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Bifurcation analysis of an autonomous epidemic predator–prey model with delay

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Abstract In this paper, a class of an autonomous epidemic predator–prey model with delay is considered. Its linear stability and Hopf bifurcation are investigated. Applying the normal form theory and center manifold theory, the explicit formulas for determining the stability and the direction of the Hopf bifurcation periodic solutions are derived. Some numerical simulations for justifying the theoretical analysis are also provided. Finally, main conclusions are included.

Keywords Predator–prey model · Time delay · Stability · Hopf bifurcation · Periodic solution

Mathematics Subject Classification (2000) 34K20 · 34C25

1 Introduction

After the seminal work of Volterra and Lotka in the mid-1920s, the dynamics properties (including stable, unstable, persistent, and oscillatory behavior) of the predator–prey models that have significant biological background have been one of the most active areas of research

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and have attracted great attention of many researchers. Many excellent and interesting results have been obtained [4–6,10,12–14,18].

In 2009, Tian et al. [17] investigated the periodic and almost periodic solution of the following non-autonomous epidemic predator–prey system with time delay:

$$\begin{cases} \dot{X}(t) = X(t)[r_1(t) - a(t)X(t) - b_1(t)S(t) - b_2(t)I(t)], \\ \dot{S}(t) = c(t)X(t-\tau)S(t-\tau) + S(t)[-r_2(t) - d_1(t)(S(t)+I(t)) - e(t)I(t)], \\ \dot{I}(t) = I(t)[e(t)S(t) - d_2(t)(S(t)+I(t))], \end{cases} \tag{1.1}$$

where $X(t)$ denotes the density of the prey, $S(t)$ and $I(t)$ denote the density of the susceptible predator and the infected predator, respectively; $r_i(t)(i = 1, 2)$ denotes the intrinsic rate of natural increase, and the minus before $r_2(t)$ means that the susceptible predator is dependent on the prey, that is, if there is no prey, then the predator will be extinct. $a(t)$ means coefficient of the density dependence, $d_i(t)(i = 1, 2)$ means the competitive coefficient between the predator, $b_i(t)(i = 1, 2)$ means the preying capacity for the susceptible and the infected predator, $c(t)$ means the relative preying capacity of the susceptible predator, $e(t)$ means the touching rate between the susceptible predator and the infected predator, while $\tau > 0$ is the time required for the gestation of the susceptible predator. $a(t), b_i(t), c(t), d_i(t), e(t), r_i(t)(i = 1, 2)$ are continuous and strictly positive functions.

It is well known that the research on the Hopf bifurcation, especially on the stability of bifurcating periodic solutions and direction of Hopf bifurcation, is one of the most important theme on the population dynamics. To obtain a deep and clear understanding of dynamics of predator–prey system with time delay, In the present paper, we go no to investigate the model (1.1) under the following assumptions: the coefficients are independent of the time of t, that is, $r_i(t) = r_i, b_i(t) = b_i, d_i(t) = d_i(i = 1, 2), a(t) = a, c(t) = c, e(t) = e$, and $r_i, b_i, d_i, a, c, e(i = 1, 2)$ are all positive constants. Further, considering the biological meaning of model (1.1), we think that the coefficient $e(t)$ in the second equation and the coefficient $e(t)$ in the third equation of model (1.1) shall be different. So we denote the coefficient $e(t)$ in the second equation and the coefficient $e(t)$ in the third equation of model (1.1) by e_1 and e_2, respectively. Then, system (1.1) becomes the following autonomous predator–prey system:

$$\begin{cases} \dot{X}(t) = X(t)[r_1 - aX(t) - b_1S(t) - b_2I(t)], \\ \dot{S}(t) = cX(t-\tau)S(t-\tau) + S(t)[-r_2 - d_1(S(t)+I(t)) - e_1I(t)], \\ \dot{I}(t) = I(t)[e_2S(t) - d_2(S(t)+I(t))], \end{cases} \tag{1.2}$$

The more detail biological meaning of the coefficients of the system (1.2) is same as that in [17].

In this paper, we study the stability, the local Hopf bifurcation for system (1.2). We would like to mention that there are a lot of papers on the Hopf bifurcation of predator–prey models [1,3,9,16,19–24]. To the best of our knowledge, it is the first time to deal with the research of Hopf bifurcation for model (1.2). We believe that our results obtained in this paper are a good complement to the earlier publications about model (1.1).

The remainder of the paper is organized as follows. In Sect. 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Sect. 3, the direction and stability of the local Hopf bifurcation are established. In Sect. 4, numerical simulations are carried out to support analytical findings. Some main conclusions are drawn in Sect. 5.

2 Stability of the positive equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations.

It is easy to see that if the following condition:

$$\text{(H1)}\;\; 0 < K < r_1,\; e_1 > d_1,\; e_2 > d_2$$

holds, where

$$K = \frac{r_1[d_1d_2 + (d_1 + e_1)(e_1 - d_1)] + r_2[b_1d_2 + b_2(e_2 - d_2)]}{a[d_1d_2 + (d_1 + e_1)(e_1 - d_1)] + c[b_1d_2 + b_2(e_2 - d_2)]},$$

then system (1.2) has an unique positive equilibrium $E_0(X^*, S^*, I^*)$, where

$$X^* = \frac{r_1[d_1d_2 + (d_1 + e_1)(e_1 - d_1)] + r_2[b_1d_2 + b_2(e_2 - d_2)]}{a[d_1d_2 + (d_1 + e_1)(e_1 - d_1)] + c[b_1d_2 + b_2(e_2 - d_2)]},$$
$$S^* = \frac{d_2(r_1 - aX^*)}{b_1d_2 + b_2(e_2 - d_2)}, \quad I^* = \frac{e_2 - d_2}{d_2} S^*. \tag{2.1}$$

