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Quasilinear parabolic P.D.E.s with discontinuous hysteresis

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Abstract This work deals with an initial- and boundary-value problem for a quasilinear parabolic equation that includes a possibly discontinuous *hysteresis operator,* $\mathcal{F}$:

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] - \Delta u = f.$$

In particular the case of $\mathcal{F}$ equal to a so-called *relay operator* is studied. Well-posedness is proved, as well as regularity of the solution and its robustness w.r.t. perturbations of $\mathcal{F}$. The large-time behaviour is studied; asymptotic stability and compactness are shown. For a time-periodic f, existence of a periodic solution is also established.

Keywords Hysteresis · Parabolic equations · Weak formulation · Entropy condition

Mathematics Subject Classification (2000) 35K60, 35R35, 47J40

Introduction

Hysteresis occurs in several phenomena. In [6] (see also [7]), Krasnosel'skiĭ and co-workers introduced the notion of *hysteresis operator* to represent hysteresis effects. An operator $\mathcal{F}$ of this class establishes a relation between two functions of time, u and w, such that at any instant t, $w(t)$ depends not only on $u(t)$ but also on the previous evolution of u. The operator $\mathcal{F}$ is also assumed to be *rate-independent;* by this we mean that, for any increasing diffeomorphism φ, if $w = \mathcal{F}(u)$ then $w \circ \varphi = \mathcal{F}(u \circ \varphi)$.

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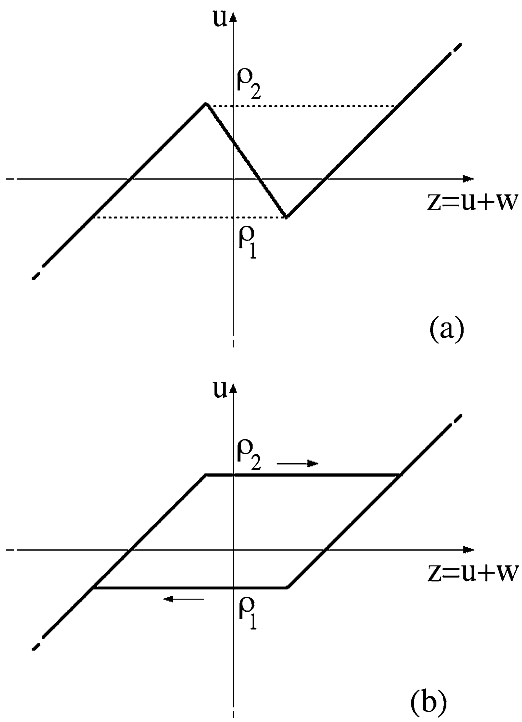

Fig. 1 Nonmonotone curve in part **a**, corresponding hysteresis loop in part **b**

An Equation with Hysteresis. In this paper we deal with an initial- and boundary-value problem for a quasilinear parabolic equation that contains an either continuous or discontinuous hysteresis operator, $\mathcal{F}$:

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] - \Delta u = f. \tag{1}$$

An equation like this may arise in diffusion processes. Let us assume that a phenomenon is described by the equation $\partial z/\partial t - \Delta u = f$, and that the variables u, z are related by a nonmonotone function: $u = \alpha(z)$, cf. Fig. 1. The decreasing branch of the graph of α being unstable, the pair (z, u) is expected to move along a discontinuous hysteresis loop, as it is indicated by the arrows; this issue is discussed in [18] and references therein. This discontinuous relation between z and u can be represented in the form $z = u + \mathcal{F}(u)$, where $\mathcal{F}$ is a relay operator. As $\mathcal{F}$ is not closed, in the framework of a weak formulation we replace it by the closure w.r.t. to natural topologies.

For $N = 1$, the Eq. (1) can model processes in a univariate ferromagnetic metal. It can be derived from the Maxwell–Ohm equations, neglecting the displacement current and assuming that coefficients are normalized, cf. [19]; the operator $\mathcal{F}$ then represents the relation between the magnetization, M, and the magnetic field, H. Here we deal with (1) for any $N \geq 1$, although the physical case is just $N = 1$; relevant modifications are needed to extend the above interpretation to multivariate ferromagnetism, see [21]. The Eq. (1) may also be regarded as a crude model of phase transitions with undercooling and superheating; in this case u represents the temperature, $\mathcal{F}(u)$ the phase function, and z the density of internal energy. Because of the discontinuity of $\mathcal{F}$, (1) may account for occurrence of moving fronts, corresponding to $u = \rho_1$ or $u = \rho_2$. The location of these interfaces is a priori unknown; namely, they are *free boundaries*.

Independently from these and other applications, here we are interested into (1) for this is a typical example of quasilinear P.D.E. with hysteresis. We also aim to develop new techniques, in view of application to other equations with hysteresis.

Main Results. In the last years research on hysteresis has been progressing, see e.g. the monographs [2, 7, 10, 13, 17]. Quasilinear parabolic equations like (1) have been studied for more than twenty years by now, for either continuous and discontinuous hysteresis operators; see e.g. [8, 15–17]. Semilinear parabolic equations have also been investigated, see e.g. [17], as well as first- and second-order quasilinear hyperbolic equations, see [9, 10, 19, 21, 22].

So far the main concern has been devoted to Eq. (1) with a continuous hysteresis operator, set on a bounded Euclidean domain, Ω. In this paper we use a weak formulation of the relay operator, and deal with a possibly unbounded Ω. We prove existence of a solution, study its regularity, its robustness w.r.t. perturbations of $\mathcal{F}$. We also deal with its large-time behaviour, and prove existence of a time-periodic solution whenever the forcing term f is periodic; this extends results known for continuous hysteresis operators, cf. [5, 8], and is achieved via a unified treatment of the cases of continuous and discontinuous hysteresis.

After [4] it is known that for a continuous hysteresis operator $\mathcal{F}$ the solution of (1) is unique. For discontinuous $\mathcal{F}$, well-posedness of a rather weak notion of solution was also established via a formulation based on the theory of contraction-semigroups, see ([17]; Chap. IX). In presence of an either continuous or discontinuous relay operator, here we prove that the solution depends Lipschitz-continuously and monotonically on the data, and is thus unique. Our argument is based on the derivation of an *entropy-like* condition, and on a procedure analogous to that introduced by Kružkov in [11, 12]. However for discontinuous $\mathcal{F}$ this proof is restricted to the case of $\Omega = \mathbf{R}^N$ and $f \equiv 0$. Recently this technique was also used in [22], for the hyperbolic equation: $\partial[u + \mathcal{F}(u)]/\partial t + \partial u/\partial x = f$, $\mathcal{F}$ being a relay operator.

Although the relay is a rather special example of discontinuous hysteresis operator, relays can be combined to generate a large class of either continuous or discontinuous operators via the classic Preisach model [14]. Due to the linearity of this construction, our results might easily be extended to this much more general class of hysteresis operators.

Other classes of partial differential equations with hysteresis might also be studied similarly, including the degenerate equation

$$\frac{\partial}{\partial t}\mathcal{F}(u) - \Delta u = f. \tag{2}$$

Open questions include a more general uniqueness result for the case of discontinuous hysteresis, and a deeper analysis of the large-time behaviour of the solution. The use of time-discretization makes our approach prone to numerical approximation, but so far this issue has only been addressed in few papers.

Here is the plan of the paper. In Sect. 1 we review the relay operator and its weak representation. In Sect. 2 we formulate an initial- and boundary-value problem for Eq. (1), and in Sect. 3 we prove existence of a solution and its robustness w.r.t. perturbations of $\mathcal{F}$. In Sect. 4 we derive some regularity results, that we use in Sect. 5 to study the large-time behaviour of the solution; in particular we prove its asymptotic stability and compactness. In Sect. 6 we deal with the Lipschitz-continuous and monotone dependence of the solution on the data, whence its uniqueness. In Sect. 7 we show existence of a time-periodic solution under periodic forcing, and finally in Sect. 8 we draw conclusions.

1 Continuous and discontinuous hysteresis

The so-called *(delayed) relay operator* is the most simple model of discontinuous hysteresis. In this section we review its definition, specify the functional framework along the lines of [17], and provide a weak formulation in view of coupling it with a P.D.E., cf. [19]. We also introduce a regularization of this operator.

Let us fix any pair $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$, with $\rho_1 < \rho_2$. For any continuous function $u : [0, T] \to \mathbf{R}$ and any $\xi \in \{-1, 1\}$, let us set $X_u(t) := \{\tau \in]0, t] : u(\tau) = \rho_1 \text{ or } \rho_2\}$ and

$$w(0) := \begin{cases} -1 & \text{if } u(0) \le \rho_1 \\ \xi & \text{if } \rho_1 < u(0) < \rho_2 \\ 1 & \text{if } u(0) \ge \rho_2, \end{cases} \tag{1.1}$$

$$w(t) := \begin{cases} w(0) & \text{if } X_u(t) = \emptyset \\ -1 & \text{if } X_u(t) \ne \emptyset \text{ and } u(\max X_u(t)) = \rho_1 \\ 1 & \text{if } X_u(t) \ne \emptyset \text{ and } u(\max X_u(t)) = \rho_2 \end{cases} \qquad \forall t \in]0, T]; \tag{1.2}$$

cf. Fig. 2. Any continuous function $u : [0, T] \to \mathbf{R}$ is uniformly continuous, hence it may have at most a finite number of oscillations between the two thresholds ρ_1, ρ_2; therefore w may jump at most a finite number of times between -1 and 1, and thus $w \in BV(0, T)$. By setting $h_\rho(u, \xi) := w$, a single-valued operator $h_\rho : C^0([0, T]) \times \{-1, 1\} \to BV(0, T)$ is thus defined.

For any diffeomorphism $\varphi : [0, T] \to [0, T]$, if $w = h_\rho(u, \xi)$ then $w \circ \varphi = h_\rho(u \circ \varphi, \xi)$, that is, h_ρ is *rate-independent;* thus it is a *hysteresis operator* in the sense of [17].

Closure. It is easy to see that $h_\rho(\cdot, \xi)$ is not closed as an operator $C^0([0, T]) \to L^1(0, T)$. We then introduce the multi-valued *completed relay operator, k_ρ*, we define as follows. For any $u \in C^0([0, T])$ and any $\xi \in [-1, 1]$, we set $w \in k_\rho(u, \xi)$ if and only if w is measurable in $]0, T[$,

$$w(0) := \begin{cases} -1 & \text{if } u(0) < \rho_1 \\ \xi & \text{if } \rho_1 \le u(0) \le \rho_2 \\ 1 & \text{if } u(0) > \rho_2, \end{cases} \tag{1.3}$$

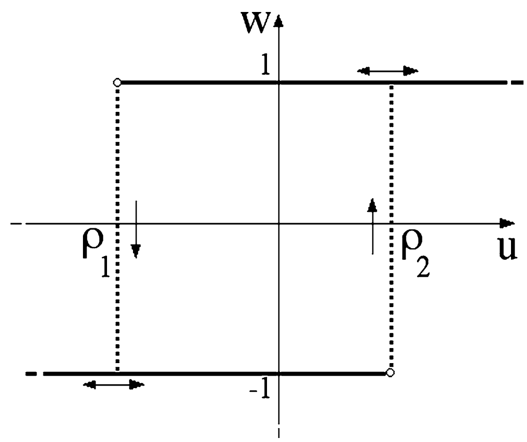

Fig. 2 Relay operator

and, for any $t \in]0, T]$,

$$w(t) \in \begin{cases} \{-1\} & \text{if } u(t) < \rho_1 \\ [-1, 1] & \text{if } \rho_1 \le u(t) \le \rho_2 \\ \{1\} & \text{if } u(t) > \rho_2, \end{cases} \tag{1.4}$$

$$\begin{cases} \text{if } u(t) \ne \rho_1, \rho_2, & \text{then } w \text{ is constant in a neighbourhood of } t \\ \text{if } u(t) = \rho_1, & \text{then } w \text{ is nonincreasing in a neighbourhood of } t \\ \text{if } u(t) = \rho_2, & \text{then } w \text{ is nondecreasing in a neighbourhood of } t; \end{cases} \tag{1.5}$$

cf. Fig. 3a. (1.4) will be referred to as the *confinement condition,* since it restricts the pair (u, w) to stay either in the rectangle $[\rho_1, \rho_2] \times [-1, 1]$ or on the two half-lines $]-\infty, \rho_1[\times\{-1\}$ and $]\rho_2, +\infty[\times\{1\}$. On the other hand (1.5) concerns the dynamics, and will be named the *dissipation condition,* for reasons that will be clear in the sequel.

For any $u \in C^0([0, T])$, $w \in BV(0, T)$ by the argument we saw for h_ρ; thus

$$k_\rho =: C^0([0, T]) \times [-1, 1] \to \mathcal{P}(BV(0, T)).$$

This operator is the closure of h_ρ w.r.t. suitable topologies, see ([17], Sect. VI.1). In the analysis of P.D.E.s it is especially convenient to replace h_ρ by this closure.

