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Curve shortening in a Riemannian manifold

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Abstract In this paper, we study the curve shortening flow in a general Riemannian manifold. We have many results for the global behavior of the flow. In particular, we show the following results: let M be a compact Riemannian manifold. (1) If the curve shortening flow exists for infinite time, and $\lim_{t\to\infty} L(\gamma_t) > 0$, then for every $n > 0$, $\lim_{t\to\infty} \sup \left(\left|\frac{D^n T}{\partial s^n}\right|\right) = 0$. Furthermore, the limiting curve exists and is a closed geodesic in M. (2) In $M \times S^1$, if γ_0 is a ramp, then we have a global flow which converges to a closed geodesic in C^∞ norm. As an application, we prove the theorem of Lyusternik and Fet.

Keywords Curve shortening flow · Global flow · Ramp flow · Closed geodesic · Lyusternik-Fet theorem

Mathematics Subject Classification (1991) 53C44

1 Introduction

Very recently, the curve shortening flow in a closed Riemannian manifold has been used by Perelman [10] to study the Ricci-Hamilton flow. Motivated by his work, we study the curve shortening flow in a general Riemannian manifold in this paper.

By definition, our curve shortening flow in a Riemannian manifold (M, g) is evolving the initial closed curve γ_0 along the flow

$$\frac{\partial \gamma_t}{\partial t} = \frac{DT}{\partial s}, \tag{1}$$

where s is the time-dependent arc-length parameter of γ_t, T is the unit tangent vector of γ_t, and $\frac{DT}{\partial s}$ is the covariant derivative of T with respect to T in the space (M, g).

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It has been shown that there always exists a short time flow for the curve shortening problem in any Riemannian manifold (see Sect. 2 in [6]).

Curve shortening flows have been studied intensively by many authors since 1980s, for example, Gage and Hamilton [6], Grayson [7,8], etc. Their work solves the so-called Jordan conjecture concerning the embedded curves in the plane. Actually they proved

Theorem A *Equation* (1) *shrinks embedded curves in the plane R^2 to points. As they shrink, they become convex, and then become circular in the sense that*

(a) *the ratio of the inscribed radius to the circumscribed radius approaches* 1*,*
(b) *the radio of the maximum curvature to the minimum curvature approaches* 1*, and*
(c) *the higher order derivatives of the curvature converges to* 0 *uniformly.*

Later, Grayson [8] generalized this results to surfaces and proved

Theorem B *Let M be a smooth Riemannian surface which is convex at infinity. For any embedded curve, along the curve shortening flow, either it shrinks to a point in finite time, or its curvature converges to zero in the C^∞ norm as time $t \to \infty$.*

Moreover, as an application of Theorem B, he proved the following

Theorem C (Lusternik-Schnirelmann) *A two-sphere with a smooth Riemannian metric has at least three simple closed geodesic.*

If the initial curve is not embedded, then singularities can develop in the form of loops which pinch off to form what are conjectured to be cusps. Abresch and Langer [1] studied non-embedded curves in the plane which evolve by homothety. These curves develop complex singularities at the moment that they disappear. Later, Altschuler [2] and Altschuler and Grayson [3] studied the formation of singularities of the curve shortening flow for space curves. In this case, singularities can develop even if the initial curve is embedded. It is an interesting problem to systematically investigate the singularities of the curve shortening flow in a general Riemannian manifold. But we will not cover that in present paper.

In this work, we shall study the global behavior of the curve shortening flow. To state our result, we introduce a class of special curves–ramps in a product space.

Definition *We shall call a curve $\gamma : S^1 \to M \times S^1$ a* ramp *if there exists a unit tangent vector field U to S^1 such that*

$$\left\langle (\pi_{S^1})_*(T), U \right\rangle_{S^1} > 0$$

along γ, where T is the unit tangent vector of γ as before and $\pi_{S^1} : M \times S^1 \to S^1$ is the projection.

Our main result is the following

Theorem D *Let M be a compact Riemannian manifold.*

1. *If the curve shortening flow exists for infinite time, and $\lim_{t\to\infty} L(\gamma_t) > 0$, then for every $n > 0$, $\lim_{t\to\infty} \sup \left(\left| \frac{D^n T}{\partial s^n} \right| \right) = 0$. Furthermore, the limiting curve exists and is a closed geodesic in M.*

2. *In $M \times S^1$, if γ_0 is a ramp, then we have a global flow which converges to a closed geodesic in C^∞ norm.*

Theorem D will be proved in Sects. 4 and 5. In fact, Theorem D will be separated as two results: Theorems 10 and 15 below. As an application of Theorem D, we can prove the following

Theorem E (Lyusternik-Fet) *Each compact Riemannian manifold contains a nontrivial closed geodesic.*

We also obtain many other results for the curve shortening problem. They will be exhibited in the sequent sections. We remark that in many cases, we do not require M is compact and only assume that it satisfies some curvature conditions (see the beginning of Sect. 3).

Curve shortening flow is of some interest in differential geometry, for instance in the problem of finding geodesic and minimal surfaces. Because of the huge literature on this area, we shall not list all of them here. The interest is not only theoretical since the motions by curvature appear in the modeling of various phenomena as crystal growth, flame propagation and interfaces between phases. More recently, this flow has also appeared in the young field of image progressing where it provides an efficient way to smooth curves representing the contours of the objects. For further information, one may read [4].

This paper is organized as follows:

In Sect. 2, we compute the first order variation of curve length, and then introduce the concept of curve shortening flow naturally. For later use, we include some fundamental computations.

In Sect. 3, after assuming that the target manifold M satisfies some curvature conditions, we deduce the important Bernstein type estimates. Our main tool is the classical maximum principle for heat equation.

In Sect. 4, we use the Bernstein type estimates to prove the convergence result for the global flow in compact manifolds. To do that, we first bound the L^2-norm of the curvature and its each order derivative, and then control their L^∞-norm by using Sobolev inequality.

In Sect. 5, we introduce the concept of ramp, and prove the ramp flow is a global flow.

In Sect. 6, through constructing ramp flows, we prove the theorem of Lyusternik and Fet.

In Sect. 7, we analyze in details the evolution of the curve shortening flow in $S^3(1)$, which requires us to solve a system of second order ordinary differential equations.

2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold, and ∇ is its Riemannian connection. Assume that

$$\gamma : S^1 \times (a, b) \to M$$

is an evolving immersed curve. Denote by γ_t the associated trajectory, i.e.,

$$\gamma_t(\cdot) = \gamma(\cdot, t).$$

Then the length of γ_t is

$$L(\gamma_t) = \int\limits_{S^1} \left| \frac{\mathrm{d}}{\mathrm{du}} \gamma_t \right| \mathrm{d}u = \int\limits_{S^1} \left| \frac{\partial \gamma}{\partial u} \right| \mathrm{d}u = \int\limits_{S^1} v \, \mathrm{d}u,$$

where $v = |\frac{\partial \gamma}{\partial u}|$. We define the arc-length parameter s by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u},$$

which implies $\mathrm{d}s = v \, \mathrm{d}u$. As usual, we denote by T the unit tangent vector, i.e.,

$$T = \frac{\partial \gamma}{\partial s} = \frac{1}{v} \frac{\partial \gamma}{\partial u}.$$

We now compute the time derivative of the length functional:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} L(\gamma_t) &= \int\limits_{S^1} \frac{\partial v}{\partial t} \mathrm{d}u \\
&= \int\limits_{S^1} \left\langle \nabla_t \frac{\partial \gamma}{\partial u}, T \right\rangle \mathrm{d}u \\
&= \int\limits_{S^1} \left\langle \nabla_u \frac{\partial \gamma}{\partial t}, T \right\rangle \mathrm{d}u \\
&= \int\limits_{S^1} \left\{ \frac{\partial}{\partial u} \left\langle \frac{\partial \gamma}{\partial t}, T \right\rangle - \left\langle \frac{\partial \gamma}{\partial t}, \nabla_u T \right\rangle \right\} \mathrm{d}u \\
&= -\int\limits_{S^1} \left\langle \frac{\partial \gamma}{\partial t}, \nabla_u T \right\rangle \mathrm{d}u \\
&= -\int\limits_{S^1} \left\langle \frac{\partial \gamma}{\partial t}, \frac{DT}{\partial s} \right\rangle \mathrm{d}s.
\end{aligned}$$

If γ evolves according to Eq. (1), then we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\gamma_t) = -\int\limits_{S^1} \kappa^2 \mathrm{d}s \le 0,$$

where $\kappa = \left| \frac{DT}{\partial s} \right|$ is the non-negative curvature of the curve γ_t. This leads us to give the following

Definition 1 *A curve shortening flow is a family of evolving immersed curves* $\gamma(\cdot, t)$ *satisfying Eq.* (1).

