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On the geometry of maximal spacelike hypersurfaces immersed in a generalized Robertson–Walker spacetime

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Abstract In this paper, we establish new characterizations of totally geodesic spacelike hypersurfaces immersed in a generalized Robertson–Walker spacetime, which is supposed to obey the null convergence condition. As applications, we get nonparametric results concerning to entire maximal vertical graphs in a such ambient spacetime. Proceeding, we obtain a lower estimate of the index of relative nullity of complete r-maximal spacelike hypersurfaces immersed in Robertson–Walker spacetimes of constant sectional curvature. In particular, we prove a sort of weak extension of the classical Calabi–Bernstein theorem.

Keywords Generalized Robertson–Walker spacetimes · Complete r-maximal spacelike hypersurfaces · Totally geodesic spacelike hypersurfaces · Null convergence condition · Entire vertical graphs · Index of relative nullity

Mathematics Subject Classification (2000) Primary 53C42; Secondary 53B30 · 53C50

1 Introduction

Spacelike hypersurfaces immersed with constant mean curvature in a Lorentzian manifold are objects worthy of a big amount of interest, from both physical and mathematical points of view. In particular, the study of maximal spacelike hypersurfaces (that is, with zero mean curvature) immersed in Lorentzian manifolds is an important topic in the theory of

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semi-Riemannian geometry. This is justified, for instance, by the fact that they are solutions of questions concerning existence and uniqueness of hypersurfaces, as the Calabi–Bernstein-type results.

In [19], Ishihara showed that the only complete maximal spacelike hypersurfaces immersed in a Lorentz manifold with nonnegative constant curvature are the totally geodesic ones. For the case of ambient spacetimes with negative constant curvature, he obtained a sharp estimate for the norm of the second fundamental form of a maximal spacelike hypersurface.

More recently, the first author jointly with Camargo has obtained in [12] rigidity results for complete maximal spacelike hypersurfaces in the anti-de Sitter space, imposing suitable conditions on both the norm of the second fundamental form and a certain height function naturally attached to the hypersurface.

In this paper, we are interested in the study of complete maximal spacelike hypersurfaces immersed in generalized Robertson–Walker (GRW) spacetimes. By GRW spacetimes, we mean Lorentzian warped products $-I \times_f M^n$ with Riemannian fiber M^n and warping function f. In particular, when the Riemannian fiber M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a Robertson–Walker (RW) spacetime (for the details, see Sect. 2).

Many authors have approached problems in this subject. We may cite, for instance, the works [9, 10] and [25], where Romero et al. have obtained rigidity and uniqueness results for the spacelike slices and complete maximal surfaces immersed in three-dimensional GRW spacetimes obeying either the *null convergence condition* or the *timelike convergence condition*. Let us recall that a spacetime obeys the null (time) convergence condition if its Ricci curvature is nonnegative on null or lightlike (timelike) directions.

Related to the compact case, Alías et al. [6] proved that in a GRW spacetime satisfying the timelike convergence condition, every compact spacelike hypersurface of constant mean curvature must be totally umbilical. In this setting, they also showed how their result solves a certain Bernstein-type problem. Later on, Alías and Colares [4] studied the problem of uniqueness for compact spacelike hypersurfaces immersed with constant higher-order mean curvature in GRW spacetimes. In order to establish one of their main results (cf. Theorem 9.2 of [4]), they supposed that the ambient spacetime obeys a new notion of convergence condition, the so-called *strong null convergence condition* that corresponds to a suitable restriction on the sectional curvature of the Riemannian fiber of the GRW spacetime.

Here, we deal with complete noncompact maximal spacelike hypersurfaces immersed in a GRW spacetime. In this setting, by assuming that the ambient spacetime obeys the null convergence condition, we obtain the following characterization of the totally geodesic spacelike hypersurfaces of such spacetime (cf. Theorem 4.6; see also Theorem 4.3, where we assume that the ambient spacetime obeys the strong null convergence condition):

Let Σ^n be a complete maximal spacelike hypersurface contained in a timelike bounded region of a GRW spacetime which obeys the null convergence condition. Suppose that the second fundamental form of Σ^n is bounded. If the gradient of the height function of Σ^n has Lebesgue integrable norm, then Σ^n is totally geodesic.

Our approach is based on a maximum principle at the infinity due to Yau in [27] (cf. Lemma 4.5). Moreover, in order to prove the result just stated above, a key assumption on the maximal spacelike hypersurface is that it is contained in a timelike bounded region of the ambient spacetime. Compared with compactness, such hypothesis is a weaker condition.

In Sect. 5, we also establish nonparametric results concerning to entire maximal vertical graphs in such ambient spacetime (cf. Corollaries 5.1 and 5.2).

Concerning the case of the higher-order mean curvatures, we obtain a lower estimate of the index of relative nullity of complete r-maximal spacelike hypersurfaces (that is, with zero $(r+1)$-th mean curvature) in a RW spacetime (cf. Theorem 6.2). Furthermore, using the ideas developed by Caminha in [13], we obtain the following weak extension of the Cheng–Yau theorem in [15] (cf. Theorem 6.5):

Let Σ^n be a complete spacelike hypersurface immersed in a static RW spacetime of constant sectional curvature, with bounded second fundamental form. Suppose that the $(r+1)$-th and $(r+2)$-th mean curvatures of Σ^n do not change sign. If the gradient of the height function of Σ^n has Lebesgue integrable norm, then its index of relative nullity is at least $n-r$. Moreover, if the ambient spacetime is the Minkowski space $\mathbb{L}^{n+1}$, then through every point of Σ^n, there passes an $(n-r)$-hyperplane of $\mathbb{L}^{n+1}$ totally contained in Σ^n.

