Annali di Matematica (2010) 189:643–660
DOI 10.1007/s10231-010-0129-2

Inequalities for eigenvalues of elliptic operators in divergence form on Riemannian manifolds

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Received: 28 May 2009 / Accepted: 21 January 2010 / Published online: 9 February 2010

Abstract In this paper, we study eigenvalues of elliptic operators in divergence form on compact Riemannian manifolds with boundary (possibly empty) and obtain a general inequality for them. By using this inequality, we prove universal inequalities for eigenvalues of elliptic operators in divergence form on compact domains of complete submanifolds in a Euclidean space, and of complete manifolds admitting special functions which include the Hadamard manifolds with Ricci curvature bounded below, a class of warped product manifolds, the product of Euclidean spaces with any complete manifold and manifolds admitting eigenmaps to a sphere.

Keywords Universal bounds · Eigenvalues · Elliptic operator · Payne-Pólya-Weinberger-Yang type inequalities · Submanifolds · Hypersurfaces in space forms · Warped manifolds

Mathematics Subject Classification (2000) 35P15 · 53C20 · 53C42 · 58G25

1 Introduction

Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded connected domain in M. Let Δ be the Laplace operator acting on functions on M. The study of the

M. P. do Carmo was partially supported by FAPERJ; Q. Wang was partially supported by CNPq; C. Xia was partially supported by CNPq and FAPDF.

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spectrum of Δ is an important topic, and many works have been done in this area during the past years (see, e.g., [5,10,30] and the references therein). Consider the *Dirichlet eigenvalue problem* or the *fixed membrane problem* given by

$$\Delta u = -\lambda u \quad \text{in}\,\Omega, \quad u|_{\partial\Omega} = 0. \tag{1.1}$$

Let

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

denote the successive eigenvalues of (1.1). Here, each eigenvalue is repeated according to its multiplicity. When M is a Euclidean space $\mathbf{R}^n$, Payne, Pólya and Weinberger proved in a seminal paper [26] that

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{k=1}^{k} \lambda_i, \quad k = 1, 2, \ldots, \tag{1.2}$$

In 1980, Hile and Protter [23] generalized (1.2) and proved

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad \text{for}\, k = 1, 2, \ldots. \tag{1.3}$$

In 1991, Yang [31] obtained the following much stronger inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n}\right) \lambda_i \right) \leq 0, \quad \text{for}\, k = 1, 2, \ldots. \tag{1.4}$$

The inequalities on the higher eigenvalues of the Laplacian on a connected bounded domain in $\mathbf{R}^n$ obtained by Payne-Pólya-Weinberger, Hile-Protter, Yang have also been extended to Riemannian manifolds (cf. [6,7,11–14,16–21,24,28,29,32], etc.). In particular, for the Dirichlet eigenvalue problem on a complete Riemannian manifold isometrically immersed in a Euclidean space, Chen–Cheng [11] and El Soufi–Harrell–Ilias [16] have proved, independently,

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \frac{n^2}{4} H_0^2 \right), \tag{1.5}$$

where $H_0 = \max_{x\in\Omega} |\mathbf{H}|$ and $\mathbf{H}$ is the mean curvature vector of M. When M is a unit n-sphere, the above inequality has also been obtained in [11]. When M is an n-dimensional hypersurface in $\mathbf{R}^{n+1}$, Harrell [18] has also proved the above inequality. Harrell-Stuble [19] realized that many of the original Payne-Pólya-Weinberger's results rely on facts involving operators, their commutators, and traces. Generalizing Yang's inequality (1.4), Ashbaugh [6] considered eigenvalues of Schrödinger operators with weight on bounded domains in $\mathbf{R}^n$ and obtained universal bounds for them. Ashbaugh-Hermi [7] presented a new proof which unifies the classical inequalities of Payne-Pólya-Weinberger, Hile–Protter and Yang. El Soufi–Harrell–Ilias [16] obtained universal bounds on eigenvalues of Schrödinger operators on bounded domains in a submanifold of Euclidean spaces.

The purpose of this paper is to study the Dirichlet eigenvalues of elliptic operators in divergence form with weight on Riemannian manifolds. Such operators include the Lapacian operator as special case. We will prove a general inequality for them. By using this inequality, we prove universal inequalities for eigenvalues of elliptic operators in divergence form on compact domains of complete submanifolds in a Euclidean space, and of complete manifolds admitting special functions which include the Hadamard manifolds with Ricci curvature bounded below, a class of warped product manifolds containing the hyperbolic space, manifolds admitting spherical eigenmaps and the product of Euclidean spaces with any complete manifold. It should be mentioned that the problem we study include the eigenvalue problem of the linearized operator L_r of the r-th mean curvature of a hypersurface which is closely related to the stability problem of hypersurfaces of constant r-th mean curvature in a space form. The first non-zero eigenvalue of L_r has been studied recently (see e.g., [1–4,9,15] and the reference therein). To the authors' knowledge, very little is known about its higher order eigenvalues.

2 A general inequality for eigenvalues of elliptic operators in divergence form on compact manifolds

In this section, we prove a general result for eigenvalues of elliptic operator in divergence form with weight on compact Riemannian manifolds.