We like to regard $\gamma_t(S^1)$ as a one-dimensional sub-manifold of M. With the induced metric from M, its mean curvature vector field of $\gamma_t(S^1)$ in M is

$$H = (\nabla_T T)^{\perp}.$$

Note that $\langle T, T\rangle \equiv 1$. Then we have $\langle \nabla_T T, T\rangle \equiv 0$. This implies $\nabla_T T \perp T.\gamma_t$. So

$$H = \nabla_T T.$$

It follows that the curve shortening flow is a mean curvature flow of higher co-dimension in the manifold M.

In the remaining part of this section, we shall make some elementary computations and derive some useful formulae (see also [8]).

Lemma 2 *The evolution of v is*

$$\frac{\partial v}{\partial t} = -\kappa^2 v.$$

Proof By definition, we have

$$v^2 = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle.$$

Differentiating it with respect to t, we get

$$\begin{aligned} 2v\frac{\partial v}{\partial t} &= 2\left\langle \nabla_t \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle \\ &= 2\left\langle \nabla_u \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial u} \right\rangle \\ &= 2v^2\left\langle \nabla_T \frac{DT}{\partial s}, T \right\rangle \\ &= -2v^2\left\langle \frac{DT}{\partial s}, \frac{DT}{\partial s} \right\rangle \\ &= -2\kappa^2 v^2. \end{aligned}$$

□

Lemma 3 *Covariant differentiations with respect to s and t are related by the equation*

$$\nabla_t \nabla_s = \nabla_s \nabla_t + \kappa^2 \nabla_s + R\left(T, \frac{DT}{\partial s}\right),$$

where R is the curvature operator on M.

Proof We have (see [5]) that

$$\nabla_t \nabla_u = \nabla_u \nabla_t + R\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right).$$

Note that $\nabla_s = \frac{1}{v}\nabla_u$. Using Lemma 2, we get

$$\begin{aligned} \nabla_t \nabla_s &= \frac{\partial}{\partial t}\left(\frac{1}{v}\right)\nabla_u + \frac{1}{v}\nabla_t \nabla_u \\ &= \kappa^2 \frac{1}{v}\nabla_u + \frac{1}{v}\nabla_u \nabla_t + \frac{1}{v} R\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right) \\ &= \nabla_s \nabla_t + \kappa^2 \nabla_s + R\left(T, \frac{DT}{\partial s}\right). \end{aligned}$$

□

Lemma 4 *The covariant derivative of T with respect to time t is*

$$\nabla_t T = \kappa^2 T + \frac{D^2 T}{\partial s^2}.$$

Proof The proof is a straightforward calculation.

$$\nabla_t T = \nabla_t \left(\frac{1}{v} \frac{\partial \gamma}{\partial u} \right) = \kappa^2 \frac{1}{v} \frac{\partial \gamma}{\partial u} + \frac{1}{v} \nabla_u \frac{\partial \gamma}{\partial t} = \kappa^2 T + \frac{D^2 T}{\partial s^2}.$$

□

3 Bernstein type estimates

From now on, we shall assume that the Riemannian manifold M satisfies the following conditions: $\forall i \geq 0$, $\exists \lambda_i > 0$, such that

$$\nabla^i R(Y_1, \ldots, Y_{i+4}) \leq \lambda_i \prod_{j=1}^{i+4} |Y_j|$$

for all $Y_j \in T.M, j = 1, \ldots, i+4$. Here R is the curvature tensor of M which has order 4, and $\nabla^i R$ is its ith covariant differential which is a tensor of order $i+4$.

Remark 5 If M is compact, then it satisfies the above conditions autonomously.

With these assumptions, we can give the following Bernstein type estimates for the curve shortening flow. Similar estimates appeared in [2] for the curve shortening flow in R^3.

Theorem 6 *Fix $t_0 \in [0, \infty)$. Let $M_{t_0} = \max \kappa^2(\cdot, t_0)$. If $M_{t_0} < \infty$, then there exist constants $c_l < \infty$ independent of t_0 such that for $t \in \left(t_0, t_0 + \frac{1}{2\lambda_0} \log\left(1 + \frac{\lambda_0}{4M_{t_0}+\lambda_0+1}\right)\right)$, we have*

$$\left| \frac{D^l T}{\partial s^l} \right|^2 \leq \frac{c_l M_{t_0}}{(t - t_0)^{l-1}}.$$

Proof Without loss of generality, we may assume that $t_0 = 0$, and then translate the estimates.

First, let us compute the time derivative of $\left| \frac{D^l T}{\partial s^l} \right|^2$ as follows:

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\left| \frac{D^l T}{\partial s^l} \right|^2 \right) &= 2 \left\langle \frac{D}{\partial t} \frac{D^l T}{\partial s^l}, \frac{D^l T}{\partial s^l} \right\rangle \\
&= 2 \left\langle \frac{D}{\partial s} \frac{D}{\partial t} \frac{D^{l-1} T}{\partial s^{l-1}} + \kappa^2 \frac{D^l T}{\partial s^l} + R\left(T, \frac{DT}{\partial s}\right) \frac{D^{l-1} T}{\partial s^{l-1}}, \frac{D^l T}{\partial s^l} \right\rangle \\
&= 2 \left\langle \frac{D}{\partial s} \frac{D}{\partial t} \frac{D^{l-1} T}{\partial s^{l-1}}, \frac{D^l T}{\partial s^l} \right\rangle + 2\kappa^2 \left| \frac{D^l T}{\partial s^l} \right|^2 + 2R\left(T, \frac{DT}{\partial s}, \frac{D^{l-1} T}{\partial s^{l-1}}, \frac{D^l T}{\partial s^l}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\left\langle \frac{D}{\partial s}\left(\frac{D}{\partial s}\frac{D}{\partial t}\frac{D^{l-2}T}{\partial s^{l-2}} + \kappa^2\frac{D^{l-1}T}{\partial s^{l-1}} + R\left(T, \frac{DT}{\partial s}\right)\frac{D^{l-2}T}{\partial s^{l-2}}\right), \frac{D^l T}{\partial s^l}\right\rangle \\
&\quad + 2\kappa^2\left|\frac{D^l T}{\partial s^l}\right|^2 + 2R\left(T, \frac{DT}{\partial s}, \frac{D^{l-1}T}{\partial s^{l-1}}, \frac{D^l T}{\partial s^l}\right) \\
&= 2\left\langle \frac{D^2}{\partial s^2}\frac{D}{\partial t}\frac{D^{l-2}T}{\partial s^{l-2}}, \frac{D^l T}{\partial s^l}\right\rangle + 2\left\langle \frac{D}{\partial s}\left(\kappa^2\frac{D^{l-1}T}{\partial s^{l-1}}\right), \frac{D^l T}{\partial s^l}\right\rangle + 2\kappa^2\left|\frac{D^l T}{\partial s^l}\right|^2 \\
&\quad + 2\left\langle \frac{D}{\partial s}\left(R\left(T, \frac{DT}{\partial s}\right)\frac{D^{l-2}T}{\partial s^{l-2}}\right), \frac{D^l T}{\partial s^l}\right\rangle + 2R\left(T, \frac{DT}{\partial s}, \frac{D^{l-1}T}{\partial s^{l-1}}, \frac{D^l T}{\partial s^l}\right) \\
&= \cdots \\
&= 2\left\langle \frac{D^l}{\partial s^l}\frac{DT}{\partial t}, \frac{D^l T}{\partial s^l}\right\rangle + 2\sum_{i=0}^{l-1}\left\langle \frac{D^i}{\partial s^i}\left(\kappa^2\frac{D^{l-i}T}{\partial s^{l-i}}\right), \frac{D^l T}{\partial s^l}\right\rangle \\
&\quad + 2\sum_{i=0}^{l-1}\left\langle \frac{D^i}{\partial s^i}\left(R\left(T, \frac{DT}{\partial s}\right)\frac{D^{l-1-i}T}{\partial s^{l-1-i}}\right), \frac{D^l T}{\partial s^l}\right\rangle. \qquad (2)
\end{aligned}
$$

