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Quasistatic crack growth in finite elasticity with Lipschitz data

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Abstract We extend the recent existence result of Dal Maso and Lazzaroni (Ann Inst H Poincaré Anal Non Linéaire 27:257–290, 2010) for quasistatic evolutions of cracks in finite elasticity, allowing for boundary conditions and external forces with discontinuous first derivatives.

Keywords Variational models · Energy minimization · Free-discontinuity problems · Polyconvexity · Quasistatic evolution · Rate-independent processes · Brittle fracture · Crack propagation · Griffith's criterion · Finite elasticity · Non-interpenetration

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0 Introduction

The purpose of this paper is to generalize a recent result [10], concerning the quasistatic evolution of cracks in finite elasticity, in order to cover the case of boundary conditions and external forces that are not smooth in time or space.

The physics of the problem relies on Griffith's principle [19] that the propagation of a crack is the result of the competition between the elastic energy released when the crack opens and the energy spent to produce new crack. The elastic body is represented by a bounded open set $\Omega \subset \mathbb{R}^n$, and the state of the system is described by a pair of variables (u, Γ), where u is the deformation of Ω and Γ is the crack. The internal energy is defined as

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$$\mathcal{E}^{\text{int}}(u,\Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma), \tag{0.1}$$

where $\mathcal{K}(\Gamma)$ is the energy dissipated to open the crack Γ, and $\mathcal{W}(u) = \int_\Omega W(\nabla u(x))\mathrm{d}x$ is the elastic energy stored in the body under the deformation u; it depends on the strain ∇u, according to the hypothesis of hyperelasticity. The body is subject to external forces, dependent on the time instant $t \in [0,1]$, with potential $\mathcal{E}^{\text{ext}}(t,u)$. Hence, the total energy is

$$\mathcal{E}(t,u,\Gamma) := \mathcal{E}^{\text{int}}(u,\Gamma) - \mathcal{E}^{\text{ext}}(t,u). \tag{0.2}$$

Moreover, a time-dependent boundary condition $u = \psi(t)$ can be imposed on a part of $\partial\Omega$.

The variational model, developed by Francfort and Marigo [15], is based on a process of time discretization, which gives rise to some incremental problems, solved through global minimization. In particular, the crack path need not be prescribed a priori, but it is determined by the energy criterion. For an account of the results obtained on this argument, we refer to [5].

In the first existence theorems in the literature, Ω is contained in $\mathbb{R}^2$, the crack Γ is supposed to be a closed set, and the deformation u is represented by a Sobolev function on the domain $\Omega\backslash\Gamma$: this was studied by Dal Maso and Toader [12] and Chambolle [7]. Instead, in the formulation of Francfort and Larsen [14], the functional setting for the deformation is the space of special functions of bounded variation $SBV(\Omega)$, while the crack is a rectifiable set containing the jump $S(u)$: this allows them to consider the case of arbitrary space dimension. All these results were obtained in the case of linearized elasticity, when $W(A) = |A - I|^2$. They were generalized by Dal Maso et al. in [9], where the energy density W is only assumed to be a quasiconvex function with a condition of polynomial growth of the type $c\,|A|^p \le W(A) \le C\,|A|^p$ (here, $c, C > 0$, and $p > 1$).

The usual hypothesis in finite elasticity is that the strain energy diverges as the determinant of the deformation gradient vanishes:

$$\mathcal{W}(u) = +\infty \quad \text{if } \det\nabla u \le 0 \qquad \text{and} \qquad \mathcal{W}(u) \to +\infty \quad \text{if } \det\nabla u \to 0^+. \tag{0.3}$$

This ensures the "physical" feature that the deformations with finite energy preserve orientation, i.e.,

$$\det\nabla u(x) > 0 \quad \text{for a.e. } x \in \Omega. \tag{0.4}$$

Unfortunately, (0.3) is incompatible with polynomial growth, which is a basic tool in the above mentioned articles for proving semicontinuity and controlling energy from above. The previous results were extended in [10] under some general assumptions compatible with finite elasticity, introduced in Ball [4], Francfort and Mielke [16], and Fusco et al. [17].

In [10], we work in spaces of SBV functions, thanks to the hypothesis that the body Ω is confined in a compact set $K \subset \mathbb{R}^n$ where all the deformations take place. We prove the existence of quasistatic evolutions $t \mapsto (u(t), \Gamma(t))$ minimizing (0.2) and satisfying an energy-dissipation balance law, which states that the time derivative of the internal energy $\mathcal{E}^{\text{int}}(u(t),\Gamma(t))$ equals the power of the external forces $\mathcal{E}^{\text{ext}}(t,u(t))$. These are the two fundamental properties of the variational approach to *rate-independent processes* introduced by Mielke (see [21] and the references therein). Moreover, a strong non-interpenetration requirement, called Ciarlet-Nečas condition [8], can be imposed on the solutions, which not only preserve orientation as in (0.4), but are globally invertible, too; this property was studied in the SBV context by Giacomini and Ponsiglione [18].

The key point for the energy estimates is replacing polynomial controls by a bound from above which is compatible with (0.3): namely, we suppose that for every $A \in GL_n^+$

$$\left|A^{\mathrm{T}} \mathrm{D}_A W(A)\right| \leq c_W^1 \left(W(A) + c_W^0\right), \tag{0.5}$$

where $c_W^0 \geq 0$ and $c_W^1 > 0$ are two constants. The *multiplicative stress estimate* (0.5) is well studied in mechanics [4]; in order to exploit it, we use a method introduced in [16] and manipulate the solutions in a multiplicative way. More precisely, we look for minimizers to (0.2) of the form

$$u = \psi(t) \circ z, \tag{0.6}$$

where z coincides with the identity function on the Dirichlet part of $\partial\Omega$. This can be done provided that the boundary datum $\psi(t)$ is extended to a function defined on the whole set K (which contains Ω) and is a diffeomorphism of K onto itself.

Following [16], in [10] we suppose both $\psi(t)(x)$ and its spatial gradient $\nabla\psi(t)(x)$ to be of class C^1 in (t, x), and the same for the spatial inverse $\phi(t) := \psi(t)^{-1}$. These hypotheses, which were made for the sake of simplicity, are not satisfactory for two reasons:

- the spatial smoothness is a strong requirement (while the solutions are only SBV);
- the class of data is not invariant under Lipschitz time reparametrizations.

In the present paper, we assume that $\psi, \phi \in W^{1,\infty}([0, 1]; W^{1,\infty}(K; K))$, which implies they are Lipschitz in both variables, but not necessarily C^1 (see Sect. 1.7 for the detailed definition of this space). Hence, we consider a wider class of data, which is invariant under Lipschitz reparametrizations of time: this is important from the point of view of rate-independent processes.

Due to the lack of regularity, the chain rule is non-trivial when deriving (0.6): indeed, if z is SBV it may happen that the counterimage through z of the set of points of non-differentiability of $\psi(t)$ is a set of positive measure. This does not occur in our case because det ∇z is a.e. positive, as well as det ∇u (see Remark 1.2 and Lemma 2.1 for the details). Notice that this property follows from (0.4) and does not require global invertibility.

Following [9], in this work we introduce also volume and surface forces ($\mathcal{E}^{\text{ext}}$ in (0.2)), which were not present in [10]. As we employ the multiplicative splitting (0.6), the minimal hypotheses on the external forces are strictly related with those on the boundary data. The assumptions we make here (see Sect. 1.6) are compatible with Lipschitz reparametrizations of time; moreover, they hold in the case of *dead loads* (Example 1.1).

The multiplicative splitting method leads us to an alternative formulation of the problem, where the time dependence of the boundary conditions is transferred to the volume energy terms (see Sect. 2.2). As in [10], we are interested in *incrementally approximable quasistatic evolutions* (Definition 3.2). The proof of the global minimality and of the energy balance (Theorem 3.3) requires some remarks about the consequences of (0.5), stated in Sect. 2.3, and some results concerning the approximation of Lebesgue integrals with Riemann sums (Lemmas 3.2 and 3.3).

The structure of the article is the following. In Sect. 1, we introduce the hypotheses on the geometry of the body, on the strain energy, on the external forces, and on the prescribed deformations. In Sect. 2, we present the multiplicative splitting method and the auxiliary formulation with time-independent boundary data, with the properties of the energy terms after the change of variables (0.6). Section 3 is devoted to the definition of quasistatic evolution and to the proof of the main results.

1 Setting of the problem

1.1 Notation

Throughout the paper, $n \geq 2$ is fixed; the symbol $\cdot$ stands for the Euclidean scalar product on $\mathbb{R}^n$ and $|\cdot|$ for the corresponding norm. The $n \times n$ real matrices are denoted by $\mathbb{M}^{n\times n}$, the ones with positive determinant by GL_n^+, and the rotation matrices by SO_n; the symbol I stands for the identity matrix. The space $\mathbb{M}^{n\times n}$ is endowed with the scalar product $A : B := \operatorname{tr} AB^{\mathrm{T}}$; we denote by $|\cdot|$ the corresponding norm. Given $A \in \mathbb{M}^{n\times n}$, we define $\operatorname{adj}_j A$ as the vector composed of the minors of A of order j.

We call *modulus of continuity* a non-decreasing function $\omega\colon [0, 1] \to [0, +\infty)$, such that $\omega(h) \to 0$ as $h \to 0$.

Henceforth, $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$, while $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The expression *almost everywhere* (*a.e.*) refers to $\mathcal{L}^n$ unless otherwise specified. Given two sets A and B in $\mathbb{R}^n$, we say that $A \,\tilde{\subset}\, B$ when $\mathcal{H}^{n-1}(A\backslash B) = 0$ and that $A \,\tilde{=}\, B$ when $\mathcal{H}^{n-1}(A \triangle B) = 0$, where $A \triangle B$ stands for the symmetric difference $(A\backslash B) \cup (B\backslash A)$.

We refer to [2] for all the following definitions. For a bounded open set $U \subset \mathbb{R}^n$ and $m \geq 1$, $BV(U; \mathbb{R}^m)$ is the space of *functions of bounded variation* and $SBV(U; \mathbb{R}^m)$ the subspace of *special functions of bounded variation*. The symbol $\mathrm{D}u$ stands for the *gradient* of u, $|\mathrm{D}u|(U)$ for its total variation, ∇u for its *absolutely continuous part*, and $\mathrm{D}^j u$ for its *jump part*. We denote the *jump set* of u by $S(u)$ and its *unit normal vector field* by ν_u. For $p > 1$, we consider the subspace

$$SBV^p(U; \mathbb{R}^m) := \left\{u \in SBV(U; \mathbb{R}^m)\colon \nabla u \in L^p(U; \mathbb{M}^{m\times n})\right\},$$

endowed with the norm

$$\|u\|_{SBV^p(U;\mathbb{R}^m)} := \int_U |u| \,\mathrm{d}x + \left(\int_U |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + |\mathrm{D}u|(U).$$

In $SBV^p(U; \mathbb{R}^m)$, we provide the following notion of weak* convergence.

Definition 1.1 A sequence u_k *converges to* u *weakly* in* $SBV^p(U; \mathbb{R}^m)$ if

- $u_k, u \in SBV^p(U; \mathbb{R}^m)$;
- $u_k \to u$ in measure;
- $\|u_k\|_{L^\infty(U;\mathbb{R}^m)}$ is bounded uniformly with respect to k;
- $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(U; \mathbb{M}^{m\times n})$;
- $\mathcal{H}^{n-1}(S(u_k))$ is bounded uniformly with respect to k.

1.2 The geometry of the body

As in [10], we will consider the deformations of an elastic body, whose *reference configuration* is the closure $\overline{\Omega}$ of a bounded open set $\Omega \subset \mathbb{R}^n$. We suppose that $\Omega \subset K$, i.e., the body is confined in a *container* K, the closure of a bounded open set. The body has a *brittle part* $\overline{\Omega}_B$, the closure of an open subset Ω_B of Ω. We fix an open set Ω_D with $\Omega \subset \Omega_D \subset K$: the set $\Omega_D\backslash\Omega$ is an *unbreakable body*, whose deformation is known, in contact with Ω. We will assume that K, Ω_D, Ω, and Ω_B have Lipschitz boundaries. The *Dirichlet part* of the boundary of Ω is $\partial_D\Omega := \partial\Omega \cap \Omega_D$, while $\partial_N\Omega := \partial\Omega\backslash\partial_D\Omega$ is the *Neumann part*. Moreover, a surface force is acting on a closed set $\partial_S\Omega \subset \partial_N\Omega$. We need a technical requirement:

$$\overline{\Omega}_B \cap \partial_D\Omega = \emptyset \quad \text{and} \quad \overline{\Omega}_B \cap \partial_S\Omega = \emptyset; \tag{1.1}$$

this means that $\Omega\backslash\overline{\Omega}_B$ is a *layer of unbreakable material* where the surface deformations are impressed. We refer to [10] for further comments.

1.3 Admissible cracks and deformations

The state of the system is described by a pair of variables (u, Γ), where u is the deformation of the domain and Γ is its fracture. More precisely, the *admissible cracks* of Ω are given by

$$\mathcal{R} := \left\{\Gamma\colon \text{countably } (\mathcal{H}^{n-1}, n-1)\text{-rectifiable}, \ \Gamma \mathrel{\tilde{\subset}} \overline{\Omega}_B \cap \Omega_D, \mathcal{H}^{n-1}(\Gamma) < +\infty\right\}, \tag{1.2}$$

while the deformations of Ω_D are represented by functions in $SBV(\Omega_D; K)$, which is defined as the set of functions $u \in SBV(\Omega_D; \mathbb{R}^n)$ such that $u(x) \in K$ for a.e. $x \in \Omega_D$. The deformation and the crack are related by the inclusion $S(u) \mathrel{\tilde{\subset}} \Gamma$.

Furthermore, we require a condition of non-interpenetration of matter in the sense of Ciarlet and Nečas [8]; the definition was proposed in [18].