Let $\bar{X}(t) = X(t) - X^*$, $\bar{S}(t) = S(t) - S^*$, $\bar{I}(t) = I(t) - I^*$ and still denote $\bar{X}(t)$, $\bar{S}(t)$, $\bar{I}(t)$ by $X(t)$, $S(t)$, $I(t)$, respectively, then (1.2) becomes

$$\begin{cases} \dot{X}(t) = m_1X(t) + m_2S(t) + m_3I(t) - aX^2(t) - b_1X(t)S(t) - b_2X(t)I(t), \\ \dot{S}(t) = n_1S(t) + n_2I(t) + n_3X(t-\tau) + n_4S(t-\tau) \\ \qquad - d_1S^2(t) - (d_1 + e_1)S(t)I(t) + cX(t-\tau)S(t-\tau), \\ \dot{I}(t) = l_1S(t) + l_2I(t) + (e_2 - d_2)S(t)I(t) - d_2I^2(t), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} m_1 &= r - 2aX^* - b_1S^* - b_2I^*,\; m_2 = -b_1X^*, m_3 = -b_2X^*, \\ n_1 &= -\left[d_1S^* + r_2 + d_2(S^* + I^*) + e_1I^*\right],\; n_2 = -(d_1 + e_1)S^*, \\ n_3 &= cS^*,\; n_4 = cX^*,\; l_1 = e_2 - d_2,\; l_2 = e_2S^* - 2d_2I^* - d_2S^*. \end{aligned}$$

The linearization of Eq. (2.2) at (0, 0, 0) is

$$\begin{cases} \dot{X}(t) = m_1X(t) + m_2S(t) + m_3I(t), \\ \dot{S}(t) = n_1S(t) + n_2I(t) + n_3X(t-\tau) + n_4S(t-\tau), \\ \dot{I}(t) = l_1S(t) + l_2I(t) \end{cases} \tag{2.3}$$

whose characteristic equation is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 - \left(B_1\lambda^2 + B_2\lambda + B_3\right)e^{-\lambda\tau} = 0, \tag{2.4}$$

where

$$\begin{aligned} A_1 &= -(m_1 + l_2 + n_1), \\ A_2 &= n_1(m_1 + l_2) + m_1l_2 - l_1n_2, \\ A_3 &= m_1n_2l_1 - m_1n_1l_2, \\ B_1 &= n_4, \\ B_2 &= m_2n_3 - n_4(m_1 + l_2), \\ B_3 &= m_1l_2n_4 + m_3n_3l_1 - m_2n_3l_2. \end{aligned}$$

Denote

$$P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,\ Q(\lambda) = -(B_1\lambda^2 + B_2\lambda + B_3).$$

Then, (2.4) takes the form

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0. \tag{2.5}$$

In order to investigate the distribution of roots of the transcendental equation (2.5), the following Lemma is useful.

Lemma 2.1 [15] *For the transcendental equation*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ *in the open right half complex plane can change, if and only if a zero appears on or crosses the imaginary axis.*

For $\tau = 0$, (2.5) becomes

$$\lambda^3 + (A_1 - B_1)\lambda^2 + (A_2 - B_2)\lambda + A_3 - B_3 = 0. \tag{2.6}$$

A set of necessary and sufficient conditions for all roots of (2.6) to have a negative real part is given by the well-known Routh–Hurwitz criteria in the following form:

$$\text{(H2)}\ \ (A_1 - B_1)(A_2 - B_2) - (A_3 - B_3) > 0,\ A_3 - B_3 > 0.$$

Assume that $i\omega(\omega > 0)$ is a root of (2.5). Following the line of Beretta and Kuang [2], ω must be the solution of the system of equations:

$$\begin{cases} \sin\omega\tau = \mathrm{Im}\left(\frac{P(i\omega)}{Q(i\omega)}\right), \\ \cos\omega\tau = -Re\left(\frac{P(i\omega)}{Q(i\omega)}\right), \end{cases} \tag{2.7}$$

Namely, ω must be a positive root of the function

$$F(\omega) = |P(i\omega)|^2 - Q(i\omega)|^2. \tag{2.8}$$

We denote the root of $F(\omega) = 0$ by ω_k. Then, the characteristic roots $\lambda = \pm i\omega_k$ occur at the delay values

$$\tau_k^{(j)} = \frac{\theta_k + 2j\pi}{\omega_k}, \quad j = 0, 1, 2, \ldots, \tag{2.9}$$

where $\theta_k \in [0, 2\pi)$ is the solution of

$$\begin{cases} \sin\theta_k = \mathrm{Im}\left(\frac{P(i\omega_k)}{Q(i\omega_k)}\right), \\ \cos\theta_k = -Re\left(\frac{P(i\omega_k)}{Q(i\omega_k)}\right). \end{cases} \tag{2.10}$$

Define

$$\tau_0 = \tau_{k0}^{(0)} = \min_{k\in\{1,2,3\}}\left\{\tau_k^{(0)}\right\}, \quad \omega_0 = \omega_{k0}. \tag{2.11}$$

In view of (2.8), we have

$$\omega^6 + \left(A_1^2 - 2A_2 - B_1^2\right)\omega^4 + \left(A_2^2 - 2A_1A_3 + 2B_1B_3 - B_2^2\right)\omega^2 + A_3^2 - B_3^2 = 0. \quad (2.12)$$

Let $z = \omega^2$, then (2.13) become

$$z^3 + r_1 z^2 + r_2 z + r_3 = 0, \quad (2.13)$$

where

$$\begin{aligned} r_1 &= A_1^2 - 2A_2 - B_1^2, \\ r_2 &= -A_2^2 - 2A_1A_3 + 2B_1B_3 - B_2^2, \\ r_3 &= A_3^2 - B_3^2. \end{aligned}$$

Denote

$$h(z) = z^3 + r_1 z^2 + r_2 z + r_3 = 0, \quad (2.14)$$

where $r = r_2 - \frac{1}{3}r_1^2$, $q = \frac{2}{27}r_1^3 - \frac{1}{3}r_1 r_2 + r_3$. Then,

$$h^{'}(z) = 3z^2 + 2r_1 z + r_2. \quad (2.15)$$

According to Beretta and Kuang [2], we derive

$$\text{sign}\left[\frac{dRe\lambda}{d\tau}\Big|_{\tau=\tau_k^{(j)}}\right] = \text{sign}\left[\frac{dF(\omega)}{d\omega}\Big|_{\omega=\omega_k}\right] = \text{sign}[h^{'}(z_k)], \quad (2.16)$$

where $\tau_k^{(j)}$ are the delay values (2.9) at which the characteristic roots $\lambda = \pm i\omega_k$ $\left(\omega_k = \sqrt{z_k} > 0, k = 1, 2, 3\right)$ occur. We assume that

$$\text{(H3)} \quad h^{'}(z_k) \neq 0.$$

Without loss of generality, we assume that (2.14) has three distinct positive roots, say z_1, z_2, z_3 such that

$$\omega_1 = \sqrt{z_1} < \omega_2 = \sqrt{z_2} < \omega_3 = \sqrt{z_3}.$$

Since $\lim_{z\to+\infty} h(z) = +\infty$, the only possibility for the signs of $h^{'}(z)$ at the roots z_k is that $h^{'}(z_1) > 0$, $h^{'}(z_2) < 0$, $h^{'}(z_3) > 0$. In the following, we assume that the θ_k values are such that

$$\tau_3^{(0)} = \frac{\theta_3}{\omega_3} < \tau_2^{(0)} = \frac{\theta_2}{\omega_2} < \tau_1^{(0)} = \frac{\theta_1}{\omega_1} \quad \text{and} \quad \tau_1^{(0)} < \tau_3^{(1)}.$$