Using Lemma 4, we get

$$
\begin{aligned}
2\left\langle \frac{D^l}{\partial s^l}\frac{DT}{\partial t}, \frac{D^l T}{\partial s^l}\right\rangle &= 2\left\langle \frac{D^{l+2}T}{\partial s^{l+2}}, \frac{D^l T}{\partial s^l}\right\rangle + 2\left\langle \frac{D^l}{\partial s^l}\left(\kappa^2 T\right), \frac{D^l T}{\partial s^l}\right\rangle \\
&= \frac{\partial^2}{\partial s^2}\left(\left|\frac{D^l T}{\partial s^l}\right|^2\right) - 2\left|\frac{D^{l+1}T}{\partial s^{l+1}}\right|^2 + 2\left\langle \frac{D^l}{\partial s^l}\left(\kappa^2 T\right), \frac{D^l T}{\partial s^l}\right\rangle. \qquad (3)
\end{aligned}
$$

Substituting (2) into (3), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\left|\frac{D^l T}{\partial s^l}\right|^2\right) &= \frac{\partial^2}{\partial s^2}\left(\left|\frac{D^l T}{\partial s^l}\right|^2\right) - 2\left|\frac{D^{l+1}T}{\partial s^{l+1}}\right|^2 + 2\sum_{i=0}^{l}\left\langle \frac{D^i}{\partial s^i}\left(\kappa^2\frac{D^{l-i}T}{\partial s^{l-i}}\right), \frac{D^l T}{\partial s^l}\right\rangle \\
&\quad + 2\sum_{i=0}^{l-1}\left\langle \frac{D^i}{\partial s^i}\left(R\left(T, \frac{DT}{\partial s}\right)\frac{D^{l-1-i}T}{\partial s^{l-1-i}}\right), \frac{D^l T}{\partial s^l}\right\rangle. \qquad (4)
\end{aligned}
$$

It is easy to see that

$$
\frac{D^i}{\partial s^i}\left(\kappa^2\frac{D^{l-i}T}{\partial s^{l-i}}\right) = \sum_{j=0}^{i}\sum_{k=0}^{j} C_{ijk}\left\langle \frac{D^{k+1}T}{\partial s^{k+1}}, \frac{D^{j-k+1}T}{\partial s^{j-k+1}}\right\rangle \frac{D^{l-j}T}{\partial s^{l-j}}, \qquad (5)
$$

where $C_{ijk} = C_i^j C_j^k$. Substituting (5) into (4), we obtain

$$\frac{\partial}{\partial t}\left(\left|\frac{D^l T}{\partial s^l}\right|^2\right) = \frac{\partial^2}{\partial s^2}\left(\left|\frac{D^l T}{\partial s^l}\right|^2\right) - 2\left|\frac{D^{l+1} T}{\partial s^{l+1}}\right|^2 + 2\sum_{i=0}^{l}\sum_{j=0}^{i}\sum_{k=0}^{j} C_{ijk}\left\langle \frac{D^{k+1}T}{\partial s^{k+1}}, \frac{D^{j-k+1}T}{\partial s^{j-k+1}}\right\rangle\left\langle \frac{D^{l-j}T}{\partial s^{l-j}}, \frac{D^l T}{\partial s^l}\right\rangle + 2\sum_{i=0}^{l-1}\left\langle \frac{D^i}{\partial s^i}\left(R\left(T, \frac{DT}{\partial s}\right)\frac{D^{l-1-i}T}{\partial s^{l-1-i}}\right), \frac{D^l T}{\partial s^l}\right\rangle. \tag{6}$$

In the following, we shall finish the proof by induction.

(1) For $l = 1$, we have

$$\frac{\partial}{\partial t}\left(\left|\frac{DT}{\partial s}\right|^2\right) = \frac{\partial^2}{\partial s^2}\left(\left|\frac{DT}{\partial s}\right|^2\right) - 2\left|\frac{D^2 T}{\partial s^2}\right|^2 + 4\kappa^4 + 2R\left(T, \frac{DT}{\partial s}, T, \frac{DT}{\partial s}\right)$$
$$\le \frac{\partial^2}{\partial s^2}\left(\left|\frac{DT}{\partial s}\right|^2\right) - 2\left|\frac{D^2 T}{\partial s^2}\right|^2 + 4\kappa^4 + 2\lambda_0\kappa^2.$$

Since

$$\left|\frac{D^2 T}{\partial s^2}\right| \ge \left|\frac{D\,|DT/ds|}{\partial s}\right| = \left|\frac{D\kappa}{\partial s}\right|,$$

we get at the maximum point of $\kappa^2(\cdot, t)$ that

$$\frac{\mathrm{d}}{\mathrm{dt}} M_t \le 4M_t^2 + 2\lambda_0 M_t.$$

It implies that M_t satisfies

$$\log\frac{M_t}{\frac{2}{\lambda_0}M_t + 1} - \log\frac{M_0}{\frac{2}{\lambda_0}M_0 + 1} \le 2\lambda_0 t.$$

If $t \le \frac{1}{2\lambda_0}\log\left(1 + \frac{\lambda_0}{4M_0 + \lambda_0 + 1}\right)$, then $M_t \le 2M_0$. So we may choose $c_1 = 2$.

(2) For $l = 2$, we have

$$\frac{\partial}{\partial t}\left(\left|\frac{D^2 T}{\partial s^2}\right|^2\right) = \frac{\partial^2}{\partial s^2}\left(\left|\frac{D^2 T}{\partial s^2}\right|^2\right) - 2\left|\frac{D^3 T}{\partial s^3}\right|^2 + 2\sum_{i=0}^{2}\sum_{j=0}^{i}\sum_{k=0}^{j} C_{ijk}\left\langle \frac{D^{k+1}T}{\partial s^{k+1}}, \frac{D^{j-k+1}T}{\partial s^{j-k+1}}\right\rangle\left\langle \frac{D^{2-j}T}{\partial s^{2-j}}, \frac{D^2 T}{\partial s^2}\right\rangle + 2\sum_{i=0}^{1}\left\langle \frac{D^i}{\partial s^i}\left(R\left(T, \frac{DT}{\partial s}\right)\frac{D^{1-i}T}{\partial s^{1-i}}\right), \frac{D^2 T}{\partial s^2}\right\rangle. \tag{7}$$