2 Generalized Robertson–Walker spacetimes

In this section, we introduce some basic notations and facts that will appear along the paper.

In what follows, if $\overline{M}^{n+1}$ is a connected semi-Riemannian manifold with metric $\overline{g} = \langle\ ,\ \rangle$, let $\mathcal{D}(\overline{M})$ denote the ring of smooth functions $\phi : \overline{M}^{n+1} \to \mathbb{R}$, and $\mathfrak{X}(\overline{M})$, the algebra of smooth vector fields on $\overline{M}^{n+1}$. We also write $\overline{\nabla}$ for the Levi–Civita connection of $\overline{M}^{n+1}$.

Let M^n be a connected, n-dimensional ($n \geq 2$) oriented Riemannian manifold; $I \subseteq \mathbb{R}$, a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$); and $f : I \to \mathbb{R}$, a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the factors I and M^n, respectively.

A particular class of Lorentzian manifolds (*spacetimes*) is the one obtained by furnishing $\overline{M}^{n+1}$ with the metric

$$\langle v, w\rangle_p = -\langle(\pi_I)_*v, (\pi_I)_*w\rangle + (f \circ \pi_I)\,(p)^2\langle(\pi_M)_*v, (\pi_M)_*w\rangle,$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$. Following the terminology introduced in [6], such a space is called a *generalized Robertson–Walker* (GRW) spacetime, f is known as the warping function, and we shall write $\overline{M}^{n+1} = -I \times_f M^n$ to denote it. In particular, when the Riemannian fiber M^n has constant sectional curvature, then $-I \times_f M^n$ is classically called a *Robertson–Walker* (RW) spacetime, and it is a spatially homogeneous spacetime (cf. [22]).

Remark 2.1 As it was observed in [5], we note that spatial homogeneity, which is reasonable as a first approximation of the large-scale structure of the universe, may not be realistic when one considers a more accurate scale. For that reason, GRW spacetimes could be suitable spacetimes to model universes with inhomogeneous spacelike geometry. Besides, small deformations of the metric on the fiber of RW spacetimes fit into the class of GRW spacetimes (see, for instance, [18] and [24]).

It follows from Proposition 7.42 of [22] that a GRW spacetime $-I \times_f M^n$ has constant sectional curvature $\overline{\kappa}$ if, and only if, its Riemannian fiber M^n has constant sectional curvature κ (that is, $-I \times_f M^n$ is in fact a RW spacetime) and the warping function f satisfies the following differential equations

$$\frac{f''}{f} = \overline{\kappa} = \frac{(f')^2 + \kappa}{f^2}. \tag{2.1}$$

From now on, we deal with spacelike hypersurfaces immersed in a GRW spacetime. We recall that a smooth immersion $\psi : \Sigma^n \to -I \times_f M^n$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted for $\langle , \rangle$. In that case, since

$$\partial_t = (\partial/\partial_t)_{(t,x)}\,, \quad (t,x) \in -I \times_f M^n,$$

is a unitary timelike vector field globally defined on the ambient spacetime, then there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n, which is in the same time orientation as ∂_t. By using Cauchy–Schwarz inequality, we get

$$\langle N, \partial_t \rangle \leq -1 < 0 \quad \text{on } \Sigma^n. \tag{2.2}$$

We will refer to that normal vector field N as the future-pointing Gauss map of the spacelike hypersurface Σ^n.

Let $\overline{\nabla}$ and ∇ denote the Levi–Civita connections in $-I \times_f M^n$ and Σ^n, respectively. Then, the Gauss and Weingarten formulas for the spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N \tag{2.3}$$

and

$$AX = -\overline{\nabla}_X N, \tag{2.4}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the shape operator (or Weingarten endomorphism) of Σ^n with respect to the future-pointing Gauss map N.

As in [22], the curvature tensor R of the spacelike hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

A well-known fact is that the curvature tensor R of the spacelike hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of the ambient spacetime $\overline{M}^{n+1}$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top - \langle AX, Z \rangle AY + \langle AY, Z \rangle AX, \tag{2.5}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\,)^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n.

Associated with the shape operator A, there are n algebraic invariants given by

$$S_r(p) = \sigma_r\left(\kappa_1(p), \ldots, \kappa_n(p)\right), \quad 1 \leq r \leq n,$$

where $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_r\left(x_1, \ldots, x_n\right) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$$

and $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ^n.

The rth mean curvature H_r of the spacelike hypersurface Σ^n is then defined by

$$\binom{n}{r} H_r = (-1)^r\, S_r(\kappa_1, \ldots, \kappa_n) = S_r(-\kappa_1, \ldots, -\kappa_n).$$

In particular, when $r = 1$, $H_1 = -\frac{1}{n}\sum_{i=1}^n \kappa_i = -\frac{1}{n}\operatorname{tr}(A) = H$ is the mean curvature of Σ^n, which is the main extrinsic curvature of the hypersurface. The choice of the sign $(-1)^r$ in the definition of H_r is justified by the fact that in this case, the mean curvature vector is given by $\overrightarrow{H} = HN$ and, therefore, $H(p) > 0$ at a point $p \in \Sigma^n$ if, and only if, $\overrightarrow{H}(p)$ is in the same time orientation as $N(p)$ (in the sense that $\langle \overrightarrow{H}, N\rangle_p < 0$).