Definition 1.2 A function $u \in SBV(\Omega_D; K)$ satisfies the *Ciarlet-Nečas non-interpenetration condition* if the following hold:

(CN1) *u preserves orientation*, i.e., for a.e. $x \in \Omega_D$, $\det \nabla u(x) > 0$;
(CN2) u is *a.e.-injective*, i.e., there exists a set $N \subset \Omega_D$, with $\mathcal{L}^n(N) = 0$, such that u is injective on $\Omega_D\backslash N$.

In the following remarks, we state some consequences of (CN1), which will be fundamental in the sequel.

Remark 1.1 Arguing as in [18], one can see that if $u \in SBV(\Omega_D; K)$ satisfies (CN1), then for any $E \subset \Omega_D$

$$\int_E |\det \nabla u| \, \mathrm{d}x = \int_{\mathbb{R}^n} m(\bar{u}, y, E) \, \mathrm{d}y, \tag{1.3}$$

where

$$m(\bar{u}, y, E) := \operatorname{card}\{x \in E\colon \bar{u}(x) = y\},$$

while $\bar{u}$ is a representative of u which coincides with the approximate limit of u on the set of points of approximate differentiability and is zero elsewhere.

Remark 1.2 It is possible to prove that, if u satisfies (CN1) and $\mathcal{L}^n(F) = 0$, then $\mathcal{L}^n(u^{-1}(F)) = 0$ (independently on the choice of the representative of u).

Indeed, by (CN1) and (1.3) with $E = \bar{u}^{-1}(F)$, we get $\mathcal{L}^n(\bar{u}^{-1}(F)) = 0$ ($\bar{u}$ is the representative of u introduced in the previous remark). If $\tilde{u}$ is another representative of u, $\tilde{u}^{-1}(F)$ differs from $\bar{u}^{-1}(F)$ by a set of null measure, so $\mathcal{L}^n(\tilde{u}^{-1}(F)) = 0$, too.

Remark 1.3 Every function u satisfying (CN1) has the following property: given a measurable set M, the preimage $u^{-1}(M)$ is measurable.

Indeed, we can write $M = B \cup M_0$, with B Borel and M_0 negligible; then, $u^{-1}(B)$ is measurable and, by Remark 1.2, $u^{-1}(M_0)$ has null measure. This implies that $u^{-1}(M) = u^{-1}(B) \cup u^{-1}(M_0)$ is measurable.

The prescribed deformation of $\Omega_D \backslash \overline{\Omega}$ is given by a function $\psi \in W^{1,1}(\Omega_D \backslash \overline{\Omega}; K)$. The Dirichlet condition takes the form $u = \psi$ a.e. in $\Omega_D \backslash \overline{\Omega}$; on $\partial_D \Omega$ the equality $u = \psi$ is satisfied in the sense of traces, because by (1.1) u is of class $W^{1,1}$ in the neighbourhood $\Omega_D \backslash \overline{\Omega}_B$ of $\partial_D \Omega$ (recall that $S(u) \mathrel{\tilde{\subset}} \Gamma$). We refer to [10] for further comments.

The *admissible deformations*, corresponding to a crack $\Gamma \in \mathcal{R}$ and a Dirichlet datum $\psi \in W^{1,1}(\Omega_D \backslash \overline{\Omega}; K)$, are

$$AD(\psi, \Gamma) := \Big\{ u \in SBV(\Omega_D; K) \colon u \text{ satisfies (CN1), (CN2)}, \\ u|_{\Omega_D \backslash \overline{\Omega}} = \psi, \text{ and } S(u) \mathrel{\tilde{\subset}} \Gamma \Big\}. \tag{1.4}$$

At each time $t \in [0, 1]$, given ψ and Γ, we are looking for deformations $u \in AD(\psi, \Gamma)$ minimizing the total energy

$$\mathcal{E}(t, u, \Gamma) := \mathcal{E}^{\mathrm{el}}(t, u) + \mathcal{K}(\Gamma), \tag{1.5}$$

with

$$\mathcal{E}^{\mathrm{el}}(t, u) := \mathcal{W}(u) - \mathcal{G}(t, u) - \mathcal{S}(t, u), \tag{1.6}$$

where $\mathcal{W}$ represents the bulk energy, $\mathcal{K}$ is the energy spent to produce the crack, $\mathcal{G}$ is the potential of the volume forces, and $\mathcal{S}$ is the potential of the surface forces. Their properties are stated in the following sections.

1.4 Bulk energy

We quote from [10] the hypotheses on the bulk energy, which were presented in [4,16,17]. They are compatible with the setting of finite elasticity, in particular with the case of Ogden materials (see [10, Example 1.8]).

The *bulk energy* on Ω of any deformation $u \in SBV(\Omega_D; K)$ is

$$\mathcal{W}(u) := \int_{\Omega} W(x, \nabla u(x))\, \mathrm{d}x, \tag{1.7}$$

where $W \colon \Omega \times \mathbb{M}^{n \times n} \to [0, +\infty]$ satisfies the following properties:

(W0) *Frame indifference:* for every $(x, A) \in \Omega \times \mathbb{M}^{n \times n}$, $W(x, QA) = W(x, A)$ for every $Q \in SO_n$;

(W1) *Polyconvexity:* there exists a function $\widetilde{W} \colon \Omega \times \mathbb{R}^{\tau} \to [0, +\infty]$ such that $x \mapsto \widetilde{W}(x, \xi)$ is $\mathcal{L}^n$-measurable on Ω for every $\xi \in \mathbb{R}^{\tau}$, $\xi \mapsto \widetilde{W}(x, \xi)$ is continuous and convex on $\mathbb{R}^{\tau}$ for every $x \in \Omega$, and $W(x, A) = \widetilde{W}(x, M(A))$ for every $(x, A) \in \Omega \times \mathbb{M}^{n \times n}$, where $M(A) := (\mathrm{adj}_1 A, \ldots, \mathrm{adj}_n A)$ is the vector (of dimension $\tau := \tau_1 + \cdots + \tau_n$) composed of all minors of A;

(W2) *Finiteness and regularity:* for every $x \in \Omega$ we have $W(x, A) < +\infty$ if and only if $A \in GL_n^+$; moreover, $A \mapsto W(x, A)$ is of class C^1 on GL_n^+.

Furthermore, we require that there exist a function $c_W^0 \in L^1_+(\Omega)$, some constants $c_W^1 > 0$, $\beta_W^0 \geq 0$, $\beta_W^1, \ldots, \beta_W^n > 0$ and some exponents $p_1, p_2, \ldots, p_n$, such that for every $x \in \Omega$ the following hold:

(W3) *Bound at identity:* we have $W(x, I) \leq c_W^0(x)$;

(W4) *Lower growth condition:* for every $A \in \mathbb{M}^{n\times n}$

$$W(x, A) \geq \sum_{j=1}^{n} \beta_W^j \left|\mathrm{adj}_j A\right|^{p_j} - \beta_W^0,$$

with $p_1 \geq 2$, $p_j \geq p_1' := \frac{p_1}{p_1-1}$ for $j = 2, \dots, n-1$, and $p_n > 1$;

(W5) *Multiplicative stress estimate:* for every $A \in GL_n^+$

$$\left|A^{\mathrm{T}}\, \mathrm{D}_A W(x, A)\right| \leq c_W^1 \left(W(x, A) + c_W^0(x)\right);$$

(W6) *Continuity of Kirchhoff stress:* for every $\varepsilon > 0$, there exists $\delta > 0$, independent of x, such that for every $A \in GL_n^+$ and $B \in GL_n^+$ with $|B - I| < \delta$

$$\left|\mathrm{D}_A W(x, BA)\,(BA)^{\mathrm{T}} - \mathrm{D}_A W(x, A)\, A^{\mathrm{T}}\right| \leq \varepsilon \left(W(x, A) + c_W^0(x)\right).$$

Henceforth, we will set $p := p_1$.

Remark 1.4 If a deformation $u \in SBV(\Omega_D; K)$ is such that $\mathcal{W}(u) < +\infty$, then by (W2) u preserves orientation as in (CN1); moreover, by (W4) u belongs to the space $SBV^p(\Omega_D; K)$. Hypotheses (W1) and (W4) guarantee the semicontinuity of $\mathcal{W}$ with respect to the weak* convergence in $SBV^p(\Omega_D; K)$, thanks to [17, Theorem 3.5]; see also [10, Theorem 3.1]. Properties (W5) and (W6) will be used for the proof of the global stability and of the energy balance. Notice that (W6) can be avoided in the case of a pure Neumann problem (or in the case of a Dirichlet problem with time-independent boundary conditions): see [11] for the details.

1.5 The crack energy and the σ^p-convergence

In this section, we define the energy spent to produce a crack and highlight its semicontinuity properties. Beforehand, we give a notion of convergence in the set of admissible cracks, introduced in [9].

Definition 1.3 A sequence Γ_k *σ^p-converges to* Γ if Γ_k and Γ are contained in Ω_D, $\mathcal{H}^{n-1}(\Gamma_k)$ is bounded uniformly with respect to k, and the following conditions are satisfied:

- if u_j converges weakly* to u in $SBV^p(\Omega_D)$ and $S(u_j) \,\tilde{\subset}\, \Gamma_{k_j}$ for some sequence $k_j \to \infty$, then $S(u) \,\tilde{\subset}\, \Gamma$;
- there exist a function $u \in SBV^p(\Omega_D)$ and a sequence u_k converging to u weakly* in $SBV^p(\Omega_D)$ such that $S(u) \,\tilde{=}\, \Gamma$ and $S(u_k) \,\tilde{\subset}\, \Gamma_k$ for every k.

According to Griffith's theory [19], we assume that the *energy spent to produce the crack* $\Gamma \in \mathcal{R}$ is given by

$$\mathcal{K}(\Gamma) := \int_{\Gamma} \kappa(x, \nu_\Gamma(x))\, \mathrm{d}\mathcal{H}^{n-1}(x), \tag{1.8}$$

where ν_Γ is a unit normal vector field on Γ and $\kappa\colon (\overline{\Omega}_B \cap \Omega_D) \times \mathbb{R}^n \to \mathbb{R}$ is a locally bounded Borel function. We suppose that

(K1) for every $\varepsilon > 0$ there exists an open set A of 1-capacity $C_1(A) < \varepsilon$ such that $x \mapsto \kappa(x, \nu)$ is lower semicontinuous on $(\overline{\Omega}_B \cap \Omega_D)\setminus A$ for every $\nu \in \mathbb{R}^n$,

(K2) $\nu \mapsto \kappa(x, \nu)$ is a norm on $\mathbb{R}^n$ for every $x \in \overline{\Omega}_B \cap \Omega_D$, (K3) $\kappa_1 |\nu| \leq \kappa(x,\nu) \leq \kappa_2 |\nu|$ for every $(x,\nu) \in (\overline{\Omega}_B \cap \Omega_D) \times \mathbb{R}^n$,

for some constants $\kappa_1 > 0$ and $\kappa_2 > 0$. As a consequence, we have

$$\kappa_1 \mathcal{H}^{n-1}(\Gamma) \leq \mathcal{K}(\Gamma) \leq \kappa_2 \mathcal{H}^{n-1}(\Gamma). \tag{1.9}$$

To simplify the exposition of auxiliary results, we extend κ to $\Omega_D \times \mathbb{R}^n$ by setting $\kappa(x,\nu) := \kappa_2 |\nu|$ if $x \in \Omega_D \setminus \overline{\Omega}_B$, and we define $\mathcal{K}(\Gamma)$ by (1.8) for every countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable subset Γ of $\mathbb{R}^n$.

The crack energy is lower semicontinuous with respect to the σ^p-convergence: this fact can be deduced by [1, Theorem 3.3], adapting the result as in [9, Theorems 2.8 and 4.3].

Theorem 1.1 (Semicontinuity) *Let κ satisfy* (K1–3)*, let Γ_0, Γ_k, and Γ be countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable subsets of Ω_D with $\mathcal{H}^{n-1}(\Gamma_0) < +\infty$, and let E be an $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(E) < +\infty$. If Γ_k σ^p-converges to Γ, then*

$$\int_{(\Gamma \cup \Gamma_0)\setminus E} \kappa(x,\nu)\, \mathrm{d}\mathcal{H}^{n-1}(x) \leq \liminf_{k\to\infty} \int_{(\Gamma_k \cup \Gamma_0)\setminus E} \kappa(x,\nu_k)\, \mathrm{d}\mathcal{H}^{n-1}(x), \tag{1.10}$$

where ν and ν_k are unit normal vector fields on $\Gamma \cup \Gamma_0$ and $\Gamma_k \cup \Gamma_0$, respectively.