By the definition of the covariant differential of a tensor [5], we have

$$\begin{aligned}
&\nabla R\left(T,\frac{DT}{\partial s},T,\frac{D^2T}{\partial s^2},T\right)\\
&\quad= T\left(R\left(T,\frac{DT}{\partial s},T,\frac{D^2T}{\partial s^2}\right)\right)-R\left(\frac{DT}{\partial s},\frac{DT}{\partial s},T,\frac{D^2T}{\partial s^2}\right)-R\left(T,\frac{D^2T}{\partial s^2},T,\frac{D^2T}{\partial s^2}\right)\\
&\qquad -R\left(T,\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{D^2T}{\partial s^2}\right)-R\left(T,\frac{DT}{\partial s},T,\frac{D^3T}{\partial s^3}\right)\\
&\quad=\left\langle\frac{D}{\partial s}\left(R\left(T,\frac{DT}{\partial s}\right)T\right),\frac{D^2T}{\partial s^2}\right\rangle-R\left(\frac{DT}{\partial s},\frac{DT}{\partial s},T,\frac{D^2T}{\partial s^2}\right)\\
&\qquad -R\left(T,\frac{D^2T}{\partial s^2},T,\frac{D^2T}{\partial s^2}\right)-R\left(T,\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{D^2T}{\partial s^2}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\left\langle\frac{D}{\partial s}\left(R\left(T,\frac{DT}{\partial s}\right)T\right),\frac{D^2T}{\partial s^2}\right\rangle &= \nabla R\left(T,\frac{DT}{\partial s},T,\frac{D^2T}{\partial s^2},T\right)+R\left(\frac{DT}{\partial s},\frac{DT}{\partial s},T,\frac{D^2T}{\partial s^2}\right)\\
&\quad +R\left(T,\frac{D^2T}{\partial s^2},T,\frac{D^2T}{\partial s^2}\right)+R\left(T,\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{D^2T}{\partial s^2}\right).
\end{aligned}\tag{8}$$

Substituting (8) into (7) and repeating to use the Peter-Paul inequality, i.e., $ab \le \epsilon a^2+\frac{1}{4\epsilon}b^2$, allows us to obtain that

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\left|\frac{D^2T}{\partial s^2}\right|^2\right)\le{}& \frac{\partial^2}{\partial s^2}\left(\left|\frac{D^2T}{\partial s^2}\right|^2\right)-\left|\frac{D^3T}{\partial s^3}\right|^2+\left(19\kappa^2+2\lambda_0+1\right)\left|\frac{D^2T}{\partial s^2}\right|^2\\
&+\left(9\lambda_0^2+\lambda_1^2\right)\kappa^2.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(t\left|\frac{D^2T}{\partial s^2}\right|^2+2\left|\frac{DT}{\partial s}\right|^2\right)\\
&\quad\le\frac{\partial^2}{\partial s^2}\left(t\left|\frac{D^2T}{\partial s^2}\right|^2+2\left|\frac{DT}{\partial s}\right|^2\right)-t\left|\frac{D^3T}{\partial s^3}\right|^2+\left[t\left(19\kappa^2+2\lambda_0+1\right)-3\right]\left|\frac{D^2T}{\partial s^2}\right|^2\\
&\qquad+8\kappa^4+\left[t\left(9\lambda_0^2+\lambda_1^2\right)+4\lambda_0\right]\kappa^2.
\end{aligned}$$

Since $0<t\le\frac{1}{2\lambda_0}\log\left(1+\frac{\lambda_0}{4M_0+\lambda_0+1}\right)\le\frac{1}{2(4M_0+\lambda_0+1)}<1$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(t\left|\frac{D^2T}{\partial s^2}\right|^2+2\left|\frac{DT}{\partial s}\right|^2\right)\\
&\quad\le\frac{\partial^2}{\partial s^2}\left(t\left|\frac{D^2T}{\partial s^2}\right|^2+2\left|\frac{DT}{\partial s}\right|^2\right)+32M_0^2+2\left[t\left(9\lambda_0^2+\lambda_1^2\right)+4\lambda_0\right]M_0.
\end{aligned}$$

Thus it follows that

$$t\left|\frac{D^2T}{\partial s^2}\right|^2+2\left|\frac{DT}{\partial s}\right|^2\leq\frac{49+\lambda_1^2}{4}M_0,$$

and we may conclude on this time interval that

$$\left|\frac{D^2T}{\partial s^2}\right|^2\leq\frac{\frac{49+\lambda_1^2}{4}M_0}{t}.$$

So we may choose $c_2=\frac{49+\lambda_1^2}{4}$.

Note that by using the definition of the covariant derivatives of tensors repeatedly, we can write

$$\begin{aligned}&\left\langle\frac{D^i}{\partial s^i}\left(R\left(T,\frac{DT}{\partial s}\right)\frac{D^{m-1-i}T}{\partial s^{m-1-i}}\right),\frac{D^mT}{\partial s^m}\right\rangle\\&\quad=\nabla^iR\left(T,\frac{DT}{\partial s},\frac{D^{m-1-i}T}{\partial s^{m-1-i}},\frac{D^mT}{\partial s^m},T,\ldots,T\right)+P,\end{aligned}\tag{9}$$

where P only involves lower order covariant derivatives of R. For example, if $m=3$, $i=2$, then we have

$$\begin{aligned}&\left\langle\frac{D^2}{\partial s^2}\left(R\left(T,\frac{DT}{\partial s}\right)T\right),\frac{D^3T}{\partial s^3}\right\rangle\\&\quad=\nabla^2R\left(T,\frac{DT}{\partial s},T,\frac{D^3T}{\partial s^3},T,T\right)+2\nabla R\left(\frac{DT}{\partial s},\frac{DT}{\partial s},T,\frac{D^3T}{\partial s^3},T\right)\\&\quad\quad+2\nabla R\left(T,\frac{D^2T}{\partial s^2},T,\frac{D^3T}{\partial s^3},T\right)+2\nabla R\left(T,\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{D^3T}{\partial s^3},T\right)\\&\quad\quad+\nabla R\left(T,\frac{DT}{\partial s},T,\frac{D^3T}{\partial s^3},\frac{DT}{\partial s}\right)+2R\left(\frac{DT}{\partial s},\frac{D^2T}{\partial s^2},T,\frac{D^3T}{\partial s^3}\right)\\&\quad\quad+2R\left(\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{DT}{\partial s},\frac{D^3T}{\partial s^3}\right)+2R\left(T,\frac{D^2T}{\partial s^2},\frac{DT}{\partial s},\frac{D^3T}{\partial s^3}\right)\\&\quad\quad+R\left(\frac{D^2T}{\partial s^2},\frac{DT}{\partial s},T,\frac{D^3T}{\partial s^3}\right)+R\left(T,\frac{D^3T}{\partial s^3},T,\frac{D^3T}{\partial s^3}\right)\\&\quad\quad+R\left(T,\frac{DT}{\partial s},\frac{D^2T}{\partial s^2},\frac{D^3T}{\partial s^3}\right).\end{aligned}$$

Substituting (9) into (6), then the induction hypothesis and repeated use of the Peter-Paul inequality allow us to find constants a_i and A,B on our time interval such that

$$\frac{\partial}{\partial t}\left(\sum_{i=1}^m a_it^{i-1}\left|\frac{D^iT}{\partial s^i}\right|^2\right)\leq AM_0^2+BM_0.$$

Thus we obtain c_m as before. □

We remark that these Bernstein type estimates give us the long time existence result of the flow. That is, as long as the curvature remains uniform bounded on finite time interval $[0,\alpha)$, one can define a smooth limit for the tangent vector T at time α. Thus, by integrating the tangent vector T, one can obtain a smooth limit curve.

4 Convergence results about global flows

In this section, we assume that the curve shortening flow exists globally. We want to prove convergence results of the global flow. Similar results can be found in [8] where the author considered the curve shortening flow in a two-dimensional Riemannian manifold.

