{-nH}{(n - 1)\sqrt{1 - \left(\frac{nH}{n-1}\right)^2}} + f'(0)y + o(y)$$

near 0, we can conclude that $\lambda(y)$ has a log behavior when $y \to 0$. Now, we study the behavior of $\lambda(y)$ near a zero of $\sqrt{1 - g^2}$. We notice that these zeroes only exist for $d > 0$. For simplicity purposes, we introduce the variable $v = d\xi^{n-1}$, $d > 0$. In this new variable, the integral given by (1.5) becomes

$$\int\limits_*^{dy^{n-1}} \frac{\sqrt{1 + l^2 \left(\frac{v}{d}\right)^{2/(n-1)}}\,((n - 1)v - nH)}{v(n - 1)^2\sqrt{1 - \left(v - \frac{nH}{n-1}\right)^2}}\,dv. \tag{2.3}$$

Near one of the zeroes of $\sqrt{1 - \left(v - \frac{nH}{n-1}\right)^2}$, say $v_0 = (1 + \frac{nH}{n-1})$, we can rewrite the integral as

$$\int\limits_*^{dy^{n-1}} \frac{h(v)}{\sqrt{v_0 - v}}\,dv,$$

with a function $h$ that is differentiable at $v_0$. We can then see that this integral, and *a fortiori* the integral in (1.5), converges to a finite value near this zero of $\sqrt{1 - g^2}$. The same happens for the other zero. Hence, the graph of $\lambda(y, d, l)$ is vertical at both zeroes of $\sqrt{1 - g^2}$. A computation shows that the (Euclidean) curvature of the curve in the $yt$-vertical 2-plane is finite at these points.

**Proposition 2.2** *Any H-CMC or minimal horizontal graph invariant by parabolic transla-tions ($l = 0$) is stable.*

*Proof* Let us consider the killing field (for a definition, see for instance [6], p. 52) generated by the homotheties in each slice $\mathbb{H}^n \times \{t\}$ w.r.t. the origin of the slice (a hyperbolic isometry of the slice). Notice that the generating curve is a horizontal smooth graph in the $yt$-vertical 2-plane ($x_1 = x_2 = \cdots = x_{n-1} \neq 0$). Then, the trajectories of the killing field cut the graph transversally at one point at most, proving that it is a killing graph and, a fortiori, proving the proposition, since killing graphs are stable (see [3], Theorem 2.7). □

## 3 The minimal examples

In this section, we consider the particular case $H = 0$. We find some explicit examples given by elementary formulas or integral formulas. Preserving the notation and the method stated in the preliminaries, we set our first result.

**Proposition 3.1** (Minimal vertical graphs)
*The minimal vertical graphs invariant by parabolic screw motions generated by a curve $t = \lambda(y)$ are, up to translations or symmetries, of one of the following types:*

(a) $t = l_1 x_1 + \cdots + l_{n-1} x_{n-1}, \quad y > 0.$

(b) $t = \lambda(y, d, l) + l_1 x_1 + \cdots + l_{n-1} x_{n-1}, \quad y \in (0, (1/d)^{\frac{1}{n-1}})$, *where $\lambda$ is increasing and strictly convex in the interval and is vertical at $y = \left(\frac{1}{d}\right)^{\frac{1}{n-1}}$ (see Fig. 1). For the particular case $l = 0$, we obtain the graph given by the elementary formula*

$$t = \frac{1}{n-1} \arcsin(dy^{n-1}) + \kappa.$$

**Fig. 1** $d = 1, n = 3, H = 0,$
$l = 0$

d=1, n= 3, H=0, l=0
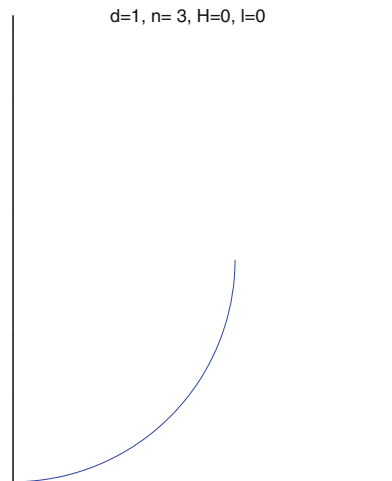
*Proof* The case $H = 0$ and $d = 0$:

If we impose in (1.4) that $H$ and $d$ vanish, we obtain that $\lambda \equiv \kappa = $ constant. In this case, the vertical graphs obtained are

$$t = \kappa + l_1 x_1 + \cdots, l_{n-1} x_{n-1}, \quad y > 0.$$

The case $H = 0$ and $d \neq 0$:

In this case, (1.5) gives, up to vertical translations or symmetries,

$$\lambda(y, d, l) = \int\limits_0^y \frac{d\xi^{n-2}\sqrt{1 + \xi^2 l^2}}{\sqrt{1 - (d\xi^{n-1})^2}} d\xi,$$

with $d > 0$. Changing for the variable $v = d\xi^{n-1}$, we obtain

$$\lambda(y, d, l) = \frac{1}{n-1} \int\limits_0^{dy^{n-1}} \frac{\sqrt{1 + \left(l(\frac{v}{d})^{\frac{1}{n-1}}\right)^2}}{\sqrt{1 - v^2}} dv. \tag{3.1}$$

We then have

$$t = \lambda(y, d, l) + l_1 x_1 + \cdots + l_{n-1} x_{n-1}, \quad y \in (0, (1/d)^{\frac{1}{n-1}}),$$

with $\lambda$ given by (3.1). By the results of Sect. 2 applied to this case, we conclude that it is increasing and strictly convex in the interval $(0, (1/d)^{\frac{1}{n-1}})$ and is vertical at $y = (\frac{1}{d})^{\frac{1}{n-1}}$.

For the special case $l = 0$, we can integrate the above formula and obtain an explicit family of examples, namely, the graphs of

$$t = \frac{1}{n-1} \arcsin(dy^{n-1}) + \kappa.$$

$\square$

**Theorem 3.2** *The curves of Proposition* 3.1 *(b) generate a family of non-entire horizontal minimal complete graphs in* $\mathbb{H}^n \times \mathbb{R}$ *invariant by parabolic screw motions. The asymptotic boundary of each graph of this family is formed by two parallel* $(n - 1)$*-planes (see Fig.* 2*).*

*Proof* We know that the function $\lambda(y, d, l)$ given by b) of Proposition (3.1) is increasing and convex in the interval $(0, (1/d)^{\frac{1}{n-1}})$ and is vertical at $y = (\frac{1}{d})^{\frac{1}{n-1}}$. We also know that the Euclidean curvature is finite at $y = (1/d)^{\frac{1}{n-1}}$. If we set $t_0 = \lambda((1/d)^{\frac{1}{n-1}}, d, l)$, we can then glue together the graph

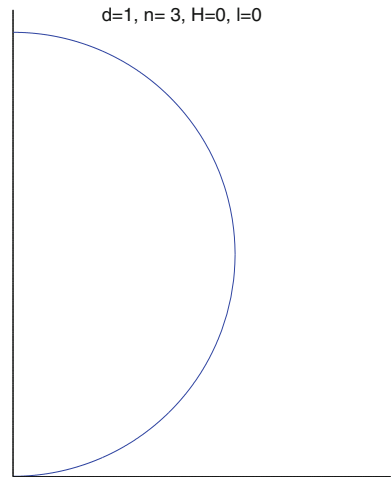$$t = \lambda(y, d, l) + l_1 x_1 + \cdots + l_{n-1} x_{n-1}$$

with its Schwarz reflection given by

$$2t_0 - \lambda(y, d, l) + l_1 x_1 + \cdots + l_{n-1} x_{n-1},$$

in order to obtain a horizontal minimal complete graph invariant by parabolic screw motions defined over $\{(x_1, \ldots, x_{n-1}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} | l_1 x_1 + \cdots + l_{n-1} x_{n-1} \leq t \leq 2t_0 + l_1 x_1 + \cdots + l_{n-1} x_{n-1}\}$. The asymptotic boundary of this example is formed by two parallel hyperplanes.