Then under the assumption (H3), it is easy to know that all the characteristic roots λ of (2.5) have $Re\lambda < 0$ in the delay interval $[0, \tau_0 = \tau_3^{(0)})$. Because of $h^{'}(z_3) > 0$, in the interval $(\tau_0 = \tau_3^{(0)}, \tau_2^{(0)})$ there will be two characteristic roots with positive real parts that, thanks to $h^{'}(z_2) < 0$, will cross the imaginary axis toward negative real parts at $\tau_2^{(0)}$, whereas at $\tau_1^{(0)}$, because of $h^{'}(z_1) > 0$, other two characteristic roots cross the imaginary axis assuming positive real parts. Similar analysis to $\tau_3^{(j)}, \tau_2^{(j)}, \tau_1^{(j)}$, we have the following Theorem 2.1 by the results of Kuang [11] and Hale [7].

Theorem 2.1 *If $(H1)$ and $(H2)$ hold, then the equilibrium $E_0(X^*, S^*, I^*)$ of system (1.2) is asymptotically stable for $\tau \in [0, \tau_0)$. In addition to the conditions $(H1)$ and $(H2)$, we further assume that $(H3)$ holds, then the positive equilibrium $E_0(X^*, S^*, I^*)$ is asymptotically stable when*

$$\left[0, \tau_3^{(0)}\right) \cup \left(\tau_2^{(0)}, \tau_1^{(0)}\right)$$

and unstable when

$$\left[\tau_3^{(0)}, \tau_2^{(0)}\right) \cup \left(\tau_1^{(0)}, +\infty\right).$$

The stability switches occur at $\tau_3^{(0)}, \tau_2^{(0)}, \tau_1^{(0)}$ and system (1.2) undergoes a Hopf bifurcation at the positive equilibrium $E_0(X^*, S^*, I^*)$ when $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, ldots$.

3 Direction and stability of the Hopf bifurcation

In the previous section, we have obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \ldots$. In this section, we shall derive the explicit formulae for determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E_0(X^*, S^*, I^*)$ at these critical value of τ, by using techniques from normal form and center manifold theory [8]. Throughout this section, we always assume that system (1.2) undergoes Hopf bifurcation at the positive equilibrium $E_0(X^*, S^*, I^*)$ for $\tau = \tau_k^{(j)}, k = 1, 2, 3; j = 0, 1, 2, \ldots,$ and then $\pm i\omega_0$ are corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E_0(X^*, S^*, I^*)$.

For convenience, let $x_1(t) = X(\tau t), x_2(t) = S(\tau t), x_3(t) = I(\tau t)$, and $\tau = \tau_k^{(j)} + \mu$, where $\tau_k^{(j)}$ is defined by (2.10) and $\mu \in R$, then system (2.2) can be written as an FDE in $C = C([-1, 0]), R^3)$ as

$$\dot{x}(t) = L_\mu x_t + f(\mu, x_t), \tag{3.1}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in C$, and $x_t(\theta) = x(t+\theta) = (x_1(t+\theta), x_2(t+\theta), x_3(t+\theta))^T \in C$, and $L_\mu : C \to R$, $f : R \times C \to R$ are given by

$$\begin{aligned} L_\mu \phi = {} & \left(\tau_k^{(j)} + \mu\right) \begin{pmatrix} m_1 & m_2 & m_3 \\ 0 & n_1 & n_2 \\ 0 & l_1 & l_2 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ & + \left(\tau_k^{(j)} + \mu\right) \begin{pmatrix} 0 & 0 & 0 \\ n_3 & n_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \end{aligned} \tag{3.2}$$

and

$$f(\mu, \phi) = \left(\tau_k^{(j)} + \mu\right) \begin{pmatrix} -a\phi_1^2(0) - b_1\phi_1(0)\phi_2(0) - b_2\phi_1(0)\phi_3(0) \\ -d_1\phi_2^2(0) - (d_1 + e_1)\phi_2(0)\phi_3(0) + c\phi_1(-1)\phi_2(-1) \\ (e_2 - d_2)\phi_2(0)\phi_3(0) - d_2\phi_3^2(0) \end{pmatrix}, \tag{3.3}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$.

From the discussion in Sect. 2, we know that if $\mu = 0$, then system (3.1) undergoes a Hopf bifurcation at the positive equilibrium $E_0(X^*, S^*, I^*)$ and the associated characteristic equation of system (3.1) has a pair of simple imaginary roots $\pm\omega_0\tau_k^{(j)}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu), \theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^{0} \mathrm{d}\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C. \tag{3.4}$$

In fact, we can choose

$$\begin{aligned}\eta(\theta, \mu) = \left(\tau_k^{(j)} + \mu\right) &\begin{pmatrix} m_1 & m_2 & m_3 \\ 0 & n_1 & n_2 \\ 0 & l_1 & l_2 \end{pmatrix} \delta(\theta) \\ &+ \left(\tau_k^{(j)} + \mu\right) \begin{pmatrix} 0 & 0 & 0 \\ n_3 & n_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1),\end{aligned} \tag{3.5}$$

where δ is the Dirac delta function.

For $\phi \in C([-1, 0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} \mathrm{d}\eta(s, \mu)\phi(s), & \theta = 0 \end{cases} \tag{3.6}$$

and

$$R\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu, \phi), & \theta = 0. \end{cases} \tag{3.7}$$

Then (3.1) is equivalent to the abstract differential equation

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{3.8}$$

where $x_t(\theta) = x(t + \theta), \theta \in [-1, 0]$.