1.6 Forces

The body is subjected to a conservative volume force, depending on time, with potential $G\colon [0,1] \times \Omega \times K \to \mathbb{R}$. We suppose that for every $t \in [0,1]$, $(x,y) \mapsto G(t,x,y)$ is $\mathcal{L}^n(\Omega)$-measurable in x and continuous in y, so that we can define the work of the body force under any deformation $u \in L^\infty(\Omega; K)$

$$\mathcal{G}(t,u) := \int_\Omega G(t,x,u(x))\, \mathrm{d}x. \tag{1.11}$$

As for the regularity in time and space, following [9] we prefer to prescribe hypotheses on the functional $\mathcal{G}$ rather than on the integrand G. We assume that there is an exponent $q \geq 1$ such that the following hold:

(G1) there is a constant $c_G > 0$ such that, for every $t \in [0,1]$, every $u \in L^\infty(\Omega; K)$, and every $v, w \in L^\infty(\Omega; \mathbb{R}^n)$ such that $u+v, u+w, u+v+w \in L^\infty(\Omega; K)$, we have

$$|\mathcal{G}(t,u)| \leq c_G,$$
$$|\mathcal{G}(t,u+v) - \mathcal{G}(t,u)| \leq c_G \|v\|_{L^q},$$
$$|\mathcal{G}(t,u+v+w) - \mathcal{G}(t,u+v) - \mathcal{G}(t,u+w) + \mathcal{G}(t,u)| \leq c_G \|v\|_{L^q} \|w\|_{L^q};$$

(G2) there is a function $a_G \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u \in L^\infty(\Omega; K)$

$$|\mathcal{G}(t_2,u) - \mathcal{G}(t_1,u)| \leq \int_{t_1}^{t_2} a_G(s)\, \mathrm{d}s;$$

(G3) there is a function $b_G \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u_1, u_2 \in L^\infty(\Omega; K)$

$$|\mathcal{G}(t_2,u_1) - \mathcal{G}(t_1,u_1) - \mathcal{G}(t_2,u_2) + \mathcal{G}(t_1,u_2)| \leq \int_{t_1}^{t_2} b_G(s)\, \mathrm{d}s\, \|u_1 - u_2\|_{L^q}.$$

In the previous formulas, the symbol $\|\cdot\|_{L^q}$ stands for $\|\cdot\|_{L^q(\Omega;\mathbb{R}^n)}$. Thanks to (G2), the function $t \mapsto \mathcal{G}(t,u)$ is absolutely continuous on [0, 1] for every $u \in L^\infty(\Omega; K)$, so that $\mathrm{D}_t\mathcal{G}(t,u)$ is defined $\mathcal{L}^1$-a.e.; hence, (G3) is equivalent to requiring that for every $u_1, u_2 \in L^\infty(\Omega; K)$

$$|\mathrm{D}_t\mathcal{G}(t,u_1) - \mathrm{D}_t\mathcal{G}(t,u_2)| \leq b_G(t)\, \|u_1 - u_2\|_{L^q(\Omega;\mathbb{R}^n)} \quad \text{for } \mathcal{L}^1\text{−}a.e. \quad t \in [0,1],$$

where $b_G(t)$ denotes the approximate limit of b_G at the Lebesgue points. Analogously, (G1) provides estimates on $\mathrm{D}_u\mathcal{G}$ and $\mathrm{D}^2_u\mathcal{G}$, if they exist.

We impose also a boundary force, with potential $S\colon [0,1] \times \partial_S\Omega \times K \to \mathbb{R}$, ($\mathcal{H}^{n-1}$-measurable in the second variable and continuous in the third), so that the work of the surface force for a deformation $u \in L^1(\partial_S\Omega; K)$ is

$$\mathcal{S}(t,u) := \int_{\partial_S\Omega} S(t,x,u(x))\, \mathrm{d}\mathcal{H}^{n-1}(x). \tag{1.12}$$

We impose these conditions on $\mathcal{S}$:

(S1) there is a constant $c_S > 0$ such that, for every $t \in [0,1]$, every $u \in L^\infty(\partial_S\Omega; K)$, and every $v, w \in L^\infty(\partial_S\Omega; \mathbb{R}^n)$ such that $u{+}v, u{+}w, u{+}v{+}w \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{S}(t,u)| \leq c_S,$$
$$|\mathcal{S}(t,u{+}v) - \mathcal{S}(t,u)| \leq c_S\, \|v\|_{L^q},$$
$$|\mathcal{S}(t,u{+}v{+}w) - \mathcal{S}(t,u{+}v) - \mathcal{S}(t,u{+}w) + \mathcal{S}(t,u)| \leq c_S\, \|v\|_{L^q}\, \|w\|_{L^q};$$

(S2) there is a function $a_S \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{S}(t_2,u) - \mathcal{S}(t_1,u)| \leq \int_{t_1}^{t_2} a_S(s)\, \mathrm{d}s;$$

(S3) there is a function $b_S \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $u_1, u_2 \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{S}(t_2,u_1) - \mathcal{S}(t_1,u_1) - \mathcal{S}(t_2,u_2) + \mathcal{S}(t_1,u_2)| \leq \int_{t_1}^{t_2} b_S(s)\, \mathrm{d}s\, \|u_1 - u_2\|_{L^q}.$$

In the previous formulas, the symbol $\|\cdot\|_{L^q}$ stands for $\|\cdot\|_{L^q(\partial_S\Omega;\mathbb{R}^n)}$. Also in this case, the function $t \mapsto \mathcal{S}(t,u)$ is absolutely continuous on [0, 1] for every $u \in L^\infty(\partial_S\Omega; K)$, and the time derivative exists $\mathcal{L}^1$-a.e.

Notice that if $u \in AD(\psi, \Gamma)$ for some $\psi \in W^{1,1}(\Omega_D\backslash\overline{\Omega}; K)$ and some $\Gamma \in \mathcal{R}$, since by (1.1) u is of class $W^{1,1}$ in the neighbourhood $\Omega_D\backslash\overline{\Omega}_B$ of $\partial_S\Omega$, one can define its trace on $\partial_S\Omega$. Moreover, by the confinement condition, the trace takes values in K, so that $\mathcal{S}(t,u)$ is well defined.

Remark 1.5 In the case of a pure Neumann problem (or in the case of a Dirichlet problem with time-independent boundary conditions), the last estimates of (G1) and of (S1) can be avoided: see Sect. 2.4 for the details.

Remark 1.6 If (G1–3) and (S1–3) are satisfied for an exponent q, then they hold even substituting q with any $r \geq q$. So, the bigger is the exponent, the weaker are the assumptions.

Remark 1.7 Properties (G1–3) are satisfied if we assume the following requirements on the integrand $G(t, x, y)$:

- there exists a non-negative function $\alpha_G^1 \in L^1(\Omega)$ such that for every $(t, x, y) \in [0, 1] \times \Omega \times K$

$$|G(t, x, y)| \le \alpha_G^1(x);$$

- there exists a non-negative function $\alpha_G^2 \in L^{\frac{q}{q-1}}(\Omega)$ such that for every $(t, x, y) \in [0, 1] \times \Omega \times K$ and every $y' \in \mathbb{R}^n$ such that $y+y' \in K$

$$\left|G(t, x, y+y') - G(t, x, y)\right| \le \alpha_G^2(x) \left|y'\right|;$$

- there exists a non-negative function $\alpha_G^3 \in L^{\frac{q}{q-2}}(\Omega)$ such that for every $(t, x, y) \in [0, 1] \times \Omega \times K$ and every $y', y'' \in \mathbb{R}^n$ such that $y+y', y+y'' \in K$

$$\left|G(t, x, y+y'+y'') - G(t, x, y+y') - G(t, x, y+y'') + G(t, x, y)\right| \le \alpha_G^3(x) \left|y'\right| \left|y''\right|;$$

- there exists a non-negative function $\alpha_G^4 \in L^1([0, 1]; L^1(\Omega))$ such that for every $(x, y) \in \Omega \times K$ and every $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$

$$|G(t_2, x, y) - G(t_1, x, y)| \le \int_{t_1}^{t_2} \alpha_G^4(s)(x)\, \mathrm{d}s;$$

- there exists a non-negative function $\alpha_G^5 \in L^1([0, 1]; L^{\frac{q}{q-1}}(\Omega))$ such that for every $(x, y) \in \Omega \times K$, every $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, and every $y' \in \mathbb{R}^n$ such that $y+y' \in K$

$$\left|G(t_2, x, y+y') - G(t_1, x, y+y') - G(t_2, x, y) + G(t_1, x, y)\right| \le \left|y'\right| \int_{t_1}^{t_2} \alpha_G^5(s)(x)\, \mathrm{d}s.$$

Analogous hypotheses can be made on $S(t, x, y)$.

Example 1.1 The properties we assumed are compatible with the case of *dead loads*, where the density of the forces per unit volume in the reference configuration does not depend on the deformation. Let $r > 1$; if $g(t, \cdot) \in L^r(\Omega; \mathbb{R}^n)$ and $s(t, \cdot) \in L^r(\partial_S\Omega; \mathbb{R}^n)$ are the densities of the body and surface force at time t, we set $G(t, x, y) := g(t, x) \cdot y$ and $S(t, x, y) := s(t, x) \cdot y$. If we suppose that $t \mapsto g(t, \cdot)$ and $t \mapsto s(t, \cdot)$ are absolutely continuous into $L^r(\Omega; \mathbb{R}^n)$ and $L^r(\partial_S\Omega; \mathbb{R}^n)$, respectively, then (G1–3) and (S1–3) are satisfied with $q = r' := \frac{r}{r-1}$.

Remark 1.8 We have seen that by (G2), for every $u \in L^\infty(\Omega; K)$, there is an $\mathcal{L}^1$-negligible set N_u such that $\mathrm{D}_t\mathcal{G}(t, u)$ exists for $t \notin N_u$. We would like to redefine this derivative in such a way that the exceptional set does not depend on u.

Fix a countable set D, dense in $L^\infty(\Omega; K)$ with respect to the norm of $L^q(\Omega; \mathbb{R}^n)$. Let $N_D := \left(\bigcup_{u\in D} N_u\right) \cup N_G$, where N_G is an $\mathcal{L}^1$-negligible set such that each $t \notin N_G$ is a Lebesgue point for the function b_G of (G3). For $u \in D$, define

$$\mathrm{D}_t^*\mathcal{G}(t, u) := \begin{cases} \mathrm{D}_t\mathcal{G}(t, u) & \text{if } t \notin N_D, \\ 0 & \text{if } t \in N_D. \end{cases}$$

By (G3), we have for every $u_1, u_2 \in D$ and every t

$$\left|\mathrm{D}_t^*\mathcal{G}(t, u_1) - \mathrm{D}_t^*\mathcal{G}(t, u_2)\right| \le b_G(t)\, \|u_1 - u_2\|_{L^q(\Omega;\mathbb{R}^n)}\,.$$

Then we can extend $\mathrm{D}_t^*\mathcal{G}(t,\cdot)$ to a $L^q(\Omega;\mathbb{R}^n)$-Lipschitz function on $L^\infty(\Omega; K)$.

Let $u \in L^\infty(\Omega; K)$ and $u_k \in D$ such that u_k converges to u in $L^q(\Omega;\mathbb{R}^n)$. If $t \notin N_u \cup N_D$, we have by (G3)

$$\left|\mathrm{D}_t\mathcal{G}(t, u) - \mathrm{D}_t^*\mathcal{G}(t, u_k)\right| \le b_G(t)\, \|u - u_k\|_{L^q(\Omega;\mathbb{R}^n)}\,,$$

so that, passing to the limit as $k \to \infty$, we get $\mathrm{D}_t^*\mathcal{G}(t,u) = \mathrm{D}_t\mathcal{G}(t,u)$. We have proven that for every $t \in [0,1]$ there exists a $L^q(\Omega;\mathbb{R}^n)$-Lipschitz function $\mathrm{D}_t^*\mathcal{G}(t,\cdot)$ such that for every $u \in L^\infty(\Omega; K)$ we have $\mathrm{D}_t^*\mathcal{G}(t,u) = \mathrm{D}_t\mathcal{G}(t,u)$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$.

Arguing in the same way, we can find a function $\mathrm{D}_t^*\mathcal{S}(t,\cdot)$ with analogous properties. In the following integral formulas, we will identify $\mathrm{D}_t\mathcal{G}(t,\cdot)$ and $\mathrm{D}_t\mathcal{S}(t,\cdot)$ with $\mathrm{D}_t^*\mathcal{G}(t,\cdot)$ and $\mathrm{D}_t^*\mathcal{S}(t,\cdot)$, respectively.

1.7 Prescribed deformations

At every time $t \in [0,1]$, we prescribe the deformation of $\Omega_D\backslash\Omega$, requiring that $u(x) = \psi(t,x)$ for a.e. $x \in \Omega_D\backslash\Omega$. As in [10], we suppose that $x \mapsto \psi(t,x)$ is defined for every $x \in K$, takes values in K, and has an inverse function on K, denoted by $y \mapsto \phi(t,y)$. This determines two functions

$$\psi,\ \phi\colon [0,1]\times K \to K,$$

satisfying, for every $(t,x) \in [0,1]\times K$,

$$\psi(t,\phi(t,x)) = x = \phi(t,\psi(t,x)). \tag{BC1}$$

In the present work, we weaken the hypotheses on the prescribed deformations made in [10]. As for the space dependence, we assume that, for every $t \in [0,1]$, $\psi(t) := \psi(t,\cdot)$ and $\phi(t) := \phi(t,\cdot)$ are Lipschitz functions of K in itself. To be more precise, consider the Sobolev space $W^{1,\infty}(\mathring{K}; K)$; since ∂K is Lipschitz, by standard results every function $v \in W^{1,\infty}(\mathring{K}; K)$ admits a Lipschitz continuous representative $\bar{v}$. We extend each function $\bar{v}$ to K and, with a slight abuse of notation, denote by $W^{1,\infty}(K;K)$ the space of all such extensions $\bar{v}$, endowed with the complete norm $\|v\|_{W^{1,\infty}(K;K)} := \sup_K |v| + \sup_K |\nabla v|$.

As for the time dependence, we require that

$$\psi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K)) \tag{BC2}$$

and

$$\phi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K)), \tag{BC3}$$

where $W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$ denotes the space of Sobolev functions valued in $W^{1,\infty}(K;K)$. By definition, this means that $\psi,\phi \in C^0([0,1]; W^{1,\infty}(K;K))$ and there exist two functions $\dot\psi, \dot\phi \in L^\infty([0,1]; W^{1,\infty}(K;K))$ such that for every $t \in [0,1]$

$$\psi(t) = \psi(0) + \int_0^t \dot\psi(s)\,\mathrm{d}s \quad\text{and}\quad \phi(t) = \phi(0) + \int_0^t \dot\phi(s)\,\mathrm{d}s, \tag{1.13}$$

where the integrals are defined in the sense of Bochner, with respect to the topology of $W^{1,\infty}(K;K)$. In particular, the Jacobian matrices $\nabla\psi$, $\nabla\phi$, $\nabla\dot{\psi}$, and $\nabla\dot{\phi}$ are defined a.e. in K. For an overview about the spaces of Sobolev functions valued in a Banach space, we refer to [6, Appendix].