$\square$

**Fig. 2** $d = 1, n = 3,$
$H = 0, l = 0$



d=1, n= 3, H=0, l=0

## 4 The H-CMC examples

In this section, we treat the case $H \neq 0$. We recall that we are using the orientation given by the upper unit normal. We first deal with the hypothesis $d = 0$.

The case $H \neq 0$ and $d = 0$:

Imposing that $d = 0$, (1.5) becomes

$$\lambda(y, l) = \frac{-nH}{(n-1)\sqrt{1 - \left(\frac{nH}{n-1}\right)^2}} \int_*^y \frac{\sqrt{1 + \xi^2 l^2}}{\xi} \mathrm{d}\xi \tag{4.1}$$

and naturally imposes the restriction $|H| < \frac{n-1}{n}$.

Integrating (4.1), we obtain

$$t = \frac{-nH}{(n-1)\sqrt{1 - \left(\frac{nH}{n-1}\right)^2}} \left[ \sqrt{1 + l^2 y^2} + \ln\left(\frac{\sqrt{1 + l^2 y^2} - 1}{\sqrt{1 + l^2 y^2} + 1}\right)^{1/2} \right] + \kappa, \quad l \neq 0,$$

which give a explicit family of entire vertical graphs of mean curvature $H$ ($|H| < \frac{n-1}{n}$), defined over $\mathbb{H}^n \times \mathbb{R}$, that are also complete horizontal graphs (see Fig. 3).

For the case $l = 0$, we obtain the explicit examples (see Fig. 4)

$$t = \frac{-nH}{(n-1)\sqrt{1 - \left(\frac{nH}{n-1}\right)^2}} \ln(y) + \kappa.$$

The case $H \neq 0$ and $d > 0$:

We can distinguish four cases, depending on the values of $H$, as one can see below. For each case, we come back to Sect. 2 and try to understand the behavior of the curve $t = \lambda(y, d, l)$. By observing that both $y$ and $1 - g^2$ should be positive in (2.1), we see that $y$, besides being positive, should be in the interval $(y_0, y_1)$, where $y_0 = (\tau_0 - \frac{1}{d})^{1/n-1}$ and $y_1 = (\tau_0 + \frac{1}{d})^{1/n-1}$.

Now we deal with each case separately.

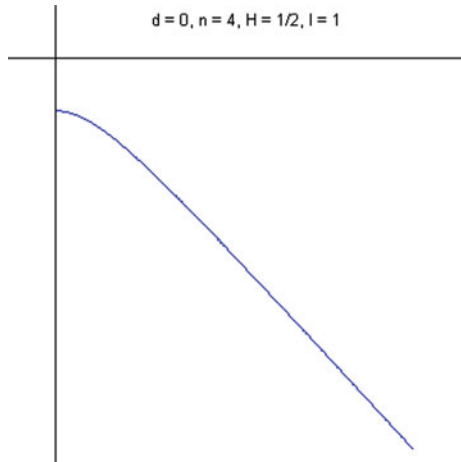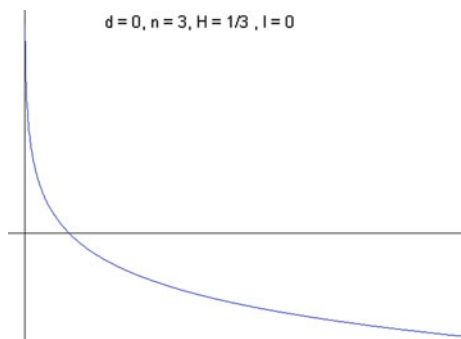**Fig. 3** $d = 0, n = 4$,
$H = 1/2, l = 1$



d = 0, n = 4, H = 1/2, l = 1

**Fig. 4** $d = 0, n = 3$,
$H = 1/3, l = 0$



d = 0, n = 3, H = 1/3 , l = 0

- $H \leq -\frac{(n-1)}{n}$: There is no solution, since $\tau_0 + \frac{1}{d} \leq 0$.
- $-\frac{(n-1)}{n} < H < 0$: Since, for this case, $\tau_0 - \frac{1}{d} < 0$ we see that $\lambda(y, d, l)$ is defined on $(0, y_1)$. We also know that $\lambda$ is increasing (see (2.2)), is vertical at $y_1$, and has a log behavior near 0.