When $r = 2$, it follows from the Gauss equation that

$$H_2 = \overline{R} - R + \frac{2}{n(n-1)}\overline{\mathrm{Ric}}(N, N),$$

where $\overline{R}$ and R are, respectively, the normalized scalar curvatures of $-I \times_f M^n$ and Σ^n and $\overline{\mathrm{Ric}}$ stands for the Ricci tensor of the ambient GRW spacetime. In particular, when the ambient GRW spacetime has constant sectional curvature $\overline{\kappa}$, one gets that

$$R = \overline{\kappa} - H_2. \tag{2.6}$$

For $0 \leq r \leq n$, one defines the rth Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r I + A P_{r-1}. \tag{2.7}$$

A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + (-1)^r A^r),$$

so that Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ that diagonalize A at $p \in \Sigma^n$ also diagonalize all of the P_r at p. Let $\{e_k\}$ be such a basis.

The following formulas hold (cf. [7], Lemma 2.1):

(a) $\mathrm{tr}(P_r) = (n-r)\binom{n}{r}H_r$;
(b) $\mathrm{tr}(AP_r) = -(n-r)\binom{n}{r}H_{r+1}$;
(c) $\mathrm{tr}(A^2P_r) = \binom{n}{r+1}(nH_1H_{r+1} - (n-r-1)H_{r+2})$.

Associated with each Newton transformation P_r, one has the second-order linear differential operator $L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$, given by

$$L_r(f) = \mathrm{tr}(P_r \circ \mathrm{Hess}\, f).$$

We observe, by taking a local orthonormal frame $\{E_1, \ldots, E_n\}$ on Σ^n, that

$$\begin{aligned} L_r(f) &= \mathrm{tr}(P_r\, \mathrm{Hess}\, f) = \sum_{k=1}^{n} \langle P_r(\nabla_{E_k}\nabla f), E_k\rangle \\ &= \sum_{k=1}^{n}\langle \nabla_{E_k}\nabla f, P_r(E_k)\rangle = \sum_{k=1}^{n}\langle \nabla_{P_r(E_k)}\nabla f, E_k\rangle \\ &= \mathrm{tr}(\mathrm{Hess}\, f \circ P_r). \end{aligned}$$

3 The strong null convergence condition

In what follows, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed into a GRW spacetime $-I \times_f M^n$, namely the (vertical) height function $h = (\pi_I)|_\Sigma$ and the support function $\langle N, \partial_t\rangle$, where we recall that N denotes the future-pointing Gauss map of Σ^n and ∂_t is the coordinate vector field induced by the universal time on $-I \times_f M^n$.

A simple computation shows that the gradient of π_I on $-I \times_f M^n$ is given by

$$\overline{\nabla}\pi_I = -\langle\overline{\nabla}\pi_I, \partial_t\rangle = -\partial_t,$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \langle N, \partial_t\rangle N. \tag{3.1}$$

Thus, we get

$$|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1, \tag{3.2}$$

where $|\ |$ denotes the norm of a vector field on Σ^n.

Now, set $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \dots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from (2.5) that the *Ricci curvature* Ric of Σ^n is given by

$$\begin{aligned}\mathrm{Ric}(X, X) &= \sum_i \langle\overline{R}(X, E_i)X, E_i\rangle + nH\langle AX, X\rangle + \langle AX, AX\rangle \\ &= \sum_i \langle\overline{R}(X, E_i)X, E_i\rangle + \left|AX + \frac{nH}{2}X\right|^2 - \frac{n^2H^2}{4}|X|^2.\end{aligned} \tag{3.3}$$

Consequently, we get

$$\mathrm{Ric}(X, X) \geq \sum_i \langle\overline{R}(X, E_i)X, E_i\rangle - \frac{n^2H^2}{4}|X|^2.$$

Thus, if the mean curvature H is supposed to be bounded, then $\mathrm{Ric}(X, X)$ is bounded from below if and only if $\sum_i \langle\overline{R}(X, E_i)X, E_i\rangle$ is bounded from below.

On the other hand, by using the equation (33) of [4] (or Proposition 7.42 of [22]) and taking into account Eq. (3.1), we get

$$\begin{aligned}\sum_i \langle\overline{R}(X, E_i)X, E_i\rangle = \sum_i &\langle R_M(X^*, E_i^*)X^*, E_i^*\rangle \\ &+(n-1)((\log f)')^2|X|^2 \\ &-(n-2)(\log f)''\langle X, \nabla h\rangle^2 \\ &-(\log f)''|\nabla h|^2|X|^2.\end{aligned} \tag{3.4}$$

where $E_i^* = (\pi_M)_*(E_i)$ and $X^* = (\pi_M)_*(X)$.

By computing the first parcel of the right side of (3.4), we get

$$\begin{aligned}\sum_i \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle \geq \frac{1}{f^2}&((n-1)|X|^2 + |\nabla h|^2|X|^2 \\ &+(n-2)\langle X, \nabla h\rangle^2)\min_i K_M(X^*, E_i^*).\end{aligned}$$

According to the terminology established in [4], we suppose valid the *strong null convergence condition*

$$K_M \geq \sup_I (f^2(\log f)''), \tag{3.5}$$

where K_M denotes the sectional curvatures of M^n.

Remark 3.1 The strong null convergence condition is a slight change on the so-called *null convergence condition*, namely we say that a GRW spacetime $-I \times_f M^n$ obeys the null convergence condition if

$$\mathrm{Ric}_M \geq (n-1)\sup_I (f^2(\log f)'')\langle , \rangle_M, \tag{3.6}$$

where Ric_M and $\langle , \rangle_M$ are, respectively, the Ricci and metric tensors of the fiber M^n. We observe that the null convergence condition (3.6) is equivalent to the Ricci curvature of $\overline{M}^{n+1}$ be nonnegative on null or lightlike directions (cf. [20]). Moreover, we also note that when the ambient space is a RW spacetime, the convergence conditions (3.5) and (3.6) are equivalent.