Weak formulation of the relay operator. We reformulate the completed relay operator, k_ρ, in view of coupling it with P.D.E.s. The conditions (1.4) and (1.5) are respectively equivalent to

$$|w| \le 1, \quad \begin{cases} (w-1)(u-\rho_2) \ge 0 \\ (w+1)(u-\rho_1) \ge 0 \end{cases} \quad \text{a.e. in }]0, T[, \tag{1.6}$$

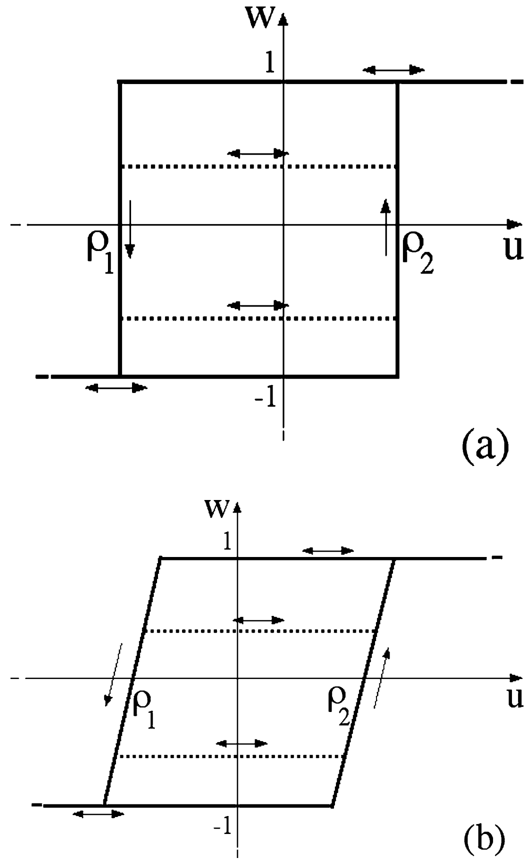

Fig. 3 The graph of the completed relay operator is outlined in part **a**. Any point of the rectangle $[\rho_1, \rho_2] \times [-1, 1]$ is accessible to the pair (u, w). If $u(t) = \rho_1$ ($u(t) = \rho_2$, respectively) then w is locally nonincreasing (nondecreasing, respectively); if $\rho_1 < u(t) < \rho_2$ then w is locally constant. The graph of the corresponding regularized relay operator is represented in part **b**

$$\int_0^t u\,dw = \int_0^t \rho_2 dw^+ - \int_0^t \rho_1 dw^- = \frac{\rho_2 + \rho_1}{2}[w(t) - w(0)] \\ + \frac{\rho_2 - \rho_1}{2} \int_0^t |dw| =: \Psi_0(w; [0, t]) \quad \forall t \in]0, T] \tag{1.7}$$

(these are Stieltjes integrals), cf. [19]. The condition (1.4) entails that

$$u\,dw \leq \rho_2 dw^+ - \rho_1 dw^-, \qquad \text{whence} \qquad \int_0^t u\,\mathrm{d}w \leq \Psi_0(w; [0, t]),$$

independently from the dynamics of the pair (u, w) through the rectangle $[\rho_1, \rho_2] \times [-1, 1]$; the opposite inequality is then equivalent to the equality (1.7).

In conclusion, the system (1.4), (1.5) is equivalent to (1.6) coupled with the inequality

$$\int_0^t u\,dw \geq \Psi_0(w;[0,t]) \quad \forall t \in]0,T]. \tag{1.8}$$

Notice that by (1.7)

$$\Psi_0(w;[0,t]) \geq \frac{\rho_2-\rho_1}{2}\int_0^t |\mathrm{d}w| - (\rho_1+\rho_2). \tag{1.9}$$

Regularized relay operator. Now we approximate the completed relay operator, k_ρ, of Fig. 3a as it is outlined in Fig. 3b. For any $\varepsilon > 0$, the two vertical segments of Fig. 3a are here replaced by two segments of slope $1/\varepsilon$, that intersect the u-axis at $(\rho_1, 0)$ and $(\rho_2, 0)$, resp. The dynamics illustrated by the arrows defines a mapping $u \mapsto w$ for any piecewise monotone input function u. After ([17]; Chap. II) this mapping is Lipschitz-continuous w.r.t. the metric of $C^0([0,T])$; it can then be extended by continuity to a continuous hysteresis operator

$$k_\rho^\varepsilon : C^0([0,T]) \times [-1,1] \to C^0([0,T]) \cap BV(0,T). \tag{1.10}$$

In passing we notice that k_ρ^ε can be represented as a *Preisach model:* k_ρ^ε is the average of a family of relays that are uniformly distributed with thresholds $\{(\rho_1 - s, \rho_2 + s) : -\varepsilon \leq s \leq \varepsilon\}$, cf. [14], [17; Chap. IV]. Moreover, as $\varepsilon \to 0$

$$k_\rho^\varepsilon(v,\xi) \to k_\rho(v,\xi) \quad \text{weakly star in } BV(0,T), \forall (v,\xi) \in C^0([0,T]) \times [-1,1];$$

see [17; Chap. VI]. Notice that

$$w = k_\rho^\varepsilon(u,\xi) \quad \Leftrightarrow \quad w \in k_\rho(u - \varepsilon w, \xi). \tag{1.11}$$

Thus $k_\rho = k_\rho^0$. The regularized relay operator, k_ρ^ε may then also be set in weak form. By (1.6) and (1.7), the latter inclusion is equivalent to the following system of inequalities:

$$|w| \leq 1, \quad \begin{cases} (w-1)(u-\varepsilon w - \rho_2) \geq 0 \\ (w+1)(u - \varepsilon w - \rho_1) \geq 0 \end{cases} \quad \text{a.e. in }]0,T[, \tag{1.12}$$

$$\int_0^t (u - \varepsilon w)\,dw \geq \Psi_0(w;[0,t]) \quad \forall t \in]0,T]. \tag{1.13}$$

The latter inequality also reads

$$\int_0^t u\,dw \geq \Psi_0(w;[0,t]) + \frac{\varepsilon}{2}[w(t)^2 - w(0)^2] =: \Psi_\varepsilon(w;[0,t]) \quad \forall t \in]0,T]. \tag{1.14}$$

2 Weak formulation

In this section we provide a weak formulation in the framework of Sobolev spaces for an initial- and boundary-value problem for the Eq. (1), in the case that $\mathcal{F}$ is either the (completed) relay operator k_ρ or the regularized operator k^ε_ρ. In order to achieve a unified treatment, we deal with k^ε_ρ for any $\varepsilon \geq 0$.

We assume that Ω is a (possibly unbounded) uniformly Lipschitz domain of $\mathbf{R}^N$, fix any $T > 0$, and set $\Omega_t := \Omega \times]0, t[$ for any $t > 0$. We also fix any smooth subset Γ_0 of $\partial\Omega$ of possibly vanishing Hausdorff $(N-1)$-dimensional measure ($\Omega = \mathbf{R}^N$ is not excluded), set

$$V := \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ a.e. on } \Gamma_0\} \quad (\gamma_0 := \text{trace operator}), \tag{2.1}$$

and identify the space $H := L^2(\Omega)$ with its topological dual H'; as V is a dense subspace of H with continuous injection, in turn H' can be identified with a dense subspace of V'. Thus

$$V \subset H = H' \subset V' \qquad \text{with continuous and dense injections.}$$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V, and define the linear and continuous operator

$$A : V \to V', \quad \langle Au, v \rangle := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V.$$

We fix any $\varepsilon \geq 0$, any $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$ with $\rho_1 < \rho_2$, define Ψ_ε as in (1.14), and assume that

$$u^0 \in H, \quad w^0 \in L^\infty(\Omega), \quad |w^0| \leq 1, \tag{2.2}$$

$$f = f_1 + f_2, \quad f_1 \in L^1(0, T; H), \quad f_2 \in L^2(0, T; V'). \tag{2.3}$$

We now formulate our initial- and boundary-value problem for any $\varepsilon \geq 0$.

Problem 2.1$_{\varepsilon,T}$. Find $u_\varepsilon \in L^\infty(0, T; H) \cap L^2(0, T; V)$ and $w_\varepsilon \in L^\infty(\Omega_T)$ such that

$$|w_\varepsilon| \leq 1 \quad \text{a.e. in } \Omega_T, \quad \frac{\partial w_\varepsilon}{\partial t} \in C^0(\overline{\Omega_T})', \tag{2.4}$$

$$\begin{aligned} &\iint_{\Omega_T} \left((u^0 - u_\varepsilon + w^0 - w_\varepsilon) \frac{\partial v}{\partial t} + \nabla u_\varepsilon \cdot \nabla v \right) dx \, dt \\ &= \iint_{\Omega_T} f_1 v \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \langle f_2, v \rangle \, dt \\ &\forall v \in H^1(0, T; H) \cap L^2(0, T; V), \ v(\cdot, T) = 0 \text{ a.e. in } \Omega, \end{aligned} \tag{2.5}$$

$$\begin{cases} (w_\varepsilon - 1)(u_\varepsilon - \varepsilon w_\varepsilon - \rho_2) \geq 0 \\ (w_\varepsilon + 1)(u_\varepsilon - \varepsilon w_\varepsilon - \rho_1) \geq 0 \end{cases} \quad \text{a.e. in } \Omega_T, \tag{2.6}$$

$$\begin{aligned} &\frac{1}{2} \int_\Omega (u_\varepsilon(x,t)^2 - u^0(x)^2) dx + \int_\Omega \Psi_\varepsilon(w_\varepsilon(x, \cdot); [0, t]) \, dx + \iint_{\Omega_t} |\nabla u_\varepsilon|^2 \, dx \, d\tau \\ &\leq \iint_{\Omega_t} f_1 u_\varepsilon \, dx \, d\tau + \int_0^t \langle f_2, u_\varepsilon \rangle \, d\tau \quad \text{for a.a. } t \in]0, T[. \end{aligned} \tag{2.7}$$

$$w_\varepsilon(x,0) = w^0(x) \quad \text{in } \mathbf{R}. \tag{2.8}$$

Interpretation. The initial condition (2.8) makes sense because of the second part of (2.4).

The Eq. (2.5) yields

$$\frac{\partial}{\partial t}(u_\varepsilon + w_\varepsilon) + Au_\varepsilon = f \quad \text{in } V', \text{ for a.a. } t \in]0,T[, \tag{2.9}$$

whence $u_\varepsilon + w_\varepsilon \in W^{1,1}(0,T;V')$. The initial condition

$$(u_\varepsilon + w_\varepsilon)|_{t=0} = u^0 + w^0 \quad \text{in } V' \tag{2.10}$$

then follows. Let us now set $\Gamma_1 := \Gamma \setminus \Gamma_0$ and denote by $\partial/\partial\nu$ the trace of the outward normal derivative on Γ_1. If, for instance,

$$g \in L^2(\Gamma_1 \times]0,T[), \quad \langle f_2, v\rangle := \int_{\Gamma_1} g\,\gamma_0 v\, \mathrm{d}\sigma \quad \forall v \in V, \text{ a.e. in }]0,T[, \tag{2.11}$$

then (2.9) is a weak formulation of the boundary-value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(u_\varepsilon + w_\varepsilon) - \Delta u_\varepsilon = f_1 & \text{in } \mathcal{D}'(\Omega_T) \\ \gamma_0 u_\varepsilon = 0 & \text{on } \Gamma_0 \times]0,T[\\ \dfrac{\partial u_\varepsilon}{\partial \nu} = g & \text{on } \Gamma_1 \times]0,T[. \end{cases} \tag{2.12}$$

Multiplying (2.9) by u_ε and integrating in space and time, we see that (2.7) is equivalent to

$$\int_0^t \left\langle \frac{\partial}{\partial \tau}(u_\varepsilon + w_\varepsilon), u \right\rangle d\tau \geq \frac{1}{2}\int_\Omega [u_\varepsilon(x,t)^2 - u^0(x)^2]dx + \int_\Omega \Psi_\varepsilon(w_\varepsilon; [0,t])\, dx$$
$$\text{for a.a. } t \in]0,T[. \tag{2.13}$$

If $\partial u_\varepsilon/\partial t \in L^2(0,T;V')$ then

$$\int_0^t \langle \partial u_\varepsilon/\partial\tau, u\rangle\, d\tau = \frac{1}{2}\int_\Omega [u_\varepsilon(x,t)^2 - u^0(x)^2]dx;$$

moreover, as $u_\varepsilon + w_\varepsilon \in W^{1,1}(0,T;V')$, it follows that $\partial w_\varepsilon/\partial t \in L^1(0,T;V')$. The inequality (2.13) then reads

$$\int_0^t \left\langle \frac{\partial w_\varepsilon}{\partial \tau}, u \right\rangle d\tau \geq \int_\Omega \Psi_\varepsilon(w_\varepsilon; [0,t])\, dx \quad \text{for a.a. } t \in]0,T[. \tag{2.14}$$

In Sect. 4, under regularity hypotheses on the data, we shall see that $\partial u_\varepsilon/\partial t \in L^2(0,T;H)$, cf. Proposition 4.2; in that case (2.14) will be fully justified.

The inequality (2.14) extends the *dissipation condition* (1.14) to space-distributed systems. By the developments of Sect. 1, we see that (2.6), (2.8) and (2.14) stand for the hysteresis relation

$$w_\varepsilon(x,t) \in \big[k_\rho^\varepsilon(u_\varepsilon(x,\cdot), w^0(x))\big](t) \quad \text{in } [0,T], \text{ a.e. in } \Omega, \tag{2.15}$$

although here $u_\varepsilon(x,\cdot)$ need not be continuous. We have thus further weakened the formulation of the relay operator.

In conclusion, Problem $2.1_{\varepsilon,T}$ is a weak formulation of the system (2.9), (2.10), (2.15).

Remarks (i) The formulation of Problem $2.1_{\varepsilon,T}$ also applies for $T = +\infty$, provided that the second formula of (2.4) is replaced by $\partial w_\varepsilon/\partial t \in C^0(\overline{\Omega_t})'$ for any $t \in]0,+\infty[$.

(ii) In the limit case of $\rho_1 = \rho_2$, (2.6) is equivalent to

$$w_\varepsilon \in \operatorname{sign}(u_\varepsilon - \varepsilon w_\varepsilon - \rho_1) \quad \text{a.e. in } \Omega_T, \tag{2.16}$$

(2.7) can be dropped, and the regularity $\partial w_\varepsilon/\partial t \in C^0(\overline{\Omega_T})'$ is lost. In this case for $\varepsilon = 0$, Problem $2.1_{\varepsilon,T}$ is equivalent to the weak formulation of the Stefan problem.

