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On uniqueness and monotonicity of solutions of non-local reaction diffusion equation

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Abstract This article deals with the uniqueness and the behavior of solutions of non-local reaction diffusion equations. Since these equations share many properties with the usual reaction diffusion model, such as a form of maximum principle and the translation invariance, uniqueness and monotone behavior for the solution, as in the usual case, are expected. I present an elementary proof of this monotone behavior. The proof essentially uses techniques based on the maximum principle and the sliding method.

Keywords Maximum principle · Sliding method · Non-local reaction diffusion equation

1 Introduction and main result

In this note, I investigate monotonicity and uniqueness of positive solutions of the following integrodifferential problem

$$J \star u - u - cu' + f(u) = 0 \text{ in } \mathbb{R}, \tag{1.1}$$

$$u(x) \to 0 \text{ as } x \to -\infty, \tag{1.2}$$

$$u(x) \to 1 \text{ as } x \to +\infty, \tag{1.3}$$

where J is an even non-negative continuous function on $\mathbb{R}$ with $\int_{\mathbb{R}} J(z)dz = 1$, c is a real constant, $J \star u(x) = \int_{\mathbb{R}} J(x-y)u(y)\mathrm{d}y$ is the standard convolution and $f : \mathbb{R} \to \mathbb{R}$ is appropriately smooth, with $f(0) = f(1) = 0$.

Remark 1.1 The previous problem is invariant under translation, which means that for any real τ, $u_\tau := u(. + \tau)$ is still a solution of (1.1)–(1.3).

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Such a problem arises in the study of so-called *Travelling Fronts* (solutions of the form $u(x,t)=\phi(x+ct)$) of the following non-local phase-transition problem

$$\frac{\partial u}{\partial t}-(J\star u-u)=f(u)\ \text{ in }\ \mathbb{R}\times\mathbb{R}^+. \tag{1.4}$$

The constant c is called the speed of the front and is usually unknown. The operator $Lu=\int_{\mathbb{R}} J\star u-u$ can be viewed as a diffusion operator. This kind of equation was originally introduced in 1937 by Kolmogorov, Petrovskii and Piskunov [20] as a way to derive the Fisher equation (i.e 1.5 below with $f(s)=s(1-s)$)

$$\frac{\partial U}{\partial t}=U_{xx}+f(U)\quad \text{for}\quad (x,t)\ \in\mathbb{R}\times\mathbb{R}^+. \tag{1.5}$$

In the literature, much attention has been drawn to reaction–diffusion equations like (1.5), as they have proved to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, nerve propagation or epidemiology. We point the interested reader to the following articles for more informations: [5, 6, 16, 18–21, 24].

For nonlinearities f satisfying $f\in C^1(\mathbb{R})$, $f(0)=f(1)=0$ and for some $\epsilon>0$, $f'(s)\le 0$ when $s<\epsilon$ and when $1-\epsilon<s$, monotonicity and uniqueness of travelling-front solutions of the reaction–diffusion Eq. (1.5) is well-known, see [5,7–9,17,22]. By uniqueness of travelling wave solution, we mean that if (u,c) and (v,c') are travelling-wave solutions of (1.5) then $c=c'$ and $u(x)=v(x+\tau)$ for some real τ. Observe that the Fisher nonlinearity ($f(s)=s(1-s)$) does not satisfy these assumptions. For this kind of nonlinearity, it is known that several travelling-wave solutions exist, see [3, 9, 20]. However, in that case, using a precise exponential asymptotic expansion of the solutions in a neighborhood $(-\infty,M)$ of $-\infty$, Berestycki and Nirenberg obtained in [9] the monotonicity and uniqueness up to translation of the travelling-wave solutions of (1.5) i.e. If (u,c) and (v,c) are travelling-wave solutions of (1.5) then $u(x)=v(x+\tau)$ for some real τ.

For the non-local Eqs. (1.1)–(1.3), existence, uniqueness and monotonicity were first obtained by Bates, Fife, Ren and Wang [4] and later by Chen [11] for a bistable nonlinearity f, i.e. a nonlinearity $f\in C^1(\mathbb{R})$ satisfying for some $\rho>0$, $f|_{(0,\rho)}<0$, $f|_{(\rho,1)}>0$, $f(0)=f(1)=0$, $f'(0)<0$ and $f'(1)<0$. In that case, they showed the following

Theorem 1.1 **[4]** *Assume that $J\in C^1(\mathbb{R})$ is a positive, even, integrable function with unit mass. Let (u,c) and (v,c') be solutions of (1.1)–(1.3) with bistable nonlinearity and assume that (u,c) is monotone increasing, then $c=c'$. Moreover, if either u or v is continuous or if v is monotone then $v(x)=u(x+\tau)$ for some $\tau\in\mathbb{R}$.*

Note that Theorem 1.1 contains two distinct uniqueness results. Indeed, it states that the speed is unique and the profile u is unique. It also shows how rigid Problem (1.1)–(1.3) is, since it has a positive solution only for one real value of c.