Theorem 2.1 *Let $(M, \langle , \rangle)$ be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty). Let $A : M \to \mathrm{End}(TM)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of TM. Let V a non-negative continuous function on M and ρ a weight function which is positive and continuous on M. Denote by Δ and ∇ the Laplacian and the gradient operator of M, respectively. Consider the eigenvalue problem*

$$\begin{cases} -\mathrm{div}(A\nabla u) + Vu = \lambda\rho u \quad \text{in } M, \\ u|_{\partial M} = 0. \end{cases} \tag{2.1}$$

Let λ_i be the ith eigenvalue of (2.1) and u_i be the corresponding orthonormal eigenfunction, that is,

$$-\mathrm{div}(A\nabla u_i) + Vu_i = \lambda_i\rho u_i \quad \text{in } M, \tag{2.2}$$

$$\int_M \rho u_i u_j = \delta_{ij}, \quad \forall\, i, j = 1, 2, \ldots. \tag{2.3}$$

Then for any function $h \in C^2(M)$ and any integer k, we have

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 ||u_i|\nabla h|||^2 \le \delta \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_M u_i^2 \langle \nabla h, A\nabla h\rangle + \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left\| \frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla h\rangle + \frac{1}{2} u_i \Delta h \right) \right\|^2. \tag{2.4}$$

Here for a vector field X on M, $\mathrm{div}\, X$ denotes the divergence of X, $||f||^2 = \int_M f^2$ and δ is any positive constant.

Remark 2.1 One can check that when $\partial M \neq \emptyset$, the first eigenvalue λ_1 of the problem (2.1) is always positive and when $\partial M = \emptyset$, we have $\lambda_1 \ge 0$ with equality holding if and only if $V \equiv 0$. In both cases, we use the same notations $\lambda_1 \le \lambda_2 \le \cdots \to \infty$ to represent the set of the eigenvalues of the problem (2.1).

Proof of Theorem 2.1 Set $Lu = -\mathrm{div}(A\nabla u) + Vu$ and for $f, g : M \to \mathbf{R}$, let us consider the inner product given by $\langle\langle f, g\rangle\rangle = \int_M \rho f g$. If a non-trivial function ϕ on M satisfying $\phi|_{\partial M} = 0$ is orthogonal to $u_1, u_2, \ldots, u_k$ with respect to the above inner product, then the Rayleigh–Ritz inequality tells us that

$$\lambda_{k+1} \le \frac{\int_M \phi L\phi}{\int_M \rho\phi^2}. \tag{2.5}$$

For each $i = 1, \ldots, k$, we define $\phi_i : M \to \mathbf{R}$ by

$$\phi_i = hu_i - \sum_{j=1}^{k} a_{ij} u_j,$$

where

$$a_{ij} = \int_\Omega \rho h u_i u_j = a_{ji}.$$

Since

$$\phi_i|_{\partial M} = 0$$

and

$$\int_M \rho u_j \phi_i = 0, \quad \forall\, i, j = 1, \ldots, k, \tag{2.6}$$

we know from (2.5) that

$$\begin{aligned} \lambda_{k+1} \int_M \rho\phi_i^2 &\le \int_M \phi_i L\phi_i \\ &= \int_M \phi_i \left(L(hu_i) - \sum_{j=1}^{k} a_{ij}\lambda_j \rho u_j \right) \\ &= \int_M \phi_i L(hu_i). \end{aligned} \tag{2.7}$$

Using (2.2), we have

$$L(hu_i) = \lambda_i \rho h u_i - 2\langle \nabla u_i, A\nabla h\rangle - u_i\, \mathrm{div}(A\nabla h). \tag{2.8}$$

Thus, we have

$$\begin{aligned} &\lambda_{k+1} \int_M \rho\phi_i^2 \\ &\quad\le \lambda_i \int_M \phi_i \rho h u_i - \int_M \phi_i (2\langle \nabla u_i, A\nabla h\rangle + u_i\, \mathrm{div}(A\nabla h)) \\ &\quad= \lambda_i \int_M \rho\phi_i^2 - \int_M hu_i (2\langle \nabla u_i, A\nabla h\rangle + u_i\, \mathrm{div}(A\nabla h)) + \sum_{j=1}^{k} b_{ij} a_{ij}, \end{aligned} \tag{2.9}$$

where

$$b_{ij} = \int_M (2\langle \nabla u_i, A\nabla h\rangle + u_i \operatorname{div}(A\nabla h))u_j$$

Multiplying (2.2) by hu_j, one gets

$$-\operatorname{div}(A\nabla u_i)hu_j + Vhu_iu_j = \lambda_i \rho h u_i u_j \tag{2.10}$$

Changing the roles of i and j, we have

$$-\operatorname{div}(A\nabla u_j)hu_i + Vhu_iu_j = \lambda_j \rho h u_i u_j \tag{2.11}$$