  Now, we should understand the sign of $\lambda''(y)$, i.e., the sign of $\eta(y)$. It is easy to see that $\eta(y) < 0$ near 0 and $\eta(y) > 0$ near $y_1$. A computation shows that $\eta'(y) = 2l^2 g' y^2 + (1 + y^2 l^2)g'' y + 3g^2 g'$. Since $g > 0$, $g' > 0$ and $g'' \geq 0$, we see that $\eta'(y) > 0$, which implies that $\eta(y) = 0$ only once in $(0, y_1)$. Then, $\lambda(y, d, l)$ is concave near 0 and becomes convex after some point in $(0, y_1)$ (see Fig. 5).
- $0 < H \leq \frac{(n-1)}{n}$: In this case, $\lambda(y, d, l)$ is defined on $(0, y_1)$, is decreasing on $\left(0, \tau_0^{1/n-1}\right)$ and increasing on $\left(\tau_0^{1/n-1}, y_1\right)$ (see (2.2)). By Lemma 2.1, it is strictly convex and by the results of Sect. 2, $\lambda$ is vertical at $y_1$, and has a log behavior at $y = 0$ (see Fig. 6).
- $H > \frac{(n-1)}{n}$: In this case, $\lambda(y, d, l)$ is defined on $(y_0, y_1)$, is decreasing on $\left(y_0, \tau_0^{1/n-1}\right)$ and increasing on $\left(\tau_0^{1/n-1}, y_1\right)$ (see (2.2)). By Lemma 2.1, $\lambda$ is strictly convex in the interval and by the results of Sect. 2 and is vertical at $y_0$ and $y_1$. We claim that $\lambda(y_0) > \lambda(y_1)$ (see Fig. 7). In order to prove the claim, we proceed as follows. Instead of working with (1.5), we use (2.3), where the integrand is defined in an interval $(v_0, v_1)$. Let us set $\sigma_0$ for

**Fig. 5** $d = 1, n = 4, H = -1/4, l = 1$

d=1, n= 4, H=−1/4, l=1



**Fig. 6** $d = 1, n = 4, H = 1/4, l = 1$
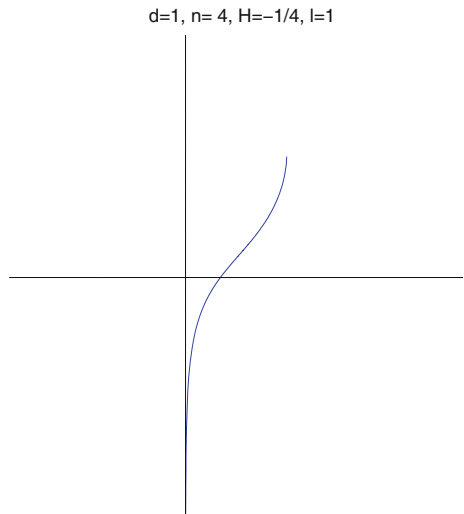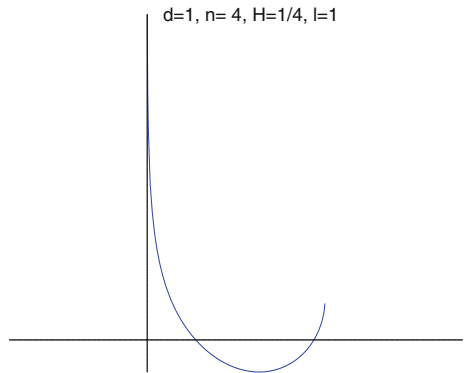
d=1, n= 4, H=1/4, l=1



the middle point of this interval. We first notice the function

$$\frac{\sqrt{1 + l^2 \left(\frac{v}{d}\right)^{2/(n-1)}}}{v}$$

has negative derivative, being therefore decreasing. Now, we explore the symmetries of the functions $((n-1)v - nH)$ and $\sqrt{1 - \left(v - \frac{nH}{n-1}\right)^2}$ to conclude that the integrand is negative in $(v_0, \sigma_0)$ and positive in $(\sigma_0, v_1)$, but with a greater modulus for the negative part. After integration, we see that the claim is true.