For $\psi \in C([0, 1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} \mathrm{d}\eta^T(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1, 0], R^3)$ and $\psi \in C([0, 1], (R^3)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \psi^T(\xi - \theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in the Sect. 2, we know that $\pm i\omega_0\tau_k^{(j)}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_k^{(j)}$ and $-i\omega_0\tau_k^{(j)}$ respectively. By direct computation, we can obtain

$$q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0\tau_k^{(j)}\theta}, \quad q^*(s) = M(1, \alpha^*, \beta^*)e^{i\omega_0\tau_k^{(j)}s}, \quad M = \frac{1}{B},$$

where

$$\begin{aligned}
\alpha &= \frac{\beta(i\omega_0 - l_2)}{l_1},\\
\beta &= \frac{(i\omega_0 - m_1)l_1}{m_2(i\omega_0 - l_2) + m_3 l_1},\\
\alpha^* &= \frac{i\omega_0 + m_1}{n_3 e^{-i\omega_0\tau_k^{(j)}}},\\
\beta^* &= \frac{n_2(i\omega_0 + m_1) - m_3 n_3 e^{-i\omega_0\tau_k^{(j)}}}{n_3(i\omega_0 + l_2)e^{-i\omega_0\tau_k^{(j)}}},\\
B &= 1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + \tau_k^{(j)}\bar{\alpha}(n_3\alpha^* + n_4\beta^*)e^{i\omega_0\tau_k^{(j)}}.
\end{aligned}$$

Furthermore, $< q^*(s), q(\theta) >= 1$ and $< q^*(s), \bar{q}(\theta) >= 0$.

Next, we use the same notations as those in Hassard et al. [8] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Eq. (3.8) when $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t\rangle, \quad W(t,\theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{3.9}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{3.10}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{3.11}$$

where

$$W_{20}(\theta) = \begin{pmatrix} W_{20}^{(1)}(\theta)\\ W_{20}^{(2)}(\theta)\\ W_{20}^{(3)}(\theta)\end{pmatrix}, \quad W_{11}(\theta) = \begin{pmatrix} W_{11}^{(1)}(\theta)\\ W_{11}^{(2)}(\theta)\\ W_{11}^{(3)}(\theta)\end{pmatrix}, \quad W_{20}(\theta) = \begin{pmatrix} W_{02}^{(1)}(\theta)\\ W_{02}^{(2)}(\theta)\\ W_{02}^{(3)}(\theta)\end{pmatrix}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (3.8),

$$\dot{z}(t) = i\omega_0\tau_k^{(j)}z + \bar{q}^*(\theta)f(0, W(z,\bar{z},\theta)) + 2Re\{zq(\theta)\} \stackrel{\text{def}}{=} i\omega_0\tau_k^{(j)}z + \bar{q}^*(\theta)f_0.$$

That is

$$\dot{z}(t) = i\omega_0\tau_k^{(j)}z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.$$

Hence, we have

$$\begin{aligned}
g(z,\bar z) &= \bar q^*(0) f_0(z,\bar z) = f(0,x_t)\\
&= \bar M\tau_k^{(j)}\Big\{-(a+b_1\alpha+b_2\beta)+\bar\alpha^*\Big[d_1+(d_1+e_1)\alpha\beta+c\alpha e^{-2i\omega_0\tau_k^{(j)}}\Big]\\
&\quad+\bar\beta^*\left[(e_2-d_2)\alpha\beta-d_2\beta^2\right]\Big\}z^2+2\bar M\tau_k^{(j)}\Big\{a+b_1Re\{\alpha\}+b_2Re\{\beta\}\\
&\quad+\bar\alpha^*\left[d_2+(d_1+e_1)Re\{\bar\alpha\beta\}+cRe\{\alpha\}\right]+\bar\beta^*\left[(e_2-d_2)Re\{\bar\alpha\beta\}+d_2|\beta|^2\right]\Big\}z\bar z\\
&\quad+\bar M\tau_k^{(j)}\Big\{a+b_1\bar\alpha+b_2\bar\beta+\bar\alpha^*\Big[d_1+(d_1+e_1)\bar\alpha\bar\beta+c\bar\alpha e^{-2i\omega_0\tau_k^{(j)}}\Big]+\bar\beta^*[(e_2-d_2)\\
&\quad\times\bar\alpha\bar\beta+d_2\bar\beta^2]\Big\}\bar z^2+\bar M\tau_k^{(j)}\Big\{a\left[W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\right]+b_1\Big[\frac{1}{2}\bar\alpha W_{20}^{(1)}(0)\\
&\quad+\frac{1}{2}W_{20}^{(2)}(0)+\alpha W_{11}^{(3)}(0)+W_{11}^{(2)}(0)\Big]+b_2\Big[\frac{1}{2}\bar\beta W_{20}^{(3)}(0)+\frac{1}{2}W_{20}^{(3)}(0)\\
&\quad+\beta W_{11}^{(1)}(0)+W_{11}^{(3)}(0)\Big]+\bar\alpha^*\Big[d_1\left(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\right)+(d_1+e_1)\\
&\quad\left(\frac{1}{2}\bar\beta W_{20}^{(2)}(0)+\frac{1}{2}\bar\alpha W_{20}^{(3)}(0)+\beta W_{11}^{(2)}(0)+\alpha W_{11}^{(3)}(0)\right)\\
&\quad+c\left(\frac{1}{2}\bar\alpha e^{i\omega_0\tau_k^{(j)}}W_{20}^{(1)}(-1)+\frac{1}{2}e^{i\omega_0\tau_k^{(j)}}W_{20}^{(2)}(-1)+\alpha e^{i\omega_0\tau_k^{(j)}}W_{11}^{(1)}(-1)\right.\\
&\quad\left.+e^{-i\omega_0\tau_k^{(j)}}W_{11}^{(2)}(-1)\right)\Big]+\bar\beta^*\Big[(e_2-d_2)\left(\frac{1}{2}\bar\beta W_{20}^{(2)}(0)+\frac{1}{2}\bar\alpha W_{20}^{(3)}(0)\beta W_{11}^{(2)}(0)\right.\\
&\quad\left.+\alpha W_{11}^{(3)}(0)\right)+d_2\left(\bar\beta W_{20}^{(3)}(0)+2\beta W_{11}^{(3)}(0)\right)\Big]\Big\}z^2\bar z+\text{h.o.t.}
\end{aligned}$$