Our first result is a generalization of the uniqueness result for the continuous profile contained in Theorem 1.1 to more general nonlinearities,

Theorem 1.2 *Assume that $J \in C^0(\mathbb{R})$ is a positive, even, integrable function with unit mass. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ be such that f satisfies for some $\epsilon > 0$, $f'(s) \leq 0$ when $s < \epsilon$ and when $1 - \epsilon < s$. Let (u, c) and (v, c) be two continuous solutions of (1.1)–(1.3) then $u(.) = v(.+\tau)$ for some real τ. Moreover, the solution u is monotone increasing.*

Note that the assumption on f in the previous theorem covers the case of bistable nonlinearities and that in the case of continuous solutions, the existence of a monotone solution u is not needed anymore.

Also observe that our theorem does not cover the case of discontinuous solutions of (1.1)–(1.3) which appears when the speed $c = 0$. However when $c = 0$, there is an example of existence of several discontinuous positive solutions of (1.1)–(1.3). A generalization of Theorem 1.2 for monotone discontinuous solutions is currently under investigation.

Our second result concerns the uniqueness of the speed c for continuous solutions when they exist. Namely, we have

Theorem 1.3 *Assume that $J \in C^0(\mathbb{R})$ is a positive even integrable function with unit integral. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ be such that for some $\epsilon > 0$, $f'(s) \leq 0$ when $s < \epsilon$ and when $1 - \epsilon < s$. Let (u, c) and (v, c') be two continuous positive solutions of (1.1)–(1.3), then $c = c'$.*

As we previously mentioned, the assumptions made on f in Theorem 1.2, do not cover the case of the Fisher nonlinearity. Our next result deals with the monotonicity of solutions in that case. With some extra assumption on the behavior of the solution in some neighborhood $(-\infty, M)$ of $-\infty$, we show that the solutions are monotone increasing. More precisely we have

Theorem 1.4 *Assume that $J \in C^0(\mathbb{R})$ is a positive even integrable function with unit mass. Let $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ be such that for some $\epsilon > 0$, $f'(s) \leq 0$ when $1 - \epsilon < s$. Let (u, c) be a positive continuous solution of (1.1)–(1.3), such that u is monotone increasing in some neighborhood $(-\infty, M)$ of $-\infty$ then u is monotone increasing in all of $\mathbb{R}$.*

Assuming that the solutions u are monotone increasing in some neighborhood $(-\infty, M)$ of $-\infty$ may seem strange, however such behavior holds true for travelling-front solutions of the reaction–diffusion Eq. (1.5). Indeed, when f is monostable, ($f \in C^1(\mathbb{R})$, such that $f(0) = f(1) = 0$, $f|_{(0,1)} > 0$) with $f'(0) > 0$, the positive solutions (u, c) of (1.5) satisfies the following expansion near $-\infty$,

$$u(x) = Ce^{\lambda_0 x} + o(e^{\lambda_0 x})$$
$$u'(x) = \lambda_0 Ce^{\lambda_0 x} + o(e^{\lambda_0 x}),$$

where C is a positive constant and λ_0 is one of the positive roots of $\lambda^2 - c\lambda + f'(0)$. u' is therefore strictly positive in a neighborhood $(-\infty, M)$ of $-\infty$, which is our needed assumption. For details on the proof of this expansion, see [1, 9]. For Eqs. (1.1)–(1.3), it seems that such exponential expansion of the solution no longer stands in general. The assumption on the monotone behavior of the solution in Theorem 1.4 fills the lack of such exponential expansion.

1.1 General remarks and comments

For the uniqueness of the speed c (Theorem 1.3), we originally required that the solutions are continuous. It appears that the proof of this result can easily be adapted to solutions u with a finite number of discontinuities. This is briefly discussed in the end of Sect. 3.

In the case of monostable nonlinearities, the uniqueness of the speed c no longer holds, see [3, 9, 13]. However, the uniqueness up to translation of the travelling-fronts of (1.5) still holds. We expect to have similar results for positive solution of (1.1)–(1.3), but we were not able to prove it.

Theorems 1.2–1.4 stand for more general linear operators than $Lu := J \star u - u - cu'$. Namely, our proofs hold for operators of the form

$$Lu := \alpha u'' + \beta \int_{\mathbb{R}} J(x-y)(u(y)-u(x))\,\mathrm{d}y - cu' - \mathrm{d}u, \tag{1.6}$$

where α, β and d are non-negative real numbers such that $\alpha + \beta > 0$ and J a positive continuous integrable kernel such that $[-b, -a] \cup [a, b] \subset \mathrm{supp}(J)$ for some $0 \le a < b$. Observe that when $\alpha \neq 0$, even in the case of stationary travelling fronts (i.e. $c = 0$), there is no need to consider discontinuous travelling fronts since the local elliptic regularity implies that solutions are smooth. Note also that the kernel J does not need to be an even function.