Let ω be the maximal existence time of the curve shortening flow. For a global flow, we mean that $\omega = \infty$. Throughout this section, we shall assume that $\lim_{t\to\infty} L(\gamma_t) > 0$. Let $l_t = L(\gamma_t)$, for $0 \le t < +\infty$, and $l_\infty = \lim_{t\to\infty} l_t$. Then along the curve-shortening flow, we have

$$l_0 \ge l_t \ge l_\infty.$$

First we have

Lemma 7 *The L^2-norm of curvature converges to zero as $t \to \infty$.*

Proof We use the argument of Grayson (see p. 105 in [8]). We first note that by our assumption, the length $L(\gamma_t)$ monotone strictly decrease to a positive constant. In fact, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\gamma_t) = -\int \kappa^2 \mathrm{d}s,$$

which implies that $\int \kappa^2 \mathrm{d}s$ approaches zero at an ϵ-dense of sufficiently large times, i.e., for any time sequence I_k approaching ∞ of any fixed length $\epsilon > 0$, we have a sequence $t_k \in I_k \to +\infty$ such that

$$\int \kappa^2 \mathrm{d}s\,(t_k) \to 0.$$

So what we need to do is to bound the time derivative of $\int \kappa^2 \mathrm{d}s$ at the large time.

$$\begin{aligned}\frac{\partial}{\partial t}\int \kappa^2 \mathrm{d}s &\le \int \left(\frac{\partial^2}{\partial s^2}\left(\left| \frac{DT}{\partial s}\right|^2 \right) - 2\left|\frac{D^2T}{\partial s^2}\right|^2 + 4\kappa^4 + 2\lambda_0\kappa^2 \right)\mathrm{d}s - \int \kappa^4 \mathrm{d}s \\ &\le -2\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s + \left(3\sup\kappa^2 + 2\lambda_0\right)\int \kappa^2 \mathrm{d}s.\end{aligned}$$

From this we obtain that

$$\begin{aligned}\sup \kappa^2 &\le \left(\inf \kappa + \int \left| \frac{\partial}{\partial s}\left|\frac{DT}{\partial s}\right|\right| \mathrm{d}s \right)^2 \\ &\le \left(\inf\kappa + \int \left|\frac{D^2T}{\partial s^2}\right| \mathrm{d}s\right)^2 \\ &\le 2\inf\kappa^2 + 2\left(\int \left|\frac{D^2T}{\partial s^2}\right|\mathrm{d}s\right)^2 \\ &\le \frac{2}{l_\infty}\int \kappa^2\mathrm{d}s + 2l_0\int\left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s.\end{aligned}$$

This implies that

$$-2\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s \le \frac{2}{l_0 l_\infty}\int \kappa^2 \mathrm{d}s - \frac{1}{l_0}\sup\kappa^2$$

Hence,

$$\frac{\partial}{\partial t}\int \kappa^2 \mathrm{d}s \le \left(2\lambda_0 + \frac{2}{l_\infty l_0}\right)\int \kappa^2 \mathrm{d}s + \sup \kappa^2 \left(3\int \kappa^2 \mathrm{d}s - \frac{1}{l_0}\right).$$

Therefore, $\int \kappa^2 \mathrm{d}s$ has at most exponential growth when it is sufficiently small. This implies that it must converge to zero at $t \to \infty$. □

Lemma 8 $\lim_{t\to\infty} \int \left|\frac{D^2 T}{\partial s^2}\right|^2 \mathrm{d}s = 0.$

Proof Suppose not. We need only consider those times when $\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s$ is sufficiently larger than $\int \kappa^2 \mathrm{d}s$. Look at the time derivative of $\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s$. The rules for differentiating yields that

$$\frac{\partial}{\partial t}\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s \le \int -\left|\frac{D^3T}{\partial s^3}\right|^2 + 19\kappa^2\left|\frac{D^2T}{\partial s^2}\right|^2 + (1+2\lambda_0)\left|\frac{D^2T}{\partial s^2}\right|^2 + \left(9\lambda_0^2+\lambda_1^2\right)\kappa^2.$$

We shall bound the last three terms in this integral by the first.

Notice that

$$\frac{\partial}{\partial s}\left\langle \frac{DT}{\partial s}, \frac{DT}{\partial s}\right\rangle = \left\langle \frac{DT}{\partial s}, \frac{D^3T}{\partial s^3}\right\rangle + \left|\frac{D^2T}{\partial s^2}\right|^2.$$

Then,

$$0 = \int \left\langle \frac{DT}{\partial s}, \frac{D^3T}{\partial s^3}\right\rangle \mathrm{d}s + \int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s.$$

Using the Cauchy–Schwartz inequality we get that

$$\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s \le \left(\int \left|\frac{DT}{\partial s}\right|^2 \mathrm{d}s \cdot \int \left|\frac{D^3T}{\partial s^3}\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}.$$

If we assume that

$$\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s > \alpha \cdot \int \kappa^2 \mathrm{d}s,$$

then we get

$$\int \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s \le \alpha^{-1} \cdot \int \left|\frac{D^3T}{\partial s^3}\right|^2 \mathrm{d}s.$$

Assume that

$$\int \kappa^2 \mathrm{d}s \le \epsilon$$

for some small $\epsilon > 0$. We estimate the second term:

$$\int \kappa^2 \left|\frac{D^2T}{\partial s^2}\right|^2 \mathrm{d}s \le \epsilon \cdot \sup \left|\frac{D^2T}{\partial s^2}\right|^2.$$

But

$$\begin{aligned}\sup\left|\frac{D^2T}{\partial s^2}\right|^2 &\le \left\{\inf\left|\frac{D^2T}{\partial s^2}\right| + \int\left|\frac{\partial}{\partial s}\left(\left|\frac{D^2T}{\partial s^2}\right|\right)\right|\mathrm{d}s\right\}^2\\ &\le \left(\inf\left|\frac{D^2T}{\partial s^2}\right| + \int\left|\frac{D^3T}{\partial s^3}\right|\mathrm{d}s\right)^2\\ &\le \frac{2}{l_\infty}\int\left|\frac{D^2T}{\partial s^2}\right|^2\mathrm{d}s + 2l_0\int\left|\frac{D^3T}{\partial s^3}\right|^2\mathrm{d}s\\ &\le \left(\frac{2}{\alpha l_\infty}+2l_0\right)\int\left|\frac{D^3T}{\partial s^3}\right|^2\mathrm{d}s.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}\int\left|\frac{D^2T}{\partial s^2}\right|^2\mathrm{d}s &\le \left[-1+19\epsilon\cdot\left(\frac{2}{\alpha l_\infty}+2l_0\right)+\frac{1+2\lambda_0}{\alpha}+\frac{9\lambda_0^2+\lambda_1^2}{\alpha^2}\right]\int\left|\frac{D^3T}{\partial s^3}\right|^2\mathrm{d}s\\ &\le -\frac{1}{2}\int\left|\frac{D^3T}{\partial s^3}\right|^2\mathrm{d}s\\ &\le -\frac{1}{2}\int\left|\frac{D^2T}{\partial s^2}\right|^2\mathrm{d}s.\end{aligned}$$

So, either $\int\left|\frac{D^2T}{\partial s^2}\right|^2\mathrm{d}s$ decays exponentially, or it is comparable to $\int\kappa^2\mathrm{d}s$. In either event, it decreases to zero.

The following Sobolev inequality is useful.

Lemma 9 *If $\|f\|_2 \le C$ and $\|f'\|_2 \le C$, then*

$$\|f\|_\infty \le \left(\frac{1}{\sqrt{2\pi}}+\sqrt{2\pi}\right)C,$$

where $\|\cdot\|_2$ is the L_2 norm and $\|\cdot\|_\infty$ is the sup norm for functions on S^1.

Note that $\left|\frac{\partial}{\partial s}\left(\left|\frac{DT}{\partial s}\right|\right)\right|^2 \le \left|\frac{D^2T}{\partial s^2}\right|^2$. So, from Lemma 8, we have

$$\lim_{t\to\infty}\int\left|\frac{\partial}{\partial s}\left(\left|\frac{DT}{\partial s}\right|\right)\right|^2\mathrm{d}s = 0.$$

Then it follows from Lemma 9 that $\sup\left(\left|\frac{DT}{\partial s}\right|\right)$ decreases to zero.

Together with Theorem 6, we have proved

Theorem 10 *If the curve shortening flow exists for infinite time, and* $\lim_{t\to\infty} L(\gamma_t) > 0$, *then for every $n > 0$,* $\lim_{t\to\infty}\sup\left(\left|\frac{D^nT}{\partial s^n}\right|\right) = 0$. *Moreover, if M is compact, then the limiting curve exists and is a closed geodesic.*

5 Ramp flows

In this section, we consider the curve shortening flow in a class of Riemannian manifolds, i.e., product manifolds $\left(M\times S^1, g+d\sigma^2\right)$. We introduce the so-called ramp flow in these special manifolds and study its properties. We shall show that the ramp flow is a global flow.