and we obtain

$$\begin{aligned}
g_{20} &= 2\bar M\tau_k^{(j)}\Big\{-(a+b_1\alpha+b_2\beta)+\bar\alpha^*\Big[d_1+(d_1+e_1)\alpha\beta+c\alpha e^{-2i\omega_0\tau_k^{(j)}}\Big]\\
&\quad+\bar\beta^*\left[(e_2-d_2)\alpha\beta-d_2\beta^2\right],\\
g_{11} &= 2\bar M\tau_k^{(j)}\Big\{a+b_1Re\{\alpha\}+b_2Re\{\beta\}+\bar\alpha^*\left[d_2+(d_1+e_1)Re\{\bar\alpha\beta\}+cRe\{\alpha\}\right]\\
&\quad+\bar\beta^*\left[(e_2-d_2)Re\{\bar\alpha\beta\}+d_2|\beta|^2\right]\Big\},\\
g_{02} &= 2\bar M\tau_k^{(j)}\Big\{a+b_1\bar\alpha+b_2\bar\beta+\bar\alpha^*\Big[d_1+(d_1+e_1)\bar\alpha\bar\beta+c\bar\alpha e^{-2i\omega_0\tau_k^{(j)}}\Big]+\bar\beta^*[(e_2-d_2)\\
&\quad\times\bar\alpha\bar\beta+d_2\bar\beta^2]\Big\},\\
g_{21} &= 2\bar M\tau_k^{(j)}\Big\{a\left[W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\right]+b_1\Big[\frac{1}{2}\bar\alpha W_{20}^{(1)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\\
&\quad+\alpha W_{11}^{(3)}(0)+W_{11}^{(2)}(0)\Big]+b_2\Big[\frac{1}{2}\bar\beta W_{20}^{(3)}(0)+\frac{1}{2}W_{20}^{(3)}(0)+\beta W_{11}^{(1)}(0)+W_{11}^{(3)}(0)\Big]
\end{aligned}$$

$$+\bar{\alpha^*}\Bigg[d_1\left(W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\right)+(d_1+e_1)\left(\frac{1}{2}\bar{\beta}W_{20}^{(2)}(0)+\frac{1}{2}\bar{\alpha}W_{20}^{(3)}(0)+\beta W_{11}^{(2)}(0)\right.$$
$$\left.+\alpha W_{11}^{(3)}(0)\right)+c\left(\frac{1}{2}\bar{\alpha}e^{i\omega_0\tau_k^{(j)}}W_{20}^{(1)}(-1)+\frac{1}{2}e^{i\omega_0\tau_k^{(j)}}W_{20}^{(2)}(-1)+\alpha e^{i\omega_0\tau_k^{(j)}}W_{11}^{(1)}(-1)\right.$$
$$\left.\left.+e^{-i\omega_0\tau_k^{(j)}}W_{11}^{(2)}(-1)\right)\Bigg]+\bar{\beta}^*\Bigg[(e_2-d_2)\left(\frac{1}{2}\bar{\beta}W_{20}^{(2)}(0)+\frac{1}{2}\bar{\alpha}W_{20}^{(3)}(0)\beta W_{11}^{(2)}(0)\right.$$
$$\left.+\alpha W_{11}^{(3)}(0)\right)+d_2\left(\bar{\beta}W_{20}^{(3)}(0)+2\beta W_{11}^{(3)}(0)\right)\Bigg]\Bigg\}.$$

For unknown $W_{20}^{(i)}(0),\ W_{11}^{(i)}(0),\ W_{20}^{(j)}(-1),\ W_{11}^{(j)}(-1),\ (i=1,2,3;\ j=1,2)$ in g_{21}, we still need to compute them.

From (3.8) and (3.9), we have

$$W^{'}=\begin{cases}AW-2Re\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1\le\theta<0,\\ AW-2Re\{\bar{q}^*(0)\bar{f}q(\theta)\}+f, & \theta=0\end{cases}\stackrel{\text{def}}{=}AW+H(z,\bar{z},\theta),\quad(3.12)$$

where

$$H(z,\bar{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots.\quad(3.13)$$

Comparing the coefficients, we obtain

$$\left(A-2i\tau_k^{(j)}\omega_0\right)W_{20}=-H_{20}(\theta),\quad(3.14)$$
$$AW_{11}(\theta)=-H_{11}(\theta).\quad(3.15)$$

We know that for $\theta\in[-1,0)$,

$$H(z,\bar{z},\theta)=-\bar{q}^*(0)f_0q(\theta)-q^*(0)\bar{f_0}\bar{q}(\theta)=-g(z,\bar{z})q(\theta)-\bar{g}(z,\bar{z})\bar{q}(\theta).\quad(3.16)$$

Comparing the coefficients of (3.16) with (3.13) gives that

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{02}\bar{q}(\theta).\quad(3.17)$$
$$H_{11}(\theta)=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta).\quad(3.18)$$

From (3.14, 3.17) and the definition of A , we get

$$\dot{W}_{20}(\theta)=2i\omega_0\tau_k^{(j)}W_{20}(\theta)+g_{20}q(\theta)+\bar{g_{02}}\bar{q}(\theta).\quad(3.19)$$

Noting that $q(\theta)=q(0)e^{i\omega_0\tau_k^{(j)}\theta}$, we have

$$W_{20}(\theta)=\frac{ig_{20}}{\omega_0\tau_k^{(j)}}q(0)e^{i\omega_0\tau_k^{(j)}\theta}+\frac{i\bar{g}_{02}}{3\omega_0\tau_k^{(j)}}\bar{q}(0)e^{-i\omega_0\tau_k^{(j)}\theta}+E_1e^{2i\omega_0\tau_k^{(j)}\theta},\quad(3.20)$$

where $E_1=\left(E_1^{(1)},E_1^{(2)},E_1^{(3)}\right)^T\in R^3$ is a constant vector.

Similarly, from (3.15, 3.18) and the definition of A, we have

$$\dot{W}_{11}(\theta)=g_{11}q(\theta)+\bar{g_{11}}\bar{q}(\theta),\quad(3.21)$$
$$W_{11}(\theta)=-\frac{ig_{11}}{\omega_0\tau_k^{(j)}}q(0)e^{i\omega_0\tau_k^{(j)}\theta}+\frac{i\bar{g}_{11}}{\omega_0\tau_k^{(j)}}\bar{q}(0)e^{-i\omega_0\tau_k^{(j)}\theta}+E_2.\quad(3.22)$$

where $E_2=\left(E_2^{(1)},E_2^{(2)},E_2^{(3)}\right)^T\in R^3$ is a constant vector.