3 Existence of a solution

In this section we prove existence of a solution of Problem $2.1_{\varepsilon,T}$ for any $\varepsilon \ge 0$ and any finite $T > 0$, and derive some uniform estimates w.r.t. T, that will be used in Sect. 5 in the study of the large-time behaviour of the solution(s).

Theorem 3.1 *(Existence) For any $\varepsilon \ge 0$ and any $T > 0$, if (2.2) and (2.3) hold, then Problem $2.1_{\varepsilon,T}$ has a solution $(u_\varepsilon, w_\varepsilon)$. Moreover the (possibly multi-valued) solution operator $(u^0, w^0, f) \mapsto (u_\varepsilon, w_\varepsilon)$ has a selection that maps bounded sets to bounded sets, uniformly w.r.t. T, ε. This also applies for $T = +\infty$.*

By the latter statement we mean that for any $M > 0$ there exists $N > 0$ such that, if

$$\|u^0\|_H + \|f\|_{L^1(0,T;H)+L^2(0,T;V')} \le M, \tag{3.1}$$

then, for any $\varepsilon \ge 0$ and any $T > 0$ ($T = +\infty$ included), there exists a solution of Problem $2.1_{\varepsilon,T}$ such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \left\|\frac{\partial w_\varepsilon}{\partial t}\right\|_{C^0(\overline{\Omega_T})'} + \|u_\varepsilon + w_\varepsilon\|_{W^{1,1}(0,T;V')} \le N. \tag{3.2}$$

This notion of *T-uniform boundedness* will be encountered repeatedly in this work, and will be used to study the large-time behaviour of the solution.

Henceforth we shall write (u, w) in place of $(u_\varepsilon, w_\varepsilon)$.

Proof This argument is based on approximation, derivation of a priori estimates and passage to the limit.

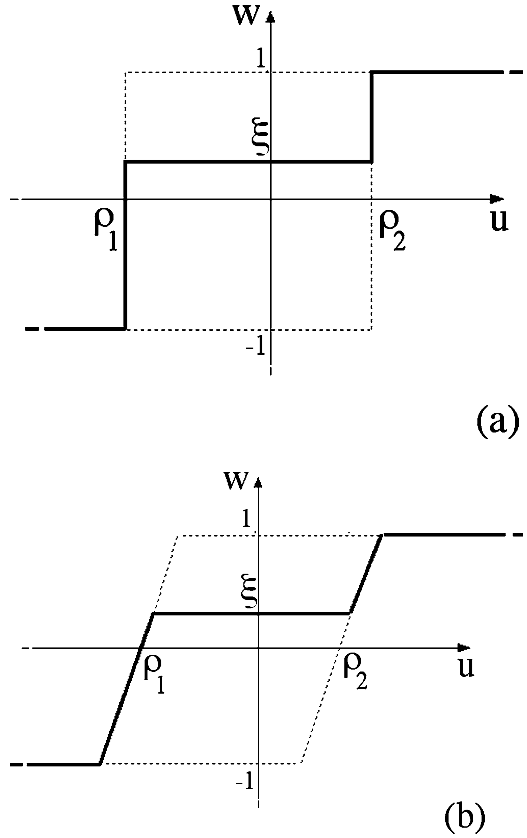

Fig. 4 Graphs of the multi-valued function $G(\cdot, \xi)$ in part **a**, and of the corresponding single-valued function $G_\varepsilon(\cdot, \xi)$ in part **b**, for any fixed $\xi \in [-1, 1]$

(i) *Approximation*. Let us fix any $m \in \mathbf{N}$, set $h := T/m$ and

$$\begin{cases} f_{1m}^n(x) := \dfrac{1}{h}\displaystyle\int_{(n-1)h}^{nh} f_1(x,t)\,\mathrm{d}t \quad \text{for a.a. } x \in \Omega, \\ f_{2m}^n := \dfrac{1}{h}\displaystyle\int_{(n-1)h}^{nh} f_2(t)\,dt \quad \text{in } V', \\ f_m^n := f_{1m}^n + f_{2m}^n, \quad u_m^0 := u^0, \quad w_m^0 := w^0, \qquad \text{for } n = 1, \dots, m, \end{cases}$$

$$G(v,\xi) := \begin{cases} \{-1\} & \text{if } v < \rho_1 \\ [-1,\xi] & \text{if } v = \rho_1 \\ \{\xi\} & \text{if } \rho_1 < v < \rho_2 \\ [\xi,1] & \text{if } v = \rho_2 \\ \{1\} & \text{if } v > \rho_2 \end{cases} \qquad \forall (v,\xi) \in \mathbf{R} \times [-1,1], \tag{3.3}$$

cf. Fig. 4a. Notice that the relation $z \in G(v - \varepsilon z, \xi)$ is equivalent to $z \in G_\varepsilon(v, \xi)$, where $G_\varepsilon(\cdot, \xi)$ is the maximal monotone function outlined in Fig. 4b.

We now approximate our problem via an implicit time-discretization scheme.

Problem 2.1$_{\varepsilon,\mathbf{T},\mathbf{m}}$. For $n = 1, \dots, m$, find $u_m^n \in V$ and $w_m^n \in H$ such that

$$w_m^n \in G_\varepsilon\big(u_m^n, w_m^{n-1}\big) \quad \text{a.e. in } \Omega, \text{ for } n = 1, \dots, m, \tag{3.4}$$

$$\frac{u_m^n - u_m^{n-1}}{h} + \frac{w_m^n - w_m^{n-1}}{h} + Au_m^n = f_m^n \quad \text{in } V', \text{ for } n = 1, \dots, m. \tag{3.5}$$

By the maximal monotonicity of $G_\varepsilon(\cdot, w_m^{n-1})$, existence and uniqueness of the approximate solution can easily be proved step by step. (Time-discretization overcomes difficulties due to the occurrence of memory: at any instant the only relevant unknown is the actual value, since the past is known and the future has no influence on the present. Notice that this remark only applies to the time-discretized problem.)

(ii) *A Priori Estimates*. Multiplying the Eq. (3.5) by u_m^n we get

$$\begin{aligned} &\int_\Omega \frac{\big(u_m^n\big)^2 - \big(u_m^{n-1}\big)^2}{2h} dx + \int_\Omega \frac{w_m^n - w_m^{n-1}}{h} u_m^n \, dx + \int_\Omega |\nabla u_m^n|^2 \, dx \\ &\quad \le \|f_{1m}^n\|_H \|u_m^n\|_H + \|f_{2m}^n\|_{V'} \|u_m^n\|_V \quad \text{for } n = 1, \dots, m. \end{aligned} \tag{3.6}$$

Defining Ψ_ε as in (1.14), by (3.4) we have

$$\begin{aligned} \sum_{n=1}^{\ell} \big(w_m^n - w_m^{n-1}\big) u_m^n &\ge \sum_{n=1}^{\ell} \big[\big(w_m^n - w_m^{n-1}\big)^+ \rho_2 - \big(w_m^n - w_m^{n-1}\big)^- \rho_1\big] \\ &= \Psi_\varepsilon(w_m; [0, \ell h]) \quad \text{a.e. in } \Omega, \text{ for } \ell = 1, \dots, m. \end{aligned} \tag{3.7}$$

Multiplying (3.6) by h and summing w.r.t. n, we then get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big[\big(u_m^\ell\big)^2 - (u^0)^2\big] dx + \int_\Omega \Psi_\varepsilon(w_m; [0, \ell h]) \, dx + h \sum_{n=1}^{\ell} \int_\Omega \big|\nabla u_m^n\big|^2 \, dx \\ &\le \|f_1\|_{L^1(0,T;H)} \max\nolimits_{n=0,\dots,\ell} \|u_m^n\|_H + \|f_2\|_{L^2(0,T;V')} \bigg(h \sum_{n=1}^{\ell} \|u_m^n\|_V^2\bigg)^{1/2} \end{aligned} \tag{3.8}$$

for $n = 1, \dots, \ell$. By (1.9), a standard calculation then yields

$$\max_{n=1,\dots,m} \|u_m^n\|_H, \; h\sum_{n=1}^{m} \|u_m^n\|_V^2, \; \sum_{n=1}^{m} \int_\Omega \big|w_m^n - w_m^{n-1}\big| \, dx \le C_1. \tag{3.9}$$

(By $C_1, C_2, \dots$ we denote suitable positive constants independent of m, T, ε.)

For any family $\{v_m^n\}_{n=1,\dots,m}$ of functions $\Omega \to \mathbf{R}$, let us set

$$\begin{aligned} &v_m := \text{ piecewise-linear interpolate of } v_m^0, \dots, v_m^m \text{ in } [0, T], \text{ a.e. in } \Omega, \\ &\bar{v}_m(\cdot, t) := v_m^n \quad \text{a.e. in } \Omega, \forall t \in](n-1)h, nh[, \text{ for } n = 1, \dots, m. \end{aligned}$$

After comparison in (3.5), the estimates (3.9) also read

$$\|u_m\|_{L^\infty(0,T;H) \cap L^2(0,T;V)}, \; \left\|\frac{\partial w_m}{\partial t}\right\|_{L^1(\Omega_T)}, \; \|u_m + w_m\|_{W^{1,1}(0,T;V')} \le C_2. \tag{3.10}$$

(iii) *Limit Procedure*. By the above estimates, there exist u, w such that, as $m \to \infty$ along a suitable sequence,

$$\begin{aligned}
&u_m \to u && \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V)\\
&w_m \to w && \text{weakly star in } L^\infty(\Omega_T)\\
&\frac{\partial w_m}{\partial t} \to \frac{\partial w}{\partial t} && \text{weakly star in } C^0(\overline{\Omega_T})'\\
&u_m + w_m \to u + w && \text{weakly in } H^a(0,T;V'), \forall a < 1/2.
\end{aligned} \tag{3.11}$$

(3.5) and (3.8) respectively read

$$\frac{\partial}{\partial t}(u_m + w_m) + A\bar{u}_m = \bar{f}_m \quad \text{in } V', \text{ a.e. in }]0,T[, \tag{3.12}$$

$$\frac{1}{2}\int_\Omega [\bar{u}_m(x,t)^2 - u^0(x)^2]dx + \int_\Omega \Psi_\varepsilon(w_m;[0,t])\,dx + \iint_{\Omega_t} |\nabla \bar{u}_m|^2\,dx d\tau \leq \int_0^t \langle \bar{f}_m, \bar{u}_m\rangle d\tau \quad \text{a.e. in }]0,T[. \tag{3.13}$$

By passing to the limit in (3.12) and to the inferior limit in (3.13), we then get (2.9) and (2.7).

(iv) *Proof of (2.6)*. Hysteresis operators are not monotone in the sense of $L^2(0,T)$, cf. [17, Sect. III.1]. Therefore to pass to the limit in the hysteresis nonlinearity we may just use a compactness argument. For any $\varphi \in \mathcal{D}(\Omega_T)$ with $\varphi \geq 0$, (3.4) entails

$$\iint_{\Omega_T} (\bar{w}_m - 1)(\bar{u}_m - \varepsilon\bar{w}_m - \rho_2)\varphi\,dx\,dt \geq 0. \tag{3.14}$$

By (3.11) for any $\varphi \in \mathcal{D}(\Omega_T)$, $\varphi \geq 0$,

$$(\bar{u}_m + \bar{w}_m)\varphi \to (u+w)\varphi \quad \text{weakly star in } L^\infty(0,T;H) \cap H^a(0,T;V'), \forall a < 1/2, \tag{3.15}$$

hence strongly in $L^2(0,T;V')$; therefore

$$\begin{aligned}
\limsup_{m\to+\infty} &\iint_{\Omega_T} \bar{w}_m\bar{u}_m\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&= \lim_{m\to+\infty}\int_0^T \langle \bar{u}_m + \bar{w}_m, \bar{u}_m\varphi\rangle\,dt - \liminf_{m\to+\infty}\iint_{\Omega_T} \bar{u}_m^2\varphi\,dx\,\mathrm{d}t\\
&\leq \int_0^T \langle u+w, u\varphi\rangle\,dt - \iint_{\Omega_T} u^2\varphi\,dx\,dt = \iint_{\Omega_T} wu\varphi\,dx\,dt.
\end{aligned}$$

By passing to the superior limit in (3.14), we then get

$$\iint_{\Omega_T} (w-1)(u - \varepsilon w - \rho_2)\varphi\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall\varphi \in \mathcal{D}(\Omega_T), \varphi \geq 0,$$

which yields $(2.6)_1$; $(2.6)_2$ can be proved similarly. □

Proposition 3.2 *(Robustness) For any $n \in \mathbf{N}$ let $\rho_{1n} < \rho_{2n}$, and for any $\varepsilon \geq 0$ let (u_n, w_n) be a corresponding solution of Problem 2.1$_{\varepsilon,T}$. If $\rho_{1n} \to \rho_1$, $\rho_{2n} \to \rho_2$ and $\rho_1 < \rho_2$, then there exists (u, w) such that, as $n \to \infty$ along a subsequence,*

$$\begin{aligned} &u_n \to u && \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V),\\ &w_n \to w && \text{weakly star in } L^\infty(\Omega_T),\\ &\frac{\partial w_n}{\partial t} \to \frac{\partial w}{\partial t} && \text{weakly star in } C^0(\overline{\Omega_T})',\\ &u_n + w_n \to u + w && \text{weakly in } H^a(0,T;V'),\ \forall a < 1/2. \end{aligned} \tag{3.16}$$

Moreover (u, w) is a solution of Problem 2.1$_{\varepsilon,T}$ corresponding to the pair (ρ_1, ρ_2).

Outline of the Proof. The argument follows the lines of the above estimate and limit procedure. In particular notice that $(3.16)_2$ and $(3.16)_3$ entail

$$\liminf_{n\to\infty} \Psi_{\rho_n}(w_n; [0,t]) \geq \Psi_\rho(w; [0,t]). \qquad \square$$

An analogous statement holds if $\rho_1 = \rho_2$; in this case however the convergence $(3.16)_3$ is lost, as well as the inequality (2.7).