Now, by using the strong null convergence condition (3.5), we obtain

$$\begin{aligned}\sum_i \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle \geq ((n-1)|X|^2 + |\nabla h|^2|X|^2 \\ +(n-2)\langle X, \nabla h\rangle^2)(\log f)''.\end{aligned} \tag{3.7}$$

Substituting (3.7) in (3.4), we get that

$$\begin{aligned}\sum_i \langle \overline{R}(X, E_i)X, E_i\rangle &\geq ((n-1)|X|^2 + (n-2)\langle X, \nabla h\rangle^2 \\ &\quad +|\nabla h|^2|X|^2)(\log f)'' + (n-1)((\log f)')^2|X|^2 \\ &\quad -(n-2)\langle X, \nabla h\rangle^2(\log f)'' - |\nabla h|^2|X|^2(\log f)'' \\ &= (n-1)\frac{f''}{f}|X|^2.\end{aligned}$$

Therefore, we obtain the following

Proposition 3.2 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime obeying the strong null convergence condition (3.5). Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface, with bounded mean curvature. If $\frac{f''}{f}$ is bounded from below on Σ^n, then the Ricci curvature of Σ^n is bounded from below.*

Remark 3.3 From Eq. (2.1), we easily see that GRW spacetimes with constant sectional curvature obey the strong null convergence condition (3.5), and hence, from the proof of Proposition 3.2, we conclude that all complete spacelike hypersurfaces immersed with bounded mean curvature in such spacetimes have Ricci curvature bounded from below.

4 Characterizations of totally geodesic hypersurfaces

The formulas collected in the following lemma are due to Alías and Colares (cf. [4], Lemma 4.1 and Corollaries 8.2 and 8.4).

Lemma 4.1 *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface immersed in a GRW spacetime. Then*

$$\begin{aligned}L_r h = -(\log f)'(h)\left((n-r)\binom{n}{r}H_r + \langle P_r \nabla h, \nabla h\rangle\right) \\ -(n-r)\binom{n}{r}H_{r+1}\langle N, \partial_t\rangle\end{aligned}$$

and

$$\begin{aligned}\Delta(f(h)\langle N, \partial_t\rangle) = nf(h)\langle\nabla H, \partial_t\rangle + nHf'(h) + f(h)\langle N, \partial_t\rangle|A|^2\\ -(n-1)f(h)\langle N, \partial_t\rangle(\log f)''(h)|\nabla h|^2\\ +f(h)\langle N, \partial_t\rangle\mathrm{Ric}_M(N^*, N^*),\end{aligned}$$

where Ric_M *stands for the Ricci tensor of the Riemannian fiber* M^n*. Moreover, if* $-I\times_f M^n$ *is a RW spacetime, then*

$$\begin{aligned}L_r(f(h)\langle N, \partial_t\rangle) = \binom{n}{r+1}f(h)\langle\nabla H_{r+1}, \partial_t\rangle + (n-r)\binom{n}{r}H_{r+1}f'(h)\\ +\binom{n}{r+1}f(h)\langle N, \partial_t\rangle\,(nHH_{r+1} - (n-r-1)H_{r+2})\\ +f(h)\langle N, \partial_t\rangle\left(\frac{\kappa}{f^2(h)} - (\ln f)''(h)\right)\\ \cdot\left((n-r)\binom{n}{r}H_r|\nabla h|^2 - \langle P_r\nabla h, \nabla h\rangle\right),\end{aligned}$$

where κ *stands for the sectional curvature of the Riemannian fiber* M^n*.*

We will also need the well-known generalized maximum principle of Omori-Yau [21,26].

Lemma 4.2 *Let* Σ^n *be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and* $u : \Sigma^n \to \mathbb{R}$ *be a smooth function that is bounded from below on* Σ^n*. Then, there is a sequence of points* $\{p_k\}$ *in* Σ^n*, such that*

$$\lim_{k\to\infty} u(p_k) = \inf u, \ \lim_{k\to\infty} |\nabla u(p_k)| = 0 \text{ and } \lim_{k\to\infty} \Delta u(p_k) \geq 0. \tag{4.1}$$

In what follows, a slab

$$[t_1, t_2]\times M^n = \left\{(t, q) \in -I\times_f M^n : t_1 \leq t \leq t_2\right\}$$

is called a *timelike bounded region* of the GRW spacetime $-I\times_f M^n$. Now, we state and prove our first theorem.

Theorem 4.3 *Let* $\overline{M}^{n+1} = -I\times_f M^n$ *be a GRW spacetime which obeys the strong null convergence condition (3.5). Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete maximal spacelike hypersurface contained in a timelike bounded region of* $\overline{M}^{n+1}$*. If* $|\nabla h|$ *is bounded on* Σ^n*, then the norm of the second fundamental form A of* Σ^n *is not globally bounded away from zero. Moreover, if* H_2 *is constant, then* Σ^n *is totally geodesic.*

Proof Let us consider the Gauss map N of Σ^n, such that ∂_t belongs to the timecone determined by N, and set the function $g : \Sigma^n \to \mathbb{R}$ given by $g = f\langle N, \partial_t\rangle$.