Remark 1.9 In particular, these hypotheses imply that there exists $l > 0$ such that for every $t, t_1, t_2 \in [0,1]$

$$\|\psi(t)\|_{W^{1,\infty}(K;K)} \le l, \quad \|\phi(t)\|_{W^{1,\infty}(K;K)} \le l, \tag{1.14}$$

$$\|\psi(t_1)-\psi(t_2)\|_{W^{1,\infty}(K;K)} \le l\,|t_1-t_2|\,, \quad \|\phi(t_1)-\phi(t_2)\|_{W^{1,\infty}(K;K)} \le l\,|t_1-t_2|\,, \tag{1.15}$$

so $t \mapsto \psi(t)$ and $t \mapsto \phi(t)$ are Lipschitz functions of $[0,1]$ in $W^{1,\infty}(K;K)$. Since the difference quotients are bounded by a constant depending on the maximum of the derivatives and on the measure of the domain, we can choose l so that

$$|\psi(t,y_1)-\psi(t,y_2)| \le l\,|y_1-y_2|\,, \tag{1.16}$$

$$|(\psi(t_1)-\psi(t_2))\,(y_1) - (\psi(t_1)-\psi(t_2))\,(y_2)| \le l\,|t_1-t_2|\,|y_1-y_2| \tag{1.17}$$

for every $t, t_1, t_2 \in [0,1]$ and every $y_1, y_2 \in K$. Moreover, employing in (1.13) the Lebesgue Differentiation Theorem [13, Theorem III.12.8], one gets the uniform convergence of the difference quotients to the derivative: for $\mathcal{L}^1$-a.e. $t \in [0,1]$, there is a modulus of continuity $\omega_t\colon [0,1] \to [0,+\infty)$ such that

$$\left\| \frac{\psi(t+h)-\psi(t)}{h} - \dot{\psi}(t) \right\|_{W^{1,\infty}(K;K)} \le \omega_t(h) \tag{1.18}$$

for every $h \in [0,1]$. It is not restrictive to assume that $\omega_t(h)$ is uniformly bounded in both t and h: indeed, we can define

$$\omega_t(h) := \sup_{h' \le h} \left\| \frac{\psi(t+h')-\psi(t)}{h'} - \dot{\psi}(t) \right\|_{W^{1,\infty}(K;K)}.$$

Since $W^{1,\infty}(K;K)$ is not separable, (1.18) may not be implied by the Lipschitz property (1.15): for more details, see [3, Chapter II, Sect. 2, Lemma 1, and Sect. 3, Example I] and Example 1.2 below.

We need also a uniform bound on the energy of the prescribed deformation: we suppose that there exists M such that for every $t \in [0,1]$

$$\mathcal{W}(\psi(t)) < M. \tag{BC4}$$

Fixed t, (BC4) and (W2) give

$$\det \nabla\psi(t,x) > 0 \quad \text{for a.e. } x \in K, \tag{1.19}$$

so that $\psi(t)$, being injective, satisfies the Ciarlet-Nečas condition; as $S(\psi(t)) = \emptyset$, this implies that $\psi(t) \in AD(\psi(t), \Gamma)$ for every $\Gamma \in \mathcal{R}$.

Remark 1.10 In these hypotheses, it is possible to find a negligible set $N_\psi \subset K$ containing ∂K, independent of t, such that, for every $t \in [0,1]$ and every $x \notin N_\psi$, the function $\psi(t,\cdot)$ is differentiable at x and $\det \nabla\psi(t,x) > 0$.

Indeed, let D be a countable dense subset of $[0,1]$; by (1.19) there is a set $N_\psi \subset K$ of null measure containing ∂K such that when $t \in D$, $\psi(t,\cdot)$ is differentiable in $\Omega \setminus N_\psi$ and

$\det \nabla\psi(t,x) > 0$ if $x \notin N_\psi$. Given $t_0 \in [0,1]$, let $t_k \in D$ such that $t_k \to t_0$; let $x_0 \notin N_\psi$. Since $\psi(t_k)$ is differentiable at x_0 and converges to $\psi(t_0)$ strongly in $W^{1,\infty}(K;K)$, $\psi(t_0)$ is also differentiable at x_0 and $\nabla\psi(t_k,x_0) \to \nabla\psi(t_0,x_0)$: this is guaranteed by Lemma 1.1, as stated below.

By convergence, we have $\det \nabla\psi(t_0,x_0) \geq 0$; we must show that $\det \nabla\psi(t_0,x_0) \neq 0$. Suppose by contradiction that $\det \nabla\psi(t_0,x_0) = 0$; then, there is a vector ξ such that $\nabla\psi(t_0,x_0)\,\xi = 0$. Take $h \neq 0$ so small that $x_0 + h\,\xi \in K$; let $y_0 := \psi(t_0,x_0)$ and $y_h := \psi(t_0, x_0 + h\,\xi)$. By the hypothesis on ξ, we have, as $h \to 0$,

$$\frac{|y_h - y_0|}{|\phi(t_0,y_h) - \phi(t_0,y_0)|} = \frac{|\psi(t_0,x_0 + h\,\xi) - \psi(t_0,x_0)|}{|h|} \to 0,$$

which is forbidden by the Lipschitz property of $\phi(t_0)$.

To conclude, we must only prove the following lemma.

Lemma 1.1 *Let v_k be a sequence converging to v strongly in $W^{1,\infty}(K;K)$. Let $x_0 \in \mathring{K}$ be such that v_k is differentiable at x_0 for every k. Then, v is differentiable at x_0 and $\nabla v_k(x_0) \to \nabla v(x_0)$.*

Proof Fixed $\varepsilon > 0$, we have for every k and j large enough

$$\left|(v_k - v_j)(x) - (v_k - v_j)(x_0)\right| \leq \varepsilon\,|x - x_0| \tag{1.20}$$

for every $x \in K$; indeed, by convergence in $W^{1,\infty}(K;K)$, the function $v_k - v_j$ is Lipschitz with vanishing constant. As $x \to x_0$, we get $\left|\nabla v_k(x_0) - \nabla v_j(x_0)\right| \leq \varepsilon$; then there exists $A_0 \in \mathbb{M}^{n\times n}$ such that, as $k \to \infty$, $\nabla v_k(x_0) \to A_0$. We deduce from (1.20) that for every $\varepsilon > 0$ there is k such that

$$\left|\frac{v_k(x) - v_k(x_0) - \nabla v_k(x_0)(x - x_0)}{|x - x_0|} - \frac{v(x) - v(x_0) - A_0(x - x_0)}{|x - x_0|}\right| \leq \varepsilon$$

for every $x \in K$. By differentiability, for every k, there is $\delta > 0$ such that for $|x - x_0| < \delta$

$$\frac{|v_k(x) - v_k(x_0) - \nabla v_k(x_0)(x - x_0)|}{|x - x_0|} \leq \varepsilon.$$

Hence, v is differentiable at x_0 with differential A_0. □

Example 1.2 We conclude the discussion about $W^{1,\infty}$ spaces by showing an example (in dimension $n = 1$) of Lipschitz function from $[0,1]$ into $W^{1,\infty}([0,1])$, which does not belong to the space $W^{1,\infty}([0,1]; W^{1,\infty}([0,1]))$. Let $\psi(t,x) := \frac{1}{2}\,|x - t|^2 \operatorname{sgn}(x - t)$ and consider the partial derivative $\mathrm{D}_x\psi(t,x) = |x - t|$. Fixed $t \in [0,1]$, the difference quotients $\frac{1}{h}(\mathrm{D}_x\psi(t + h, x) - \mathrm{D}_x\psi(t,x))$ are continuous in x and uniformly bounded with respect to h; moreover, as $h \to 0$ they converge to $1_{[0,t)} - 1_{(t,1]}$ strongly in $L^r([0,1])$ for every $r < +\infty$. Nevertheless, being the limit discontinuous in x, the convergence cannot be uniform. Therefore, ψ satisfies (1.15), while (1.18) does not hold. Notice that also the property of Remark 1.10 is not satisfied in this example.

1.8 Minimum energy configurations

Following [15], we consider evolutions of *minimum energy configurations* for $\mathcal{E}$: given a boundary datum $\psi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$, at each time $t \in [0,1]$ we look for solutions $(u(t), \Gamma(t))$, with $\Gamma(t) \in \mathcal{R}$ and $u(t) \in AD(\psi(t), \Gamma(t))$, such that the unilateral minimality condition holds:

$$\mathcal{E}(t, u(t), \Gamma(t)) \leq \mathcal{E}(t, u, \Gamma) \tag{1.21}$$

for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \tilde{\subset} \Gamma$, and every $u \in AD(\psi(t), \Gamma)$.

Hence, fixed $t \in [0,1]$ and given an initial datum $\Gamma_0 \in \mathcal{R}$, we consider the minimum problem

$$\min \left\{ \mathcal{E}(t, u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma_0 \tilde{\subset} \Gamma,\ u \in AD(\psi(t), \Gamma) \right\}. \tag{1.22}$$

The next theorem ensures that there exists at least a solution; for the proof, we refer to [10, Theorem 2.2].

Theorem 1.2 (Minimization of the total energy) *Let $\mathcal{E}$ be the energy defined in* (1.5) *and* (1.6)*, where $\mathcal{W}$ satisfies* (W0–6)*, $\mathcal{G}$ satisfies* (G1–3)*, $\mathcal{S}$ satisfies* (S1–3)*, and $\mathcal{K}$ satisfies* (K1–3)*. Consider the prescribed deformations defined in* (BC1–4)*. Then, for every $t \in [0,1]$ and $\Gamma_0 \in \mathcal{R}$, the minimum problem* (1.22) *has a solution.*

2 The auxiliary formulation

Following [10,16], we study the properties of the system described in the previous section, through a change of variables. Then, we are led to consider an auxiliary problem with time-independent prescribed deformations and time-dependent bulk energy. In this section, we pass to this auxiliary formulation via the so-called multiplicative splitting method and state the properties of the new energy terms.

2.1 The multiplicative splitting method

The formulation with time-independent prescribed deformations is obtained using the method of multiplicative splitting introduced in [16], which will allow us to employ the multiplicative estimates (W5) and (W6).

Given $\psi \in W^{1,\infty}([0,1]; W^{1,\infty}(K;K))$ and $\Gamma \in \mathcal{R}$, we look for a solution $u \in AD(\psi(t), \Gamma)$ to (1.22) of the form $u = \psi(t) \circ z$, with $z \in SBV(\Omega_D; K)$. This request implies $z \in AD(I, \Gamma)$, where I denotes the identical deformation on Ω_D. In order to express ∇u in terms of $\psi(t)$ and z, we have to check the chain rule for these functions, exploiting the non-interpenetration property of the solutions.

Lemma 2.1 *Let $v \in W^{1,\infty}(K;K)$ and $z \in SBV(\Omega_D; K)$ such that $\mathcal{L}^n(z^{-1}(F)) = 0$ whenever $\mathcal{L}^n(F) = 0$. Then $u := v \circ z \in SBV(\Omega_D; K)$ and $\nabla u(x) = \nabla v(z(x))\, \nabla z(x)$ for a.e. $x \in \Omega_D$.*

Proof The proof is obtained by modifying the one of [2, Theorem 3.99]. By [2, Theorem 3.101] we get that $u = v \circ z \in SBV(\Omega_D; K)$ and $\mathrm{D}^j u = (v(z^+) - v(z^-)) \otimes \nu_z \mathcal{H}^{n-1} \llcorner S(z)$. It is possible to approximate v by mollification with a sequence v_k; let $u_k := v_k \circ z$. By [2, Theorem 3.96] we have $\nabla u_k = \nabla v_k(z)\, \nabla z$ and $\mathrm{D}^j u_k = (v_k(z^+) - v_k(z^-)) \otimes \nu_z \mathcal{H}^{n-1} \llcorner S(z)$. As u_k converges to u uniformly and $|\mathrm{D}u_k|(\Omega_D)$ is equibounded, we get that $\mathrm{D}u_k$ converges to $\mathrm{D}u$ weakly* in the sense of measures. As $\mathrm{D}^j u_k$ converges to $\mathrm{D}^j u$ strongly, ∇u_k converges to ∇u weakly* in the sense of measures. In order to see the convergence of $\nabla v_k(z)$, let F be the set of the points that are not Lebesgue for ∇v. As ∇v_k converges to ∇v pointwise on $\Omega_D \setminus F$ and $\mathcal{L}^n(z^{-1}(F)) = 0$, we obtain that $\nabla v_k(z)$ converges to $\nabla v(z)$ a.e. in Ω_D. The conclusion follows from the dominated convergence theorem. □

Thanks to the non-interpenetration property (see Remark 1.2), we get from the previous lemma $\nabla u(x) = \nabla\psi(t, z(x))\,\nabla z(x)$ for a.e. $x \in \Omega_D$.

Recall that, by Remark 1.10, there is a negligible set N_ψ containing ∂K such that, for every $t \in [0, 1]$, $\psi(t, \cdot)$ is differentiable in $K \backslash N_\psi$, with $\det \nabla\psi(t, y) > 0$ at every $y \notin N_\psi$. This leads us to define the auxiliary volume energy density imposing the chain rule where $\nabla\psi(t, y)$ exists:

$$V(t, x, y, A) := \begin{cases} W(x, \nabla\psi(t, y)\,A) & \text{if } y \notin N_\psi, \\ W(x, A) & \text{if } y \in N_\psi. \end{cases} \tag{2.1}$$

We consider the integral functional, defined for $z \in AD(I, \Gamma)$,

$$\mathcal{V}(t, z) := \int_\Omega V(t, x, z(x), \nabla z(x))\,\mathrm{d}x. \tag{2.2}$$

Notice that, in order to study $\mathcal{V}(t, z)$, we are free to choose any value for $V(t, x, y, A)$ when $y \in N_\psi$, because $z^{-1}(N_\psi)$ has null measure. For $u = \psi(t) \circ z$ we have

$$\mathcal{W}(u) = \mathcal{V}(t, \phi(t) \circ u), \quad \mathcal{V}(t, z) = \mathcal{W}(\psi(t) \circ z).$$

As for the external forces, we set

$$\mathcal{L}(t, z) := \mathcal{G}(t, \psi(t) \circ z), \tag{2.3}$$

$$\mathcal{T}(t, z) := \mathcal{S}(t, \psi(t) \circ z). \tag{2.4}$$

Finally, we define

$$\mathcal{F}^{\mathrm{el}}(t, z) := \mathcal{V}(t, z) - \mathcal{L}(t, z) - \mathcal{T}(t, z), \tag{2.5}$$

$$\mathcal{F}(t, z, \Gamma) := \mathcal{F}^{\mathrm{el}}(t, z) + \mathcal{K}(\Gamma). \tag{2.6}$$

Hence,

$$\mathcal{E}^{\mathrm{el}}(t, u) = \mathcal{F}^{\mathrm{el}}(t, \phi(t) \circ u), \quad \mathcal{F}^{\mathrm{el}}(t, z) = \mathcal{E}^{\mathrm{el}}(t, \psi(t) \circ z). \tag{2.7}$$

The properties of the auxiliary bulk energy and of the new force terms are stated in axiomatic form in the following sections.