In what follows, we shall seek appropriate E_1,E_2 in (3.20, 3.22), respectively. It follows from the definition of A, (3.17) and (3.18) that

$$\int_{-1}^{0} \mathrm{d}\eta(\theta) W_{20}(\theta) = 2i\omega_0 \tau_k^{(j)} W_{20}(0) - H_{20}(0) \tag{3.23}$$

and

$$\int_{-1}^{0} \mathrm{d}\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{3.24}$$

where $\eta(\theta) = \eta(0, \theta)$.
From (3.12), we have

$$H_{20}(0) = -g_{20} q(0) - \bar{g_{02}} \bar{q}(0) + 2\tau_k^{(j)} (H_1, H_2, H_3)^T, \tag{3.25}$$

$$H_{11}(0) = -g_{11} q(0) - \bar{g_{11}}(0) \bar{q}(0) + 2\tau_k^{(j)} (P_1, P_2, P_3)^T, \tag{3.26}$$

where

$$\begin{aligned}
H_1 &= -(a + b_1\alpha + b_2\beta),\\
H_2 &= -\left[d_1 + (d_1 + e_1)\alpha\beta + c\alpha e^{-2i\omega_0\tau_k^{(j)}}\right],\\
H_3 &= (e_2 - d_2)\alpha\beta - d_2\beta^2,\\
P_1 &= a + b_1 Re\{\alpha\} + b_2 Re\{\beta\},\\
P_2 &= d_1 + (d_1 + e_1) Re\{\bar{\alpha}\beta\} + c Re\{\alpha\},\\
P_3 &= (e_2 - d_2) Re\{\bar{\alpha}\beta\} + d_2|\beta|^2.
\end{aligned}$$

Noting that

$$\left(i\omega_0\tau_k^{(j)} I - \int_{-1}^{0} e^{i\omega_0\tau_k^{(j)}\theta} \mathrm{d}\eta(\theta)\right) q(0) = 0,$$

$$\left(-i\omega_0\tau_k^{(j)} I - \int_{-1}^{0} e^{-i\omega_0\tau_k^{(j)}\theta} \mathrm{d}\eta(\theta)\right) \bar{q}(0) = 0$$

and substituting (3.20) and (3.25) into (3.23), we have

$$\left(2i\omega_0\tau_k^{(j)} I - \int_{-1}^{0} e^{2i\omega_0\tau_k^{(j)}\theta} \mathrm{d}\eta(\theta)\right) E_1 = 2\tau_k^{(j)} (H_1, H_2, H_3)^T.$$

That is

$$\begin{pmatrix} 2i\omega_0 - m_1 & -m_2 & -m_3 \\ -n_3 e^{-2i\omega_0\tau_k^{(j)}} & 2i\omega_0 - n_1 - n_4 e^{-2i\omega_0\tau_k^{(j)}} & -n_2 \\ 0 & -l_1 & 2i\omega_0 - l_2 \end{pmatrix} E_1 = 2(H_1, H_2, H_3)^T.$$

It follows that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \quad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \quad E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, \tag{3.27}$$

where

$$\Delta_1 = \det \begin{pmatrix} 2i\omega_0 - m_1 & -m_2 & -m_3 \\ -n_3 e^{-2i\omega_0 \tau_k^{(j)}} & 2i\omega_0 - n_1 - n_4 e^{-2i\omega_0 \tau_k^{(j)}} & -n_2 \\ 0 & -l_1 & 2i\omega_0 - l_2 \end{pmatrix},$$

$$\Delta_{11} = 2\det \begin{pmatrix} H_1 & -m_2 & -m_3 \\ H_2 & 2i\omega_0 - n_1 - n_4 e^{-2i\omega_0 \tau_k^{(j)}} & -n_2 \\ H_3 & -l_1 & 2i\omega_0 - l_2 \end{pmatrix},$$

$$\Delta_{12} = 2\det \begin{pmatrix} 2i\omega_0 - m_1 & H_1 & -m_3 \\ -n_3 e^{-2i\omega_0 \tau_k^{(j)}} & H_2 & -n_2 \\ 0 & H_3 & 2i\omega_0 - l_2 \end{pmatrix},$$

$$\Delta_{13} = 2\det \begin{pmatrix} 2i\omega_0 - m_1 & -m_2 & H_1 \\ -n_3 e^{-2i\omega_0 \tau_k^{(j)}} & 2i\omega_0 - n_1 - n_4 e^{-2i\omega_0 \tau_k^{(j)}} & H_2 \\ 0 & -l_1 & H_3 \end{pmatrix}.$$

Similarly, substituting (3.21) and (3.26) into (3.24), we have

$$\left(\int_{-1}^{0} \mathrm{d}\eta(\theta) \right) E_2 = 2\tau_k^{(j)} (P_1, P_2, P_3)^T, .$$

That is

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ n_3 & n_1 + n_4 & n_2 \\ 0 & l_1 & l_2 \end{pmatrix} E_2 = 2(-P_1, -P_2, -P_3)^T.$$

It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \quad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \quad E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2}, \tag{3.28}$$

where

$$\Delta_2 = \det \begin{pmatrix} m_1 & m_2 & m_3 \\ n_3 & n_1 + n_4 & n_2 \\ 0 & l_1 & l_2 \end{pmatrix},$$

$$\Delta_{21} = 2\det \begin{pmatrix} -P_1 & m_2 & m_3 \\ -P_2 & n_1 + n_4 & n_2 \\ -P_3 & l_1 & l_2 \end{pmatrix},$$

$$\Delta_{22} = 2\det \begin{pmatrix} m_1 & -P_1 & m_3 \\ n_3 & -P_2 & n_2 \\ 0 & -P_3 & l_2 \end{pmatrix},$$

$$\Delta_{23} = 2\det \begin{pmatrix} m_1 & m_2 & -P_1 \\ n_3 & n_1 + n_4 & -P_2 \\ 0 & l_1 & -P_3 \end{pmatrix}.$$

From (3.20,3.22,3.27) and (3.28), we can calculate g_{21} and derive the following values:

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0\tau_k^{(j)}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{Re\{c_1(0)\}}{Re\left\{\lambda'(\tau_k^{(j)})\right\}},\\
\beta_2 &= 2Re(c_1(0)),\\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\left\{\lambda'(\tau_k^{(j)})\right\}}{\omega_0\tau_k^{(j)}}.
\end{aligned}
$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (3.1) at $\tau = \tau_k^{(j)}$, $(k = 1, 2, 3; j = 0, 2, 3, \ldots)$ on the center manifold. From the discussion above, we have the following result:

Theorem 3.1 *The periodic solution is supercritical (subcritical) if $\mu_2 > 0(\mu_2 < 0)$; the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0(\beta_2 > 0)$; the periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0(T_2 < 0)$.*

Remark 3.2 A τT-periodic solution of (3.1) is a T-periodic solution of (2.2).