From Lemma 4.1, we get

$$\Delta g = g\left(\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log f)''|\nabla h|^2 + |A|^2\right).$$

On the other hand, taking into account that $N^* = N + \langle N, \partial_t\rangle\partial_t$, from Eq. (3.1), we easily verify that

$$\langle N^*, N^*\rangle_M = \frac{1}{f^2(h)}|\nabla h|^2.$$

Consequently, from the strong null convergence condition (3.5), we have

$$\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log f)''|\nabla h|^2 \geq 0.$$

Thus, since the function g is negative, we obtain that

$$\Delta g \leq g|A|^2 \leq 0. \tag{4.2}$$

Now, we want to apply Lemma 4.2 to prove our result. For this, first we note that, since we are supposing that Σ^n is contained in a timelike bounded region of $-I \times_f M^n$, Proposition 3.2 guarantees that the Ricci curvature of Σ^n is bounded from below. Moreover, the function g is bounded on Σ^n. In fact,

$$|g| = f|\langle N, \partial_t\rangle| \leq f\langle N, \partial_t\rangle^2 = f(1 + |\nabla h|^2)$$

and, since that f is bounded on some interval $[t_1, t_2] \subset I$ and $|\nabla h|$ is bounded on Σ^n, it follows the assertion. In particular, we have that the infimum $\inf_{p\in\Sigma} g(p)$ exists and is a negative number.

Hence, from Lemma 4.2 and inequality (4.2), we have that there exists a sequence of points p_k in Σ^n, such that

$$0 \leq \lim_{k\to\infty} \Delta g(p_k) \leq \inf_{p\in\Sigma} g(p) \lim_{k\to\infty} |A|^2(p_k) \leq 0.$$

Thus, $\lim_{k\to\infty} |A|^2(p_k) = 0$, and hence, $|A|$ cannot be bounded away from zero on Σ^n. Moreover, since for a maximal spacelike hypersurface we have that $|A|^2 = -n(n-1)H_2$, if H_2 is constant, then $|A|$ is also constant. By the same arguments above, it follows that $A \equiv 0$ and, therefore, Σ^n are totally geodesic. □

Remark 4.4 Taking into account the estimate of the norm of the second fundamental form of a complete maximal spacelike hypersurface immersed in the anti-de Sitter space due to Ishihara (cf. [19], Theorem 1.2) and the fact that a GRW spacetime of constant sectional curvature trivially satisfies the strong null convergence condition (see Remark 3.3), Theorem 4.3 can be considered as a natural extension of Theorem 1.1 of [12] to the context of the GRW spacetimes.

In the paper [27], Yau obtained the following version of Stokes' theorem on an n-dimensional, complete noncompact Riemannian manifold Σ^n: If $\omega \in \Omega^{n-1}(\Sigma)$ is an $n-1$ differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n, such that $B_i \subset B_{i+1}$, $\Sigma^n = \bigcup_{i\geq 1} B_i$ and

$$\lim_{i\to+\infty} \int_{B_i} d\omega = 0.$$

By applying this result to $\omega = \iota_{\nabla g}$, where $g : \Sigma^n \to \mathbb{R}$ is a smooth function, ∇g denotes its gradient and $\iota_{\nabla g}$ the contraction in the direction of ∇g, Yau established an extension of H. Hopf's theorem on a complete noncompact Riemannian manifold. In what follows, $\mathcal{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 4.5 (Corollary on p. 660 of [27]) *Let Σ^n be an n-dimensional, complete noncompact Riemannian manifold and let $g : \Sigma^n \to \mathbb{R}$ be a smooth function. If g is a subharmonic (or superharmonic) function with $|\nabla g| \in \mathcal{L}^1(\Sigma)$, then g must actually be harmonic.*

Now, we are in position to present our next result.

Theorem 4.6 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime which obeys the null convergence condition (3.6). Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete maximal spacelike hypersurface contained in a timelike bounded region of $\overline{M}^{n+1}$, and with bounded second fundamental form A. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then Σ^n is totally geodesic.*

Proof We take the Gauss map N of Σ^n such that ∂_t belongs to the timecone determined by N.

Another application of Lemma 4.1 gives us $\Delta(f\langle N, \partial_t\rangle) \le 0$ on Σ^n. Besides, if we consider $X = \nabla(f\langle N, \partial_t\rangle)$, we get $X = -fA(\partial_t^\top)$, since that the vector field $K = f\partial_t$ is closed and conformal. In fact, for any $Y \in \mathfrak{X}(M)$,

$$\begin{aligned}\langle \nabla\langle K, N\rangle, Y\rangle &= Y(\langle K, N\rangle)\\ &= \langle \overline{\nabla}_Y K, N\rangle + \langle K, \overline{\nabla}_Y N\rangle\\ &= f'\langle Y, N\rangle - \langle K, AY\rangle\\ &= -f\langle A(\partial_t^\top), Y\rangle.\end{aligned}$$

Thus,

$$|X| = |fA(\partial_t^\top)| \le f|A||\nabla h|,$$

where we have used $\nabla h = -\partial_t^\top$ to obtain the last inequality. Consequently, since we are supposing that f and $|A|$ are bounded and that $|\nabla h|$ is Lebesgue integrable on Σ^n, we conclude that X has Lebesgue integrable norm on Σ^n. Now, it is enough to apply the Lemma 4.5 for the function $g = f\langle N, \partial_t\rangle$ to conclude that $|A| \equiv 0$ and so that Σ^n must actually be totally geodesic. □

Remark 4.7 By a similar way as observed in Remark 4.4, Theorem 4.6 is an extension of Theorem 1.2 of [12].