2.2 Formulation with time-independent prescribed deformations

The previous discussion leads us to introduce a class of functions $V : [0, 1] \times \Omega \times K \times \mathbb{M}^{n\times n} \to [0, +\infty]$ satisfying the following requirements:

(V1) *Measurability:* for every $(t, A) \in [0, 1] \times \mathbb{M}^{n\times n}$, the function $(x, y) \mapsto V(t, x, y, A)$ is $\mathcal{L}^n(\Omega) \otimes \mathcal{L}^n(K)$-measurable on $\Omega \times K$, and for every $(x, y) \in \Omega \times K$, the function $(t, A) \mapsto V(t, x, y, A)$ is continuous on $[0, 1] \times \mathbb{M}^{n\times n}$.

(V2) *Finiteness:* for every $(t, x, y) \in [0, 1] \times \Omega \times K$, we have $V(t, x, y, A) < +\infty$ if and only if $A \in GL^+_n$.

Thanks to Remark 1.3, property (V1) ensures, for every $z \in AD(I, \Gamma)$, the measurability of $V(t, x, z(x), \nabla z(x))$; hence, $\mathcal{V}(t, z)$ is well defined by (2.2).

We require the following properties on this integral functional:

(V3) *Bound at identity:* there exists a constant $M > 0$ such that $\mathcal{V}(t, I) \leq M$ for every $t \in [0, 1]$;

(V4) *Semicontinuity and coercivity:* if z_k converges to z weakly* in $SBV^p(\Omega_D; K)$ and $t_k \to t$, then

$$\mathcal{V}(t, z) \le \liminf_{k\to\infty} \mathcal{V}(t_k, z_k);$$

moreover, there exist some constants $\beta_V^0, \dots, \beta_V^n > 0$ such that, for every $t \in [0, 1]$ and every $z \in AD(I, \Gamma)$,

$$\mathcal{V}(t, z) \ge \sum_{j=1}^{n} \beta_V^j \left\|\mathrm{adj}_j \nabla u\right\|^{p_j}_{L^{p_j}(\Omega_D;\mathbb{R}^{\tau_j})} - \beta_V^0,$$

where $p_1 \ge 2$, $p_j \ge p_1' := \frac{p_1}{p_1-1}$ for $j = 2, \dots, n-1$, $p_n > 1$, and τ_j is the dimension of $\mathrm{adj}_j \nabla u$.

Furthermore, we assume that there exist a constant $\gamma_V \in (0, 1)$, a function $c_V^0 \in L^1_+(\Omega)$ and a constant $c_V^1 > 0$, such that:

(V5) *Multiplicative stress estimate:* for every $(t, x, y, A) \in [0, 1] \times \Omega \times K \times GL_n^+$ and every $B \in GL_n^+$ with $|B - I| < \gamma_V$,

$$V(t, x, y, AB) + c_V^0(x) \le c_V^1 \left(V(t, x, y, A) + c_V^0(x)\right);$$

(V6) *Estimate on time increments:* for every $(t_1, x, y, A) \in [0, 1] \times \Omega \times K \times GL_n^+$ and every $t_2 \in [0, 1]$ such that $|t_1 - t_2| < \gamma_V$,

$$|V(t_1, x, y, A) - V(t_2, x, y, A)| \le c_V^1 \left(V(t_1, x, y, A) + c_V^0(x)\right) |t_1 - t_2|;$$

(V7) *Estimate on the convergence of time increments:* there exists an $\mathcal{L}^1$-negligible set N such that for $t \notin N$ the partial time derivative $\mathrm{D}_t V(t, x, y, A)$ is defined for every $(x, A) \in \Omega \times GL_n^+$ and a.e. $y \in K$, and we have for every $h > 0$ with $t \pm h \in [0, 1]$ that

$$\left|\mathrm{D}_t V(t, x, y, A) \mp \frac{V(t\pm h, x, y, A) - V(t, x, y, A)}{h}\right| \le \omega_t(h) \left(V(t, x, y, A) + c_V^0(x)\right),$$

where $\omega_t\colon [0, 1] \to [0, +\infty)$ is a modulus of continuity, depending only on t, with $t \mapsto \omega_t(h)$ in $L^\infty([0, 1])$ for every $h \in [0, 1]$;

(V8) *Estimate on spatial increments:* for every $(t, x, y, A) \in [0, 1] \times \Omega \times K \times GL_n^+$ and every $y' \in K$,

$$V(t, x, y', A) + c_V^0(x) \le c_V^1 \left(V(t, x, y, A) + c_V^0(x)\right).$$

Remark 2.1 Let $z \in SBV(\Omega_D; K)$; suppose that $\mathcal{V}(t_0, z) < +\infty$ for some $t_0 \in [0, 1]$. Then $\mathcal{V}(t, z) < +\infty$ for every $t \in [0, 1]$: indeed, by (V6)

$$V(t, x, z(x), \nabla z(x)) + c_V^0(x) \le \left(c_V^1 + 1\right) \left(V(t_0, x, z(x), \nabla z(x)) + c_V^0(x)\right).$$

Using again (V6), one sees that $t \mapsto \mathcal{V}(t, z)$ is Lipschitz, with constant depending on $\mathcal{V}(t_0, z)$. Hence, it has a derivative $\mathrm{D}_t\mathcal{V}(\cdot, z) \in L^\infty([0, 1])$, defined $\mathcal{L}^1$-a.e. in $[0, 1]$. Also, $t \mapsto V(t, x, z(x), \nabla z(x))$ is Lipschitz, so it is derivable $\mathcal{L}^1$-a.e.; more precisely, by (V7) and by Remark 1.2, we find for each $t \notin N$ a set $N_{t,z}$ of null measure such that $\mathrm{D}_t V(t, x, z(x), \nabla z(x))$ exists for every $x \notin N_{t,z}$.

We are going to establish a representation formula for $\mathrm{D}_t\mathcal{V}(\cdot, z)$. Let us extend $t \mapsto V(t, x, z(x), \nabla z(x))$ to $[0, +\infty)$ by setting its value to be $V(1, x, z(x), \nabla z(x))$ for $t > 1$. Consider the function

$$\mathrm{D}_t^* V(t,x) := \liminf_{k\to\infty} \frac{V\left(t+\frac{1}{k}, x, z(x), \nabla z(x)\right) - V(t, x, z(x), \nabla z(x))}{\frac{1}{k}},$$

which is integrable on $[0,1]\times\Omega$ by (V6). Since $\mathrm{D}_t^* V(t,x) = \mathrm{D}_t V(t, x, z(x), \nabla z(x))$ where the derivative is defined, we have for every $t_1, t_2 \in [0,1]$

$$\mathcal{V}(t_2, z) - \mathcal{V}(t_1, z) = \int_\Omega \int_{t_1}^{t_2} \mathrm{D}_t^* V(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

Finally, exchanging the order of integration, we obtain for $\mathcal{L}^1$-a.e. $t \in [0,1]$

$$\mathrm{D}_t\mathcal{V}(t,z) = \int_\Omega \mathrm{D}_t^* V(t,x)\,\mathrm{d}x.$$

Given $t \notin N$, we have $\mathrm{D}_t^* V(t,x) = \mathrm{D}_t V(t, x, z(x), \nabla z(x))$ in $\Omega\backslash N_{t,z}$: integrating (V7) we deduce that

$$\left|\mathrm{D}_t\mathcal{V}(t,z) \mp \frac{\mathcal{V}(t\pm h, z) - \mathcal{V}(t,z)}{h}\right| \le \omega_t(h)\left(\mathcal{V}(t,z) + \left\|c_V^0\right\|_{L^1(\Omega)}\right).$$

In particular, this shows that the partial time derivative $\mathrm{D}_t\mathcal{V}(\cdot, z)$ is defined out of an $\mathcal{L}^1$-negligible set independent of z.

In the following section, we prove that the auxiliary energy introduced in Sect. 2.1 satisfies the axioms stated above.

2.3 Proof of the properties of the auxiliary energy

In this section, we prove that the volume energy $\mathcal{V}$, obtained from $\mathcal{W}$ and ψ through the change of variable described in (2.1), satisfies the properties (V1–8) stated above.

We start with some consequences of hypothesis (W5).

Proposition 2.1 *Let $W(x, A)$ be a Carathéodory function satisfying* (W2) *and* (W5). *Then there exists a constant $\gamma \in (0,1)$ (depending only on n) such that, for every $(x, A) \in \Omega \times GL_n^+$ and every $B \in \mathbb{M}^{n\times n}$ with $|B - I| < \gamma$, we have $B \in GL_n^+$ and*

$$W(x, AB) + c_W^0(x) \le \frac{n}{n-1}\left(W(x,A) + c_W^0(x)\right). \tag{2.8}$$

If W satisfies also (W0)*, then for every $(x, A) \in \Omega \times GL_n^+$*

$$\left|\mathrm{D}_A W(x,A)\,A^{\mathrm{T}}\right| \le c_W^1\left(W(x,A) + c_W^0(x)\right). \tag{2.9}$$

If W satisfies (2.9)*, there exists a constant, still denoted γ, such that, for every $A \in GL_n^+$ and every $B \in \mathbb{M}^{n\times n}$ with $|B - I| < \gamma$, we have $B \in GL_n^+$ and*

$$W(x, BA) + c_W^0(x) \le \frac{n}{n-1}\left(W(x,A) + c_W^0(x)\right) \tag{2.10}$$

and

$$\left|\mathrm{D}_A W(x, BA)\,A^{\mathrm{T}}\right| \le \frac{n^2}{n-1}c_W^1\left(W(x,A) + c_W^0(x)\right). \tag{2.11}$$

Proof Argue as in [4, Sect. 2.4]. □

The Kirchhoff tensor $\mathrm{D}_A W(x, A)\, A^{\mathrm{T}}$ appearing in (2.9) is related with the "multiplicative increments" of type $W(x, BA) - W(x, A)$, because

$$\mathrm{D}_A W(x, A)\, A^{\mathrm{T}} : (B - I) = \mathrm{d}_A W(x, A)[BA - A].$$

This suggests to write (2.9) without using derivatives.

Proposition 2.2 *Let $W(x, A)$ be a Carathéodory function satisfying* (W2) *and* (2.9). *Then*

$$|W(x, BA) - W(x, A)| \leq \frac{n^2}{n-1}\, c_W^1 \left(W(x, A) + c_W^0(x)\right) |B - I| \tag{2.12}$$

for every $(x, A) \in \Omega \times GL_n^+$ and every $B \in GL_n^+$ with $|B - I| < \gamma$, where γ is the constant introduced in the previous proposition.

Proof Fixed (x, A) and B as in the statement, define for $\lambda \in [0, 1]$ the function $w(\lambda) := W(x, (1-\lambda)A + \lambda BA)$, whose derivative is $w'(\lambda) = \mathrm{D}_A W\,(x, (1-\lambda)A + \lambda BA)\; A^{\mathrm{T}} : (B - I)$. We have $W(x, BA) - W(x, A) = \int_0^1 w'(\lambda)\,\mathrm{d}\lambda$. By (2.11), we get $\left|w'(\lambda)\right| \leq \frac{n^2}{n-1}\, c_W^1 (W(x, A) + c_W^0(x))\, |B - I|$, so we conclude. □

In the next proposition, we present an estimate where multipliers need not to be near I.

Proposition 2.3 *Let $W(x, A)$ be a Carathéodory function satisfying* (W0), (W2), *and* (2.9). *Then for every $M > 0$ there exists $c_M > 0$ such that*

$$W(x, BA) + c_W^0(x) \leq c_M \left(W(x, A) + c_W^0(x)\right) \tag{2.13}$$

for every $(x, A) \in \Omega \times GL_n^+$ and every $B \in GL_n^+$ with $|B| < M$ and $\left|B^{-1}\right| < M$.

Proof Let A, B, and M be as in the statement. Consider a decomposition $B = QC$ with $Q \in SO_n$ and C symmetric and positive definite (take $C := \sqrt{B^{\mathrm{T}} B}$). We can find an integer N such that

$$\left|C^{\frac{1}{N}} - I\right| < \gamma;$$

here, N depends only on the constant γ of Proposition 2.1 and on M, which controls $|B|$ and $\left|B^{-1}\right|$. We can apply (W0) and (2.10) to get

$$W(x, BA) + c_W^0(x) = W\left(x, \left(C^{\frac{1}{N}}\right)^N A\right) + c_W^0(x) \leq \left(\frac{n}{n-1}\right)^N \left(W(x, A) + c_W^0(x)\right).$$

This concludes the proof. □

Now, we are ready to show the passage between the two formulations presented above. We will use the following fact.

Remark 2.2 By (2.9) we get for every $(x, A) \in \Omega \times GL_n^+$ and every $B \in GL_n^+$

$$\left|\mathrm{D}_A W(x, BA)\, A^{\mathrm{T}}\right| \leq c_W^1 \left(W(x, BA) + c_W^0(x)\right) \left|B^{-1}\right|. \tag{2.14}$$

Proposition 2.4 *If* (W0–6) *and* (BC1–4) *hold, then the functional $\mathcal{V}$ defined in* (2.1) *satisfies properties* (V1–8).