Subtracting (2.11) from (2.10) and integrating the resulted equality on M, we get by using the divergence theorem that

$$\begin{aligned}(\lambda_i - \lambda_j)a_{ij} &= \int_M (\operatorname{div}(A\nabla u_j)hu_i - \operatorname{div}(A\nabla u_i)hu_j)\\ &= \int_M (-\langle A\nabla u_j, \nabla(hu_i)\rangle + \langle A\nabla u_i, \nabla(hu_j)\rangle)\\ &= \int_M (-\langle \nabla u_j, A(u_i\nabla h)\rangle + \langle A\nabla u_i, u_j\nabla h\rangle)\\ &= \int_M u_j(\operatorname{div}(u_iA\nabla h) + \langle A\nabla u_i, \nabla h\rangle)\\ &= b_{ij},\end{aligned} \tag{2.12}$$

which, combining with (2.9), gives

$$\begin{aligned}&(\lambda_{k+1} - \lambda_i)\int_M \rho\phi_i^2\\ &\le -\int_M hu_i(2\langle \nabla u_i, A\nabla h\rangle + u_i \operatorname{div}(A\nabla h)) + \sum_{j=1}^k (\lambda_i - \lambda_j)a_{ij}^2.\end{aligned} \tag{2.13}$$

Setting

$$c_{ij} = \int_M u_j\left(\langle \nabla u_i, \nabla h\rangle + \frac{1}{2}u_i\Delta h\right);$$

then, one gets from the divergence theorem that

$$\begin{aligned}c_{ij} + c_{ji} &= \int_M \langle \nabla h, u_j\nabla u_i + u_i\nabla u_j\rangle + \int_M u_iu_j\Delta h\\ &= \int_M \langle \nabla h, \nabla(u_iu_j)\rangle + \int_M u_iu_j\Delta h = 0,\end{aligned} \tag{2.14}$$

and

$$\int_M (-2)\phi_i \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) = -2 \int_M h u_i \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) + 2 \sum_{j=1}^{k} c_{ij} a_{ij}$$
$$= ||u_i |\nabla h|||^2 + 2 \sum_{j=1}^{k} c_{ij} a_{ij}. \tag{2.15}$$

Multiplying (2.15) by $(\lambda_{k+1} - \lambda_i)^2$ and using the Schwarz inequality and (2.13), we get

$$\begin{aligned}
&(\lambda_{k+1} - \lambda_i)^2 \left(||u_i|\nabla h|||^2 + 2 \sum_{j=1}^{k} c_{ij} a_{ij} \right) \\
&= (\lambda_{k+1} - \lambda_i)^2 \int_\Omega (-2)\sqrt{\rho}\phi_i \left(\frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) - \sum_{j=1}^{k} c_{ij} \sqrt{\rho} u_j \right) \\
&\le \delta (\lambda_{k+1} - \lambda_i)^3 ||\sqrt{\rho}\phi_i||^2 \\
&\quad + \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \int_\Omega \left| \frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) - \sum_{j=1}^{k} c_{ij} \sqrt{\rho} u_j \right|^2 \\
&= \delta (\lambda_{k+1} - \lambda_i)^3 ||\sqrt{\rho}\phi_i||^2 \\
&\quad + \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left(\left\| \frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) \right\|^2 - \sum_{j=1}^{k} c_{ij}^2 \right) \\
&\le \delta (\lambda_{k+1} - \lambda_i)^2 \left(-\int_M h u_i (2\langle \nabla u_i, A\nabla h \rangle + u_i \operatorname{div}(A\nabla h)) + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2 \right) \\
&\quad + \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left(\left\| \frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) \right\|^2 - \sum_{j=1}^{k} c_{ij}^2 \right),
\end{aligned}$$

where δ is any positive constant. Summing over i and noticing $a_{ij} = a_{ji}$, $c_{ij} = -c_{ji}$, we infer

$$\begin{aligned}
&\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 ||u_i|\nabla h|||^2 - 2 \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j) a_{ij} c_{ij} \\
&\le \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(-\int_M h u_i (2\langle \nabla u_i, A\nabla h \rangle + u_i \operatorname{div}(A\nabla h)) \right) \\
&\quad + \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left\| \frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla h \rangle + \frac{1}{2} u_i \Delta h \right) \right\|^2 \\
&\quad - \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) \delta (\lambda_i - \lambda_j)^2 a_{ij}^2 - \sum_{i,j=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} c_{ij}^2.
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i|\nabla h|\|^2\\
&\quad\le \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(-\int_M hu_i(2\langle\nabla u_i, A\nabla h\rangle+u_i\ \mathrm{div}\ (A\nabla h))\right)\\
&\qquad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)\right\|^2. \qquad (2.16)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
-2\int_M hu_i\langle\nabla u_i, A\nabla h\rangle &= -\frac{1}{2}\int_M\langle\nabla u_i^2, A\nabla h^2\rangle\\
&=\frac{1}{2}\int_M u_i^2\mathrm{div}(A\nabla h^2)\\
&=\int_M u_i^2\langle\nabla h, A\nabla h\rangle+\int_M u_i^2 h\mathrm{div}(A\nabla h). \qquad (2.17)
\end{aligned}
$$

Substituting (2.17) into (2.16), we get (2.4). This completes the proof of Theorem 2.1. □

3 Inequalities for eigenvalues of submanifolds in Euclidean spaces

In this section, we prove inequalities for eigenvalues of elliptic operators in divergence form on submanifolds in Euclidean spaces by using Theorem 2.1.