We now collect the results we obtained for $H \neq 0$ in the following propositions and theorems.

**Proposition 4.1** (H-CMC vertical graphs, $0 < |H| \leq \frac{n-1}{n}$)
*The H-CMC vertical graphs invariant by parabolic screw motions generated by a curve $t = \lambda(y)$ are, up to translations or symmetries, of one of the following types:*

**Fig. 7** $d = 1, n = 3, H = 1,$
$l = 1$

d=1, n= 3, H=1, l=1
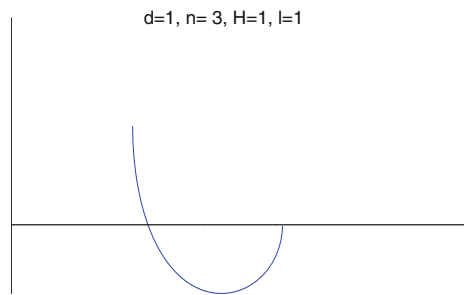
(a) $t = \dfrac{-nH}{(n-1)\sqrt{1-\left(\frac{nH}{n-1}\right)^2}}\left[\sqrt{1+l^2y^2} + \ln\left(\dfrac{\sqrt{1+l^2y^2}-1}{\sqrt{1+l^2y^2}+1}\right)^{1/2}\right] + l_1x_1 + \cdots + l_{n-1}x_{n-1},$

$l \neq 0, \quad y > 0, \quad |H| < \frac{n-1}{n}$ (see Fig. 3).

(b) $t = \dfrac{-nH}{(n-1)\sqrt{1-\left(\frac{nH}{n-1}\right)^2}}\ln(y), \quad y > 0, \quad |H| < \frac{n-1}{n}$ (see Fig. 4).

(c) $t = \lambda(y,d,l) + l_1x_1 + \cdots + l_{n-1}x_{n-1}, for -\frac{(n-1)}{n} < H < 0.$ In this case, $\lambda(y,d,l)$ is defined on $(0, y_1)$, is increasing, is vertical at $y_1$, and has a log behavior near 0. Also, $\lambda(y,d,l)$ is concave near 0 and becomes convex after some point in $(0, y_1)$ (see Fig. 5).

(d) $t = \lambda(y,d,l) + l_1x_1 + \cdots + l_{n-1}x_{n-1}, for 0 < H \leq \frac{(n-1)}{n}.$ In this case, $\lambda(y,d,l)$ is defined on $(0, y_1)$ is decreasing on $\left(0, \tau_0^{1/n-1}\right)$ and increasing on $\left(\tau_0^{1/n-1}, y_1\right)$. It is strictly convex, is vertical at $y_1$, and has a log behavior at $y = 0$ (see Fig. 6).

**Proposition 4.2** (H-CMC vertical graphs, $H > \frac{n-1}{n}$)
*The H-CMC vertical graphs invariant by parabolic screw motions generated by a curve $t = \lambda(y)$ are, up to translations or symmetries, of the form $t = \lambda(y,d,l) + l_1x_1 + \cdots + l_{n-1}x_{n-1}$, where $\lambda(y,d,l)$ is defined on $(y_0, y_1)$, is decreasing on $\left(y_0, \tau_0^{1/n-1}\right)$ and increasing on $\left(\tau_0^{1/n-1}, y_1\right)$. $\lambda$ is strictly convex in the interval and is vertical at $y_0$ and $y_1$, with $\lambda(y_0) > \lambda(y_1)$(see Fig. 7).*

**Theorem 4.3** (Complete H-CMC hypersurfaces, $0 < |H| \leq \frac{n-1}{n}$)

(a) *For each H, $0 < |H| < \frac{n-1}{n}$, the curves of Proposition 4.1 (a) generate a family of complete entire vertical graphs of mean curvature H in $\mathbb{H}^n \times \mathbb{R}$ that are also complete stable horizontal graphs (see Fig. 3).*