Proof Properties (V1–3) are given by (W1–3). After a change of variables, one sees that (V4) is a consequence of (W1) and (W4), thanks to the lower semicontinuity of $\mathcal{W}$ (see Remark 1.4).

In what follows, we will take $c_V^0 := c_W^0$, $c_V^1 \geq \frac{n}{n-1}$, and $\gamma_V \leq \gamma$, where γ is the constant introduced in Proposition 2.1. Then (V5) is implied by (2.8), because

$$W(x, \nabla\psi(t, y)AB) + c_W^0(x) \leq \frac{n}{n-1}\left(W(x, \nabla\psi(t, y)A) + c_W^0(x)\right).$$

In order to see (V6), take $\gamma_V \leq l^{-2}\gamma$, where l is the constant appearing in Remark 1.9. By (1.14) and (1.15), for a.e. $y \in K$ we have

$$|\nabla\psi(t_2, y)\nabla\phi(t_1, \psi(t_1, y)) - I| < \gamma$$

if $|t_1 - t_2| < \gamma_V$. Hence, we can apply (2.12) to get for every $A \in GL_n^+$

$$\begin{aligned}
&|W(x, \nabla\psi(t_2, y)A) - W(x, \nabla\psi(t_1, y)A)| \\
&\leq \frac{n^2}{n-1} l^2 c_W^1 \left(W(x, \nabla\psi(t_1, y)A) + c_W^0(x)\right)|t_1 - t_2|;
\end{aligned}$$

then (V6) follows for c_V^1 large enough.

The partial time derivative of V exists everywhere $\dot\psi$ is defined. By (2.1), property (V7) is trivially satisfied when $y \in N_\psi$, where N_ψ is the negligible subset of K defined in Remark 1.10. If $y \notin N_\psi$, we have $\mathrm{D}_t V(t, x, y, A) = \mathrm{D}_A W(x, \nabla\psi(t, y)A)\, A^{\mathrm{T}} : \nabla\dot\psi(t, y)$ where the derivatives exist. Given $h > 0$ small enough, using the mean value theorem we can find a convex combination B_h of $\nabla\psi(t + h, y)$ and $\nabla\psi(t, y)$ such that

$$\begin{aligned}
&\left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}} : \nabla\dot\psi(t) - \frac{W(\nabla\psi(t+h)A) - W(\nabla\psi(t)A)}{h}\right| \\
&= \left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}} : \nabla\dot\psi(t) - \mathrm{D}_A W(B_h A)\, A^{\mathrm{T}} : \frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right| \\
&\leq \left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}}\right| \left|\nabla\dot\psi(t) - \frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right| \\
&\quad + \left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}} - \mathrm{D}_A W(B_h A)\, A^{\mathrm{T}}\right| \left|\frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right|.
\end{aligned}$$

Here and henceforth, we omit the arguments x and y when they are obvious; it is understood that B_h is invertible for h small. Consider the first summand of the last expression; using (2.14) and (1.14) in the first factor and (1.18) in the second, we get

$$\left|\mathrm{D}_A W(\nabla\psi(t)A)\, A^{\mathrm{T}}\right| \left|\nabla\dot\psi(t) - \frac{\nabla\psi(t+h) - \nabla\psi(t)}{h}\right| \leq l\, c_W^1\, \omega_t(h) \left(W(\nabla\psi(t)A) + c_W^0\right),$$

where ω_t is the modulus of continuity defined in Remark 1.9. As for the second summand, we can use (1.15) to control the last factor; the remaining part is

$$\begin{aligned}
&\left|\mathrm{D}_A W(B_h A)\, A^{\mathrm{T}} - \mathrm{D}_A W(\nabla\psi(t, y)A)\, A^{\mathrm{T}}\right| \\
&\leq \left|\mathrm{D}_A W(B_h' A')\, (B_h' A')^{\mathrm{T}} - \mathrm{D}_A W(A')\, A'^{\mathrm{T}}\right| \left|B_h^{-1}\right| \\
&\quad + \left|\mathrm{D}_A W(A')\, A'^{\mathrm{T}}\right| \left|B_h^{-1} - \nabla\phi(t, \psi(t, y))\right|,
\end{aligned}$$

where $B_h' := B_h \nabla\phi(t, \psi(t, y))$ and $A' := \nabla\psi(t, y)A$. The first term is estimated by (W6), since $\left|B_h^{-1}\right|$ is bounded by (1.14); as for the second one, we use (2.9), recalling that, if h is small enough, B_h is uniformly near to $\nabla\psi(t, y)$, being a convex combination of $\nabla\psi(t, y)$ and $\nabla\psi(t+h, y)$. Hence, there is a modulus of continuity $\omega\colon [0,1] \to [0,+\infty)$ such that

$$\left|\mathrm{D}_A W(B_h A)\, A^{\mathrm{T}} - \mathrm{D}_A W(\nabla\psi(t, y)A)\, A^{\mathrm{T}}\right| \le \omega(h)\, \left(W(\nabla\psi(t)A) + c_W^0\right);$$

notice that, by (1.14) and (2.13), ω is bounded. This concludes the proof of (V7) in the case of $t+h$; the case of $t-h$ is analogous.

Finally, (V8) follows from (2.13), because the functions $\nabla\psi(t,\cdot)$ and $\nabla\phi(t,\cdot)$ are uniformly bounded in $W^{1,\infty}(K;K)$ by (1.14). □

2.4 Properties of the force terms

The volume forces in the new formulation are given by a functional $\mathcal{L}(t,z)$, defined in $[0,1] \times AD(I,\Gamma)$, where $\Gamma \in \mathcal{R}$. We assume that there is an exponent $q \ge 1$ such that the following hold:

(L1) there is a constant $c_L > 0$ such that for every $t \in [0,1]$ and every $z, z_1, z_2 \in L^\infty(\Omega;K)$

$$|\mathcal{L}(t,z)| \le c_L,$$
$$|\mathcal{L}(t,z_1) - \mathcal{L}(t,z_2)| \le c_L\, \|z_1 - z_2\|_{L^q(\Omega;\mathbb{R}^n)};$$

(L2) there is a function $a_L \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z \in L^\infty(\Omega;K)$

$$|\mathcal{L}(t_2,z) - \mathcal{L}(t_1,z)| \le \int_{t_1}^{t_2} a_L(s)\,\mathrm{d}s;$$

(L3) there is a function $b_L \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z_1, z_2 \in L^\infty(\Omega;K)$

$$|\mathcal{L}(t_2,z_1) - \mathcal{L}(t_1,z_1) - \mathcal{L}(t_2,z_2) + \mathcal{L}(t_1,z_2)| \le \int_{t_1}^{t_2} b_L(s)\,\mathrm{d}s\, \|z_1 - z_2\|_{L^q(\Omega;\mathbb{R}^n)}.$$

As for the surface forces, they are given by a functional $\mathcal{T}(t,z)$, defined in $[0,1] \times AD(I,\Gamma)$. We suppose:

(T1) there is a constant $c_T > 0$ such that for every $t \in [0,1]$ and every $z, z_1, z_2 \in L^\infty(\partial_S\Omega;K)$

$$|\mathcal{T}(t,z)| \le c_T,$$
$$|\mathcal{T}(t,z_1) - \mathcal{T}(t,z_2)| \le c_T\, \|z_1 - z_2\|_{L^q(\partial_S\Omega;\mathbb{R}^n)};$$

(T2) there is a function $a_T \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z \in L^\infty(\partial_S\Omega;K)$

$$|\mathcal{T}(t_2,z) - \mathcal{T}(t_1,z)| \le \int_{t_1}^{t_2} a_T(s)\,\mathrm{d}s;$$

(T3) there is a function $b_T \in L^1_+([0,1])$ such that for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and every $z_1, z_2 \in L^\infty(\partial_S\Omega; K)$

$$|\mathcal{T}(t_2, z_1) - \mathcal{T}(t_1, z_1) - \mathcal{T}(t_2, z_2) + \mathcal{T}(t_1, z_2)| \le \int_{t_1}^{t_2} b_T(s)\, \mathrm{d}s\, \|z_1 - z_2\|_{L^q(\partial_S\Omega;\mathbb{R}^n)}\,.$$

Thanks to (L2) and (T2), given any $z \in AD(I,\Gamma)$ the functions $t \mapsto \mathcal{L}(t,z)$ and $t \mapsto \mathcal{T}(t,z)$ are absolutely continuous on $[0,1]$, so that $\mathrm{D}_t\mathcal{L}(t,z)$ and $\mathrm{D}_t\mathcal{T}(t,z)$ exist $\mathcal{L}^1$-a.e. Arguing as in Remark 1.8, we may define for every $t \in [0,1]$ some $L^q(\Omega;\mathbb{R}^n)$-Lipschitz functions $\mathrm{D}^*_t\mathcal{L}(t,\cdot)$ and $\mathrm{D}^*_t\mathcal{T}(t,\cdot)$, such that for every $z \in AD(I,\Gamma)$ we have $\mathrm{D}^*_t\mathcal{L}(t,u) = \mathrm{D}_t\mathcal{L}(t,u)$ and $\mathrm{D}^*_t\mathcal{T}(t,u) = \mathrm{D}_t\mathcal{T}(t,u)$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$. We identify $\mathrm{D}_t\mathcal{L}(t,\cdot)$ with $\mathrm{D}^*_t\mathcal{L}(t,\cdot)$ and $\mathrm{D}_t\mathcal{T}(t,\cdot)$ with $\mathrm{D}^*_t\mathcal{T}(t,\cdot)$; we set also

$$\mathrm{D}_t\mathcal{F}^{\mathrm{el}}(t,z) := \mathrm{D}_t\mathcal{V}(t,z) - \mathrm{D}_t\mathcal{L}(t,z) - \mathrm{D}_t\mathcal{T}(t,z). \tag{2.15}$$

We will use in particular these consequences of (L1–3) and (T1–3) for $z, z_k \in AD(I,\Gamma)$ such that $z_k \to z$ in measure:

$$\text{if } t_k \to t,\quad \mathcal{L}(t_k, z_k) \to \mathcal{L}(t,z) \quad\text{and}\quad \mathcal{T}(t_k, z_k) \to \mathcal{T}(t,z); \tag{2.16}$$

$$\text{for } \mathcal{L}^1\text{-a.e. } t,\quad \mathrm{D}_t\mathcal{L}(t, z_k) \to \mathrm{D}_t\mathcal{L}(t,z) \quad\text{and}\quad \mathrm{D}_t\mathcal{T}(t, z_k) \to \mathrm{D}_t\mathcal{T}(t,z). \tag{2.17}$$

Finally, we prove that (L1–3) and (T1–3) are satisfied when $\mathcal{L}$ and $\mathcal{T}$ are given by (2.3) and (2.4).

Proposition 2.5 *If* (G1–3)*,* (S1–3)*, and* (BC1–4) *hold, then the functionals $\mathcal{L}$ and $\mathcal{T}$ defined in* (2.3) *and* (2.4) *satisfy properties* (L1–3) *and* (T1–3)*.*

Proof We show (L1–3); the proof of (T1–3) is analogous.

Property (L1) comes immediately from (G1), taking $c_L := c_G(1 \vee l)$, where l is the constant of Remark 1.9.

Henceforth, we write $\psi_1 := \psi(t_1)$ and $\psi_2 := \psi(t_2)$. As for (L2), by (G1), (G2), and (1.15) we have

$$\begin{aligned}|\mathcal{L}(t_2,z) - \mathcal{L}(t_1,z)| &\le |\mathcal{G}(t_2,\psi_1\circ z) - \mathcal{G}(t_1,\psi_1\circ z)| + |\mathcal{G}(t_1,\psi_1\circ z) - \mathcal{G}(t_1,\psi_2\circ z)|\\ &\le \int_{t_1}^{t_2} a_G(s)\,\mathrm{d}s + l\, c_G\, \mathcal{L}^n(\Omega)^{\frac{1}{q}}\,(t_2 - t_1),\end{aligned}$$

so we define $a_L(s) := a_G(s) + l\, c_G\, \mathcal{L}^n(\Omega)^{\frac{1}{q}}$.