In our analysis for linear operators L satisfying Lemma 1.1 below, we have also observed that some assumptions can be weakened, in particular the translation invariance. We summarize below the required condition on L

(H1) For all positive functions U, let $U_h(.) := U(. + h)$. Then for all $h > 0$ we have $L[U_h](x) \le L[U](x+h)\ \ \forall x \in \mathbb{R}$.
(H2) Let v a positive constant then we have $L[v] \le 0$.

Operators satisfying these two conditions are easily constructed. For example, let J be a positive, even, continuous, integrable kernel of mass one, then the operator $Lu := \int_{-r}^{+\infty} J(x-y)u(y)\mathrm{d}y - u$ where $r > 0$, is *not* translation-invariant but satisfies (H1) and (H2).

Most of the results that we obtain can be generalized to multidimensional situations. For example, Theorems 1.2–1.4 can be generalized to the following problem

$$\epsilon \Delta u + \beta \int_{\Sigma} J(x-t, y-s)(u(t,s) - u(x,y))\,\mathrm{d}t\mathrm{d}s + \gamma(y)u_x + f(u) = 0 \text{ on } \Sigma \tag{1.7}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ at } \partial\Sigma \tag{1.8}$$

$$u(x,y) \to 0 \text{ uniformly in } y \text{ as } x \to -\infty \tag{1.9}$$

$$u(x,y) \to 1 \text{ uniformly in } y \text{ as } x \to +\infty, \tag{1.10}$$

where $\Sigma := \mathbb{R} \times \Omega$ is a cylinder.

Here ϵ and β are non-negative constants, $\Omega \subset \mathbb{R}^{n-1}$, $(n \ge 2)$ is a bounded domain with a $C^{2,\alpha}$ boundary for some $\alpha > 0$ if $n > 2$, ν is the outward normal to the boundary of $\mathbb{R} \times \Omega$, $\gamma(y) : \Omega \to \mathbb{R}$ is a smooth function and the spatial coordinates are denoted by (x, y) where $x = x_1$ and $y = (x_2, \dots, x_n)$. $J(x, y)$ is a positive, continuous, integrable kernel on Σ such that the support of J contains a set of the form $([-b, -a] \cup [a, b]) \times \omega$ for some $0 \le a < b$, where ω is an open subset of Ω containing 0.

When $\beta = 0$, such kind of equations arise in combustion theory to describe the propagation of flames in a tube. The term $\gamma(y)$ is usually composed of two terms $\gamma(y) = c + \gamma_1(y)$, where c is the unknown speed of the flame and $\gamma_1(y)$ is the given drifting flow. The operator $Lu := \beta \int_\Sigma J(x-t, y-s)(u(t, s) - u(x, y))\mathrm{d}t\mathrm{d}s$ is a natural multidimensional generalization of the one dimensional diffusion operator $J \star u - u$.

1.2 Method and plan

To prove Theorems 1.2–1.4, we use a sliding technique introduced by Berestycki–Nirenberg in [9], combined with some ideas of Alikakos–Bates–Chen [2] (see also Chen [11, 17]) and Vega [22]. We also use extensively a strong maximum principle that holds for the operator $Lu := J \star u - u - cu'$:

Theorem 1.5 *Maximum Principle*
Let u be a smooth (C^1) function on $\mathbb{R}$, such that

$$L[u](x) \ge 0 \;\; (resp.\;\; L[u](x) \le 0) \;\; \text{in} \;\; \mathbb{R}.$$

Then u cannot achieve a global maximum (resp. a global minimum) without being constant.

and some property attached to our operator L:

Lemma 1.1 *Let u be a smooth (C^1) function. If u achieves a global minimum (resp. a global maximum) at some point ξ then the following holds:*

- *Either* $L[u](\xi) > 0 \;\; (resp.\;\; L[u](\xi) < 0)$
- *Or* $L[u](\xi) = 0$ *and u is identically constant.*

Remark 1.2 The assumption on the regularity of u in Theorem 1.5 and Lemma 1.1 have to be adjusted to the regularity requirements of L. More precisely, if $Lu := J \star u - u$, C^1 regularity is not really needed. Indeed in that case analog of Lemma 1.1 and the Maximum Principle stand for $u \in L^\infty$, continuous by parts and with a finite number of discontinuities. Observe that operators of the form $Lu := \alpha u'' + \beta \int_\mathbb{R} J(x-y)\big(u(y) - u(x)\big)\mathrm{d}y - cu' - du$ also satisfy Theorem 1.5 and Lemma 1.1 when $\beta \ne 0$ under the additional assumption $u \in C^2$.