We recall the definition of ramp. As before, define

$$\gamma(\cdot) : S^1 \to M \times S^1$$

is an immersed curve. Let

$$\pi_{S^1} : M \times S^1 \to S^1$$

be the canonical projection. It naturally induces a linear mapping

$$\left(\pi_{S^1}\right)_* : T.\left(M \times S^1\right) \to T_{\pi_{S^1}(\cdot)}S^1.$$

Definition 11 *We shall call γ a ramp if there exists a unit tangent vector field U to S^1 such that*

$$\left\langle \left(\pi_{S^1}\right)_* (T), U \right\rangle_{S^1} > 0$$

along γ.

From this definition, it is easy to deduce the following

Proposition 12 *An immersed curve is a* ramp *if and only if the $T.S^1$-component of its tangent vector is non-zero everywhere.*

Some authors have already studied ramps (for example, see [3,10]). As we shall see, ramps have many good properties.

Proposition 13 *For a curve shortening flow, if γ_0 is a ramp, then for all $t > 0$, γ_t is also a ramp.*

Proof By definition, there exists a unit tangent vector field $U \in TS^1$, such that

$$u = \left\langle \left(\pi_{S^1}\right)_* (T), U \right\rangle_{S^1} > 0$$

at $t = 0$. The time derivative of u is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2} + \kappa^2 u. \tag{10}$$

If we define $\mu_t = \min_{S^1} u(\cdot, t)$, then $\mu_0 > 0$. Equation (10) tells us that μ_t is non-decreasing. This completes the proof.

Proposition 14 *Assume that the sectional curvature of $M \times S^1$ has an upper bound $\Xi > 0$, and γ_0 is a ramp. Then*

$$\kappa(\cdot, t) \leq C \exp(\Xi t)$$

for all $t \geq 0$, where C is a positive constant depending only on γ_0.

Proof Since γ_0 is a ramp, Proposition 13 guarantees that γ_t is always a ramp. So we may divide κ by u. The time derivative of $\frac{\kappa}{u}$ is

$$\frac{\partial}{\partial t}\left(\frac{\kappa}{u}\right) = \frac{\partial^2}{\partial s^2}\left(\frac{\kappa}{u}\right) + 2\frac{\partial}{\partial s}\log u \frac{\partial}{\partial s}\left(\frac{\kappa}{u}\right) + \frac{\kappa}{u}\left(\kappa^2 - \left|\frac{DN}{\partial s}\right|^2\right) + \frac{\kappa}{u}R(T, N, T, N), \tag{11}$$

where N satisfies $\frac{DT}{\partial s} = \kappa N$. Note that

$$\kappa^2 = \left\langle \frac{DT}{\partial s}, N \right\rangle^2 = \left\langle T, \frac{DN}{\partial s} \right\rangle^2$$

and then

$$\kappa^2 \leq \left| \frac{DN}{\partial s} \right|^2.$$

So the third term on the right-hand side of Eq. (11) is non-positive, and the fourth term is bounded by $\Xi \frac{\kappa}{u}$ since the sectional curvature $R(T, N, T, N)$ is bounded from above by Ξ.

If we define $\Psi_t = \max_{S^1} \frac{\kappa}{u}(\cdot, t)$, then without loss of generality, we may assume $\Psi_t > 0$ for all $t \geq 0$. Equation (11) tells us that Ψ_t satisfies

$$\frac{\partial}{\partial t} \Psi_t \leq \Psi_t \left(\kappa^2 - \left| \frac{DN}{\partial s} \right|^2 \right) + \Psi_t R(T, N, T, N)$$

which implies that

$$\Psi_t^{-1} \frac{\partial}{\partial t} \Psi_t \leq \Xi$$

and

$$\Psi_t \leq \Psi_0 \exp(\Xi t).$$

Note $\Psi_0 > 0$ and $u \leq 1$. Then we can obtain the desired inequality easily.

The following theorem is a direct consequence of Theorem 10 and Proposition 14.

Theorem 15 *Let M be a compact Riemannian manifold. In* $M \times S^1$*, if* γ_0 *is a ramp, then the curve shortening flow will converge to a closed geodesic in the* C^∞ *norm.*

Proof M is compact, so is $M \times S^1$. Then the sectional curvature of $M \times S^1$ has an upper bound. For γ_0 is a ramp, Proposition 14 guarantees that the curve shortening flow will not blow-up in finite time. This means that the flow will exists for infinite time. Moreover, from the proof of Proposition 13, we know that μ_t is non-decreasing. This will guarantee that $\lim_{t \to \infty} L(\gamma_t) > 0$. Then Theorem 10 tells us that the limiting curve exists and is a closed geodesic in $M \times S^1$.

6 The theorem of Lyusternik and Fet

As an application of Theorem 14, we want to prove the following result:

Theorem 16 (Lyustrenik and Fet) *Each compact Riemannian manifold contains a non-trivial closed geodesic.*

We shall slightly modify J.Jost's proof (see [9]) by replacing his curve shortening with our curve shortening. First of all, let us give some fundamental results.

Lemma 17 *Let M be a compact Riemannian manifold. Then every homotopy class of closed curves in M contains a geodesic.*

Proof Let γ_0 be a curve in the given homotopy class. Define

$$\widetilde{\gamma}_0 : S^1 \to M \times S^1,$$
$$\theta \mapsto (\gamma_0(\theta), \theta) .$$

Obviously, $\widetilde{\gamma}_0$ is a ramp. Evolving it along the curve shortening flow, then by Theorem 15, we know that $\widetilde{\gamma}_0$ is homotopic to a geodesic, say $\widetilde{\gamma}_\infty$, in $M \times S^1$. Let $\gamma_\infty = \pi_M \circ \widetilde{\gamma}_\infty$, where $\pi_M : M \times S^1 \to M$ is the canonical projection. Then it is easy to see that γ_∞ is a geodesic in M. By our process, γ_0 is homotopic to γ_∞. This completes the proof. □

The following lemma is well-known.

Lemma 18 *Let M be a compact Riemannian manifold. Then there exists* $\rho_0 > 0$ *with the property that any two points* $p, q \in M$ *with* $d(p,q) \le \rho_0$ *can be connected by precisely one geodesic of shortest length. This geodesic depends continuously on p and q.*

Finally, we need the following result from algebraic topology. (A proof may be found e.g. in [11].)

Lemma 19 *Let M be a compact differentiable manifold of dimension n. Then there exists some i,* $1 \le i \le n$*, and a differentiable map* $h : S^i \to M$*, which is not homotopic to a constant map.*

Now we begin to prove Theorem 16.

Proof of Theorem 16 Let i be as in Lemma 19. If $i = 1$, the result is a consequence of Lemma 17. We therefore only consider the case $i \ge 2$. h from Lemma 19 then induces a continuous map H of the $(i-1)$-cell D^{i-1} into the space of differentiable curves in M, mapping ∂D^{i-1} to point curves. In order to see this, we first identify D^{i-1} with the half equator $\{x^1 \ge 0, x^2 = 0\}$ of the unit sphere S^i in R^{i+1} with coordinates $\left(x^1, \ldots, x^{i+1}\right)$. To $p \in D^{i-1} \subset S^i$, we assign that circle $c_p(\theta), \theta \in [0, 2\pi]$, parameterized proportionally to angle that starts at p orthogonally to the hyperplane $x^2 = 0$ into the half sphere $x^2 \ge 0$ with constant values of $x^3, \ldots, x^{i+1}$. For $p \in \partial D^{i-1}$, c_p then is the trivial (i.e. constant) circle $c_p(\theta) = p$. The map H is then given by $H(p)(\theta) = h \circ c_p(\theta)$. Each $q \in S^i$ then has a representation of the form $q = c_p(\theta)$ with $p \in D^{i-1}$, p is uniquely determined, and θ as well, unless $q \in \partial D^{i-1}$. A homotopy $\widetilde{H}$, i.e. a continuous map $\{\widetilde{H} : D^{i-1} \times [0,1] \to$ *closed curves in M*$\}$ that maps $\partial D^{i-1} \times [0,1]$ to point curves and satisfies $\widetilde{H}|_{D^{i-1} \times \{0\}} = H$, then induces a homotopy $\widetilde{h} : S^i \times [0,1] \to M$ of h by

$$\widetilde{h}(q,s) = \widetilde{h}\left(c_p(\theta), s\right) = \widetilde{H}(p,s)(\theta)$$

($q = c_p(\theta)$, as just described).