5 Entire vertical graphs in a GRW spacetime $-I \times_f M^n$

Let $\Omega \subseteq M^n$ be a connected domain of M^n. A vertical graph over Ω is determined by a smooth function $u \in \mathcal{C}^\infty(\Omega)$, and it is given by

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\} \subset -I \times_f M^n.$$

The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma^n(u)$ is

$$\langle , \rangle = -du^2 + f^2(u)\langle , \rangle_{M^n}. \tag{5.1}$$

The graph is said to be entire if $\Omega = M^n$. It can be easily seen that a graph $\Sigma^n(u)$ is a spacelike hypersurface if and only if $|Du|^2_{M^n} < f^2(u)$, Du being the gradient of u in Ω and $|Du|_{M^n}$ its norm, both with respect to the metric $\langle , \rangle_{M^n}$ in Ω. Observe that by Lemma 3.1 in [6], in the case where M^n is a simply connected manifold, every complete spacelike hypersurface Σ^n in $-I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph in such space. In particular, this happens for complete spacelike hypersurfaces bounded away from the infinity of $-I \times_f M^n$. However, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph in a Lorentzian spacetime is not necessarily complete, in the sense that the induced Riemannian metric (5.1) is not necessarily complete on M^n. For instance, A.L. Albujer has obtained explicit examples of noncomplete entire maximal graphs in $-\mathbb{R} \times \mathbb{H}^2$ (cf. [1], Sect. 3).

In this context, by using the ideas of [2], we obtain the following nonparametric version of Theorem 4.3.

Corollary 5.1 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime which obeys the strong null convergence condition (3.5) and whose fiber M^n is complete. Let $\Sigma^n(u)$ be an entire maximal spacelike vertical graph contained in a timelike bounded region of $\overline{M}^{n+1}$. If*

$$|Du|_{M^n} \leq \alpha f(u), \tag{5.2}$$

for some constant $0 \leq \alpha < 1$, then $\Sigma^n(u)$ is complete and the norm of its second fundamental form is not globally bounded away from zero. Moreover, if H_2 is constant, then $\Sigma^n(u)$ is totally geodesic.

Proof Observe first that, under the assumptions of the theorem, $\Sigma^n(u)$ is a complete hypersurface. In fact, from (5.1) and the Cauchy–Schwarz inequality, we get

$$\langle X, X\rangle = -\langle Du, X\rangle^2_{M^n} + f^2(u)\langle X, X\rangle_{M^n} \geq \left(f^2(u) - |Du|^2_{M^n}\right)\langle X, X\rangle_{M^n},$$

for every tangent vector field X on Σ. Therefore,

$$\langle X, X\rangle \geq c\langle X, X\rangle_{M^n}$$

for the positive constant $c = (1-\alpha^2)\inf_{\Sigma^n(u)} f^2(u)$. This implies that $L \geq \sqrt{c}L_{M^n}$, where L and L_{M^n} denote the length of a curve on $\Sigma^n(u)$ with respect to the Riemannian metrics $\langle , \rangle$ and $\langle , \rangle_{M^n}$, respectively. As a consequence, as M^n is complete by assumption, the induced metric on $\Sigma^n(u)$ from the metric of $-I \times_f M^n$ is also complete.

On the other hand, we have that

$$N = -\langle N, \partial_t\rangle\partial_t + N^*, \tag{5.3}$$

where N^* denotes the projection of N onto the fiber M^n. Consequently, from (3.1), (3.2) and (5.3), we get

$$N^{*^\top} = -\langle N, \partial_t\rangle \nabla h \tag{5.4}$$

and

$$|\nabla h|^2 = f^2(h)\langle N^*, N^*\rangle_{M^n}. \tag{5.5}$$

Moreover, with a straightforward computation, we verify that

$$N = \frac{f(u)}{\sqrt{f^2(u) - |Du|^2_{M^n}}}\left(\partial_t + \frac{1}{f^2(u)}Du\right). \tag{5.6}$$

Thus, from (5.4), (5.5) and (5.6), we obtain that

$$|\nabla h|^2 = \frac{|Du|^2_{M^n}}{f^2(u) - |Du|^2_{M^n}}. \tag{5.7}$$

Therefore, from (5.2) and (5.7) we get

$$|\nabla h|^2 \leq \frac{1}{c}|Du|^2_{M^n} \leq \frac{\alpha^2}{c}\sup_{\Sigma(u)} f^2(u) \tag{5.8}$$

and, hence, the result follows from Theorem 4.3. □

One can reason as in the proof of Corollary 5.1 to obtain the following nonparametric version of Theorem 4.6.

Corollary 5.2 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime which obeys the null convergence condition (3.6) and whose fiber M^n is complete. Let $\Sigma^n(u)$ be an entire maximal spacelike vertical graph contained in a timelike bounded region of $\overline{M}^{n+1}$, and with bounded second fundamental form A. If $|Du|_{M^n} \leq \alpha f(u)$, for some constant $0 \leq \alpha < 1$, and $|Du|_{M^n} \in \mathcal{L}^1(M^n)$, then $\Sigma^n(u)$ is complete and totally geodesic.*

Remark 5.3 In [3], the first author jointly with Albujer and Camargo obtained uniqueness results concerning to complete spacelike hypersurfaces with constant mean curvature immersed in a RW spacetime which is supposed to obey the null convergence condition (3.6). As an application of such uniqueness results for the case of vertical graphs in a RW spacetime, they also get nonparametric rigidity results.

6 Estimating the relative nullity in RW spacetimes

Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface, with second fundamental form A. According [16], for $p \in \Sigma^n$, we define the *space of relative nullity* $\Delta(p)$ of Σ^n at p by

$$\Delta(p) = \{v \in T_p\Sigma;\, v \in \ker(A_p)\},$$

where $\ker(A_p)$ denotes the kernel of A_p. The *index of relative nullity* $\nu(p)$ of Σ^n at p is the dimension of $\Delta(p)$, that is,

$$\nu(p) = \dim\left(\Delta(p)\right),$$

and the *index of minimum relative nullity* ν_0 of Σ^n is defined by

$$\nu_0 = \min_{p\in\Sigma} \nu(p).$$

In order to prove our next results, we need the following extension of Lemma 4.5 due to Caminha et al. [14].