Remark 1.3 The maximum principle for $Lu = J \star u - u$ needs that u achieves a global extrema on $\mathbb{R}$. It will be untrue if we only assume that u achieves a local extrema. For the Laplacian operator the maximum principle holds even if we assume that u achieves a local extrema. This difference is easily explained by the global/local nature of our operator and the Laplacian operator. This also implies that local analysis will fail for our operator.

Remark 1.4 For general multidimensional operators $Lu := \int_\Sigma J(x-t, y, s)$ $(u(t, s) - u(x, y))\mathrm{d}t\mathrm{d}s$ the strong maximum principle stated as in Theorem 1.5 no longer holds. However, our proofs will still hold only by assuming that at a global minimum (x_0, y_0) (resp. a global maximum), we have

- Either $L[u](x_0, y_0) > 0$ $(resp.\ L[u](x_0, y_0) < 0)$
- Or $L[u](x_0, y_0) = 0$ and $u(x, y) = u(x_0, y_0)$ on $\mathbb{R} \times \{y\}$ for some $y \in \bar{\Omega}$.

This conditions enables to consider a much greater variety of kernels.

Details of the proof of the maximum principle and the previous lemma can be found in [12, 13]. Let me describe in a few words the idea of the method. We compare translations of two solutions u and v on $\mathbb{R}$. We show that for some real τ, we have

$$u(. + \tau) \geq v(\cdot) \text{ on all } \mathbb{R}. \tag{1.11}$$

Then, using standard procedures we obtain the desired conclusion. To obtain (1.11), a global approach is needed, since we deal with non-local operators. The method used by Berestycki–Nirenberg [7, 9] and Vega [22] fails in our case because it relies on comparison results either on compact set or semi infinite cylinders, which cannot be obtained in our case.

This note is organized as follows: Sect. 2 is devoted to some preliminary results which will be used extensively in the other sections. Uniqueness and monotonicity of travelling front solution (i.e. Theorems 1.2 and 1.3) is proved in Sect. 3. Theorem 1.4 is then proved in Sect. 4. In the last section we examine some aspect of the multidimensional problem.

Remark 1.5 Even though the Laplacian (i.e. $L := \Delta$) does not satisfy Lemma 1.1, one can show that our proof of Theorems 1.2–1.3 holds for this operator.

2 Preliminary results, nonlinear comparison principle

In this section we present some useful results concerning sub and supersolutions of the problem

$$Lu + f(u) = 0 \text{ on } \mathbb{R} \tag{2.1}$$

$$u(x) \to 0 \text{ as } x \to -\infty \tag{2.2}$$

$$u(x) \to 1 \text{ as } x \to +\infty, \tag{2.3}$$

where $f \in C^1$ satisfies the set of conditions in Theorem 1.2. For the sake of simplicity, we will only consider linear translation-invariant operators L satisfying Lemma 1.1 and (H2). For convenience, we introduce the notation $u_\tau := u(. + \tau)$. As briefly mentioned in the introduction, all our proofs rely on a comparison result on translations of two solutions. So we start by showing a Nonlinear Comparison Principle which will enable us to order translations of sub and supersolutions of (2.1)–(2.3). More precisely we have the following result.

Theorem 2.1 *Nonlinear Comparison Principle*
Let f satisfy the assumptions of Theorem 1.2. Let u and v be two smooth (C^1) functions on $\mathbb{R}$, such that

$$Lu + f(u) \leq 0 \text{ on } \mathbb{R} \tag{2.4}$$

$$Lv + f(v) \geq 0 \text{ on } \mathbb{R} \tag{2.5}$$

$$\lim_{x\to-\infty} u(x) \geq 0, \quad \lim_{x\to-\infty} v(x) \leq 0 \tag{2.6}$$

$$\lim_{x\to+\infty} u(x) \geq 1, \quad \lim_{x\to+\infty} v(x) \leq 1. \tag{2.7}$$

Then there exists a positive real τ such that $u_\tau \geq v$. Moreover, either $u_\tau > v$ or $u_\tau \equiv v$.

Remark 2.1 Observe that by the Maximum Principle (i.e. Theorem 1.5) and since $f(s) \geq 0 \ \ \forall s \leq 0$, the supersolution u is necessarily positive. Similarly, since $f(s) \leq 0 \ \ \forall s \geq 1$, the Maximum Principle implies that $v < 1$.

Before proving Theorem 2.1, we start with some definitions of quantities that we will use all along this section.
Let $0 < \delta \leq \frac{\epsilon}{2}$ such that

$$f'(p) \leq 0 \quad \text{for } p < \delta \quad \text{and} \quad 1 - p < \delta. \tag{2.8}$$

Choose $M > 0$ so that

$$1 - u(x) < \frac{\delta}{2} \ \ \forall x > M \tag{2.9}$$

$$\text{and} \quad v(x) < \frac{\delta}{2} \ \ \forall x < -M. \tag{2.10}$$

The proof of Theorem 2.1 is mainly based on the following technical lemma, which will be proved later on.