To prove (L3), adding and subtracting we obtain

$$\begin{aligned}&|\mathcal{L}(t_2,z_1) - \mathcal{L}(t_1,z_1) - \mathcal{L}(t_2,z_2) + \mathcal{L}(t_1,z_2)|\\ &\quad\le |\mathcal{G}(t_2,\psi_1\circ z_1) - \mathcal{G}(t_1,\psi_1\circ z_1) - \mathcal{G}(t_2,\psi_1\circ z_2) + \mathcal{G}(t_1,\psi_1\circ z_2)|\\ &\qquad + |\mathcal{G}(t_2,\psi_2\circ z_1) - \mathcal{G}(t_2,\psi_1\circ z_1) - \mathcal{G}(t_2,\psi_2\circ z_2) + \mathcal{G}(t_2,\psi_1\circ z_2)|\,.\end{aligned}$$

The first summand is controlled by $l\int_{t_1}^{t_2} b_G(s)\,\mathrm{d}s\,\|z_1 - z_2\|_{L^q(\Omega;\mathbb{R}^n)}$ thanks to (G3) and (1.16). As for the second summand, we get from (G1), (1.16), and (1.15)

$$\begin{aligned}&|\mathcal{G}(t_2,\psi_1\circ z_1 + \psi_2\circ z_2 - \psi_1\circ z_2) - \mathcal{G}(t_2,\psi_1\circ z_1) - \mathcal{G}(t_2,\psi_2\circ z_2) + \mathcal{G}(t_2,\psi_1\circ z_2)|\\ &\quad\le l^2\, c_G(t_2 - t_1)\,\|z_1 - z_2\|_{L^q(\Omega;\mathbb{R}^n)}\,.\end{aligned}$$

What remains is estimated with (G1) and (1.17):

$$|\mathcal{G}(t_2, \psi_1 \circ z_1 + \psi_2 \circ z_2 - \psi_1 \circ z_2) - \mathcal{G}(t_2, \psi_2 \circ z_1)| \leq l\, c_G (t_2 - t_1)\, \|z_1 - z_2\|_{L^q(\Omega;\mathbb{R}^n)} \,.$$

Then, we conclude taking $b_L(s) := l\, b_G(s) + l\, c_G + l^2\, c_G$. □

Remark 2.3 The time derivatives of the energies considered above have $\mathcal{L}^1$-a.e. the following form:

$$\mathrm{D}_t \mathcal{V}(t, z) = \int_\Omega \mathrm{D}_A W(x, \nabla(\psi(t) \circ z)) : \nabla \left(\dot{\psi}(t) \circ z\right) \mathrm{d}x,$$

$$\mathrm{D}_t \mathcal{L}(t, z) = \int_\Omega \mathrm{D}_y G(t, x, \psi(t) \circ z) \cdot \left(\dot{\psi}(t) \circ z\right) \mathrm{d}x + \mathrm{D}_t \mathcal{G}(t, \psi(t) \circ z),$$

$$\mathrm{D}_t \mathcal{T}(t, z) = \int_{\partial_S \Omega} \mathrm{D}_y S(t, x, \psi(t) \circ z) \cdot \left(\dot{\psi}(t) \circ z\right) \mathrm{d}\mathcal{H}^{n-1}(x) + \mathrm{D}_t \mathcal{S}(t, \psi(t) \circ z).$$

For $u = \psi(t) \circ z$, we define the power of the external forces

$$\begin{aligned}\mathcal{P}(t, u) := & \int_\Omega \mathrm{D}_A W(x, \nabla u) : \nabla \left(\dot{\psi}(t) \circ \phi(t) \circ u\right) \mathrm{d}x \\ & - \int_\Omega \mathrm{D}_y G(t, x, u) \cdot \left(\dot{\psi}(t) \circ \phi(t) \circ u\right) \mathrm{d}x \\ & - \int_{\partial_S \Omega} \mathrm{D}_y S(t, x, u) \cdot \left(\dot{\psi}(t) \circ \phi(t) \circ u\right) \mathrm{d}\mathcal{H}^{n-1}(x),\end{aligned}$$

so that the time derivative of the total energy takes the form

$$\mathrm{D}_t \mathcal{F}^{\mathrm{el}}(t, \phi(t) \circ u) = \mathcal{P}(t, u) - \mathrm{D}_t \mathcal{G}(t, u) - \mathrm{D}_t \mathcal{S}(t, u).$$

These formulas allow us to pass from the problem with fixed boundary data to the original one (see [10, Sects. 2.4 and 7]).

3 Quasistatic evolution

The goal of this work is to study *quasistatic evolutions*: namely, motions which at each time minimize the total energy and satisfy an energy-dissipation balance law. We quote from [10] the definition of *incrementally approximable quasistatic evolution*; in Theorems 3.2 and 3.3, we present the existence result and the properties of global stability and energy balance, in the weak hypotheses presented in Sect. 1.

Throughout the section, we adopt the formulation with time-independent boundary conditions, introduced in Sect. 2. All definitions and theorems presented here can be formulated in the framework with time-dependent boundary data (see Sect. 1), using Remark 2.3; for the details we refer to [10, Sects. 2.4 and 7].

3.1 Definitions and properties

We fix an initial condition (u_0, Γ_0), which is supposed to be a minimum energy configuration at time 0, i.e., $\Gamma_0 \in \mathcal{R}$, $u_0 \in AD(I, \Gamma_0)$, and

$$\mathcal{F}(0, u_0, \Gamma_0) \leq \mathcal{F}(0, u, \Gamma) \tag{3.1}$$

for every $\Gamma \in \mathcal{R}$ with $\Gamma_0 \,\tilde{\subset}\, \Gamma$ and every $u \in AD(I, \Gamma)$.

Consider a *time discretization*, i.e., a sequence of subdivisions $\{t_k^i\}_{0 \leq i \leq k}$ of the interval $[0, 1]$, with

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = 1 \quad \text{and} \quad \lim_{k \to \infty} \max_{1 \leq i \leq k} \left(t_k^i - t_k^{i-1} \right) = 0. \tag{3.2}$$

For a given subdivision, we define a corresponding incremental approximate solution.

Definition 3.1 Fix $k \in \mathbb{N}$. An *incremental approximate solution* corresponding to the time subdivision $\{t_k^i\}_{0 \leq i \leq k}$ with initial datum (u_0, Γ_0) is a function $t \mapsto (u_k(t), \Gamma_k(t))$, such that

(a) $(u_k(0), \Gamma_k(0)) = (u_0, \Gamma_0)$;
(b) $u_k(t) = u_k\left(t_k^i\right)$ and $\Gamma_k(t) = \Gamma_k\left(t_k^i\right)$ for $t \in \left[t_k^i, t_k^{i+1}\right)$ and $i = 0, \ldots, k-1$;
(c) for $i = 1, \ldots, k$, $\left(u\left(t_k^i\right), \Gamma\left(t_k^i\right)\right)$ is a solution of

$$\min \left\{ \mathcal{F}\left(t_k^i, u, \Gamma\right) : \Gamma \in \mathcal{R},\ \Gamma_k^{i-1} \,\tilde{\subset}\, \Gamma,\ u \in AD(I, \Gamma) \right\}. \tag{3.3}$$

If (u_k, Γ_k) satisfies the previous definition, by the minimality and by (V3) we have $\mathcal{F}(t, u_k(t), \Gamma_k(t)) < +\infty$ for every t, hence $u_k \in SBV^p(\Omega_D; K)$ by (V4). The existence of incremental approximate solutions is guaranteed by the following theorem, which is the counterpart of Theorem 1.2; for the proof, we refer to [10, Theorem 2.10].

Theorem 3.1 (Minimization of the total energy) *Let $\mathcal{F}$ be the energy defined in* (2.1)–(2.6)*, where $\mathcal{V}$ satisfies* (V1–8)*, $\mathcal{L}$ satisfies* (L1–3)*, $\mathcal{T}$ satisfies* (T1–3)*, and $\mathcal{K}$ satisfies* (K1–3)*. Then, for every $t \in [0, 1]$ and $\Gamma_0 \in \mathcal{R}$, the minimum problem*

$$\min \left\{ \mathcal{F}(t, u, \Gamma) \colon \Gamma \in \mathcal{R},\ \Gamma_0 \,\tilde{\subset}\, \Gamma,\ u \in AD(I, \Gamma) \right\} \tag{3.4}$$

has a solution.

To find an incrementally approximable quasistatic evolution, we take a sequence of incremental approximate solutions and pass to the limit as the time step vanishes. In the passage to the limit, we use the weak* convergence in $SBV^p(\Omega_D; K)$ for the deformations (see Sect. 1.1) and the σ^p-convergence for the cracks (see Sect. 1.5).

Definition 3.2 A function $t \mapsto (u(t), \Gamma(t))$ from $[0, 1]$ in $SBV^p(\Omega_D; K) \times \mathcal{R}$ is an *incrementally approximable quasistatic evolution* of minimum energy configurations with initial datum (u_0, Γ_0), if there exist an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, a time discretization $\{t_k^i\}_{0 \leq i \leq k}$, and a corresponding sequence of incremental approximate solutions $(u_k(t), \Gamma_k(t))$ with the same initial datum, such that for every $t \in [0, 1]$

(a) $\Gamma_k(t)$ σ^p-converges to $\Gamma^*(t)$ and $\Gamma(t) = \Gamma^*(t) \cup \Gamma_0$;
(b) there is a subsequence $u_{k_j}(t)$, depending on t, such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $SBV^p(\Omega_D; K)$; moreover, for $\mathcal{L}^1$-a.e. $t \in [0, 1]$, $\lim_{k \to \infty} \theta_{k_j}(t) = \limsup_{k \to \infty} \theta_k(t)$, where

$$\theta_k(t) := \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(t, u_k(t)). \tag{3.5}$$

We state the existence result for measurable incrementally approximable quasistatic evolutions; the proof can be done as in [10, Theorem 2.13, Theorem 6.1, and Corollary 6.2], with minor modifications due to the presence of the forces (see (3.10) for the discrete energy inequality).

Theorem 3.2 (Existence of quasistatic evolutions) *Let $\mathcal{F}$ be the energy defined in* (2.1)–(2.6)*, where $\mathcal{V}$ satisfies* (V1–8)*, $\mathcal{L}$ satisfies* (L1–3)*, $\mathcal{T}$ satisfies* (T1–3)*, and $\mathcal{K}$ satisfies* (K1–3). *Let (u_0, Γ_0) be a minimum energy configuration at time* 0 *as in* (3.1). *Then there exists an incrementally approximable quasistatic evolution $t \mapsto (u(t), \Gamma(t))$ with initial datum (u_0, Γ_0), such that the function $t \mapsto u(t)$ is strongly measurable, regarded as a function from* $[0, 1]$ *into $SBV^p(\Omega_D; \mathbb{R}^n)$.*

The main result of this work is the proof of the following properties, which characterize quasistatic evolutions as *rate-independent processes* (see [21] and the references therein). We refer to [10, Remark 2.16] for further comments on the energy balance rule.

Theorem 3.3 (Properties of quasistatic evolutions) *For every incrementally approximable quasistatic evolution $(u(t), \Gamma(t))$, the following hold*:

1. Global stability: *for every $t \in [0, 1]$, the pair $(u(t), \Gamma(t))$ is a minimum energy configuration at time t, i.e., $\Gamma(t) \in \mathcal{R}$, $u(t) \in AD(I, \Gamma(t))$, and*
$$\mathcal{F}(t, u(t), \Gamma(t)) \leq \mathcal{F}(t, v, \Gamma) \tag{3.6}$$
for every $\Gamma \in \mathcal{R}$, with $\Gamma(t) \mathrel{\tilde{\subset}} \Gamma$, and every $v \in AD(I, \Gamma)$;
2. Energy balance: *the function $F(t) := \mathcal{F}(t, u(t), \Gamma(t))$ is absolutely continuous on* $[0, 1]$ *and its time derivative satisfies*
$$\dot{F}(t) = \mathrm{D}_t\mathcal{F}^{\mathrm{el}}(t, u(t)) \quad \textit{for } \mathcal{L}^1\textit{-a.e. } t \in [0, 1]. \tag{3.7}$$

In the next section, we provide the proof of Theorem 3.3, which is based on the arguments of [9] and [10].

3.2 Proof of Theorem 3.3

Let $(u(t), \Gamma(t))$ be an incrementally approximable quasistatic evolution. Then there exist an increasing set function $t \mapsto \Gamma^*(t) \in \mathcal{R}$, a time discretization $\{t_k^i\}_{0 \leq i \leq k}$ such that (3.2) holds, and a sequence of incremental approximate solutions $(u_k(t), \Gamma_k(t))$ with the same initial datum (u_0, Γ_0), which fulfil properties (a) and (b) of Definition 3.2. Let $\theta_k(t)$ be as in (3.5); set $\tau_k(t) := t_k^i$ and $\mathcal{F}_k(t, \cdot) := \mathcal{F}(t_k^i, \cdot)$ for $t \in [t_k^i, t_k^{i+1})$.

Global stability

The proof of the global stability can be done as in [10, Sect. 4], with obvious adaptations to treat the case where volume and surface forces are added. The properties of V presented before are sufficient to repeat the procedure of [10]; in particular, the properties of [10, Remark 2.8] used in the Crack Transfer Lemma [10, Lemma 4.1] can be substituted by the weaker ones (V6) and (V8) stated here.

Fixed $t \in [0, 1]$, by Definition 3.2 there is a subsequence $u_{k_j}(t)$ converging to $u(t)$ weakly* in $SBV^p(\Omega_D; K)$. Arguing as in [10, Remark 4.4], one can see that
$$\mathcal{V}(\tau_{k_j}(t), u_{k_j}(t)) \to \mathcal{V}(t, u(t)). \tag{3.8}$$

Discrete energy inequality

Let now $\left(u_k^i, \Gamma_k^i\right) := \left(u_k\left(t_k^i\right), \Gamma_k\left(t_k^i\right)\right)$. Taking $(u, \Gamma) = \left(I, \Gamma_k^{i-1}\right)$ in (3.3), we get $\mathcal{F}^{\text{el}}\left(t_k^i, u_k^i\right) \leq \mathcal{F}^{\text{el}}\left(t_k^i, I\right)$. Hence, by (V3), (L1), and (T1)

$$\mathcal{F}^{\text{el}}\left(t_k^i, u_k^i\right) < M + c_L + c_T, \tag{3.9}$$

so that $\left\|\nabla u_k^i\right\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})}$ is bounded uniformly in k and i by coercivity. As $u_k^{i-1} \in AD\left(I, \Gamma_k^{i-1}\right)$, by (3.3) we have $\mathcal{F}\left(t_k^i, u_k^i, \Gamma_k^i\right) \leq \mathcal{F}\left(t_k^i, u_k^{i-1}, \Gamma_k^{i-1}\right)$. By (V6), (3.9), (L2), and (T2), the function $t \mapsto \mathcal{F}^{\text{el}}\left(t, u_k^{i-1}\right)$ is absolutely continuous; therefore,

$$\mathcal{F}^{\text{el}}\left(t_k^i, u_k^{i-1}\right) - \mathcal{F}^{\text{el}}\left(t_k^{i-1}, u_k^{i-1}\right) = \int_{t_k^{i-1}}^{t_k^i} \mathrm{D}_t\mathcal{F}^{\text{el}}\left(t, u_k^{i-1}\right)\,\mathrm{d}t.$$

Summing up, we obtain for every $t \in [0, 1]$ the discrete energy inequality

$$\mathcal{F}_k(t, u_k(t), \Gamma_k(t)) \leq \mathcal{F}(0, u_0, \Gamma_0) + \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s. \tag{3.10}$$

By (V6), (3.9), (L2), (T2), and (3.10), $\mathcal{F}_k(t, u_k(t), \Gamma_k(t))$ is bounded uniformly with respect to k and t. The non-negativity of V, (L1), (T1), and (1.9) give a bound also on $\mathcal{H}^{n-1}(\Gamma_k(t))$, uniform in k and t.