Theorem 3.1 *Let M be an n-dimensional compact submanifold with boundary ∂M (possibly empty) immersed in $\mathbf{R}^m$. Let $A : M \to \mathrm{End}(TM)$ be a smooth symmetric positive definite section of the bundle of all endomorphisms of TM. Assume that the eigenvalues of A are bounded below by ξ_1 and that $tr(A)\le n\xi_2$ throughout M, ξ_1 and ξ_2 being two positive constants. Let ρ be a continuous function on M satisfying $\rho_1\le\rho(x)\le\rho_2$, $\forall x\in M$, for some positive constants ρ_1 and ρ_2. Let $\mathbf{H}$ be the mean curvature vector of M in $\mathbf{R}^m$ and denote by $\{\lambda_i\}_{i=1}^{\infty}$ the eigenvalues of the eigenvalue problem*

$$
\begin{cases} -div(A\nabla u)+Vu=\lambda\rho u & in\ M,\\ u|_{\partial M}=0. \end{cases} \qquad (3.1)
$$

Then we have

$$
\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\le\frac{4\xi_2\rho_2^2}{n\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i-\frac{V_0}{\rho_2}\right)+\frac{n^2H_0^2}{4\rho_1}\right), \qquad (3.2)
$$

where $H_0=\max_{x\in M}|\mathbf{H}(x)|$ and $V_0=\min_{x\in M}V(x)$.

Proof of Theorem 3.1 Let $x_1,\dots,x_m$ be the standard coordinate functions of $\mathbf{R}^m$ and let $\{u_i\}_{i=1}^{\infty}$ be the orthonormal eigenfunctions of the problem (3.1) corresponding to $\{\lambda_i\}_{i=1}^{\infty}$.

Taking $h = x_\alpha$ in (2.4) and summing over α, we get

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \sum_{\alpha=1}^{m} ||u_i|\nabla x_\alpha|||^2 \\ \leq \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \sum_{\alpha=1}^{m} \int_M u_i^2 \langle \nabla x_\alpha, A\nabla x_\alpha\rangle \\ + \sum_{i=1}^{k} \frac{(\lambda_{k+1}-\lambda_i)}{\delta} \sum_{\alpha=1}^{m} \left\| \frac{1}{\sqrt{\rho}} \left(\langle \nabla u_i, \nabla x_\alpha\rangle + \frac{1}{2} u_i \Delta x_\alpha \right) \right\|^2 .\end{aligned} \tag{3.3}$$

Since M is isometrically immersed in $\mathbf{R}^m$, we have

$$\Delta(x_1, \ldots, x_m) \equiv (\Delta x_1, \ldots, \Delta x_m) = n\mathbf{H}, \tag{3.4}$$

$$\sum_{\alpha=1}^{m} \langle \nabla x_\alpha, \nabla u_i\rangle^2 = \sum_{\alpha=1}^{m} (\nabla u_i(x_\alpha))^2 = |\nabla u_i|^2, \tag{3.5}$$

and

$$\sum_{\alpha=1}^{m} |\nabla x_\alpha|^2 = n. \tag{3.6}$$

It follows that

$$\sum_{\alpha=1}^{m} ||u_i|\nabla x_\alpha||| = n \int_\Omega u_i^2 \geq \frac{n}{\rho_2}. \tag{3.7}$$

Observe that for any tangent vector field Z locally defined on M, we have

$$Z = Z(x_1, \ldots, x_m) \equiv (Zx_1, \ldots, Zx_m). \tag{3.8}$$

Let $e_1, \ldots, e_n$ be orthonormal tangent vector fields locally defined on M; then one gets from (3.8) that

$$\begin{aligned}\sum_{\alpha=1}^{n+1} \langle A\nabla x_\alpha, \nabla x_\alpha\rangle &= \sum_{\alpha=1}^{n+1} \left\langle A\left(\sum_{i=1}^{n} \langle \nabla x_\alpha, e_i\rangle e_i\right), \sum_{j=1}^{n} \langle \nabla x_\alpha, e_j\rangle e_j \right\rangle \\ &= \sum_{i,j=1}^{n} \sum_{\alpha=1}^{n+1} (e_i x_\alpha)(e_j x_\alpha) \langle Ae_i, e_j\rangle \\ &= \sum_{i,j=1}^{n} \langle e_i, e_j\rangle \langle Ae_i, e_j\rangle = \sum_{i=1}^{n} \langle Ae_i, e_i\rangle = tr(A)\end{aligned}$$

and so

$$\sum_{\alpha=1}^{m} \int_M u_i^2 \langle \nabla x_\alpha, A\nabla x_\alpha\rangle \leq \int_M u_i^2\, tr(A) \leq \frac{n\xi_2}{\rho_1}. \tag{3.9}$$

For any $i = 1, \ldots, k$, we know from (3.4) and (3.8) that

$$\sum_{\alpha=1}^{m} \Delta x_\alpha \langle \nabla x_\alpha, \nabla u_i\rangle = \sum_{\alpha=1}^{m} \Delta x_\alpha \nabla u_i(x_\alpha) = \langle n\mathbf{H}, \nabla u_i\rangle = 0.$$