Lemma 6.1 *Let Σ^n be a complete spacelike hypersurface immersed in a RW spacetime of constant sectional curvature, with bounded second fundamental form. If $g : \Sigma^n \to \mathbb{R}$ is a smooth function such that $|\nabla g| \in \mathcal{L}^1(\Sigma)$ and $L_r g$ does not change sign on Σ^n, then $L_r g = 0$ on Σ^n.*

Our next result establishes a lower estimate of the index of minimum relative nullity in the case of complete r-maximal spacelike hypersurfaces (that is, with zero $(r+1)$-th mean curvature) in a RW spacetime.

Theorem 6.2 *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a complete r-maximal spacelike hypersurface contained in a timelike bounded region of a RW spacetime of constant sectional curvature. Suppose that the second fundamental form of Σ^n is bounded and that H_{r+2} does not change sign on Σ^n. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then the index of minimum relative nullity ν_0 of Σ^n is at least $n - r$.*

Proof Initially, since we are supposing that our ambient RW spacetime $-I \times_f M^n$ has constant sectional curvature, we observe that equations (2.1) guarantee that

$$\frac{\kappa}{f^2(h)} - (\ln f)''(h) \equiv 0,$$

where κ stands for the sectional curvature of the Riemannian fiber M^n. Hence, from Lemma 4.1, we get

$$L_r\left(f(h)\langle N, \partial_t\rangle\right) = -(n-r)\binom{n}{r}H_{r+2}f(h)\langle N, \partial_t\rangle.$$

Thus, the hypothesis that H_{r+2} does not change sign on Σ^n assures that $L_r\left(f(h)\langle N, \partial_t\rangle\right)$ also does not change sign on Σ^n. Consequently, by applying Lemma 6.1, we get that $H_{r+2} = 0$ on Σ^n.

On the other hand, since Σ^n is r-maximal, we have $H_{r+1} = 0$ on Σ^n. Consequently, from Proposition 1(c) of [13], $H_j = 0$ for all $j \geq r+1$, and hence, we conclude that $\nu_0 \geq n-r$. □

Remark 6.3 We observe that the case $r = 0$ of the Theorem 6.2 corresponds to the Theorem 4.6 when the ambient spacetime is a RW spacetime of constant sectional curvature. In this sense, Theorem 6.2 can be seen as a natural extension of Theorem 4.6 for the context of the higher-order mean curvatures.

By considering each one of the three standard RW models for (suitable open sets in) de Sitter space $\mathbb{S}^{n+1}_1$ (see [20], Sect. 4), that is,

$$-\mathbb{R}\times_{\cosh t}\mathbb{S}^n, \quad -\mathbb{R}\times_{e^t}\mathbb{R}^n \quad \text{and} \quad -(0,+\infty)\times_{\sinh t}\mathbb{H}^n,$$

from Theorem 6.2, we get

Corollary 6.4 *Let $\psi : \Sigma^n \to \mathbb{S}^{n+1}_1$ be a complete 1-maximal spacelike hypersurface contained in a timelike bounded region of $\mathbb{S}^{n+1}_1$. Suppose that Σ^n has nonzero constant mean curvature H and that H_3 does not change sign on Σ^n. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then Σ^n is a rotation hypersurface of $\mathbb{S}^{n+1}_1$.*

Proof By our previous theorem, it follows that $\nu_0 \geq n-r$. Therefore, Proposition 1.2 of [8] guarantees that Σ^n is a rotation hypersurface. □

Let us recall that a RW spacetime $-I\times_f M^n$ is said to be *static* when the warping function f is constant. In this case, we can suppose without loss of generality that $f \equiv 1$. Following the ideas of Caminha [13] and using the static RW model of the Minkowski space

$$\mathbb{L}^{n+1} \simeq -\mathbb{R}\times\mathbb{R}^n,$$

we obtain a weak extension of Calabi–Bernstein theorem (see [11], for $n \leq 4$, and [15], for arbitrary n).

Theorem 6.5 *Let $\psi : \Sigma^n \to -I\times M^n$ be a complete spacelike hypersurface immersed in a static RW spacetime of constant sectional curvature, with bounded second fundamental form. Suppose that, for some $r = 0,\ldots,n-2$, H_{r+1} and H_{r+2} do not change sign on Σ^n. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then the index of minimum relative nullity ν_0 of Σ^n is at least $n-r$. Moreover, when the ambient spacetime is the Minkowski space $\mathbb{L}^{n+1}$, if H_r does not vanish on Σ^n then through every point of Σ^n there passes an $(n-r)$-hyperplane of $\mathbb{L}^{n+1}$ totally contained in Σ^n.*

Proof From Lemma 4.1 we have

$$L_r h = -(n-r)\binom{n}{r}H_{r+1}\langle N, \partial_t\rangle.$$

Thus, the hypothesis that H_{r+1} does not change sign on Σ^n assures that $L_r h$ also does not change sign on Σ^n. Consequently, from Lemma 6.1, we conclude that $H_{r+1} = 0$ on Σ^n. With analogous arguments, we also conclude that $H_{r+2} = 0$ on Σ^n. From [13], Proposition 1(c), $H_j = 0$ for all $j \geq r+1$ and, hence, $\nu_0 \geq n-r$.