Remarks (i) The above existence result can be extended in several directions; in particular the Laplace operator can be replaced by more general, possibly time-dependent and possibly non-self-adjoint, elliptic operators. The relay operator can also be replaced by a more general *Preisach operator.*

(ii) Problem 2.1$_{\varepsilon,T}$ and the above existence theorem can easily be extended to the initial- and boundary-value problems for degenerate equations like

$$\frac{\partial}{\partial t} k^\varepsilon_\rho(u, w^0) - \Delta u = f \quad \text{in } \Omega_T \ (\varepsilon \geq 0); \tag{3.17}$$

of course in this case in general the regularity $u \in L^\infty(0,T;H)$ is lost. For instance existence of a solution can be proved for the equation

$$\frac{\partial}{\partial t} k^\varepsilon_\rho(u, w^0) - \Delta u - \frac{\partial}{\partial z} k^\varepsilon_\rho(u, w^0) = f \quad \text{in } \Omega_T, \tag{3.18}$$

that arises as a simplified model of porous medium filtration (with normalized coefficients); here u represents the water pressure, $k^\varepsilon_\rho(u, w^0)$ the medium saturation, and z is the vertical coordinate; cf. [20].

(iii) The problem with discontinuous hysteresis, Problem 2.1$_{0,T}$, can also be approximated by passing to the limit as $\varepsilon \to 0$ in Problem 2.1$_{\varepsilon,T}$. The argument is analogous to that above, since the estimates we derived are uniform w.r.t. ε.

4 Regularity of solutions

In this section we derive some regularity properties for the solution(s) of Problem 2.1$_{\varepsilon,T}$ for any $\varepsilon \geq 0$, in view of the study of the large-time behaviour. (We still write (u, w) in place of $(u_\varepsilon, w_\varepsilon)$.)

Proposition 4.1 *Let $\varepsilon \geq 0$, $p \in\,]2, +\infty]$, and let (2.2) be fulfilled. If*

$$u^0 \in L^p(\Omega), \quad f \in L^1(0, T; L^p(\Omega)), \tag{4.1}$$

then Problem 2.1$_{\varepsilon,T}$ has a solution such that

$$u \in L^\infty(0, T; L^p(\Omega)). \tag{4.2}$$

Moreover the (possibly multi-valued) solution operator $(u^0, f) \mapsto (u, w)$ has a selection that maps bounded subsets of $L^p(\Omega) \times L^1\big(0, T; L^p(\Omega)\big)$ to bounded subsets of $L^\infty\big(0, T; L^p(\Omega)\big)$, uniformly w.r.t. T, ε. This also applies for $T = +\infty$.

Proof (i) First let us assume that $2 < p < +\infty$, and set

$$a_{m,p-1}(v) := \min\{|v|^{p-2}, m\}v, \quad A_{m,p}(v) := \int_0^v a_{m,p-1}(\xi)\,\mathrm{d}\xi \quad \forall v \in \mathbf{R}, \forall m \in \mathbf{N}.$$

Let us multiply (3.5) by $a_{m,p-1}(u_m^n)$ $(\in V)$; by (3.4) there exists a constant $C \geq 0$ such that

$$\sum_{n=1}^{\ell} \big(w_m^n - w_m^{n-1}\big)\, a_{m,p-1}\big(u_m^n\big) \geq -C \qquad \text{a.e. in } \Omega, \text{ for } \ell = 1, \ldots, m.$$

We then get

$$\begin{aligned}
&\int_\Omega \big[A_{m,p}\big(u_m^\ell\big) - A_{m,p}\big(u^0\big)\big]\mathrm{d}x + h \sum_{n=1}^{\ell} \int_\Omega a'_{m,p-1}\big(u_m^n\big) \big|\nabla u_m^n\big|^2 \,\mathrm{d}x \\
&\quad \leq \|f\|_{L^1(0,T;L^p(\Omega))} \max_{n=1,\ldots,\ell} \big\|a_{m,p-1}\big(u_m^n\big)\big\|_{L^{p/(p-1)}(\Omega)} + C \\
&\quad \text{for } \ell = 1, \ldots, m.
\end{aligned} \tag{4.3}$$

It is then easy to see that u_m is bounded in $L^\infty\big(0, T; L^p(\Omega)\big)$ uniformly w.r.t. m, T, ε. The thesis thus holds for any $p < +\infty$.

If $p = \infty$, let us set $M := \max\{\|u^0\|_{L^\infty(\Omega)}, \rho_1, \rho_2\}$, and multiply the approximate Eq. (3.5) by $z_m^n := (u_m^n - M)^+ - (u_m^n - M)^-$. As $(w_m^n - w_m^{n-1})z_m^n \geq 0$, a standard procedure yields $z_m^n = 0$ a.e. in Ω for any n. Hence $|u| \leq M$ a.e. in Ω_T. □

Now we show that a stronger regularity of the inital data and of the second member improve the regularity of the solution.

Proposition 4.2 *Let $\varepsilon \geq 0$. If (2.2) hold and*

$$u^0 \in V, \quad f = f_1 + f_2, \quad f_1 \in L^2(0, T; H), \quad f_2 \in W^{1,1}(0, T; V'), \tag{4.4}$$

then Problem 2.1$_{\varepsilon,T}$ has a solution such that

$$u \in H^1(0, T; H) \cap L^\infty(0, T; V). \tag{4.5}$$

Moreover the (possibly multi-valued) solution operator $(u^0, f) \mapsto (u, w)$ has a selection that maps bounded sets to bounded sets, uniformly w.r.t. T, ε. This also applies for $T = +\infty$.

By (4.5) and by the Lipschitz continuity of k_ρ^ε, if $\varepsilon > 0$ then $w \in H^1(0, T; H)$.

Proof (i) Let us multiply the time-discretized Eq. (3.5) by $u_m^n - u_m^{n-1}$ and sum for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$. By the monotonicity of G_ε,

$$\big(w_m^n - w_m^{n-1}\big)\big(u_m^n - u_m^{n-1}\big) \geq 0 \quad \text{a.e. in } \Omega, \text{ for } n = 1, \dots, m, \tag{4.6}$$

whence

$$\begin{aligned} &h \int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \big(|\nabla u_m^n|^2 - \big|\nabla u_m^{n-1}\big|^2\big)\, dx \\ &\leq \int_\Omega f_{1m}^n \big(u_m^n - u_m^{n-1}\big)\, \mathrm{d}x + \big\langle f_{2m}^n, u_m^n - u_m^{n-1}\big\rangle \\ &\leq \frac{h}{2}\big\| f_{1m}^n\big\|_H^2 + \frac{h}{2}\int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + \big\langle f_{2m}^n, u_m^n\big\rangle \\ &\qquad -\big\langle f_{2m}^{n-1}, u_m^{n-1}\big\rangle - \big\langle f_{2m}^n - f_{2m}^{n-1}, u_m^{n-1}\big\rangle \quad \forall n \in \mathbf{N}. \end{aligned} \tag{4.7}$$

Setting $z_m^n := \frac{1}{2}\int_\Omega |\nabla u_m^n|^2\, \mathrm{d}x - \langle f_{2m}^n, u_m^n\rangle$ for any n, we get

$$\begin{aligned} &\frac{h}{2}\int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + z_m^n - z_m^{n-1} \\ &\leq \frac{h}{2}\big\| f_{1m}^n\big\|_H^2 + \big\| f_{2m}^n - f_{2m}^{n-1}\big\|_{V'}\big\|u_m^{n-1}\big\|_V \quad \text{for } n = 1, \dots, m. \end{aligned} \tag{4.8}$$

A simple calculation then yields

$$\|u_m\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C_3, \tag{4.9}$$

and passing to the limit in m the thesis follows. □

In view of the analysis of the large-time behaviour of the solution, we introduce the following cut-off function

$$\beta(t) := \min\{t, 1\} \quad \forall t > 0, \tag{4.10}$$

and use it to derive some regularity properties independently from the initial data.

Proposition 4.3 *Let $\varepsilon \geq 0$, let (2.2) and (2.3) be fulfilled, and assume that*

$$\sqrt{\beta(t)}\, f_1 \in L^2(0, T; H), \quad \sqrt{\beta(t)}\, f_2 \in H^1(0, T; V'). \tag{4.11}$$

Then Problem $2.1_{\varepsilon,T}$ has a solution such that

$$\sqrt{\beta(t)}\frac{\partial u}{\partial t} \in L^2(0, T; H), \quad \sqrt{\beta(t)}\, u \in L^\infty(0, T; V). \tag{4.12}$$

Moreover there exists a selection of the (possibly multi-valued) solution operator $(u^0, f) \mapsto (u, w)$ that maps bounded sets to bounded sets, uniformly w.r.t. T and ε in the following sense: for any $M > 0$ there exists $N > 0$ such that, if

$$\begin{aligned} &\|u^0\|_H, \|f_1\|_{L^1(0,T;H)}, \|\sqrt{\beta(t)} f_1\|_{L^2(0,T;H)}, \\ &\|f_2\|_{L^2(0,T;V')}, \big\|\sqrt{\beta(t)}\, f_2\big\|_{L^\infty(0,T;V')\cap H^1(0,T;V')} \leq M, \end{aligned} \tag{4.13}$$

then, for any $T > 0$ and any $\varepsilon \geq 0$, there exists a solution of Problem 2.1$_{\varepsilon,T}$ such that

$$\Big\| \sqrt{\beta(t)}\frac{\partial u}{\partial t} \Big\|_{L^2(0,T;H)} + \|\sqrt{\beta(t)}\, u\|_{L^\infty(0,T;V)} \leq N. \tag{4.14}$$

This also applies for $T = +\infty$.

Proof Multiplying (3.5) by $\beta(nh)\,(u_m^n - u_m^{n-1})$, by (4.6) we get

$$h\beta(nh)\int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + \frac{\beta(nh)}{2}\int_\Omega \big(|\nabla u_m^n|^2 - |\nabla u_m^{n-1}|^2\big)\,\mathrm{d}x$$
$$\leq \beta(nh)\int_\Omega f_{1m}^n\big(u_m^n - u_m^{n-1}\big)\,\mathrm{d}x + \beta(nh)\big\langle f_{2m}^n, u_m^n - u_m^{n-1}\big\rangle \text{ for } n = 1, \ldots, m, \tag{4.15}$$

that is,

$$h\beta(nh)\int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + \frac{\beta(nh)}{2}\int_\Omega |\nabla u_m^n|^2\,\mathrm{d}x - \frac{\beta(nh-h)}{2}\int_\Omega |\nabla u_m^{n-1}|^2\,\mathrm{d}x$$
$$\leq \frac{\beta(nh) - \beta(nh-h)}{2}\int_\Omega |\nabla u_m^{n-1}|^2\,\mathrm{d}x + \beta(nh)\int_\Omega f_{1m}^n\big(u_m^n - u_m^{n-1}\big)\,\mathrm{d}x$$
$$+ \beta(nh)\big\langle f_{2m}^n, u_m^n\big\rangle - \beta(nh-h)\big\langle f_{2m}^{n-1}, u_m^{n-1}\big\rangle$$
$$- \big\langle \beta(nh) f_{2m}^n - \beta(nh-h) f_{2m}^{n-1}, u_m^{n-1}\big\rangle. \tag{4.16}$$

Notice that

$$\frac{\beta(nh) - \beta(nh-h)}{2}\int_\Omega |\nabla u_m^{n-1}|^2\,\mathrm{d}x \leq \frac{h}{2}\|u_m^{n-1}\|_V^2,$$

$$\beta(nh)\int_\Omega f_{1m}^n\big(u_m^n - u_m^{n-1}\big)\,\mathrm{d}x \leq \frac{h\beta(nh)}{2}\int_\Omega |f_{1m}^n|^2\,\mathrm{d}x + \frac{h\beta(nh)}{2}\int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x,$$

$$\beta(nh)\big\langle f_{2m}^n, u_m^n\big\rangle \leq \frac{\sqrt{\beta(nh)}}{2}\|f_{2m}^n\|_{V'}^2 + \frac{\sqrt{\beta(nh)}}{2}\|u_m^n\|_V^2,$$

$$\big\langle \beta(nh) f_{2m}^n - \beta(nh-h) f_{2m}^{n-1}, u_m^{n-1}\big\rangle$$
$$\leq \frac{1}{2}\Big\|\frac{\beta(nh) f_{2m}^n - \beta(nh-h) f_{2m}^{n-1}}{h}\Big\|_{V'}^2 + \frac{h}{2}\|u_m^{n-1}\|_V^2,$$

and $\|\beta(t) f_2\|_{H^1(0,T;V')} \leq \|\sqrt{\beta(t)} f_2\|_{H^1(0,T;V')}$. By (4.16) and by these inequalities, recalling that $h\sum_{n=1}^m \|u_m^n\|_V^2$ is uniformly bounded (cf. (3.9)), we infer that there exists a real sequence $\{g_m^n\}$ such that $\sum_{n=1}^\infty g_m^n$ converges uniformly w.r.t. m, T, and such that

$$\frac{h\beta(nh)}{2}\int_\Omega \Big|\frac{u_m^n - u_m^{n-1}}{h}\Big|^2 \mathrm{d}x + \frac{\beta(nh)}{2}\int_\Omega |\nabla u_m^n|^2\,\mathrm{d}x$$
$$- \frac{\beta(nh-h)}{2}\int_\Omega |\nabla u_m^{n-1}|^2\,\mathrm{d}x \leq g_m^n \quad \forall n. \tag{4.17}$$

We then conclude that

$$\left\| \sqrt{\beta(t)} \frac{\partial u_m}{\partial t} \right\|_{L^2(0,T;H)} + \|\sqrt{\beta(t)}\, u_m\|_{L^\infty(0,T;V)} \le C_4. \tag{4.18}$$

The latter constant is independent of m, T and depends on the data u^0, f only through the norms of (4.13). Therefore Problem $2.1_{\varepsilon,T}$ has a solution that fulfils (4.12), and the solution operator has the stated boundedness property. □

L^1**-Type Results.** First let us set

$$s_0(\eta) := -1 \quad \text{if } \eta < 0, \quad s_0(0) := 0, \quad s_0(\eta) := 1 \quad \text{if } \eta > 0.$$

In the next three results we assume that $\varepsilon > 0$.