4 Numerical examples

In this section, we present some numerical results of system (1.2) to verify the analytical predictions obtained in the previous section. From Sect. 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

$$
\begin{cases}
\dot{X}(t) = X(t)[2 - 0.5X(t) - 0.6S(t) - 0.82I(t)],\\
\dot{S}(t) = 2X(t-\tau)S(t-\tau) + S(t)[-3 - 0.4(S(t) + I(t)) + 0.9I(t)],\\
\dot{I}(t) = I(t)[4S(t) - 2(S(t) + I(t))],
\end{cases}
\tag{4.1}
$$

which has a positive equilibrium $E_0(X^*, S^*, I^*) \approx (1.4559, 0.8818, 0.8818)$ and satisfies the conditions indicated in Theorem 2.1. When $\tau = 0$, the positive equilibrium $E_0 \approx (1.4559, 0.8818, 0.8818)$ is asymptotically stable. Take $j = 0$, for example, by some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 0.8940, \tau_0 \approx 0.2551, \lambda'(\tau_0) \approx 1.0038 - 6.7531i$. Thus, we can calculate the following values:

$$
c_1(0) \approx -2.1230 - 6.1422i, \quad \mu_2 \approx 2.1150, \quad \beta_2 \approx -4.2460, \quad T_2 \approx 89.5441.
$$

Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the positive equilibrium $E_0 \approx (1.4559, 0.8818, 0.8818)$ is stable when $\tau < \tau_0$ as is illustrated by the computer simulations (see Fig. 1). When τ passes through the critical value τ_0, the positive equilibrium $E_0 \approx (1.4559, 0.8818, 0.8818)$ loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from the positive equilibrium $E_0 \approx (1.4559, 0.8818, 0.8818)$.

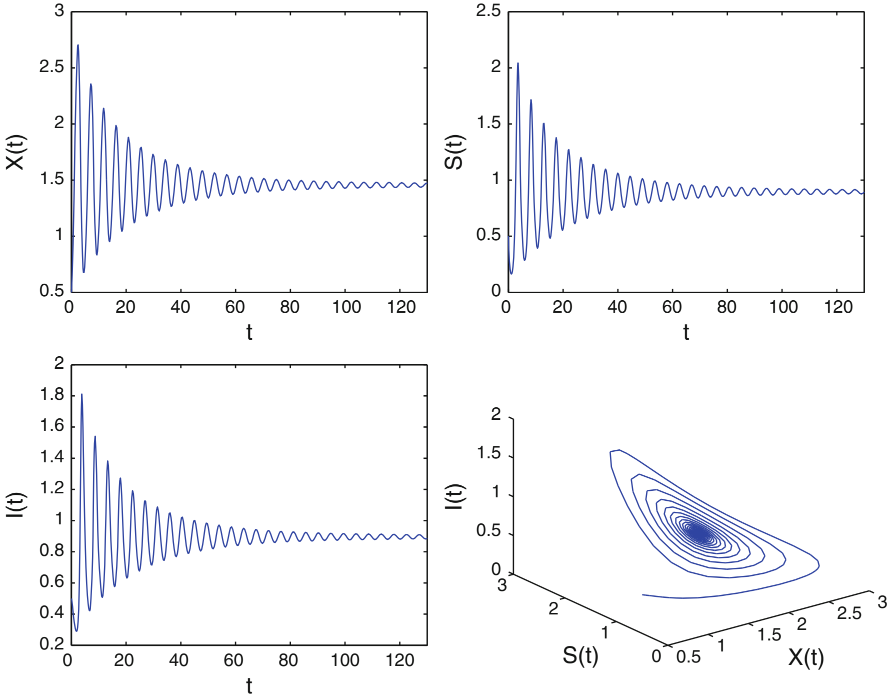

Fig. 1 Behavior and phase portrait of system (4.1) with $\tau = 0.08 < \tau_0 \approx 0.2551$. The positive equilibrium $E_0 \approx (1.4559, 0.8818, 0.8818)$ is asymptotically stable. The initial value is (0.5,0.5,0.5)

Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $E_0 \approx (1.4559, 0.8818, 0.8818)$ at τ_0 are stable, which are depicted in Fig. 2.

5 Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E_0(X^*, S^*, I^*)$ and local Hopf bifurcation in an autonomous epidemic predator–prey model with delay. We have showed that if the conditions (H1) and (H2) hold, the positive equilibrium $E_0(X^*, S^*, I^*)$ of system (1.2) is asymptotically stable for all $\tau \in [0, \tau_0)$. This means that the density of the prey, the density of the susceptible predator and the infected predator will tend to be stable, that is, the density of the prey, the density of the susceptible predator and the infected predator will tend to X^*, S^*, I^*, respectively, for all $\tau \in [0, \tau_0)$. Under the conditions (H1) and (H2), if the condition (H3) holds, as the delay τ increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(X^*, S^*, I^*)$, that is, a family of periodic orbits bifurcates from the the positive equilibrium $E_0(X^*, S^*, I^*)$. This shows that the density of the prey, the density of the susceptible predator and the infected predator may keep in an oscillatory mode near the positive equilibrium $E_0(X^*, S^*, I^*)$. Applying the normal form theory and the center manifold theorem, the direction of Hopf bifurcation and the stability of the bifurcating