Now, suppose that the ambient RW spacetime is the Minkowski space $\mathbb{L}^{n+1}$. By Theorem 5.3 of [16] (see also [17]), since we are supposing that H_r does not vanish on Σ^n, the distribution $p \mapsto \Delta(p)$ of minimal relative nullity of Σ^n is smooth and integrable with complete leaves, totally geodesic in Σ^n and in $\mathbb{L}^{n+1}$. Therefore, the result follows from the characterization of complete totally geodesic submanifolds of $\mathbb{L}^{n+1}$ as spacelike hyperplanes of suitable dimension. □

Corollary 6.6 *Let $\psi : \Sigma^n \to -I \times M^n$ be a complete spacelike hypersurface immersed in a static RW spacetime of constant sectional curvature $\overline{\kappa}$, with bounded second fundamental form. Suppose that the mean curvature H does not change sign and that the normalized scalar curvature R satisfies $R \leq \overline{\kappa}$ or $R \geq \overline{\kappa}$. If $|\nabla h| \in \mathcal{L}^1(\Sigma)$, then Σ^n is totally geodesic. Moreover, if the ambient spacetime is the Minkowski space $\mathbb{L}^{n+1}$, then Σ^n is a hyperplane.*

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References

1. Albujer, A.L.: New examples of entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$. Diff. Geom. Its Appl. **26**, 456–462 (2008)
2. Albujer, A.L., Camargo, F., de Lima, H.F.: Complete spacelike hypersurfaces with constant mean curvature in $-\mathbb{R} \times \mathbb{H}^n$. J. Math. Anal. Appl. **368**, 650–657 (2010)
3. Albujer, A.L., Camargo, F., de Lima, H.F.: Complete spacelike hypersurfaces in a Robertson-Walker spacetime. Math. Proc. Camb. Philos. Soc. **151**, 271–282 (2011)
4. Alías, L.J., Colares, A.G.: Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in Generalized Robertson-Walker spacetimes. Math. Proc. Camb. Philos. Soc. **143**, 703–729 (2007)
5. Alías, L.J., Montiel, S.: Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes. Differential Geometry, Valencia, (2001), 59–69, World Sci. Publ., River Edge (2002)
6. Alías, L.J., Romero, A., Sánchez, M.: Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes. Gen. Relativ. Gravit. **27**, 71–84 (1995)
7. Barbosa, J.L.M., Colares, A.G.: Stability of hypersurfaces with constant r-mean curvature. Ann. Glob. Anal. Geom. **15**, 277–297 (1997)
8. Brasil, A. Jr., Colares, A.G., Palmas, O.: Complete spacelike hypersurfaces with constant mean curvature in the de Sitter space: a gap theorem. Ill. J. Math. **47**, 847–866 (2003)
9. Caballero, M., Romero, A., Rubio, R.M.: Constant mean curvature spacelike surfaces in three-dimensional generalized Robertson Walker spacetimes. Lett. Math. Phys. **93**, 85–105 (2010)
10. Caballero, M., Romero, A., Rubio, R.M.: Uniqueness of maximal surfaces in generalized Robertson-Walker spacetimes and Calabi-Bernstein type problems. J. Geom. Phys. **60**, 394–402 (2010)
11. Calabi, E.: Examples of Bernstein problems for some nonlinear equations. Proc. Sympos. Pure Math. **15**, 223–230 (1970)
12. Camargo, F., de Lima, H.F.: New characterizations of totally geodesic hypersurfaces in the anti-de Sitter space $\mathbb{H}_1^{n+1}$. J. Geom. Phys. **60**, 1326–1332 (2010)
13. Caminha, A.: On spacelike hypersurfaces of constant sectional curvature lorentz manifolds. J. Geom. Phys. **56**, 1144–1174 (2006)
14. Caminha, A., Sousa, P., Camargo, F.: Complete foliations of space forms by hypersurfaces. Bull. Braz. Math. Soc. **41**, 339–353 (2010)
15. Cheng, S.Y., Yau, S.T.: Maximal spacelike hypersurfaces in the LorentzMinkowski space. Ann. Math. **104**, 407–419 (1976)

16. Dajczer, M. et al.: Submanifolds and Isometric Immersions. Publish or Perish, Houston (1990)
17. Ferus, D.: On the completeness of nullity foliations. Mich. Math. J. **18**, 61–64 (1971)
18. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Spacetime. Cambridge University Press, Cambridge (1973)
19. Ishihara, T.: Maximal spacelike submanifolds of a pseudoriemannian space of constant curvature. Mich. Math. J. **35**, 345–352 (1988)
20. Montiel, S.: Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes. Math. Ann. **314**, 529–553 (1999)
21. Omori, H.: Isometric immersions of Riemannian manifolds. J. Math. Soc. Jpn. **19**, 205–214 (1967)
22. O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, London (1983)
23. Parente, U.L.: Alguns Resultados Tipo-Bernstein em Variedades Semi-Riemannianas. PhD Thesis, Universidade Federal do Ceará, Brazil (2011)
24. Rainer, M., Schmidt, H.-J.: Inhomogeneous cosmological models with homogeneous inner hypersurface geometry. Gen. Relativ. Gravit. **27**, 1265–1293 (1995)
25. Romero, A., Rubio, R.M.: On the mean curvature of spacelike surfaces in certain three-dimensional Robertson Walker spacetimes and CalabiBernstein type problems. Ann. Glob. Anal. Geom. **37**, 21–31 (2010)
26. Yau, S.T.: Harmonic functions on complete Riemannian manifolds. Comm. Pure Appl. Math. **28**, 201–228 (1975)
27. Yau, S.T.: Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. Indiana Univ. Math. J. **25**, 659–670 (1976)