Energy inequality

By Theorem 1.1, we have for every $t \in [0, 1]$

$$\mathcal{K}(\Gamma(t)) = \mathcal{K}(\Gamma^*(t) \cup \Gamma_0) \leq \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t) \cup \Gamma_0) = \liminf_{k\to\infty} \mathcal{K}(\Gamma_k(t)); \tag{3.11}$$

moreover, Fatou's lemma implies that the function

$$\theta_\infty(t) := \limsup_{k\to\infty} \theta_k(t) \tag{3.12}$$

belongs to $L^1([0, 1])$ and

$$\limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s \leq \int_0^t \theta_\infty(s)\,\mathrm{d}s. \tag{3.13}$$

Fixed $s \in [0, 1]$, by Definition 3.2 there is a subsequence $(u_{k_j}(s), \Gamma_{k_j}(s))$ such that

$$u_{k_j}(s) \rightharpoonup u(s) \text{ weakly}^* \text{ in } SBV^p(\Omega_D; K) \tag{3.14}$$

and

$$\theta_\infty(s) = \lim_{k\to\infty} \theta_{k_j}(s). \tag{3.15}$$

By (3.8), (V6), and (2.16) we have

$$\mathcal{V}(s, u_{k_j}(s)) \to \mathcal{V}(s, u(s)), \quad \mathcal{L}(s, u_{k_j}(s)) \to \mathcal{L}(s, u(s)), \quad \mathcal{T}(s, u_{k_j}(s)) \to \mathcal{T}(s, u(s)). \tag{3.16}$$

In order to pass to the limit as $k_j \to \infty$ in (3.10), we employ Lemma 3.1, which is based on the following consequence of (V7).

Remark 3.1 From Remark 2.1, we deduce that for every $s \in [0, 1]$ and $M > 0$ there exists a modulus of continuity $\omega_s^M : [0, 1] \to [0, +\infty)$, with $s \mapsto \omega_s^M(h)$ in $L^\infty([0, 1])$ for every $h \in [0, 1]$, such that

$$\left| \mathrm{D}_t \mathcal{V}(s, v) \mp \frac{\mathcal{V}(s \pm h, v) - \mathcal{V}(s, v)}{h} \right| \leq \omega_s^M(h) \tag{3.17}$$

for every $v \in SBV^p(\Omega_D; K)$ such that $\mathcal{V}(0, v) \leq M$ and every $h > 0$ with $s \pm h \in [0, 1]$, provided that $\mathrm{D}_t \mathcal{V}$ is defined.

Lemma 3.1 *Let* $\mathcal{V} : [0, 1] \times SBV^p(\Omega_D; K) \to [0, +\infty]$ *be lower semicontinuous with respect to the weak* convergence in* $SBV^p(\Omega_D; K)$ *and satisfying* (3.17)*. Let* u_j *be a sequence converging to* u_∞ *weakly* in* $SBV^p(\Omega_D; K)$*; fix* $s \in [0, 1]$ *where* $\mathrm{D}_t \mathcal{V}$ *is defined. Assume that* $\mathcal{V}(s, u_j) \to \mathcal{V}(s, u_\infty) < +\infty$*. Then* $\mathrm{D}_t \mathcal{V}(s, u_j) \to \mathrm{D}_t \mathcal{V}(s, u_\infty)$*.*

Proof This lemma was shown in [16, Proposition 3.3]; from that proof, it is clear that ω_t need not be uniform with respect to t. □

Applying Lemma 3.1, from (3.14) and (3.16) we deduce that for $\mathcal{L}^1$-a.e. $s \in [0, 1]$

$$\mathrm{D}_t \mathcal{V}(s, u_{k_j}(s)) \to \mathrm{D}_t \mathcal{V}(s, u(s)).$$

The convergence of the derivatives of the force terms is given by (2.17). Hence, by (2.5), (3.5), and (3.15), we conclude that for $\mathcal{L}^1$-a.e. $s \in [0, 1]$

$$\theta_\infty(s) = \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s)). \tag{3.18}$$

By (3.8), (2.16), and (3.11) we have

$$\mathcal{F}(t, u(t), \Gamma(t)) \leq \liminf_{j\to\infty} \mathcal{F}_{k_j}(t, u_{k_j}(t), \Gamma_{k_j}(t)) \leq \limsup_{k\to\infty} \mathcal{F}_k(t, u_k(t), \Gamma_k(t)).$$

From (3.10), (3.12), (3.13), and (3.18) we obtain

$$\limsup_{k\to\infty} \mathcal{F}_k(t, u_k(t), \Gamma_k(t)) \leq \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s))\, \mathrm{d}s.$$

Then we get the energy inequality

$$\mathcal{F}(t, u(t), \Gamma(t)) \leq \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s))\, \mathrm{d}s. \tag{3.19}$$

Finally, comparing $(u(t), \Gamma(t))$ with $(I, \Gamma(t))$, by (3.6), (V3), (V4), (L1), and (T1), we find a constant $C > 0$ such that

$$\mathcal{V}(t, u(t)) \leq C, \quad \|\nabla u(t)\|_{L^p(\Omega_D; \mathbb{M}^{n\times n})} \leq C, \quad \mathcal{H}^{n-1}(S(u(t))) \leq C \tag{3.20}$$

uniformly in t.

Approximation with Riemann sums

For the next point, we will use the approximation of Lebesgue integrals with suitable Riemann sums [20]. Let $\mathcal{C}_1$ a countable subset of $L^\infty(\Omega; K)$, dense for the norm of $L^q(\Omega; \mathbb{R}^n)$, and $\mathcal{C}_2$ a countable subset of $L^\infty(\partial_S\Omega; K)$, dense for the norm of $L^q(\partial_S\Omega; \mathbb{R}^n)$. By [9, Lemma 4.12 and Remark 4.13], we can find a sequence of subdivisions $\{s_k^i\}_{0\le i\le i_k}$ satisfying:

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{i_k-1} < s_k^{i_k} = t, \quad \lim_{k\to\infty} \max_{1\le i\le i_k} \left(s_k^i - s_k^{i-1}\right) = 0, \tag{3.21}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| b_L\left(s_k^i\right) - b_L(s) \right| \mathrm{d}s = 0, \tag{3.22}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| b_T\left(s_k^i\right) - b_T(s) \right| \mathrm{d}s = 0, \tag{3.23}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t\mathcal{F}^{\mathrm{el}}\left(s_k^i, u\left(s_k^i\right)\right) - \mathrm{D}_t\mathcal{F}^{\mathrm{el}}(s, u(s)) \right| \mathrm{d}s = 0, \tag{3.24}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t\mathcal{L}\left(s_k^i, v\right) - \mathrm{D}_t\mathcal{L}(s, v) \right| \mathrm{d}s = 0 \quad \text{for every } v \in \mathcal{C}_1, \tag{3.25}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t\mathcal{T}\left(s_k^i, v\right) - \mathrm{D}_t\mathcal{T}(s, v) \right| \mathrm{d}s = 0 \quad \text{for every } v \in \mathcal{C}_2, \tag{3.26}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \omega_{s_k^i}^C\left(\frac{1}{m}\right) - \omega_s^C\left(\frac{1}{m}\right) \right| \mathrm{d}s = 0 \quad \text{for every } m \in \mathbb{N}, \tag{3.27}$$

where ω_s^C is defined in Remark 3.1 and C is the constant of (3.20). In the previous formulas, it is understood that all time derivatives are well defined at s_k^i.

We can deduce the following lemma.

Lemma 3.2 *In the previous assumptions,*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t\mathcal{V}\left(s_k^i, u\left(s_k^i\right)\right) - \mathrm{D}_t\mathcal{V}\left(s, u\left(s_k^i\right)\right) \right| \mathrm{d}s = 0. \tag{3.28}$$

Proof Fixed $m \in \mathbb{N}$, we have $\max_i (s_k^i - s_k^{i-1}) \leq 1/m$ for k large. Comparing the derivatives with the difference quotients and employing twice (3.17), we get

$$\int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{V}\left(s_k^i, u\left(s_k^i\right)\right) - \mathrm{D}_t \mathcal{V}\left(s, u\left(s_k^i\right)\right) \right| \mathrm{d}s \leq \int_{s_k^{i-1}}^{s_k^i} \left[\omega_{s_k^i}^C \left(\frac{1}{m}\right) + \omega_s^C \left(\frac{1}{m}\right) \right] \mathrm{d}s$$

for every $s \in [s_k^{i-1}, s_k^i)$. We deduce that

$$\begin{aligned} &\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{V}\left(s_k^i, u\left(s_k^i\right)\right) - \mathrm{D}_t \mathcal{V}\left(s, u\left(s_k^i\right)\right) \right| \mathrm{d}s \\ &\leq \lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left[\omega_{s_k^i}^C \left(\frac{1}{m}\right) + \omega_s^C \left(\frac{1}{m}\right) \right] \mathrm{d}s \leq 2 \int_0^t \omega_s^C \left(\frac{1}{m}\right) \mathrm{d}s, \end{aligned}$$

where in the last estimate we used (3.27). Passing to the limit as $m \to \infty$, we conclude by dominated convergence, thanks to the uniform bound on ω_s^C. □

As for the approximation of the force terms, we follow [9, Lemma 5.7].

Lemma 3.3 *In the previous assumptions,*

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{L}\left(s_k^i, u\left(s_k^i\right)\right) - \mathrm{D}_t \mathcal{L}\left(s, u\left(s_k^i\right)\right) \right| \mathrm{d}s = 0, \tag{3.29}$$

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{T}\left(s_k^i, u\left(s_k^i\right)\right) - \mathrm{D}_t \mathcal{T}\left(s, u\left(s_k^i\right)\right) \right| \mathrm{d}s = 0. \tag{3.30}$$

Proof Consider the set H of all functions $v \in SBV^p(\Omega_D; K)$ such that

$$\|\nabla v\|_{L^p(\Omega_D;\mathbb{M}^{n\times n})} \leq C \text{ and } \mathcal{H}^{n-1}(S(v)) \leq C,$$

where C is the constant appearing in (3.20). By the SBV compactness theorem [2, Theorem 4.8], H is compact in $L^\infty(\Omega; K)$ with respect to the norm of $L^q(\Omega; \mathbb{R}^n)$. Fix $\varepsilon > 0$; there exists a finite number of functions $v_1, \ldots, v_h \in \mathcal{C}_1$ such that for every $v \in H$ there exists j with $\left\|v - v_j\right\|_{L^q(\Omega;\mathbb{R}^n)} < \varepsilon$. By (L3), we have

$$\left|\mathrm{D}_t \mathcal{L}(s)(v) - \mathrm{D}_t \mathcal{L}(s)(v_j)\right| \leq \varepsilon\, b_L(s)$$

for $\mathcal{L}^1$-a.e. $s \in [0,1]$ (including the points s_k^i). Then,

$$\sum_{i=1}^{i_k} \sup_{v \in H} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{L}\left(s_k^i, v\right) - \mathrm{D}_t \mathcal{L}(s, v) \right| \mathrm{d}s$$

$$\leq \sum_{j=1}^{h} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{L}\left(s_k^i, v_j\right) - \mathrm{D}_t \mathcal{L}(s, v_j) \right| \mathrm{d}s + \varepsilon \sum_{i=1}^{i_k} \int_0^t \left[b_L\left(s_k^i\right) + b_L(s) \right] \mathrm{d}s.$$

First, we pass to the lim sup as $k \to \infty$, then we let $\varepsilon \to 0$; recalling (3.22) and (3.25) we find that the left hand side in the previous expression is vanishing. Hence, (3.29) follows. The proof of (3.30) is analogous. □

Summing up (3.24), (3.28), (3.29), and (3.30), we obtain

$$\lim_{k \to \infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \mathrm{D}_t \mathcal{F}^{\mathrm{el}}\left(s, u\left(s_k^i\right)\right) - \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s)) \right| \mathrm{d}s = 0. \tag{3.31}$$

Energy equality

The converse of (3.19) is a consequence of the stability property, via a discretization argument.

For $i = 1, \ldots, i_k$, $\left(u\left(s_k^{i-1}\right), \Gamma\left(s_k^{i-1}\right)\right)$ and $\left(u\left(s_k^i\right), \Gamma\left(s_k^i\right)\right)$ are competitors in (3.6): as $u\left(s_k^i\right) \in AD\left(I, \Gamma\left(s_k^i\right)\right)$ and $\Gamma\left(s_k^{i-1}\right) \subset \Gamma\left(s_k^i\right)$, we get

$$\mathcal{F}\left(s_k^{i-1}, u\left(s\left(_k^{i-1}\right), \Gamma\left(s_k^{i-1}\right)\right) \leq \mathcal{F}\left(s_k^{i-1}, u\left(s_k^i\right), \Gamma\left(s_k^i\right)\right).$$

Arguing as in the proof of the discrete energy inequality, by (3.20), (V6), (L2), and (T2) we obtain

$$\mathcal{F}\left(s_k^{i-1}, u\left(s_k^i\right), \Gamma\left(s_k^i\right)\right) = \mathcal{F}\left(s_k^i, u\left(s_k^i\right), \Gamma\left(s_k^i\right)\right) - \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{F}^{\mathrm{el}}\left(s, u\left(s_k^i\right)\right) \mathrm{d}s.$$

Summing up,

$$\mathcal{F}(t, u(t), \Gamma(t)) \geq \mathcal{F}(0, u_0, \Gamma_0) + \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \mathrm{D}_t \mathcal{F}^{\mathrm{el}}\left(s, u\left(s_k^i\right)\right) \mathrm{d}s.$$

By (3.19) and (3.31) we have

$$\mathcal{F}(t, u(t), \Gamma(t)) = \mathcal{F}(0, u_0, \Gamma_0) + \int_0^t \mathrm{D}_t \mathcal{F}^{\mathrm{el}}(s, u(s))\, \mathrm{d}s,$$

which implies (2). The proof of Theorem 3.3 is concluded. □

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