Lemma 4.4 *Hilpert's Inequality [4]) Let $\varepsilon > 0$. For any $(u_i, \xi_i) \in W^{1,1}(0,T) \times \mathbf{R}$ $(i = 1, 2)$, setting $\tilde{w} := k_\rho^\varepsilon(u_1, \xi_1) - k_\rho^\varepsilon(u_2, \xi_2)$,*

$$\frac{d\tilde{w}}{dt} s_0(u_1 - u_2) \ge \frac{d}{dt}|\tilde{w}| \quad \textit{a.e. in }]0, T[. \tag{4.19}$$

For the proof of this statement we refer the reader to [4], [17; Sect. III.2].

In view of the next result, let us fix any direction $\hat{\xi}$, and for any $\eta > 0$ let us denote by $\delta_{\hat{\xi},\eta}$ the space-increment operator in the direction $\hat{\xi}$, that is, $\delta_{\hat{\xi},\eta} z(x) := z(x + \eta\hat{\xi}) - z(x)$ for any function $z : \mathbf{R}^N \to \mathbf{R}$ and any $x \in \mathbf{R}^N$.

Lemma 4.5 *Assume that (2.2) and (2.3) hold and that $f \in L^1(\Omega_T)$. Let Ω_1, Ω_2 be bounded subsets of Ω such that $\bar{\Omega}_1 \subset \Omega_2$, and set $\Omega_{i,t} := \Omega_i \times]0, t[$ for $i = 1, 2$. Then there exists a constant $C_7 > 0$ such that, for any $\varepsilon > 0$ and any solution (u, w) of Problem $2.1_{\varepsilon,T}$,*

$$\begin{aligned} &\int_{\Omega_1} \left(\left|\delta_{\hat{\xi},\eta} u\right| + \left|\delta_{\hat{\xi},\eta} w\right|\right)(x,t)\,\mathrm{d}x \\ &\quad \le \int_{\Omega_2} \left(\left|\delta_{\hat{\xi},\eta} u^0\right| + \left|\delta_{\hat{\xi},\eta} w^0\right|\right)\mathrm{d}x + \iint_{\Omega_{2,t}} \left|\delta_{\hat{\xi},\eta} f\right| \mathrm{d}x\mathrm{d}\tau + C_7\eta \\ &\quad\quad \forall \eta > 0 \textit{ such that } \Omega_1 + \eta\hat{\xi} \subset \Omega_2 \quad \textit{for a.a. } t \in]0, T[. \end{aligned} \tag{4.20}$$

Proof Let us approximate the function s_0 by setting

$$s_j(\zeta) := \max\{\min\{j\zeta, 1\}, -1\} \quad \forall \zeta \in \mathbf{R}, \forall j \in \mathbf{N}. \tag{4.21}$$

Let $\varphi : \Omega \to [0, 1]$ be of class C^2 and such that

$$0 \le \varphi \le 1 \quad \text{in } \Omega, \quad \varphi = 1 \quad \text{in } \Omega_1, \quad \varphi = 0 \quad \text{in } \Omega \setminus \Omega_2. \tag{4.22}$$

Setting

$$\Lambda_j(v) := \int_0^v s_j(\zeta)\, d\zeta \quad \forall v \in \mathbf{R}, \forall j \in \mathbf{N}, \tag{4.23}$$

for suitable constants C_5, C_6, C_7 that depend on Ω_1 and Ω_2 via φ, we have

$$\begin{aligned}
&\int_0^T \langle A\delta_{\hat{\xi},\eta}u, s_j(\delta_{\hat{\xi},\eta}u)\varphi\rangle dt \\
&= \iint_{\Omega_{2,T}} (s_j'(\delta_{\hat{\xi},\eta}u)|\nabla\delta_{\hat{\xi},\eta}u|^2\varphi + s_j(\delta_{\hat{\xi},\eta}u)\nabla\delta_{\hat{\xi},\eta}u\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \iint_{\Omega_{2,T}} \nabla\Lambda_j(\delta_{\hat{\xi},\eta}u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = -\iint_{\Omega_{2,T}} \Lambda_j(\delta_{\hat{\xi},\eta}u)\cdot\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\geq -C_5\iint_{\Omega_{2,T}} |\delta_{\hat{\xi},\eta}u|\,\mathrm{d}x\,\mathrm{d}t \geq -C_6\eta\|u\|_{L^2(0,T;V)} \geq -C_7\eta.
\end{aligned}$$

Applying $\delta_{\hat{\xi},\eta}$ to the Eq. (2.9) and multiplying it by $s_j(\delta_{\hat{\xi},\eta}u)\varphi$, we get

$$\iint_{\Omega_t}\Big(\frac{\partial\delta_{\hat{\xi},\eta}u}{\partial\tau}+\frac{\partial\delta_{\hat{\xi},\eta}w}{\partial\tau}\Big)s_j(\delta_{\hat{\xi},\eta}u)\varphi\,\mathrm{d}x\mathrm{d}\tau \leq \iint_{\Omega_t}|\delta_{\hat{\xi},\eta}f|\varphi\,\mathrm{d}x\mathrm{d}\tau + C_7\eta \qquad \text{for a.a. } t\in\,]0,T[,\ \forall\eta>0.$$

As $s_j(\delta_{\hat{\xi},\eta}u) \to s_0(\delta_{\hat{\xi},\eta}u)$ a.e. in Ω_T as $j\to\infty$, by passing to the limit in this inequality we get

$$\iint_{\Omega_t}\Big(\frac{\partial\delta_{\hat{\xi},\eta}u}{\partial\tau}+\frac{\partial\delta_{\hat{\xi},\eta}w}{\partial\tau}\Big)s_0(\delta_{\hat{\xi},\eta}u)\varphi\,\mathrm{d}x\mathrm{d}\tau \leq \iint_{\Omega_t}|\delta_{\hat{\xi},\eta}f|\varphi\,\mathrm{d}x\mathrm{d}\tau + C_7\eta \qquad \text{for a.a. } t\in\,]0,T[.$$

Note that

$$\frac{\partial\delta_{\hat{\xi},\eta}u}{\partial\tau}s_0(\delta_{\hat{\xi},\eta}u) = \frac{\partial}{\partial\tau}|\delta_{\hat{\xi},\eta}u| \quad \text{a.e. in } \Omega_t.$$

Moreover, as $\partial\delta_{\hat{\xi},\eta}u/\partial\tau \in L^2(\Omega_T)$ we can apply the Hilpert inequality (4.19), getting

$$\frac{\partial\delta_{\hat{\xi},\eta}w}{\partial\tau}s_0(\delta_{\hat{\xi},\eta}u) \geq \frac{\partial}{\partial\tau}|\delta_{\hat{\xi},\eta}w| \quad \text{a.e. in } \Omega_t.$$

We then get

$$\iint_{\Omega_t}\Big[\frac{\partial}{\partial\tau}(|\delta_{\hat{\xi},\eta}u|+|\delta_{\hat{\xi},\eta}w|)\Big]\varphi\,\mathrm{d}x\mathrm{d}\tau \leq \iint_{\Omega_t}|\delta_{\hat{\xi},\eta}f|\varphi\,\mathrm{d}x\mathrm{d}\tau + C_7\eta \qquad \text{for a.a. } t\in\,]0,T[,$$

and this yields (4.20). □

Proposition 4.6 *Assume that (2.2) and (2.3) hold and that $f\in L^1(\Omega_T)$. For any $\varepsilon>0$ let $(u_\varepsilon, w_\varepsilon)$ be a solution of Problem $2.1_{\varepsilon,T}$. Then there exists $(u,w)\in L^1_{\mathrm{loc}}(\Omega_T)^2$ such that, as $\varepsilon\to 0$ along a suitable sequence,*

$$u_\varepsilon \to u, \quad w_\varepsilon \to w \quad \textit{strongly in } L^1_{\mathrm{loc}}(\Omega_T). \tag{4.24}$$

Moreover, (u,w) is a solution of Problem $2.1_{0,T}$.

Outline of the Proof. By (3.2) and (4.20), the classic Fréchet–Riesz–Kolmogorov compactness criterion yields (4.24). The limit procedure can then be performed as for Theorem 3.1. □

Proposition 4.7 *(Local Space Regularity) Let $\varepsilon \geq 0$ and the hypotheses of Theorem 3.1 be fulfilled. Let Ω_1, Ω_2 be bounded subsets of Ω such that $\bar{\Omega}_1 \subset \Omega_2$, and assume that*

$$\nabla u^0, \nabla w^0 \in (C^0(\bar{\Omega}_2)^N)', \quad \nabla f \in L^1(0, T; (C^0(\bar{\Omega}_2)^N)'). \tag{4.25}$$

Then Problem $2.1_{\varepsilon,T}$ has a solution such that

$$\nabla u_\varepsilon, \nabla w_\varepsilon \in L^\infty(0, T; (C^0(\bar{\Omega}_1)^N)'). \tag{4.26}$$

This also applies for $T = +\infty$. Moreover

$$\int_{\Omega_1} (|\nabla u_\varepsilon| + |\nabla w_\varepsilon|)(\cdot, t)\,\mathrm{d}x \leq \int_{\Omega_2} (|\nabla u^0| + |\nabla w^0|)\,\mathrm{d}x + \iint_{\Omega_2 \times]0,T[} |\nabla f|\,\mathrm{d}x\,\mathrm{d}t + C_7. \tag{4.27}$$

Proof For any $\varepsilon > 0$ (4.26) follows from (4.20). Passing to the limit as $\varepsilon \to 0$ we then get the result for $\varepsilon = 0$, too. □

So far we dealt with a relay operator; the above results can be generalized to include *Preisach operators* (namely, linear combinations of a possibly-infinite family of relays).

5 Uniform asymptotic stability and compactness

In this section we deal with the large-time behaviour of the solution(s) of Problem $2.1_{\varepsilon,T}$. We already pointed out that this problem is meaningful also for $T = +\infty$; however here we introduce a slightly different formulation, still for any $\varepsilon \geq 0$.

Problem $5.1_{\varepsilon,\infty}$. Find

$$u \in (L^\infty(0, +\infty; H) \cap L^2(0, +\infty; V)) + V, \quad w \in L^\infty(\Omega_\infty) \tag{5.1}$$

such that for any $T \in]0, +\infty[$ the restriction of (u, w) to $\Omega \times]0, T[$ solves Problem $2.1_{\varepsilon,T}$.

Theorem 5.1 *(Large-Time Existence and Uniform Asymptotic Stability in H) Assume that (2.2) holds, and that*

Ω *is comprised between two parallel hyperplanes of* $\mathbf{R}^N$,
Γ_0 $(\subset \partial\Omega)$ *has positive Hausdorff* $(N-1)$*-dimensional measure,* (5.2)

$$f = f_2 + f_\infty, \quad f_2 \in L^2(0, +\infty; V'), \quad f_\infty \in V'. \tag{5.3}$$

Then for any $\varepsilon \geq 0$ *there exists a solution* (u, w) *of Problem* $5.1_{\varepsilon,\infty}$ *such that, setting* $u_\infty := A^{-1} f_\infty$ $(\in V)$,

$$u \in (L^\infty(0, +\infty; H) \cap L^2(0, +\infty; V)) + u_\infty. \tag{5.4}$$

Moreover, possibly after redefining $u(\cdot, t)$ *on a subset of* $\mathbf{R}^+$ *of vanishing measure,*

$$u(\cdot, t) \to u_\infty \quad \text{strongly in } H\text{, as } t \to +\infty, \tag{5.5}$$

uniformly as u^0 *ranges in any bounded subset of* H.

The latter statement means that

$$\forall R > 0, \forall \delta > 0, \exists \tilde{t} > 0 :$$
$$\|u^0\|_H \leq R \;\Rightarrow\; \|u(\cdot, t) - u_\infty\|_H \leq \delta \quad \text{for a.a. } t > \tilde{t}. \tag{5.6}$$

Any neighbourhood of u_∞ *in* H *is thus absorbing.*

Proof We fix any $T > 0$, for any $m \in \mathbf{N}$ consider the time-discretizated scheme of Problem $2.1_{\varepsilon,\mathbf{T},\mathbf{m}}$ of Sect. 3, but here we let n range through the whole $\mathbf{N}$. Setting $\tilde{u}_m^n := u_m^n - u_\infty$, (3.5) also reads

$$\frac{\tilde{u}_m^n - \tilde{u}_m^{n-1}}{h} + \frac{w_m^n - w_m^{n-1}}{h} + A\tilde{u}_m^n = f_{2m}^n \quad \text{in } V', \forall n \in \mathbf{N}. \tag{5.7}$$

Multiplying this equation by $\tilde{u}_m^n$ we get

$$\int_\Omega \frac{\left(\tilde{u}_m^n\right)^2 - \left(\tilde{u}_m^{n-1}\right)^2}{2h} \mathrm{d}x + \int_\Omega \frac{w_m^n - w_m^{n-1}}{h} \tilde{u}_m^n \,\mathrm{d}x + \int_\Omega \left|\nabla \tilde{u}_m^n\right|^2 \mathrm{d}x$$
$$\leq \frac{1}{4a} \left\| f_{2m}^n \right\|_{V'}^2 + a \left\| \tilde{u}_m^n \right\|_V^2 \quad \forall n \in \mathbf{N}, \forall a > 0. \tag{5.8}$$

By (5.3) we then get a uniform estimate like (3.9) for $\tilde{u}_m^n$, and conclude that there exists a pair (u, w) that solves Problem $2.1_{\varepsilon,T}$ for any $T > 0$, with u as in (5.4).