Lemma 2.1 *Let u and v be as in Theorem 2.1 and satisfy Conditions (2.9) and (2.10) above. If there exists a positive constant b such that u and v satisfy:*

$$u(x+b) > v(x) \ \ \forall x \in [-M-1, M+1] \tag{2.11}$$

$$\text{and} \quad u(x+b) + \frac{\delta}{2} > v(x) \ \ \forall x \in \mathbb{R}, \tag{2.12}$$

then we have $u(x+b) \geq v(x) \ \forall x \in \mathbb{R}$.

Proof of Theorem 2.1 Note that if $\inf_{\mathbb{R}} u > \max_{\mathbb{R}} v$, the theorem trivially holds. In the sequel, we assume that $\inf_{\mathbb{R}} u \leq \max_{\mathbb{R}} v$. Assume for a moment that Lemma 2.1 holds. To prove Theorem 2.1, by construction of M and δ, we just have to find an appropriate constant b which satisfies (2.11) and (2.12). Since u and v satisfy (2.6)–(2.7) using Remark 2.1, there exists a positive constant D such that on the compact set $[-M-1, M+1]$, we have for every $b \geq D$

$$u(x+b) > v(x) \ \ \forall x \in [-M-1, M+1].$$

Now, we claim that there exists $b \geq D$ such that $u(x+b) + \frac{\delta}{2} > v(x) \ \ \forall x \in \mathbb{R}$.
If not then we have,

$$\forall b \geq D \ \ \text{there exists} \ \ x(b) \ \ \text{such that} \ \ u(x(b)+b) + \frac{\delta}{2} \leq v(x(b)). \tag{2.13}$$

Since u is non-negative and v satisfies 2.6 there exists a positive constant A such that

$$u(x+b)+\frac{\delta}{2}>v(x) \quad \text{for all } b>0 \quad \text{and} \quad x\le -A. \tag{2.14}$$

Take now a sequence $(b_n)_{n\in\mathbb{N}}$ which tends to $+\infty$. Let $x(b_n)$ be the point defined by (2.13). Thus we have for that sequence

$$u(x(b_n)+b_n)+\frac{\delta}{2}\le v(x(b_n)). \tag{2.15}$$

According to (2.14) we have $x(b_n)\ge -A$. Therefore the sequence $x(b_n)+b_n$ converges to $+\infty$. Pass to the limit in (2.15) to get

$$1+\frac{\delta}{4}\le \lim_{n\to+\infty} u(x(b_n)+b_n)+\frac{\delta}{2}\le \limsup_{n\to+\infty} v(x(b_n))\le 1,$$

which is a contradiction. Therefore there exists a $b>D$ such that

$$u(x+b)+\frac{\delta}{2}>v(x) \;\; \forall x\in\mathbb{R}.$$

Since we have found our appropriate constant b, we can apply Lemma 2.1 to obtain

$$u(x+\tau)\ge v(x) \;\; \forall x\in\mathbb{R},$$

with $\tau=b$. It remains to prove that either $u_\tau>v$ or $u_\tau\equiv v$. We argue as follows. Let $w:=u_\tau-v$, then either $w>0$ or w achieves a non-negative minimum at some point $x_0\in\mathbb{R}$. If such x_0 exists then at this point we have $w(x)\ge w(x_0)=0$ and

$$0\le Lw(x_0)\le f(v(x_0))-f(u(x_0+\tau))=f(v(x_0))-f(v(x_0))=0. \tag{2.16}$$

Then using Lemma 1.1, we obtain $w\equiv 0$, which means $u_\tau\equiv v$. This ends the proof of Theorem 2.1. □

Remark 2.2 Note that the construction of b still stands if we only assume that u is continuous and $v<1$ has a finite number of discontinuities.

We now turn our attention to the proof of Lemma 2.1.

Proof of Lemma 2.1 Let u and v be respectively a super and a subsolution of (2.1)–(2.3) satisfying (2.9) and (2.10). Let $a>0$ be such that

$$u(x+b)+a>v(x)\;\forall x\in\mathbb{R}. \tag{2.17}$$

Note that for b defined by (2.11) and (2.12), any $a\ge\frac{\delta}{2}$ satisfies (2.17). Define

$$a^*=\inf\{a>0 \mid u(x+b)+a>v(x)\;\forall x\in\mathbb{R}\}. \tag{2.18}$$

We claim that

Claim 2.1 $a^*=0$.