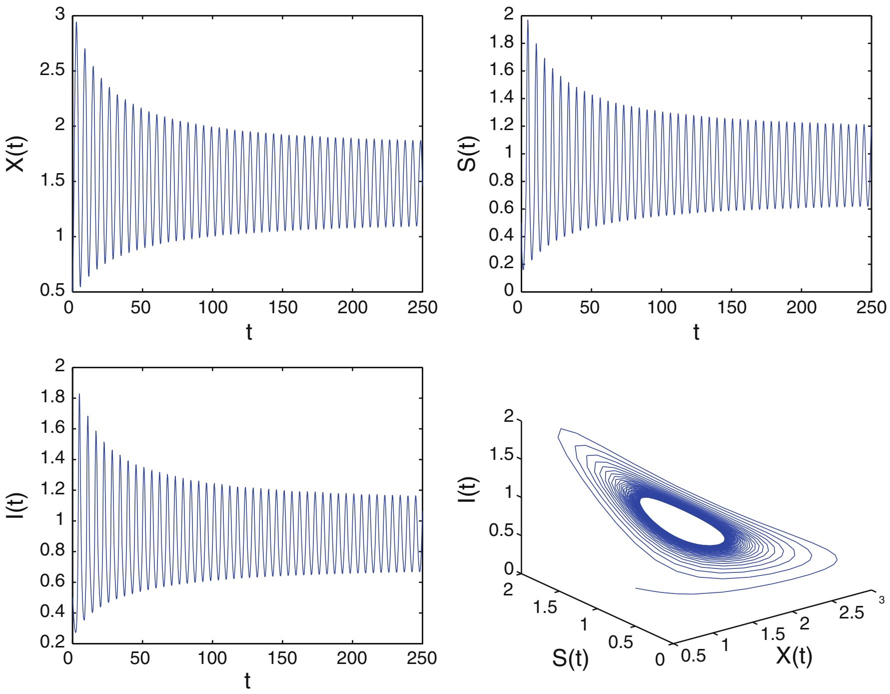

Fig. 2 Behavior and phase portrait of system (4.1) with $\tau = 0.3 > \tau_0 \approx 0.2551$. Hopf bifurcation occurs from the positive equilibrium $E_0 \approx (1.4559, 0.8818, 0.8818)$. The initial value is (0.5,0.5,0.5)

periodic orbits are discussed. A numerical example verifying our theoretical results is also included.

References

1. Bairagi, N., Jana, D.: On the stability and Hopf bifurcation of a delay-induced predator-prey system with habitat complexity. Appl. Math. Model. **35**(7), 3255–3267 (2011)
2. Beretta, E., Kuang, Y.: Geometric stability switch criteria in delay differential system with delay dependent parameters. SIAM J. Math. Anal. **33**(5), 1144–1165 (2002)
3. Chakraborty, K., Chakraborty, M., Kar, T.K.: Bifurcation and control of a bioeconomic model of a prey-predator system with a time delay. Nonlinear Anal. Hybrid Syst. **5**(4), 613–625 (2011)
4. Das, K.P., Kundu, K., Chattopadhyay, J.: A predator-prey mathematical model with both the populations affected by diseases. Ecol. Complex. **8**(1), 68–80 (2011)
5. Duque, C., Lizana, M.: On the dynamics of a predator-prey model with nonconstant death rate and diffusion. Nonlinear Anal. Real World Appl. **12**(4), 2198–2210 (2011)
6. Duque, C., Lizana, M.: Stable periodic traveling waves for a predator-prey model with non-constant death rate and delay. Appl. Math. Comput. **217**(23), 9717–9722 (2011)
7. Hale, J.: Theory of Functional Differential Equation. Springer, Berlin (1977)
8. Hassard, B., Kazarino, D., Wan, Y.: Theory and Applications of Hopf Bifurcation. Cambridge University Press, Cambridge (1981)
9. Kar, T.K., Ghorai, A.: Dynamic behaviour of a delayed predator-prey model with harvesting. Appl. Math. Comput. **217**(22), 9085–9104 (2011)
10. Ko, W., Ryu, K.: Analysis of diffusive two-competing-prey and one-predator systems with Beddington-Deangelis functional response. Nonlinear Anal. Theory Methods Appl. **71**(9), 4185–4202 (2009)

11. Kuang, Y.: Delay Differential Equations with Applications in Populations Dynamics. Academic Press, INC, Massachusetts (1993)
12. Kuang, Y., Takeuchi, Y.: Predator-prey dynamics in models of prey dispersal in two-patch environments. Math. Biosci. **120**(1), 77–98 (1994)
13. Li, B., Wang, M.X.: Stationary patterns of the stage-structured predator- prey model with diffusion and cross-diffusion. Math. Comput. Model. **54**(5–6), 1380–1393 (2011)
14. Oeda, K.: Effect of cross-diffusion on the stationary problem of a prey-predator model with a protection zone. J Differ. Equ. **250**(10), 3988–4009 (2011)
15. Ruan, S.G., Wei, J.J.: On the zero of some transcendential functions with applications to stability of delay differential equations with two delays. Dyn. Contin. Discret. Impuls. Syst. Ser. A **10**(2), 863–874 (2003)
16. Song, Y.L., Wei, J.J.: Local Hopf bifurcation and global existence of periodic solutions in a delayed predator-prey system. J. Math. Anal. Appl. **301**(1), 1–21 (2005)
17. Tian, B.D., Qiu, Y.H., Chen, N.: Periodic and almost periodic solution for a non-autonomous epidemic predator-prey system with time-delay. Appl. Math. Comput. **215**(2), 779–790 (2009)
18. Vayenas, D.V., Aggelis, G., Tsagou, V., Pavlou, S.: Dynamics of a two-prey-one-predator system with predator switching regulated by a catabolic repression control-like mode. Ecol. Model. **186**(3), 345–357 (2005)
19. Xu, C.J., Liao, M.X., He, X.F.: Stability and Hopf bifurcation analysis for a Lokta-Volterra predator-prey model with two delays. Int. J. Appl. Math. Comput. Sci. **21**(1), 97–107 (2011)
20. Xu, C.J., Tang, X.H., Liao, M.X.: Stability and bifurcation analysis of a delayed predator-prey model of prey dispersal in two-patch environments. Appl. Math. Comput. **216**(10), 2920–2936 (2010)
21. Xu, C.J., Tang, X.H., Liao, M.X., He, X.F.: Bifurcation analysis in a delayed Lokta-Volterra predator-prey model with two delays. Nonlinear Dyn. **66**(1–2), 169–183 (2011)
22. Xu, R., Ma, Z.E.: Stability and Hopf bifurcation in a ratio-dependent predator-prey system with stage structure. Chaos Solitons Fract. **38**(3), 669–684 (2008)
23. Yan, X.P., Zhang, C.H.: Hopf bifurcation in a delayed Lokta-Volterra predator-prey system. Nonlinear Anal. Real World Appl. **9**(1), 114–127 (2008)
24. Zhou, X.Y., Shi, X.Y., Song, X.Y.: Analysis of non-autonomous predator-prey model with nonlinear diffusion and time delay. Appl. Math. Comput. **196**(1), 129–136 (2008)