Thus,

$$\begin{aligned}
&\sum_{\alpha=1}^{m}\left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i,\nabla x_\alpha\rangle+\frac{1}{2}u_i\Delta x_\alpha\right)\right\|^2\\
&=\int_M\frac{1}{\rho}\sum_{\alpha=1}^{m}\left(\langle\nabla u_i,\nabla x_\alpha\rangle^2+\langle u_i\nabla u_i,\Delta x_\alpha\nabla x_\alpha\rangle+\frac{1}{4}u_i^2(\Delta x_\alpha)^2\right)\\
&=\int_M\frac{1}{\rho}\left(|\nabla u_i|^2+\frac{n^2|\mathbf{H}|^2}{4\rho}\rho u_i^2\right)\\
&\le\frac{1}{\rho_1}\int_M\left(|\nabla u_i|^2+\frac{n^2H_0^2}{4\rho_1}\rho u_i^2\right)\\
&\le\frac{1}{\rho_1}\left(\int_M|\nabla u_i|^2+\frac{n^2H_0^2}{4\rho_1}\right)
\end{aligned}\tag{3.10}$$

Multiplying the equation

$$-\operatorname{div}(A\nabla u_i)+Vu_i=\lambda_i\rho u_i$$

by u_i and integrating on M, we get

$$\begin{aligned}
\lambda_i&=\int_M u_i(-\operatorname{div}(A\nabla u_i)+Vu_i)\\
&=\int_M(\langle A\nabla u_i,\nabla u_i\rangle+Vu_i^2)\\
&\ge\xi_1\int_M|\nabla u_i|^2+\frac{V_0}{\rho_2}
\end{aligned}\tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\begin{aligned}
&\sum_{\alpha=1}^{m}\left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i,\nabla x_\alpha\rangle+\frac{1}{2}u_i\Delta x_\alpha\right)\right\|^2\\
&\le\frac{1}{\rho_1}\left(\frac{1}{\xi_1}\left(\lambda_i-\frac{V_0}{\rho_2}\right)+\frac{n^2H_0^2}{4\rho_1}\right)
\end{aligned}\tag{3.12}$$

Introducing (3.7), (3.9) and (3.12) into (3.3), we infer

$$\begin{aligned}
&\frac{n}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\
&\le\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\cdot\frac{n\xi_2}{\rho_1}+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\frac{1}{\rho_1}\left(\frac{1}{\xi_1}\left(\lambda_i-\frac{V_0}{\rho_2}\right)+\frac{n^2H_0^2}{4\rho_1}\right).
\end{aligned}\tag{3.13}$$

Taking

$$\delta = \frac{\left\{\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left(\frac{1}{\xi_1} \left(\lambda_i - \frac{V_0}{\rho_2} \right) + \frac{n^2 H_0^2}{4\rho_1} \right) \right\}^{1/2}}{\left\{ n\xi_2 \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \right\}^{1/2}},$$

we obtain (3.2). This completes the proof of Theorem 3.1. □

Remark 3.1 If we take $V \equiv 0$, $\rho \equiv 1$, and A to be the identity map, then the inequality (3.2) becomes (1.5) obtained by Chen–Cheng and El Soufi–Harrell–Ilias.

Remark 3.2 Let M be an n-dimensional closed (i.e., M is compact, connected and $\partial M = \emptyset$) orientable manifold, isometrically immersed in a complete simply connected Riemannian manifold $\mathbf{Q}^{n+1}(c)$ of constant curvature c and let A be the shape operator associated with a globally defined unit normal vector field ν on M. Denote by S_r the r-th elementary symmetric function of the eigenvalues $\kappa_1, \ldots, \kappa_n$ of A, i.e., $S_0 = 1$, $S_1 = \kappa_1 + \cdots + \kappa_n$, $S_n = \kappa_1 \ldots \kappa_n$. The r-th mean curvature of M is defined as

$$H_r = \frac{1}{\binom{n}{r}} S_r.$$

Let T_r be the r-th Newton transformation arising from A, that is, $T_0 = I$, $T_r = S_r I - AT_{r-1}$. Each P_r is a self-adjoint operator whose trace is $c(r)H_r$, where $c(r) = (n-r)\binom{n}{r}$ (see [9, Lemma 2.1]). Consider the operator $L_r : C^\infty(M) \to C^\infty(M)$ given by

$$L_r = \operatorname{div}(T_r \nabla f).$$

We call L_r the linearized operator of the r-th mean curvature of M. There are some conditions to guarantee that L_r is an elliptic operator (in particular, if the eigenvalues of T_r are all of the same sign, L_r is elliptic). In this case, some optimal upper bounds for the first non-zero eigenvalue of the eigenvalue problem

$$L_r f = -\lambda f, \ \ f \in C^\infty(M)$$

have been obtained (Cf. [1–4,15]). However, to the authors' knowledge, there are no known estimates for the higher order eigenvalues of L_r.

Assume now that the eigenvalues of T_r are bounded below by ξ_1 and that H_r is bounded above by ξ_2 throughout M, ξ_1 and ξ_2 being two positive constants. If M is a hypersurface in $\mathbf{R}^{n+1}$, from Theorem 3.1, for any positive integer k, one can derive an upper bound for the $(k+1)$-th eigenvalue in terms of the first k eigenvalues of L_r, the dimension n, ξ_1, ξ_2, and an up bound of the norm of the mean curvature vector of M in $\mathbf{R}^{n+1}$. Observe that if M is a hypersurface in a unit sphere $\mathbf{S}^{n+1}$, then M is a submanifold of $\mathbf{R}^{n+2}$ and the norm of the mean curvature vector of M in $\mathbf{R}^{n+2}$ is bounded above by a constant which depends on the dimension n and an upper bound of the mean curvature of M in $\mathbf{S}^{n+1}$. Hence, in this case, one can also get a similar upper bound for the $(k+1)$-th eigenvalue of L_r as in the case of M being a hypersurface in $\mathbf{R}^{n+1}$.