Observe that Claim 2.1 implies that $u(x+b)\ge v(x)\;\forall x\in\mathbb{R}$, which is the desired conclusion. □

Proof of Claim 2.1 We argue by contradiction. If $a^* > 0$, since

$$\lim_{x \to \pm\infty} u(x+b) + a^* - v(x) \geq a^* > 0,$$

there exists $x_0 \in \mathbb{R}$ such that $u(x_0 + b) + a^* = v(x_0)$. Let $w(x) := u(x+b) + a^* - v(x)$, then

$$0 = w(x_0) = \min_{\mathbb{R}} w(x). \tag{2.19}$$

Observe that w also satisfies the following equations:

$$Lw \leq f(v(x)) - f(u(x+b)) \tag{2.20}$$

$$w(+\infty) \geq a^* \tag{2.21}$$

$$w(-\infty) \geq a^*. \tag{2.22}$$

Since $w \geq_{\not\equiv} 0$, by Lemma 1.1

$$Lw(x_0) > 0. \tag{2.23}$$

By our assumption, $u(x+b) > v(x)$ on $[-M-1, M+1]$. Hence $|x_0| > M+1$. Let us define

$$Q(x) := f(v(x)) - f(u(x+b)). \tag{2.24}$$

We now have to consider the two following cases:

- $x_0 < -M - 1$:
  At x_0 we have

  $$Q(x_0) = f(v(x_0)) - f(v(x_0) - a^*) \leq 0, \tag{2.25}$$

  since f is non-increasing for $s \leq \epsilon$, $a^* > 0$ and $v \leq \frac{\delta}{2} < \epsilon$ for $x < -M$. Now, combining (2.20), (2.23) and (2.25) yields the following contradiction

  $$0 < Lw(x_0) \leq Q(x_0) \leq 0.$$

- $x_0 > M + 1$:
  We argue similarly for that case. At x_0 we have

  $$Q(x_0) = f(u(x_0+b) + a^*) - f(u(x_0+b)) \leq 0, \tag{2.26}$$

  since f is non-increasing for $s \geq 1-\epsilon$, $a^* > 0$ and $1 - \epsilon < 1 - \frac{\delta}{2} \leq u$ for $x > M$. Again, combining (2.20), (2.23) and (2.26) yields the contradiction

  $$0 < Lw(x_0) \leq Q(x_0) \leq 0.$$

Hence $a^* = 0$, which ends the proof of Claim 2.1. □

Remark 2.3 One can observe that the proof of Lemma 2.1 still holds for any $\delta < \frac{\epsilon}{2}$ and M such that (2.8)–(2.10) hold. In particular, since u and v satisfies (2.6)–(2.7), Lemma 2.1 holds if we increase M.

General remarks and comments:

One can observe that most of the arguments used in the above two proofs hold for the Laplacian operator ($L = \Delta$). Only the final argument in the alternative fails. We can still obtain a contradiction, in this case, by arguing as follows:
Set $z := u(x + b) + a^*$ and $\Omega^- = \{x < -M - 1 | w(x) = 0\}$.
If $x_0 < -M - 1$, we have $\Omega^- \neq \emptyset$ and

$$Q(x) := f(v(x)) - f(z(x) - a^*) = f'(\theta(x))(v - z) + a^* f'(\theta(x)),$$

for some $\theta(x) \in [\min\{v(x), z(x) - a^*\}, \max\{v(x), z(x) - a^*\}]$.
Since $x_0 < -M - 1$, we have

$$z(x_0) - a^* < v(x_0) < \frac{\delta}{2} \leq \frac{\epsilon}{4}.$$

Therefore on a small neighborhood $V(x_0)$ of x_0, $z(x) - a^* < \frac{\epsilon}{2}$ and $v(x) < \frac{\epsilon}{2}$. Hence, on $V(x_0)$ we have $f(\theta(x)) \leq 0$. By observing that $w = z - v$, from (2.20)–(2.22), we then have on $V(x_0)$

$$Lw + f'(\theta(x))w \leq a^* f'(\theta(x)) \leq 0 \quad \text{on } V(x_0) \tag{2.27}$$

$$w \geq 0 \quad \text{on } V(x_0). \tag{2.28}$$

Apply now the usual Strong Maximum Principle (i.e. Theorem 1.5) to obtain $w \equiv 0$ on $V(x_0)$. Observe that the previous computation holds for any $x \in \Omega^-$, therefore Ω^- is an open subset of $(-\infty, -M - 1)$. Since w is continuous, Ω^- is obviously a closed subset of $(-\infty, -M - 1)$. By connectedness, we then have $\Omega^- = (-\infty, -M - 1)$, which is a contradiction since $lim_{x \to -\infty} w(x) \geq a^* > 0$. A similar argument can be used for the case $x_0 > M + 1$.

In the case of a continuous supersolution u and a subsolution v with a finite number of discontinuities, the first part of Theorem 2.1 holds. Similarly, this result also holds if the subsolution v is continuous and the supersolution u has a finite number of discontinuities.