By (5.2) we can apply the Poincaré inequality. For suitable constants $a, c > 0$, (5.8) yields

$$\int_\Omega \frac{\left(\tilde{u}_m^n\right)^2 - \left(\tilde{u}_m^{n-1}\right)^2}{2h} \mathrm{d}x + \int_\Omega \frac{w_m^n - w_m^{n-1}}{h} \tilde{u}_m^n \,\mathrm{d}x$$
$$+ c \int_\Omega \left(\tilde{u}_m^n\right)^2 \mathrm{d}x \leq \frac{1}{4a} \left\| f_{2m}^n \right\|_{V'}^2, \forall n \in \mathbf{N}. \tag{5.9}$$

Setting $\hat{\rho} := (\rho_2 - \rho_1)/2$, we have

$$\sum_{n=1}^{\infty} \int_\Omega \left(w_m^n - w_m^{n-1}\right) \tilde{u}_m^n \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_\Omega \left(w_m^n - w_m^{n-1}\right)\left(u_m^n - \hat{\rho}\right) \mathrm{d}x$$
$$- \sum_{n=1}^{\infty} \int_\Omega \left(w_m^n - w_m^{n-1}\right)(u_\infty - \hat{\rho}) \,\mathrm{d}x.$$

By (3.4) the first sum of the right member is nonnegative; moreover

$$\begin{aligned}\sum_{n=1}^{\infty}\int_{\Omega}\big(w_m^n-w_m^{n-1}\big)(u_\infty-\hat\rho)\,\mathrm{d}x &\le \int_{\Omega}\bigg|\sum_{n=1}^{\infty}\big(w_m^n-w_m^{n-1}\big)\bigg||u_\infty-\hat\rho|\,\mathrm{d}x\\ &\le 2\int_{\Omega}|u_\infty-\hat\rho|\,\mathrm{d}x<\infty.\end{aligned}$$

Applying Lemma 5.2 below with $y_n := \int_\Omega (u_m^n)^2\,\mathrm{d}x$ for any n, by (5.9) we then infer that

$$\|\tilde u_m(t)\|_H \to 0 \quad \text{as } t\to+\infty, \text{ uniformly w.r.t. } m;$$

possibly after redefining $u(\cdot,t)$ on a subset of $\mathbf{R}^+$ of vanishing measure, (5.6) then follows. □

Lemma 5.2 *Let c, C be positive constants, and $g:\mathbf{R}^+\to\mathbf{R}^+$ be a continuous integrable function. For any $h\in\,]0,1]$ let $y_h := \{y_h^n\}_{n\in\mathbf{N}}$ be a nonnegative real sequence such that*

$$y_h^0\le C,\qquad \frac{y_h^n-y_h^{n-1}}{h}+cy_h^n\le g(nh)\quad \forall n\in\mathbf{N}, \tag{5.10}$$

and set $\tilde y_h(t) := y_n$ for any $t\in](n-1)h,nh]$ and any $n\in\mathbf{N}$. Then

$$\tilde y_h(t)\to 0 \quad \textit{as } t\to+\infty, \textit{ uniformly w.r.t. } h\in]0,1] \textit{ and w.r.t. } y_h^0\le C. \tag{5.11}$$

Proof The homogeneous difference equation $z_n - z_{n-1} + chz_n = 0$ generates the discrete semigroup $z_0 \mapsto z_n = (1+ch)^{-n}z_0$ for any $n\in\mathbf{N}$. Let us set $M_h := h\sum_{j=1}^{\infty} g(jh)$ for any $h>0$, and note that $M_h\to\int_0^{+\infty} g(\tau)\,d\tau$ as $h\to 0$. The discrete formula of variation of constants yields □

$$\begin{aligned} y_n &\le (1+ch)^{-n}y_0 + h\sum_{j=1}^{n}(1+ch)^{j-n}g(jh)\le C+M_h \quad \forall n\in\mathbf{N},\\ y_n &\le (1+ch)^{m-n}y_m + h\sum_{j=m}^{n}(1+ch)^{j-n}g(jh)\\ &\le (1+ch)^{m-n}(C+M_h)+h\sum_{j=m+1}^{n} g(jh)\quad \forall n>m, \forall m\in\mathbf{N}. \end{aligned}\tag{5.12}$$

Without loss of generality, we can assume that $1/h\in\mathbf{N}$; setting $T := mh$ and $t := nh$ we then have

$$\begin{aligned}\tilde y_h(t) = y_{t/h} &\le (1+ch)^{(T-t)/h}(C+M_h)+h\sum_{j=(T/h)+1}^{t/h} g(jh)\\ &\le e^{T-t}(C+M_h)+h\sum_{j=(T/h)+1}^{\infty} g(jh)\quad \forall t\ge T, \forall T\ge 0, \forall h.\end{aligned}$$

For any $\delta > 0$, there exist $\tilde{t}, T > 0$ such that

$$e^{T-t}(C + M_h) \le \delta \quad \forall t > \tilde{t}, \quad h \sum_{j=(T/h)+1}^{\infty} g(jh) \le \delta,$$

whence $\tilde{y}_h(t) \le 2\delta$; (5.11) thus holds. □

Proposition 5.3 *Let (2.2) and (5.3) hold and assume that, setting* $\beta(t) := \min\{t, 1\}$,

$$\sqrt{\beta(t)}\, f_2 \in H^1(0, +\infty; V') \cap L^\infty(0, +\infty; V'). \tag{5.13}$$

For any $\varepsilon \ge 0$ *then there exists a solution* (u, w) *of Problem* $5.1_{\varepsilon,\infty}$ *such that*

$$\sqrt{\beta(t)} \frac{\partial u}{\partial t} \in L^2(0, +\infty; H), \quad \sqrt{\beta(t)}\, u \in L^\infty(0, +\infty; V). \tag{5.14}$$

Moreover the (possibly multi-valued) solution operator $(u^0, f) \mapsto (u, w)$ *has a bounded selection, in the following sense. For any* $M > 0$ *there exists* $N > 0$ *such that, if*

$$\|f_2\|_{L^2(0,+\infty;V')}, \|\sqrt{\beta(t)}\, f_2\|_{H^1(0,+\infty;V') \cap L^\infty(0,+\infty;V')}, \|f_\infty\|_{V'} \le M, \tag{5.15}$$

then there exists a solution of Problem $5.1_{\varepsilon,\infty}$ *such that*

$$\left\| \sqrt{\beta(t)} \frac{\partial u}{\partial t} \right\|_{L^2(0,+\infty;H)} + \|\sqrt{\beta(t)}\, u\|_{L^\infty(0,+\infty;V)} \le N. \tag{5.16}$$

Proof Let us set $\tilde{u}_m^n := u_m^n - u_\infty$ as above. As $\tilde{u}_m^n - \tilde{u}_m^{n-1} = u_m^n - u_m^{n-1}$, by applying the argument of Proposition 4.3 one gets (4.17) with $\tilde{u}_m^n$ in place of u_m^n. The regularity (5.14) and the stated boundedness then follow. □

The next theorem is a simple consequence of (5.5) and of Proposition 5.3, and states the compactness of the transformation that maps the data to the solution(s) of Problem $5.1_{\varepsilon,\infty}$.

Theorem 5.4 *(Asymptotic Compactness) Assume (5.2). For any* $M > 0$ *and any neighbourhood* U *of the origin in* H*, then there exist* $\tilde{t} > 0$ *and a set* $K \subset U \cap V$ *such that the following holds. For any* u_0, w_0, f *that fulfil (2.2), (5.3) and (5.15), there exists a solution of Problem* $5.1_{\varepsilon,\infty}$ *such that*

$$u(t) \in K + u_\infty \quad \text{for a.a. } t \ge \tilde{t}. \tag{5.17}$$

6 Uniqueness of the solution

In this section we prove the uniqueness of the solution of Problem $2.1_{\varepsilon,T}$ for any $\varepsilon \geq 0$, under appropriate retrictions. First we deal with $\varepsilon > 0$; in this case for the equivalent formulation in terms of hysteresis operator this result is already known to hold; here we extend it to the case of unbounded Ω. In the next statement we replace the notation f by g, so that we can write g_1 and g_2 without raising any ambiguity.

Theorem 6.1 *(Dependence on the Data and Uniqueness for $\varepsilon > 0$, after Hilpert [4]) Let $\varepsilon > 0$,*

$$u_i^0 \in V, \quad w_i^0 \in H, \quad g_i \in L^2(0,T;H) + W^{1,1}(0,T;V') \quad (i = 1,2), \tag{6.1}$$

assume that (2.2) is fulfilled and that $g_1 - g_2 \in L^1(\Omega_T)$. For $i = 1, 2$, let (u_i, w_i) be corresponding solutions of Problem $2.1_{\varepsilon,T}$ such that $u_i \in H^1(0,T;H) \cap L^\infty(0,T;V)$ (these exist after Proposition 4.2). Then

$$\int_\Omega (|u_1 - u_2| + |w_1 - w_2|)(x,t)\,\mathrm{d}x \leq \int_\Omega \left(\left|u_1^0 - u_2^0\right| + \left|w_1^0 - w_2^0\right|\right)\mathrm{d}x + \int_0^t d\tau \int_\Omega |g_1 - g_2|(x,\tau)\,\mathrm{d}x \quad \text{for a.a. } t \in\,]0,T[. \tag{6.2}$$

Therefore the solution of Problem $2.1_{\varepsilon,T}$ is unique.

Proof This argument extends to unbounded domains that of [4]; see also [17; Sect. IX.2]. Let $\varphi : \Omega \to [0,1]$ be of class C^2 and such that, for some $0 < r < R$,

$$0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R}^N, \quad \varphi(x) = 1 \quad \text{if } |x| < r, \quad \varphi(x) = 0 \quad \text{if } |x| > R. \tag{6.3}$$

Let us also approximate s_0 as in (4.21), write (2.9) for $i = 1, 2$, take the difference of these equations, multiply it by $s_j(u_1 - u_2)\varphi(x)$, and integrate it in Ω_t for a.a. $t \in\,]0,T[$. By the monotonicity of s_j,

$$\begin{aligned}\langle Au_1 - Au_2, s_j(u_1 - u_2)\varphi\rangle &= \int_\Omega (s_j'(u_1 - u_2)|\nabla u_\varepsilon|^2\varphi \\ &\quad + s_j(u_1 - u_2)\nabla(u_1 - u_2)\cdot\nabla\varphi)\,\mathrm{d}x \\ &\geq -\int_\Omega |\nabla(u_1 - u_2)|\,|\nabla\varphi|\,\mathrm{d}x =: -C_\varphi,\end{aligned} \tag{6.4}$$

the constant C_φ depending on the cut-off function φ. Thus we get

$$\begin{aligned}&\iint_{\Omega_t} \Big(\frac{\partial}{\partial\tau}(u_1 - u_2) + \frac{\partial}{\partial\tau}(w_1 - w_2)\Big) s_j(u_1 - u_2)\,\varphi\,\mathrm{d}x\mathrm{d}\tau \\ &\quad \leq \int\int_{\Omega_t} |g_1 - g_2|\,\mathrm{d}x\mathrm{d}\tau + C_\varphi \quad \text{for a.a. } t \in\,]0,T[.\end{aligned}$$

As $j \to \infty$, $s_j(u_1 - u_2) \to s_0(u_1 - u_2)$ a.e. in Ω_T. By passing to the limit in the latter formula we then get

$$\iint_{\Omega_t} \Big(\frac{\partial}{\partial \tau}(u_1 - u_2) + \frac{\partial}{\partial \tau}(w_1 - w_2)\Big) s_0(u_1 - u_2)\varphi(x)\,\mathrm{d}x\mathrm{d}\tau \\ \le \iint_{\Omega_t} |g_1 - g_2|\,\mathrm{d}x\mathrm{d}\tau + C_\varphi \qquad \forall t \in]0, T]. \tag{6.5}$$

Notice that

$$\left(\frac{\partial}{\partial \tau}(u_1 - u_2)\right) s_0(u_1 - u_2) = \frac{\partial}{\partial \tau}|u_1 - u_2| \quad \text{a.e. in } \Omega_t;$$

moreover, as $\partial(w_1 - w_2)/\partial\tau \in L^2(\Omega_T)$, we can apply the Hilpert inequality (4.19):

$$\left(\frac{\partial}{\partial \tau}(w_1 - w_2)\right) s_0(u_1 - u_2) \ge \frac{\partial}{\partial \tau}|w_1 - w_2| \quad \text{a.e. in } \Omega_t.$$

The inequality (6.5) then yields

$$\iint_{\Omega_t} \frac{\partial}{\partial \tau}(|u_1 - u_2| + |w_1 - w_2|)\varphi(x)\,\mathrm{d}x\mathrm{d}\tau \\ \le \iint_{\Omega_t} |g_1 - g_2|\,\mathrm{d}x\mathrm{d}\tau + C_\varphi \qquad \text{for a.a. } t \in]0, T[.$$

Let us now φ vary along a sequence $\{\varphi_n\}$ such that the corresponding sequences $\{r_n\}$ and $\{R_n\}$ diverge; this entails that $\varphi_n \to 1$ a.e. in Ω_T. A suitable choice of the φ_n's also yields $C_{\varphi_n} \to 0$; the inequality (6.1) then follows. □

Let us now come to the case of $\varepsilon = 0$. Here the lack of time regularity of w prevents us from using the above argument. In order to select a unique solution of Problem $2.1_{0,T}$, we append a further condition in the formulation of the problem, show that any limit of solutions of the above time-dicretized problems fulfils that additional condition, and use it to prove uniqueness. This procedure can be compared with the classic one that Kružkov introduced for quasilinear first-order equations without hysteresis [11, 12].