4 Eigenvalues on manifolds admitting special functions

In this section, using (2.4), we prove universal inequalities for eigenvalues of the elliptic operators in divergence form on manifolds admitting special functions. Our main result is as follows.

Theorem 4.1 *Let $(M, \langle,\rangle)$ be an n-dimensional complete Riemannian manifold and Ω be a bounded connected domain in M. Let $A : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric positive definite section of the bundle of all endomorphisms of $T\Omega$ and assume that $\xi_1 I \le A \le \xi_2 I$ in the sense that the eigenvalues of A lie in the interval $[\xi_1, \xi_2]$ throughout Ω, ξ_1 and ξ_2 being two positive constants. Let ρ be a continuous function on Ω satisfying $\rho_1 \le \rho(x) \le \rho_2$, $\forall x \in \Omega$, for some positive constants ρ_1 and ρ_2. Denote by Δ the Laplacian of M and let $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of the problem*

$$\begin{cases} -div(A\nabla u) + Vu = \lambda\rho u & in\, \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

i) *If there exists a function $\phi : \Omega \to \mathbf{R}$ such that*

$$|\nabla\phi| = 1, \ |\Delta\phi| \le A_0, \quad on\ \Omega, \tag{4.2}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{8\xi_2\rho_2^2}{\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) + \frac{A_0^2}{4\rho_1}\right), \tag{4.3}$$

ii) *Assume that $\rho \equiv 1$ and that there exists a function $\psi : \Omega \to \mathbf{R}$ such that*

$$|\nabla\psi| = 1, \ \Delta\psi = B_0, \quad on\ \Omega, \tag{4.4}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le 4\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) - \frac{B_0^2}{4}\right); \tag{4.5}$$

iii) *If there exist l functions $\phi_p : \Omega \to \mathbf{R}$ such that*

$$\langle\nabla\phi_p, \nabla\phi_q\rangle = \delta_{pq}, \ \Delta\phi_p = 0, \quad on\ \Omega, \ p, q = 1, \ldots, l, \tag{4.6}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4\xi_2\rho_2^2}{l\xi_1\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i - \frac{V_0}{\rho_2}\right); \tag{4.7}$$

iv) *If M admits an eigenmap $f = (f_1, f_2, \ldots, f_{m+1}) : \Omega \to \mathbf{S}^m$ corresponding to an eigenvalue μ, that is, $\sum_{\alpha=1}^{m+1} f_\alpha^2 = 1$, $\Delta f_\alpha = -\mu f_\alpha$, $\alpha = 1, \ldots, m+1$, then*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4\xi_2\rho_2^2}{\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) + \frac{\mu}{4\rho_1}\right); \tag{4.8}$$

In the above, A_0 and B_0 are constants, $\mathbf{S}^m$ is the unit m-sphere and $V_0 = \min_{x\in\Omega} V(x)$.

Proof of Theorem 4.1 Similar calculations as in the last section give (Cf. (3.7), (3.11))

$$\int_\Omega |\nabla u_i|^2 \le \frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right), \tag{4.9}$$

$$\frac{1}{\rho_2} \le \int_\Omega u_i^2 \le \frac{1}{\rho_1}. \tag{4.10}$$

i) Taking $h = \phi$ in (2.4), using (4.2), (4.9), (4.10), $\langle \nabla\phi, A\nabla\phi\rangle \le \xi_2|\nabla\phi|^2$ and the Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\
&\le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2||u_i||^2\\
&\le \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\langle\nabla\phi, A\nabla\phi\rangle\\
&\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left\|\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i,\nabla\phi\rangle+\frac{\Delta\phi}{2}u_i\right)\right\|^2\\
&\le \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega \xi_2 u_i^2|\nabla\phi|^2\\
&\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega\frac{2}{\rho_1}\left(\langle\nabla u_i,\nabla\phi\rangle^2+\frac{(\Delta\phi)^2}{4}u_i^2\right)\\
&\le \frac{\delta\xi_2}{\rho_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\
&\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega\frac{2}{\rho_1}\left(|\nabla u_i|^2+\frac{A_0^2}{4}u_i^2\right)\\
&\le \frac{\delta\xi_2}{\rho_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\
&\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\frac{2}{\rho_1}\left(\frac{1}{\xi_1}\left(\lambda_i-\frac{V_0}{\rho_2}\right)+\frac{A_0^2}{4\rho_1}\right)
\end{aligned}\tag{4.11}$$

Taking

$$\delta=\frac{\left\{2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i-\frac{V_0}{\rho_2}\right)+\frac{A_0^2}{4\rho_1}\right)\right\}^{1/2}}{\left\{\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right\}^{1/2}},$$

one gets (4.3).