Recall that u and v satisfy (2.9)–(2.10) for a positive M. Let us assume that v is discontinuous, the proof in the other case is similar. Since v has finite discontinuities, we can increase M further if necessary so that all the point of discontinuities of v are in $(-M, M)$. By doing so, $w := u(. + b) + a - v$ is then continuous on $(-\infty, -M] \cup [M, +\infty)$ for all positive a, b. Using Remark 2.2, there exists b such that u and v satisfy (2.11)–(2.12). As in the proof of Lemma 2.1 we can define

$$a^* = \inf\{a > 0 \mid u(x + b) + a > v(x) \; \forall x \in \mathbb{R}\}.$$

If $a^* > 0$, since $w > a^*$ on $(-M-1, M+1)$ and w is continuous on $(-\infty, -M] \cup [M, +\infty)$, w achieves a global minimum at some point $x_0 \in \mathbb{R} \setminus [-M - 1, M + 1]$. Since Lemma 1.1 holds for discontinuous functions with a finite number of discontinuities, arguing as in the continuous case we end up with $u(. + b) \geq v$. If u is discontinuous and v continuous, we choose M such that all the points of discontinuity of u are in $(-M, M)$. We argue as above, with the function $\widetilde{w} := u - v(. - b) - a$ instead of w.

Theorem 2.1 and Lemma 2.1 will be used extensively in other proofs.

3 Uniqueness and monotonicity of solutions of the integrodifferential equation on $\mathbb{R}$

In this section we present the proof of Theorems 1.2 and 1.3. We show that smooth positive solutions of the following problem are unique up to translation and are always monotone.

$$Lu + f(u) = 0 \text{ on } \mathbb{R} \tag{3.1}$$

$$u(x) \to 0 \text{ as } x \to -\infty \tag{3.2}$$

$$u(x) \to 1 \text{ as } x \to +\infty, \tag{3.3}$$

where $f \in C^1$ satisfies the assumptions of Theorem 1.2. For a sake of simplicity, in the sequel we will only consider continuous solutions and linear translation-invariant operators L satisfying Lemma 1.1 and (H2). Using the comparison principle and the translation invariance, without loss of generality, we may also assume that the solutions satisfy

$$0 < u < 1. \tag{3.4}$$

We break down this section in three subsections. In the first two subsections, we show that the solution is unique up to translation and monotone, which proves Theorem 1.2. The last subsection deals with non-existence of the solution of (3.1)–(3.4), and as a corollary, we obtain the uniqueness of the speed c of a travelling wave, which proves Theorem 1.3.

3.1 Uniqueness up to translation:

Let u and v be two solutions of (3.1)–(3.4).

First we define the following real number:

$$\tau^* = \inf\{\tau \geq 0 |\ u_\tau(x) \geq v(x)\ \ \forall x \in \mathbb{R}\}. \tag{3.5}$$

Since u and v are solutions of (3.1)–(3.4), they satisfy the assumptions of Theorem 2.1 and therefore τ^* is well defined and has an upper bound. Since v is positive and u satisfies (3.2), there exists $\tau_0 > 0$ such that $u(-\tau) < v(0)\ \ \forall \tau \geq \tau_0$. Therefore τ^* is bounded from above. By continuity, we have at τ^*, $u_{\tau^*} \geq v$. We claim the following

Lemma 3.1 $u_{\tau^*}(x) = v(x)$, *for all* $x \in \mathbb{R}$.

Proof: We argue by contradiction and assume that $w := u_{\tau^*} - v \geq_{\not\equiv} 0$. We will show that for ϵ small enough, we have

$$u_{\tau^*-\epsilon}(x) \geq v(x) \text{ for all } x \in \mathbb{R}, \tag{3.6}$$

which will contradict the definition of τ^*. Let us start the construction of our desired ϵ. We first show that $w > 0$. Assume that there exists x_0 in $\mathbb{R}$ such that w achieves a non-negative minimum at this point. Then we have $w(x) \geq w(x_0) = 0$ and

$$0 \leq Lw(x_0) = f(v(x_0)) - f(u(x_0 + \tau^*)) = f(v(x_0)) - f(v(x_0)) = 0. \tag{3.7}$$

Using Lemma 1.1, we obtain $w \equiv 0$, which contradicts $u_{\tau^*} \geq_{\not\equiv} v$. Therefore, we must have $w > 0$.

Choose $M > 0$ and $\delta < \frac{\epsilon}{2}$ as in Sect. 2 such that u and v satisfy 2.9 and 2.10. By continuity and since $u_{\tau^*} > v$, we can then find $\epsilon_1 > 0$ such that

$$\forall \epsilon \in [0, \epsilon_1) \;\; u(x + (\tau^* - \epsilon)) > v(x) \;\; \text{for all } x \in [-M - 1, M + 1]. \tag{3.8}$$

We claim the following □

Claim 3.1 *There exists $\epsilon \in (0, \epsilon_1]$ such that u and v satisfy*

$$u(x + (\tau^* - \epsilon)) + \frac{\delta}{2} > v(x) \;\; \textit{for all} \;\;\; x \in \mathbb{R}. \tag{3.9}$$