For *technical reasons* we are able to perform this program only assuming that $\Omega = \mathbf{R}^N$ and that the source term vanishes in the differential equation, i.e. $f \equiv 0$.

Let us denote by $\mathcal{L}$ the *hysteresis region* of the relay, namely, the union of the rectangle $[\rho_1, \rho_2] \times [-1, 1]$ and of the two half-lines $]-\infty, \rho_1[\times\{-1\}$ and $]\rho_2, +\infty[\times\{1\}$. Note that, trivially, $\hat\theta \in k^0_\rho(\theta, \hat\theta)$ for any $(\theta, \hat\theta) \in \mathcal{L}$.

Theorem 6.2 *Assume that (2.2), is fulfilled and that $f \equiv 0$, $\varepsilon \ge 0$. Then there exists a solution (u, w) of Problem $2.1_{\varepsilon,T}$ such that*

$$\iint_{\Omega_T} \left[(|u - \theta| + |w - \hat\theta|)\frac{\partial v}{\partial t} - \nabla|u - \theta|\cdot\nabla v\right]\mathrm{d}x\,\mathrm{d}t \ge 0 \\ \forall(\theta, \hat\theta) \in \mathcal{L}, \forall v \in \mathcal{D}(\mathbf{R}_T),\ v \ge 0. \tag{6.6}$$

Moreover the regularity results of Sect. 4 apply to this solution.

Whenever $u \in L^\infty(\Omega_T)$, taking $\theta = \pm\|u\|_{L^\infty(\Omega_T)}$ one easily sees that (6.6) entails (2.9).

Proof Let us first assume that $(\theta, \hat\theta)$ does not stay on the boundary of the rectangle $[\rho_1, \rho_2]\times[-1, 1]$. Then there exists $\varepsilon > 0$ such that k_ρ^ε maps θ to $\hat\theta$. Once we prove our statement for any pair $(\theta, \hat\theta)$ like this, an obvious approximation procedure will provide it for any $(\theta, \hat\theta) \in \mathcal{L}$.

Still for $\varepsilon > 0$, let us approximate s_0 as in (4.21), consider the solution $(u_\varepsilon, w_\varepsilon)$ of Problem $2.1_{\varepsilon,T}$, and multiply the corresponding Eq. (2.9) by $s_j(u_\varepsilon - \theta)v$ ($\in L^2(0, T; V)$) for any nonnegative $v \in \mathcal{D}(\mathbf{R}_T)$ and any $j \in \mathbf{N}$. Notice that, defining Λ_j as in (4.22),

$$\begin{aligned}\langle Au_\varepsilon, s_j(u_\varepsilon - \theta)v\rangle &= \int_\Omega (s_j'(u_\varepsilon - \theta)|\nabla u_\varepsilon|^2 v + s_j(u_\varepsilon - \theta)\nabla u_\varepsilon\cdot\nabla v)\,\mathrm{d}x\\ &\ge \int_\Omega \nabla\Lambda_j(u_\varepsilon - \theta)\cdot\nabla v\,\mathrm{d}x.\end{aligned}$$

As $\varepsilon > 0$, by the Lipschitz continuity of the regularized hysteresis operator k_ρ^ε we have $u_\varepsilon + w_\varepsilon \in H^1(0, T; H)$; we can then pass to the limit as $j \to \infty$, getting

$$\iint_{\Omega_T}\left[\frac{\partial(u_\varepsilon + w_\varepsilon)}{\partial\tau}s_0(u_\varepsilon - \theta)\ v + \nabla|u_\varepsilon - \theta|\cdot\nabla v\right]\mathrm{d}x\,\mathrm{d}\tau \le 0.$$

Now notice that

$$\frac{\partial u_\varepsilon}{\partial\tau}s_0(u_\varepsilon - \theta) = \frac{\partial}{\partial\tau}|u_\varepsilon - \theta| \quad \text{a.e. in } \Omega_T,$$

and that k_ρ^ε maps u_ε to w_ε and (as we saw) θ to $\hat\theta$. As $\partial w_\varepsilon/\partial\tau \in L^2(\Omega_T)$, by the Hilpert inequality (4.19), we have

$$\frac{\partial w_\varepsilon}{\partial\tau}s_0(u_\varepsilon - \theta) \ge \frac{\partial}{\partial\tau}|w_\varepsilon - \hat\theta| \quad \text{a.e. in } \Omega_T.$$

We thus get

$$\iint_{\Omega_T}\left[\left(\frac{\partial}{\partial\tau}|u_\varepsilon - \theta| + \frac{\partial}{\partial\tau}|w_\varepsilon - \hat\theta|\right)v + \nabla|u_\varepsilon - \theta|\cdot\nabla v\right]\mathrm{d}x\,\mathrm{d}\tau \le 0,$$

whence, integrating by parts in time,

$$\iint_{\Omega_T}\left[(|u_\varepsilon - \theta| + |w_\varepsilon - \hat\theta|)\frac{\partial v}{\partial t} - \nabla|u_\varepsilon - \theta|\cdot\nabla v\right]\mathrm{d}x\,\mathrm{d}t \ge 0.$$

On account of Proposition 4.6, passing to the limit as $\varepsilon \to 0$ we then get (6.6) for any $\varepsilon \ge 0$.

Finally, notice that the regularity results of Sect. 4 apply to $(u_\varepsilon, w_\varepsilon)$ uniformly w.r.t ε. Therefore they also hold for solutions that fulfil (6.6). □

Theorem 6.3 *(Dependence on the Data and Uniqueness for $\varepsilon \geq 0$) Assume that $\Omega = \mathbf{R}^N$, $f \equiv 0$ and $\varepsilon \geq 0$. For $i = 1, 2$, let*

$$u_i^0, w_i^0 \in L^\infty(\mathbf{R}^N), \quad u_i^0 \in V \tag{6.7}$$

fulfil (2.2), and let $(u_i, w_i) \in L^\infty(\mathbf{R}^N)^2$ be a corresponding solution of Problem $2.1_{\varepsilon,T}$ that fulfils (6.6) (this exists after Proposition 4.1 for $p = \infty$). Then

$$u_1 - u_2, w_1 - w_2 \in L^\infty(0, T; L^1(\mathbf{R}^N)), \tag{6.8}$$

$$\int_{\mathbf{R}^N} (|u_1 - u_2| + |w_1 - w_2|)(x, t)\,\mathrm{d}x \leq \int_{\mathbf{R}^N} (|u_1^0 - u_2^0| + |w_1^0 - w_2^0|)\mathrm{d}x \quad \text{for a.a. } t \in]0, T[, \tag{6.9}$$

$$\int_{\mathbf{R}^N} [(u_1 - u_2)^+ + (w_1 - w_2)^+](x, t)\,\mathrm{d}x \leq \int_{\mathbf{R}^N} \big[\big(u_1^0 - u_2^0\big)^+ + \big(w_1^0 - w_2^0\big)^+\big]\mathrm{d}x \quad \text{for a.a. } t \in]0, T[. \tag{6.10}$$

Therefore Problem $2.1_{\varepsilon,T}$ has a unique solution $(u, w) \in L^\infty(\mathbf{R}^N)^2$ that fulfils (6.4).

Proof This argument is based on Hilpert's inequality (4.19) and on Kružkov's technique of *doubling the variables,* cf. [11, 12]. The hypothesis $\Omega = \mathbf{R}^N$ allows us to integrate by parts in space without getting any boundary term. By writing the inequality (6.6) for $(u_1(x, t), w_1(x, t))$ and $(\theta, \hat{\theta}) = (u_2(\xi, \tau), w_2(\xi, \tau))$ for any fixed $(\xi, \tau) \in \mathbf{R}_T^N$ ($:= \mathbf{R}^N \times]0, T[$), we have

$$\iint_{\mathbf{R}_T^N} \bigg[(|u_1(x, t) - u_2(\xi, \tau)| + |w_1(x, t) - w_2(\xi, \tau)|) \frac{\partial v}{\partial t}(x, t) + |u_1(x, t) - u_2(\xi, \tau)| \Delta_x v(x, t) \bigg] \mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall v \in \mathcal{D}(\mathbf{R}_T), v \geq 0; \tag{6.11}$$

on the other hand, by writing (6.6) for $(u_2(\xi, \tau), w_2(\xi, \tau))$ and $(\theta, \hat{\theta}) = (u_1(x, t), w_1(x, t))$ for any fixed $(x, t) \in \mathbf{R}_T^N$, we also get

$$\iint_{\mathbf{R}_T^N} \Big[\big(|u_2(\xi, \tau) - u_1(x, t)| + |w_2(\xi, \tau) - w_1(x, t)|\big) \frac{\partial v}{\partial \tau}(\xi, \tau) + |u_2(\xi, \tau) - u_1(x, t)| \Delta_\xi v(\xi, \tau) \Big] \mathrm{d}\xi\,\mathrm{d}\tau \geq 0 \quad \forall v \in \mathcal{D}(\mathbf{R}_T), v \geq 0. \tag{6.12}$$

In both of these inequalities let us now take any nonnegative $v = v(x, t, \xi, \tau) \in \mathcal{D}(\mathbf{R}_T^N)$. Integrating (6.11) ((6.12), resp.) w.r.t. (ξ, τ) (w.r.t. (x, t), resp.), and summing them, we get

$$\iiiint_{(\mathbf{R}_T^N)^2} \bigg[(|u_1(x, t) - u_2(\xi, \tau)| + |w_1(x, t) - w_2(\xi, \tau)|) \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial \tau} \right) + |u_1(x, t) - u_2(\xi, \tau)| \cdot (\Delta_x v + \Delta_\xi v) \bigg] \mathrm{d}x\mathrm{d}t\mathrm{d}\xi\mathrm{d}\tau \geq 0. \tag{6.13}$$

Let us fix any nonnegative $\lambda \in \mathcal{D}(\mathbf{R}^N)$ and $\mu \in \mathcal{D}(0,T)$, and two mollifiers $\psi_1 \in C^\infty(\mathbf{R})$ and $\psi_N \in C^\infty(\mathbf{R}^N)$, e.g.,

$$\psi_1(t) := \pi^{-1/2}\, e^{-|t|^2} \quad \forall t \in \mathbf{R}, \quad \psi_N(x) := \pi^{-N/2}\, e^{-|x|^2} \quad \forall x \in \mathbf{R}^N. \tag{6.14}$$

For any $\eta > 0$, let us then take

$$v = v_\eta(x,t,\xi,\tau) := \frac{1}{\eta^{N+1}}\, \psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\psi_1\Big(\frac{t-\tau}{\eta}\Big)\lambda\Big(\frac{x+\xi}{2}\Big)\mu\Big(\frac{t+\tau}{2}\Big)$$

in (6.13), and pass to the limit as $\eta \to 0$. Denoting by δ_0 the Dirac measure, we have

$$\begin{aligned}\frac{\partial v_\eta}{\partial t} + \frac{\partial v_\eta}{\partial \tau} &= \frac{1}{\eta^{N+1}}\, \psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\psi_1\Big(\frac{t-\tau}{\eta}\Big)\lambda\Big(\frac{x+\xi}{2}\Big)\mu'\Big(\frac{t+\tau}{2}\Big)\\ &\to \delta_0(x-\xi)\delta_0(t-\tau)\lambda(x)\mu'(t) \quad \text{in } \mathcal{D}'\big(\mathbf{R}^N_T\big), \text{ as } \eta \to 0.\end{aligned}$$

Notice that

$$\begin{aligned}\Delta_x\Big[\psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\lambda\Big(\frac{x+\xi}{2}\Big)\Big] &= 2\nabla_x\psi_1\Big(\frac{|x-\xi|}{\eta}\Big)\cdot\nabla_x\lambda\Big(\frac{x+\xi}{2}\Big)\\ &+\lambda\Big(\frac{x+\xi}{2}\Big)\Delta_x\psi_N\Big(\frac{|x-\xi|}{\eta}\Big) + \psi_1\Big(\frac{|x-\xi|}{\eta}\Big)\Delta_x\lambda\Big(\frac{x+\xi}{2}\Big)\\ &= -2\nabla_\xi\psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\cdot\nabla_\xi\lambda\Big(\frac{x+\xi}{2}\Big)\\ &+\lambda\Big(\frac{x+\xi}{2}\Big)\Delta_\xi\psi_n\Big(\frac{|x-\xi|}{\eta}\Big) + \psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\Delta_\xi\lambda\Big(\frac{x+\xi}{2}\Big);\end{aligned}$$

by adding this formula with the analogous one for $\Delta_\xi[\psi_N((|x-\xi|)/\eta)$ $\lambda((x+\xi)/2)]$, we obtain

$$\begin{aligned}(\Delta_x + \Delta_\xi)&\Big[\psi_N\Big(\tfrac{|x-\xi|}{\eta}\Big)\lambda\Big(\tfrac{x+\xi}{2}\Big)\Big]\\ &= 2\lambda\Big(\tfrac{x+\xi}{2}\Big)\Delta_x\psi_N\Big(\tfrac{|x-\xi|}{\eta}\Big) + 2\psi_N\Big(\tfrac{|x-\xi|}{\eta}\Big)\Delta_x\lambda\Big(\tfrac{x+\xi}{2}\Big).\end{aligned}$$

Moreover, as $\Delta\psi_N(x) = (-2N + 4|x|^2)\,\pi^{-N/2}\, e^{-|x|^2}$,

$$\frac{1}{\eta^N}\Delta_x\psi_N\Big(\frac{|x-\xi|}{\eta}\Big) \to -2N\delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}^N), \text{ as } \eta \to 0.$$