(b) *For each H, $0 < |H| < \frac{n-1}{n}$, the curves of Proposition 4.1 (b) generate a family of complete entire vertical graphs of mean curvature H in $\mathbb{H}^n \times \mathbb{R}$ that are also entire stable horizontal graph (see Fig. 4).*

(c) *For each H, $-\frac{(n-1)}{n} < H < 0$, the curves of Proposition 4.1 (c) generate a family of entire horizontal H-CMC graphs in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motions (see Fig. 8).*

(d) *For each H, $0 < H \leq \frac{(n-1)}{n}$, the curves of Proposition 4.1 (d) generate a family of complete non-embedded H-CMC hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motions (see Fig. 9).*

*Proof* The proofs of (a) and (b) are trivial. For the other two cases we can, similar to what we have done in Theorem 3.2, glue the graphs given in (c) and (d) of Proposition 4.1 with

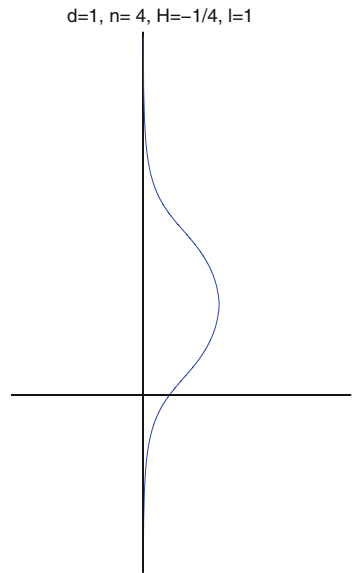**Fig. 8** $d = 1, n = 4, H = -1/4, l = 1$ and its Schwarz reflection

d=1, n= 4, H=−1/4, l=1

**Fig. 9** $d = 1, n = 4, H = 1/4, l = 1$ and its Schwarz reflection
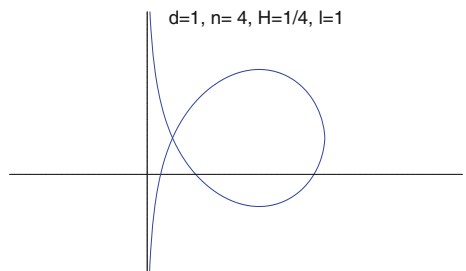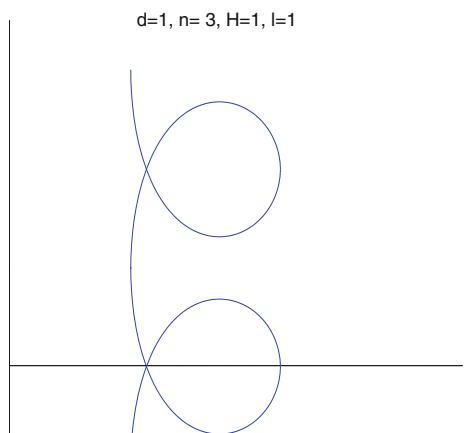
d=1, n= 4, H=1/4, l=1

**Fig. 10** $d = 1, n = 3, H = 1, l = 1$, with Schwarz reflection and translation

d=1, n= 3, H=1, l=1

their Schwarz reflection. Stability comes from the well-known fact that vertical H-CMC graphs are stable. Notice that each parabolic translation H-CMC horizontal graph is stable (Proposition 2.2). □

**Theorem 4.4** (Complete H-CMC hypersurfaces, $H > \frac{n-1}{n}$) *For each* $H$, $H > \frac{n-1}{n}$, *the curves of Proposition* (4.2) *generate a family of complete non-embedded and t-periodic H-CMC hypersurfaces in* $\mathbb{H}^n \times \mathbb{R}$ *invariant by parabolic screw motions (see Fig.* 10).

*Proof* Similar to what we have done in Theorem 3.2, we can first take the curve given by Proposition (4.2) and glue the graph with its Schwarz reflection. Then, we conveniently translate upwards and downwards the obtained curve successively in order to obtain, after gluing, a complete hypersurface. We recall that the curve obtained in Proposition (4.2) is vertical at $y_0$ and $y_1$. □

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