Now we define $\gamma_p(\theta, 0) = (H(p)(\theta), \theta)$. Then $\gamma_p(\cdot, 0)$ is a family of ramps in $M \times S^1$. As before, we evolve them along the curve shortening flow. By Theorem 15, we get a family of non-trivial closed geodesics in $M \times S^1$. We denote them by $\gamma_p(\cdot, \infty)$. Then $\pi_M \circ \gamma_p(\cdot, \infty)$ is a family of closed geodesics in M.

Claim There exists some $p_0 \in D^{i-1}$, such that $L\left(\pi_M \circ \gamma_{p_0}(\cdot, \infty)\right) > 0$.

Proof If not, then $L\left(\pi_M \circ \gamma_p(\cdot, \infty)\right) = 0, \forall p \in D^{i-1}$. We take ρ_0 as in Lemma 18. Since D^{i-1} is compact, we know that there must be some $t_{\rho_0} > 0$, such that

$\max_{p\in D^{i-1}} L\left(\pi_M \circ \gamma_p\left(\cdot, t_{\rho_0}\right)\right) < \rho_0$. Then, for every curve $c_p = \pi_M \circ \gamma_p\left(\cdot, t_{\rho_0}\right)$ and each $\theta \in [0, 2\pi]$, we have $d\left(c_p(0), c_p(\theta)\right) < \rho_0$. By Lemma 18, the shortest connection from $c_p(0)$ to $c_p(\theta)$ is uniquely determined; denote it by $q_{p,\theta}(s), s \in [0.1]$. Because of uniqueness, $q_{p,\theta}$ depends continuously on p and θ. $\overline{H}(p,s)(\theta) = q_{p,\theta}(1-s)$ then defines a homotopy between $\pi_M \circ \gamma_{\cdot}\left(\cdot, t_{\rho_0}\right)$ and a map that maps D^{i-1} into the space of point curves in M, i.e. into M. Such a map, however, is homotopic to a constant map, for example since D^{i-1} is homotopy equivalent to a point. This implies that $\pi_M \circ \gamma_{\cdot}\left(\cdot, t_{\rho_0}\right)$ is homotopic to a constant map, hence so are H and h, contracting the choice of h. So our claim is true. □

Proof of Theorem 16 (*continuous*) $\gamma_{p_0}(\cdot, \infty)$ in above claim is a non-trivial closed geodesic in M. So we are done.

7 Curve shortening flows in $S^3(1)$

In this section, we study the curve shortening problem in the three-dimensional unit sphere $S^3(1) \subset R^4$ with constant sectional curvature 1. Similarly, the analysis of curve shortening flows in other symmetric spaces is also very interesting.

Let T, N, B be the Frenet frame on the curve $\gamma \subset S^3(1)$. We have the well-known Frenet matrix:

$$\frac{D}{\partial s}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

here, s, κ, τ are γ's arc-length parameter, curvature, and torsion, respectively.

Another important fact is that the curvature operator R on $S^3(1)$ has a simple expression, i.e.,

$$R\left(X_1, X_2\right) X_3 = \left\langle X_1, X_3\right\rangle X_2 - \left\langle X_2, X_3\right\rangle X_1$$

for all $X_i \in T.S^3(1), i = 1, 2, 3$. In particular, $R(T,N)T = N$, and $R(T,N)N = -T$.

These relations will significantly simplify our computations and make the results very nice.

Lemma 20 $\nabla_t T = \frac{\partial\kappa}{\partial s} N + \tau\kappa B$.

Proof By Frenet matrix, we have

$$\begin{aligned}\frac{D^2 T}{\partial s^2} &= \frac{D}{\partial s}(\kappa N) \\ &= \frac{\partial\kappa}{\partial s} N + \kappa\left(-\kappa T + \tau B\right),\end{aligned}$$

$\Rightarrow$

$$\kappa^2 T + \frac{D^2 T}{\partial s^2} = \frac{\partial\kappa}{\partial s} N + \tau\kappa B.$$

Then, by Lemma 4, we obtain

$$\begin{aligned}\nabla_t T &= \kappa^2 T + \frac{D^2 T}{\partial s^2} \\ &= \frac{\partial\kappa}{\partial s} N + \tau\kappa B.\end{aligned}$$

Now, we can compute the evolution of curvature κ.

Lemma 21 $\frac{\partial\kappa}{\partial t} = \frac{\partial^2\kappa}{\partial s^2} + \kappa^3 - \tau^2\kappa + \kappa.$

Proof By Lemmata 3 and 20, we have

$$\begin{aligned}\nabla_t\nabla_s T &= \nabla_s\nabla_t T + \kappa^2\nabla_s T + \kappa R(T,N)T \\ &= \nabla_s\left(\frac{\partial\kappa}{\partial s}N + \tau\kappa B\right) + \kappa^3 N + \kappa N \\ &= \frac{\partial^2\kappa}{\partial s^2}N + \frac{\partial\kappa}{\partial s}\left(-\kappa T + \tau B\right) + \frac{\partial}{\partial s}\left(\tau\kappa\right)B \\ &\quad + \left(\tau\kappa\right)\left(-\tau N\right) + \kappa^3 N + \kappa N \\ &= -\kappa\frac{\partial\kappa}{\partial s}T + \left(\frac{\partial^2\kappa}{\partial s^2} + \kappa^3 - \tau^2\kappa + \kappa\right)N \\ &\quad + \left(\tau\frac{\partial\kappa}{\partial s} + \frac{\partial}{\partial s}\left(\tau\kappa\right)\right)B,\end{aligned}$$

so

$$\begin{aligned}\frac{\partial\kappa}{\partial t} &= \frac{\partial}{\partial t}\left\langle\nabla_s T, N\right\rangle \\ &= \left\langle\nabla_t\nabla_s T, N\right\rangle + \left\langle\nabla_s T, \nabla_t N\right\rangle \\ &= \frac{\partial^2\kappa}{\partial s^2} + \kappa^3 - \tau^2\kappa + \kappa.\end{aligned}$$

We also need to know the rate at which the unit normal vector to the curve rotates. This can be got directly from the proof of Lemma 21.

Corollary 22 $\nabla_t N = -\frac{\partial\kappa}{\partial s}T + \left(\frac{\partial\tau}{\partial s} + 2\frac{\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right)B.$

Proof Note that $\nabla_s T = \kappa N$, so

$$\nabla_t\nabla_s T = \nabla_t(\kappa N) = \frac{\partial\kappa}{\partial t}N + \kappa\nabla_t N.$$

With this equation, and noticing that $\langle\nabla_t N, N\rangle = 0$, we get the following relation from the proof of Lemma 21:

$$\kappa\nabla_t N = -\kappa\frac{\partial\kappa}{\partial s}T + \left(\kappa\frac{\partial\tau}{\partial s} + 2\tau\frac{\partial\kappa}{\partial s}\right)B.$$

Multiplying both sides $\frac{1}{\kappa}$, we obtain what we want. □

Now we can compute the evolution of torsion τ.