ii) Since $\rho = 1$, we have by taking $h = \psi$ in (2.4), using (4.4) and similar calculations as in (4.11) that

$$\begin{aligned}
\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 &\le \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2\\
&\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega\left(\langle\nabla u_i,\nabla\psi\rangle^2+B_0u_i\langle\nabla u_i,\nabla\psi\rangle+\frac{B_0^2}{4}u_i^2\right)
\end{aligned}$$

$$\leq \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_M\left(|\nabla u_i|^2+\frac{B_0}{2}\langle\nabla u_i^2,\nabla\psi\rangle+\frac{B_0^2}{4}u_i^2\right)$$

$$=\delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_M\left(|\nabla u_i|^2-\frac{B_0}{2}u_i^2\Delta\psi+\frac{B_0^2}{4}u_i^2\right)$$

$$\leq \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\frac{1}{\xi_1}(\lambda_i-V_0)-\frac{B_0^2}{4}\right).$$

Taking

$$\delta=\frac{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}(\lambda_i-V_0)-\frac{B_0^2}{4}\right)\right\}^{1/2}}{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2\right\}^{1/2}},$$

we get (4.5).

iii) Taking $h=\phi_p$ in (2.4), we get

$$\begin{aligned}\frac{1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 &\leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2||u_i||^2\\ &\leq \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\langle\nabla\phi_p,A\nabla\phi_p\rangle\\ &\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left|\left|\frac{1}{\sqrt{\rho}}\langle\nabla u_i,\nabla\phi_p\rangle\right|\right|^2\\ &\leq \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega \xi_2u_i^2|\nabla\phi_p|^2\\ &\quad+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left|\left|\frac{1}{\sqrt{\rho}}\langle\nabla u_i,\nabla\phi_p\rangle\right|\right|^2\end{aligned}\tag{4.12}$$

From (4.6), we know that

$$\sum_{p=1}^{l}\langle\nabla u_i,\nabla\phi_p\rangle^2\leq|\nabla u_i|^2.$$

Thus, we obtain by summing over p in (4.12) that

$$\begin{aligned}&\frac{l}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\ &\leq \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega l\xi_2u_i^2+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega\frac{1}{\rho}|\nabla u_i|^2\\ &\leq \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\frac{l\xi_2}{\rho_1}+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\frac{1}{\rho_1\xi_1}\left(\lambda_i-\frac{V_0}{\rho_2}\right)\end{aligned}$$

Taking

$$\delta = \frac{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i - \frac{V_0}{\rho_2}\right)\right\}^{1/2}}{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 l \xi_1 \xi_2\right\}^{1/2}},$$

we obtain (4.7).

iv) Taking the Laplacian of the equation

$$\sum_{\alpha=1}^{m+1} f_\alpha^2 = 1$$

and using the fact that

$$\Delta f_\alpha = -\mu f_\alpha,\ \alpha = 1, \ldots, m+1,$$

we have

$$\sum_{\alpha=1}^{m+1} |\nabla f_\alpha|^2 = \mu. \tag{4.13}$$

It then follows by taking $h = f_\alpha$ in (2.4) and summing over α that

$$\begin{aligned}
\frac{\mu}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 &\le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \sum_{\alpha=1}^{m+1} ||u_i|\nabla f_\alpha||^2 \\
&\le \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2 \sum_{\alpha=1}^{m+1}\langle \nabla f_\alpha, A\nabla f_\alpha\rangle \\
&\quad + \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\sum_{\alpha=1}^{m+1}\left\|\frac{1}{\sqrt{\rho}}\left(\langle \nabla u_i, \nabla f_\alpha\rangle - \frac{\mu}{2}u_i f_\alpha\right)\right\|^2 \\
&\le \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \int_\Omega \xi_2 u_i^2 \sum_{\alpha=1}^{m+1}|\nabla f_\alpha|^2 \\
&\quad + \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega \frac{1}{\rho}\left(\sum_{\alpha=1}^{m+1}\langle \nabla u_i, \nabla f_\alpha\rangle^2 + \frac{\mu^2}{4}u_i^2\right) \\
&\le \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \int_\Omega \xi_2 \mu u_i^2 \\
&\quad + \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega \frac{1}{\rho}\left(\sum_{\alpha=1}^{m+1}|\nabla u_i|^2|\nabla f_\alpha|^2 + \frac{\mu^2}{4}u_i^2\right) \\
&\le \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\frac{\xi_2\mu}{\rho_1} + \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega \frac{1}{\rho}\left(\mu|\nabla u_i|^2 + \frac{\mu^2}{4}u_i^2\right) \\
&\le \delta \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\frac{\xi_2\mu}{\rho_1} + \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\frac{1}{\rho_1}\left(\mu\left(\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right) + \frac{\mu^2}{4\rho_1}\right)\right).
\end{aligned}$$

Taking

$$\delta = \frac{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right)+\frac{\mu}{4\rho_1}\right)\right\}^{1/2}}{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\xi_2\right\}^{1/2}},$$

we obtain (4.8). This completes the proof of Theorem 4.1. □

Now we give some examples of manifolds admitting special functions as in Theorem 4.1.

Example 4.1 Let M be an n-dimensional complete Riemannian manifold with Ricci curvature satisfying $\mathrm{Ric}_M \geq -(n-1)c^2$, $c \geq 0$. Let $f \to \mathbf{R}$ be a smooth function with $|\nabla f| \equiv 1$. It has been proved by Sakai that $|\Delta f| \leq (n-1)c$ on M (see Theorem 3.5 in [27]). Assume further that M is a Hadamard manifold and let $\gamma : [0, +\infty) \to M$ be a geodesic ray, namely a unit speed geodesic with $d(\gamma(s), \gamma(t)) = t - s$ for any $t > s > 0$. The Busemann function b_γ corresponding to γ defined by

$$b_\gamma(x) = \lim_{t\to+\infty}(d(x, \gamma(t)) - t)$$

satisfies $|\nabla b_\gamma| \equiv 1$(Cf. [8,22]). We know from Sakai's theorem that $|\Delta b_\gamma| \leq (n-1)c$ on M. Thus, any Hadamard manifold with Ricci curvature bounded below by a negative constant admits functions satisfying (4.2).