Therefore, as $\lambda \geq 0$, there exists $\sigma \in \mathcal{D}'(\mathbf{R}^N_T)$ such that $\sigma \leq 0$ and

$$\begin{aligned}\frac{1}{\eta^N}(\Delta_x + \Delta_\xi)&\Big[\psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\lambda\Big(\frac{x+\xi}{2}\Big)\Big]\\ &\to \sigma + \frac{1}{2}\delta_0(x-\xi)\Delta\lambda(x) \quad \text{in } \mathcal{D}'(\mathbf{R}^N), \text{ as } \eta \to 0,\end{aligned}$$

$$\begin{aligned}\frac{1}{\eta^{N+1}}(\Delta_x + \Delta_\xi)&\Big[\psi_N\Big(\frac{|x-\xi|}{\eta}\Big)\psi_1\Big(\frac{t-\tau}{\eta}\Big)\lambda\Big(\frac{x+\xi}{2}\Big)\mu\Big(\frac{t+\tau}{2}\Big)\Big]\\ &\to \Big[\sigma + \frac{1}{2}\delta_0(x-\xi)\Delta\lambda(x)\Big]\delta_0(t-\tau)\mu(t) \quad \text{in } \mathcal{D}'(\mathbf{R}^N_T), \text{ as } \eta \to 0.\end{aligned}$$

By (6.13) and by Lemma 6.4 below we then infer that

$$\iint_{\mathbf{R}^N_T} (|u_1(x,t) - u_2(x,t)| + |w_1(x,t) - w_2(x,t)|)\lambda(x)\mu'(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint_{\mathbf{R}^N_T} |u_1(x,t) - u_2(x,t)|\ \Delta\lambda(x)\mu(t)\,\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{6.15}$$

For any fixed $t\ \in]0,T]$, let μ converge to the indicator function of the interval $[0,t]$. Moreover let $\lambda = \varphi_{r,R}$ be as in (6.3), and let $r, R \to +\infty$ in such a way that $M_{r,R} := \max_{\mathbf{R}^N} |\Delta\lambda_{r,R}| \to 0$. This yields (6.9), whence (6.8). Finally, let us multiply the Eq. (2.9) by $\varphi_{r,R}$, integrate in x, t, and pass to the limit as $r, R \to +\infty$ as we just indicated. This yields

$$\int_{\mathbf{R}^N} (u_1 - u_2 + w_1 - w_2)(x,t)\,\mathrm{d}x = \int_{\mathbf{R}^N} (u_1^0 - u_2^0 + w_1^0 - w_2^0)\mathrm{d}x \qquad \text{for a.a. } t\ \in]0,T[,$$

and by adding this equality to (6.9) we get (6.10). □

Lemma 6.4 *Let $\varphi \in L^\infty(]0,T[^2)$ and ψ_1 be as in (6.14). Then*

$$\iint_{]0,T[^2} \frac{1}{\eta}\,\psi_1\left(\frac{t-\tau}{\eta}\right)\varphi(t,\tau)\,dt d\tau \to \int_{]0,T[} \varphi(t,t)\,dt \tag{6.16}$$

Proof For any $\varepsilon > 0$ there exists a function $\varphi_\varepsilon \in L^1_t(0,T; C^0_\tau([0,T]))$ such that (denoting the one-dimensional Lebesgue measure by μ)

$$\|\varphi_\varepsilon(t,\cdot)\|_{C^0([0,T])} \leq \|\varphi(t,\cdot)\|_{L^\infty(]0,T[^2)} =: C$$
$$\mu(\{\tau \in]0,T[: \varphi(t,\tau) \neq \varphi_\varepsilon(t,\tau)|\}) \leq \varepsilon \qquad \text{for a.a. } t \in]0,T[.$$

Therefore

$$\left|\iint_{]0,T[^2} \frac{1}{\eta}\,\psi_1\left(\frac{t-\tau}{\eta}\right)\varphi(t,\tau)\,dt d\tau - \iint_{]0,T[^2} \frac{1}{\eta}\,\psi_1\left(\frac{t-\tau}{\eta}\right)\varphi_\varepsilon(t,\tau)\,dt d\tau\right| \leq C\varepsilon.$$

As

$$\iint_{]0,T[^2} \frac{1}{\eta}\,\psi_1\left(\frac{t-\tau}{\eta}\right)\varphi_\varepsilon(t,\tau)\,dt d\tau \to \int_{]0,T[} \varphi(t,t)\,dt,$$

we then get (6.16). □

7 Periodic problem

In this section we associate a time-periodic problem to the Eq. (1) for $\mathcal{F} = k_\rho$, and prove existence of a solution via approximation by regularization of the relay operator. In this argument we take advantage of the fact that a solution is known to exist for the corresponding regularized problem, in which k_ρ is replaced by k_ρ^ε, cf. [5]. We assume that

$$f = f_1 + f_2, \quad f_1 \in L^1(0,T;H), \quad f_2 \in L^2(0,T;V'). \tag{7.1}$$

Problem 7.1. Find $u \in C^0\big([0,T]; H\big) \cap L^2(0,T;V)$ and $w \in L^\infty(\Omega_T)$ such that $u + w \in H^1(0,T;V')$ and

$$|w| \le 1 \quad \text{a.e. in } \Omega_T, \quad \frac{\partial w}{\partial t} \in C^0(\overline{\Omega_T})', \tag{7.2}$$

$$\frac{\partial}{\partial t}(u+w) + Au = f \quad \text{in } V', \text{ for a.a. } t \in]0,T[, \tag{7.3}$$

$$\begin{cases} (w-1)(u-\rho_2) \ge 0 \\ (w+1)(u-\rho_1) \ge 0 \end{cases} \quad \text{a.e. in } \Omega_T, \tag{7.4}$$

$$\begin{aligned}\frac{1}{2}\int_\Omega [u(x,t)^2 - u(x,0)^2]\mathrm{d}x + \int_\Omega \Psi_0(w(x,\cdot);[0,t])\,\mathrm{d}x + \iint_{\Omega_t} |\nabla u|^2\,\mathrm{d}x\mathrm{d}\tau \\ \le \iint_{\Omega_t} f_1 u\,\mathrm{d}x\mathrm{d}\tau + \int_0^t \langle f_2, u\rangle\, d\tau \quad \forall t \in]0,T[,\end{aligned} \tag{7.5}$$

$$u(\cdot,0) = u(\cdot,T), \quad \text{a.e. in } \Omega, \quad (u+w)(0) = (u+w)(T) \quad \text{in } V'. \tag{7.6}$$

The interpretation of this problem is analogous to that of Problem $2.1_{\varepsilon,T}$.

Theorem 7.1 *Let (5.2) be fulfilled, and*

$$f = f_1 + f_2, \quad f_1 \in L^2(0,T;H), \quad f_2 \in H^1(0,T;V'), \quad f_2(0) = f_2(T), \tag{7.7}$$

$$\exists f_*, f^* \in V': \quad f_* \le f_2 \le f^* \quad \mathit{in}\ \mathcal{D}'(\Omega), \forall t \in [0,T]. \tag{7.8}$$

Then Problem 7.1 has a solution such that

$$u \in H^1(0,T;H) \cap L^2(0,T;V), \tag{7.9}$$

$$A^{-1} f_* \le u \le A^{-1} f^* \quad \mathit{a.e.\ in}\ \Omega_T. \tag{7.10}$$

Proof For any $\varepsilon > 0$, let us define the regularized relay operator k^ε_ρ as in Sect. 1, cf. (1.11), and consider the following corresponding regularized time-periodic problem.

Problem 7.1$_\varepsilon$. Find $u_\varepsilon, w_\varepsilon : \Omega \to C^0([0,T])$ measurable such that

$$u_\varepsilon \in L^2(0,T;V), \quad u_\varepsilon + w_\varepsilon \in H^1(0,T;V'),$$

$$w_\varepsilon(x,t) = \big[k^\varepsilon_\rho(u_\varepsilon(x,\cdot), w_\varepsilon(x,0))\big](t) \quad \forall t \in [0,T], \text{ for a.a. } x \in \Omega, \tag{7.11}$$

$$\frac{\partial}{\partial t}(u_\varepsilon + w_\varepsilon) + Au_\varepsilon = f \quad \text{in } V', \text{ for a.a. } t \in]0,T[, \tag{7.12}$$

$$u_\varepsilon(\cdot,0) = u_\varepsilon(\cdot,T) \quad \text{a.e. in } \Omega, \quad (u_\varepsilon + w_\varepsilon)(0) = (u_\varepsilon + w_\varepsilon)(T) \quad \text{in } V'. \tag{7.13}$$

After the results of [5] (see also [20]), by the Lipschitz-continuity of the hysteresis operator $k^\varepsilon_\rho(\cdot,\xi)$, for any $\varepsilon > 0$ this problem has a solution $(u_\varepsilon, w_\varepsilon)$ such that

$$u_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad w_\varepsilon \in H^1(0,T;H). \tag{7.14}$$

More precisely, the set of the solutions of Problem 7.1_ε has minimal and maximal elements w.r.t. the pointwise ordering, and for any solution

$$A^{-1} f_* \le u_\varepsilon \le A^{-1} f^* \quad \text{a.e. in } \Omega_T, \forall \varepsilon > 0.$$

Let us now multiply (7.12) by u_ε and integrate in $[0, T]$. By (3.7) and by the time-periodicity of u_ε, we have

$$\begin{aligned}\int_\Omega \Psi_\varepsilon(w_\varepsilon; [0, T])\, dx + \iint_{\Omega_T} |\nabla u_\varepsilon|^2\, dx\, dt \le \int_0^T \langle f, u_\varepsilon\rangle\, dt \\ \le \|f_1\|_{L^2(0,T;H)} \|u_\varepsilon\|_{L^2(0,T;H)} + \|f_2\|_{L^2(0,T;V')} \|u_\varepsilon\|_{L^2(0,T;V)}.\end{aligned}$$

Let us now multiply (7.12) $\partial u_\varepsilon/\partial t$ and integrate in $[0, T]$. As $\frac{\partial w_\varepsilon}{\partial t} \frac{\partial u_\varepsilon}{\partial t} \ge 0$ a.e. in Ω_T and again by the time-periodicity of u_ε, we get

$$\begin{aligned}\iint_{\Omega_T} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\, dt \le \int_0^T \left\langle f, \frac{\partial u_\varepsilon}{\partial t}\right\rangle dt[3pt] \\ \le \|f_1\|_{L^2(0,T;H)} \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2(0,T;H)} + \left\|\frac{\partial f_2}{\partial t}\right\|_{L^2(0,T;V')} \|u_\varepsilon\|_{L^2(0,T;V)}.\end{aligned}$$

By the Poincaré inequality, the two latter inequalities entail that

$$\|u_\varepsilon\|_{H^1(0,T;H)\cap L^2(0,T;V)}, \left\|\frac{\partial w_\varepsilon}{\partial t}\right\|_{C^0(\overline{\Omega_T})'} \le \text{ Constant (independent of } \varepsilon).$$

Therefore there exists u, w such that, up to subsequences,

$$\begin{aligned} &u_\varepsilon \to u && \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \\ &w_\varepsilon \to w && \text{weakly star in } L^\infty(\Omega_T) \\ &\frac{\partial w_\varepsilon}{\partial t} \to \frac{\partial w}{\partial t} && \text{weakly star in } C^0(\overline{\Omega_T})'. \end{aligned} \tag{7.15}$$

Passing to the limit in (7.12) and (7.13) we then get (7.3) and (7.6). Finally, (7.4) and (7.5) can be derived as in the proof of Theorem 7.2. □

8 Conclusions

We dealt with an initial- and boundary-value problem for a quasilinear parabolic equation that contains a possibly multi-valued hysteresis operator, $\mathcal{F}$:

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] - \Delta u = f. \tag{8.1}$$

We considered two basic examples of $\mathcal{F}$: the discontinuous relay operator k^0_ρ and the corresponding (continuous) regularized operator k^ε_ρ; we represented these operators in a way that allowed us to provide a unified formulation of both settings. We proved existence of a solution via time-discretization, derivation a priori estimates and passage to the limit. A key role was here played by the weak formulation of the relay operator that we outlined in Sect. 2; this allowed us to deal with an especially weak notion of solution, possibly nondifferentiable w.r.t. time.

We then proved regularity results, and derived uniform-in-time estimates, that allowed us to study the large-time behaviour of the solution; in particular we proved asymptotic stability and compactness of the trasformation that maps the data to the solution(s). We also showed existence of a periodic solution, in presence of a periodic forcing term. As the Eq. (7.1) arises as a model of univariate ferromagnetism and of other physical phenomena, information on the large-time behaviour of the solution and on the time-periodic problem may be of some interest.

Uniqueness of the solution was one of the main issues. After [17; Chap. VIII] it is known that a semigroup formulation of our problem is well-posed for any $\varepsilon \geq 0$. That is based on the notion of *integral solution* in the sense of Bénilan [1]; this solution is rather weak, and thus we were interested into a statement that directly refers to Problem $2.1_{\varepsilon,T}$. For the case of continuous hysteresis (i.e., $\varepsilon > 0$), it was already known that the solution is unique, because of the Hilpert inequality (4.19), cf. [4]. On the other hand for Problem $2.1_{0,T}$ uniqueness of the solution does not seem trivial, as the lack of regularity of the solution prevents one from applying the procedure we used for $\varepsilon > 0$. Indeed the Hilpert inequality is just known to hold if w is absolutely continuous w.r.t. time, and in general this fails for discontinuous hysteresis operators.

We showed that any limit of solutions of the above time-dicretized problems fulfils an extra condition, and used it to prove uniqueness of the solution. This procedure is reminiscent of the use of an entropy condition to select a unique solution of quasilinear first-order equations; actually these developments have been expired by the classic results of Kružkov [11, 12]. The analogy between our second-order equation and first-order equations goes further: it is known that Kružkov's entropic solution coincides with the mild semigroup solution, cf. [3]; the same applies to the solution of Problem $2.1_{\varepsilon,T}$ ($\varepsilon \geq 0$) fulfilling the entropy-like condition. In a work apart [22] this procedure is applied to quasilinear first-order equations with hysteresis.

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