Lemma 23 $\frac{\partial\tau}{\partial t} = \frac{\partial^2\tau}{\partial s^2} + 2\frac{1}{\kappa}\frac{\partial\kappa}{\partial s}\frac{\partial\tau}{\partial s} + 2\frac{\tau}{\kappa}\left(\frac{\partial^2\kappa}{\partial s^2} - \frac{1}{\kappa}\left(\frac{\partial\kappa}{\partial s}\right)^2 + \kappa^3\right).$

Proof We have

$$\nabla_s N = -\kappa T + \tau B$$

$\Rightarrow$

$$\nabla_t\nabla_s N = -\frac{\partial\kappa}{\partial t}T - \kappa\nabla_t T + \frac{\partial\tau}{\partial t}B + \tau\nabla_t B. \tag{12}$$

The left-hand side of (15) equals

$$\begin{aligned}
&\nabla_s\nabla_t N + \kappa^2\nabla_s N + \kappa R(T,N)N \\
&= \nabla_s\left(-\frac{\partial\kappa}{\partial s}T + \left(\frac{\partial\tau}{\partial s} + 2\frac{\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right)B\right) + \kappa^2(-\kappa T + \tau B) - \kappa T \\
&= -\frac{\partial^2\kappa}{\partial s^2}T - \kappa\frac{\partial\kappa}{\partial s}N + \left(\frac{\partial^2\tau}{\partial s^2} + 2\frac{\partial}{\partial s}\left(\frac{\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right)\right)B \\
&\quad + \left(\frac{\partial\tau}{\partial s} + 2\frac{\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right)(-\tau N) - \kappa^3 T + \tau\kappa^2 B - \kappa T.
\end{aligned}$$

The coefficient of B is

$$\frac{\partial^2\tau}{\partial s^2} + 2\frac{\partial}{\partial s}\left(\frac{\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right) + \tau\kappa^2.$$

The right-hand side of (15) equals

$$-\frac{\partial\kappa}{\partial t}T - \kappa\left(\frac{\partial\kappa}{\partial s}N + \tau\kappa B\right) + \frac{\partial\tau}{\partial t}B + \tau\nabla_t B.$$

Note that $\langle\nabla_t B, B\rangle = 0$, so the coefficient of B is

$$-\tau\kappa^2 + \frac{\partial\tau}{\partial t}.$$

Then it must be

$$\frac{\partial^2\tau}{\partial s^2} + 2\frac{\partial}{\partial s}\left(\frac{\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right) + \tau\kappa^2 = -\tau\kappa^2 + \frac{\partial\tau}{\partial t}$$

$\Rightarrow$

$$\frac{\partial\tau}{\partial t} = \frac{\partial^2\tau}{\partial s^2} + 2\frac{1}{\kappa}\frac{\partial\kappa}{\partial s}\frac{\partial\tau}{\partial s} + 2\frac{\tau}{\kappa}\left(\frac{\partial^2\kappa}{\partial s^2} - \frac{1}{\kappa}\left(\frac{\partial\kappa}{\partial s}\right)^2 + \kappa^3\right).$$

Now we consider a special case, i.e., both κ and τ only depend on time t. Then their evolutions reduce to

$$\begin{cases} \frac{\mathrm{d}\kappa}{\mathrm{d}t} = \kappa^3 - \tau^2\kappa + \kappa, \\ \frac{\mathrm{d}\tau}{\mathrm{d}t} = 2\tau\kappa^2. \end{cases} \tag{13}$$

Let

$$\begin{cases} u = \kappa^2, \\ v = \tau^2. \end{cases}$$

Then system (16) is equivalent to

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = 2u^2 + 2u - 2uv, \cdots (*) \\ \frac{\mathrm{d}v}{\mathrm{d}t} = 4uv. \cdots (**) \end{cases} \tag{14}$$

Assume that at $t = 0$,

$$\begin{cases} u(0) > 0, \\ v(0) > 0. \end{cases}$$

Now we solve this initial value problem as follows.

From Eq. $(**)$, we know that v is non-decreasing. So $v(t) > 0$ for all $t \geq 0$. Then we can divide $(*)$ by $(**)$ to get

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{u+1}{2v} - \frac{1}{2}.$$

Define $w = u + 1$, then

$$\frac{\mathrm{d}w}{\mathrm{d}v} = \frac{1}{2}\left(\frac{w}{v} - 1\right). \tag{15}$$

Moreover, we define $z = \frac{w}{v}$, then $w = zv$. Substituting it into (18), we get

$$\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{\frac{1}{2}(z-1) - z}{v} = -\frac{z+1}{2v}.$$

$\Rightarrow$

$$\frac{\mathrm{d}z}{z+1} = -\frac{\mathrm{d}v}{2v}.$$

Integrating above equation from time 0 to t, we have

$$\frac{z(v(t))+1}{z(v(0))+1} = \left(\frac{v(t)}{v(0)}\right)^{-\frac{1}{2}}.$$

Note that $z = \frac{u+1}{v}$, so

$$\frac{\frac{u(v(t))+1}{v(t)}+1}{\frac{u(v(0))+1}{v(0)}+1} = \left(\frac{v(t)}{v(0)}\right)^{-\frac{1}{2}}$$

$\Rightarrow$

$$\begin{aligned} u(v(t)) &= -1 + v(t)\left[\left(\frac{v(t)}{v(0)}\right)^{-\frac{1}{2}}\left(\frac{u(v(0))+1}{v(0)}+1\right) - 1\right] \\ &= -1 + (v(t)v(0))^{\frac{1}{2}}\left(\frac{u(v(0))+1}{v(0)}+1\right) - v(t). \end{aligned} \tag{16}$$

Substituting (19) into $(**)$, we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 4v\left(-1 + \Lambda\,(vv_0)^{\frac{1}{2}} - v\right),$$

where $v_0 = v(0), \Lambda = \frac{u(v(0))+1}{v(0)} + 1$.

Let $\tilde{\tau} = \sqrt{v}$, i.e., $v = \tilde{\tau}^2$, then

$$\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}t} = 2\tilde{\tau}\left(-1 + a\tilde{\tau} - \tilde{\tau}^2\right), \tag{17}$$

where $\tilde{\tau}_0 = \tilde{\tau}(0), a = \Lambda\tilde{\tau}_0$. Note that

$$\begin{aligned} a &= \left(\frac{u(v(0))+1}{v(0)}+1\right)\sqrt{v(0)} \\ &= \frac{u(v(0))+1}{\sqrt{v(0)}} + \sqrt{v(0)} \\ &\geq 2\sqrt{u(v(0))+1} \\ &> 2. \end{aligned} \tag{18}$$

Since v is non-decreasing, so is $\tilde{\tau}$. Then

$$\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}t} > 0.$$

By Eq. (20), it must be

$$-1 + a\tilde{\tau} - \tilde{\tau}^2 > 0$$

$\Rightarrow$

$$\frac{a - \sqrt{a^2 - 4}}{2} < \tilde{\tau} < \frac{a + \sqrt{a^2 - 4}}{2}.$$

Solving (20), we obtain

$$\tilde{\tau}\left(-\tilde{\tau} + \frac{a + \sqrt{a^2 - 4}}{2}\right)^b \left(\tilde{\tau} - \frac{a - \sqrt{a^2 - 4}}{2}\right)^c$$
$$= \tilde{\tau}_0\left(-\tilde{\tau}_0 + \frac{a + \sqrt{a^2 - 4}}{2}\right)^b \left(\tilde{\tau}_0 - \frac{a - \sqrt{a^2 - 4}}{2}\right)^c \cdot \exp(-2t), \qquad (19)$$

where

$$b = -\frac{1}{2} + \frac{a}{2\sqrt{a^2 - 4}},$$

$$c = -\frac{1}{2} - \frac{a}{2\sqrt{a^2 - 4}}.$$

By (21), we know that b is positive. Let $t \to \infty$, we find that the right-hand side of (22) tends to 0. So it must be $\tilde{\tau} \to \frac{a+\sqrt{a^2-4}}{2}$. Together with (19), we see that $u \to 0$, i.e., $\kappa \to 0$. This shows that the limiting curve is a geodesic, i.e., its trajectory is on some great circle. Moreover, the limit of torsion is not zero, which reflects the fact that the frame is twisting along the geodesic.

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