Example 4.2 Let (N, ds_N^2) be a complete Riemannian manifold and define a Riemannian manifold on $M = \mathbf{R} \times N$ by

$$\langle , \rangle = ds_M^2 = dt^2 + \eta^2(t)ds_N^2, \tag{4.14}$$

where η is a positive smooth function defined on $\mathbf{R}$ with $\eta(0) = 1$. Then, $(M, \langle , \rangle)$ is called a warped product and denoted by $M = \mathbf{R} \times_\eta N$. It is known that M is a complete Riemannian manifold.

Set $\eta = e^{-t}$ and consider the warped product $M = \mathbf{R} \times_{e^{-t}} N$. Define $\psi : M \to \mathbf{R}$ by $\psi(x, t) = t$. Let us calculate $|\nabla\psi|$ and $\Delta\psi$. Assume that $\{\overline{\omega}_2, \ldots, \overline{\omega}_n\}$ be an orthonormal coframe on N with respect to ds_N^2. If we define $\omega_1 = dt$ and $\omega_\alpha = e^{-t}\overline{\omega}_\alpha$ for $2 \leq \alpha \leq n$, then the set $\{\omega_i\}_{i=1}^n$ forms an orthonormal coframe of M with respect to ds_M^2. The connection 1-forms ω_{ij} of M are defined by

$$\begin{cases} d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_i, \\ \omega_{ij} + \omega_{ji} = 0. \end{cases}$$

Direct exterior differentiation yields

$$d\omega_1 = 0$$

and

$$\begin{aligned} d\omega_\alpha &= -e^{-t}dt \wedge \overline{\omega}_\alpha + e^{-t}\sum_{\beta=2}^n \overline{\omega}_{\alpha\beta} \wedge \overline{\omega}_\beta \\ &= \omega_\alpha \wedge \omega_1 + \sum_{\beta=2}^n \overline{\omega}_{\alpha\beta} \wedge \omega_\beta, \end{aligned}$$

where $\overline{\omega}_{\alpha\beta}$ are the connection 1-forms on N. Hence we conclude that the connection 1-forms of M are given by

$$\omega_{1\alpha} = -\omega_{\alpha 1} = \omega_\alpha, \; \omega_{\alpha\beta} = \overline{\omega}_{\alpha\beta}.$$

For any C^2 function f on M, its gradient and hessian can be calculated by the following formulas

$$df = \sum_{i=1}^{n} f_i \omega_i, \quad df_i + \sum_{j=1}^{n} f_j \omega_{ji} = \sum_{j=1}^{n} f_{ij} \omega_j. \tag{4.15}$$

From $d\psi = \omega_1$, we know that

$$\psi_i = \delta_{i1} \tag{4.16}$$

and so

$$d\psi_i = 0. \tag{4.17}$$

Taking $f = \psi$ in (4.18) and using (4.17), (4.19) and (4.20), we get

$$\psi_{1j} = 0, \quad \psi_{\alpha\beta} = \delta_{\alpha\beta}. \tag{4.18}$$

Hence, we have

$$|\nabla \psi| = 1, \quad \Delta \psi = n - 1. \tag{4.19}$$

That is, a warped product manifold $= \mathbf{R} \times_{e^{-t}} N$ admits functions satisfying (4.4).

Let $\mathbf{H}^n$ be the n-dimensional hyperbolic space with constant curvature -1. Using the upper half-space model, $\mathbf{H}^n$ is given by

$$\mathbf{R}^n_+ = \{(x_1, x_2, \ldots, x_n) | x_n > 0\} \tag{4.20}$$

with metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2} \tag{4.21}$$

One can check that the map $\Phi : \mathbf{R} \times_{e^{-t}} \mathbf{R}^{n-1} \to \mathbf{H}^n$ given by

$$\Phi(t, x) = (x, e^t)$$

is an isometry. Therefore, $\mathbf{H}^n$ admits a warped product model, $\mathbf{H}^n = \mathbf{R} \times_{e^{-t}} \mathbf{R}^{n-1}$.

Example 4.3 Let N be any complete Riemannian manifold. Let

$$M = \mathbf{R}^l \times N = \left\{(x_1, x_2, \ldots x_l, z) | (x_1, x_2, \ldots, x_l) \in \mathbf{R}^l, z \in N\right\}$$

be the product of $\mathbf{R}^l$ and N endowed with the product metric. Consider the functions $\phi_q : M \to \mathbf{R}, \; q = 1, \ldots, l$, defined by

$$\phi_q(x_1, x_2, \ldots x_l, z) = x_q.$$

It is easy to see that $\{\phi_q\}_{q=1}^l$ satisfy (4.6).

Example 4.4 Any compact homogeneous Riemannian manifold admits eigenmaps for the first positive eigenvalue of the Laplacian (Cf. [25]).

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