Fix now $\epsilon \in (0, \epsilon_1)$, such that (3.9) holds. Observe that $b := \tau^* - \epsilon$ satisfies assumptions (2.11) and (2.12) of Lemma 2.1. Therefore by Lemma 2.1 we end up with the desired contradiction

$$u(x + (\tau^* - \epsilon)) \geq v(x) \;\; \text{for all} \;\;\; x \in \mathbb{R}.$$

Assume for the moment that Claim 3.1 holds. □

Proof of Claim 3.1 We argue by contradiction. If (3.9) fails, then for all $\epsilon \in (0, \epsilon_1)$ there exists $x(\epsilon) \in \mathbb{R}$ such that

$$u(x(\epsilon) + (\tau^* - \epsilon)) + \frac{\delta}{2} \leq v(x(\epsilon)). \tag{3.10}$$

Now take a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ which tends to zero. Let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined by (3.10). Thus u satisfies for each positive integer n

$$u(x_n + (\tau^* - \epsilon_n)) + \frac{\delta}{2} \leq v(x_n). \tag{3.11}$$

Since u and v satisfy (2.9) and (2.10), $(x_n)_{n \in \mathbb{N}}$ stays in the compact [−M,M]. Therefore we can extract a subsequence of $(x_n)_{n \in \mathbb{N}}$ which converges to some $\bar{x} \in \mathbb{R}$. Letting now n go to $+\infty$ in (3.11) we end up with

$$u(\bar{x} + \tau^*) + \frac{\delta}{2} \leq v(\bar{x}), \tag{3.12}$$

which contradicts $u_{\tau^*} \geq v$. □

3.2 Monotonicity of the solution

Now, we show the second part of Theorem 1.2 on the monotone behavior of the solution of (3.1)–(3.3). More precisely we show

Theorem 3.1 *Let f be as in Theorem 1.2, then the solution u of (3.1)–(3.3) is monotone increasing.*

We break down our proof into three steps:

- first step: we prove that for any solution u of (3.1)–(3.3) there exists a positive τ such that

$$u(x+\tau) \geq u(x) \;\; \forall x \in \mathbb{R}.$$

- second step: we show that for any $\widetilde{\tau} \geq \tau$, u satisfies

$$u(x+\widetilde{\tau}) \geq u(x) \;\; \forall x \in \mathbb{R}.$$

- third step: we prove that

$$\inf\{\tau > 0 \mid \forall \widetilde{\tau} > \tau, \;\; u(x+\widetilde{\tau}) \geq u(x) \;\; \forall x \in \mathbb{R}.\} \leq 0.$$

We easily see that the last step provides the conclusion of Theorem 3.1.
Proof of Theorem 3.1:
First step: The first step is easily obtained from Theorem 2.1 by observing that u is a sub and a supersolution of (3.1)–(3.3). Therefore we have $u_\tau \geq u$ for one positive τ.
Second step: Choose $0 < \delta \leq \frac{\epsilon}{2}$ and M such that

$$f'(p) \leq 0 \quad \text{for } p < \delta \text{ and } 1-p < \delta \tag{3.13}$$

and so that u satisfies

$$1 - u(x) < \frac{\delta}{2} \qquad \forall x > M, \tag{3.14}$$

$$\text{and} \qquad u(x) < \frac{\delta}{2} \qquad \forall x < -M. \tag{3.15}$$

We achieve the second step with the following proposition.

Proposition 3.1 *Let u be a positive solution of (3.1)–(3.4) satisfying (3.14) and (3.15). If there exists $\tau > 0$ such that $u(x+\tau) \geq u(x) \;\; \forall x \in \mathbb{R}$, then for all $\widetilde{\tau} \geq \tau$ we have, $u(x+\widetilde{\tau}) \geq u(x) \;\; \forall x \in \mathbb{R}$.*

From the previous step we know that such a τ exists.
The proof of Proposition 3.1 is based on the following two technical lemmas which will be proved later on.

Lemma 3.2 *Let u be a positive solution of (3.1)–(3.4) and $\tau > 0$ such that $u(x+\tau) \geq u(x) \;\; \forall x \in \mathbb{R}$. Then, we have $u(x+\tau) > u(x) \;\; \forall x \in \mathbb{R}$.*

Lemma 3.3 *Let u be a positive solution of (3.1)–(3.4) satisfying (3.14) and (3.15) and $\tau > 0$ such that $u(x+\tau) > u(x) \;\; \forall x \in \mathbb{R}$. Then, there exists $\epsilon_0(\tau) > 0$ such that for all $\widetilde{\tau} \in [\tau, \tau+\epsilon_0]$, we have*

$$u(x+\widetilde{\tau}) > u(x) \;\; \forall x \in \mathbb{R}. \tag{3.16}$$

□

Proof of Proposition 3.1 We know from the first step that we can find a positive τ such that,

$$u(x+\tau) \geq u(x) \ \ \forall x \in \mathbb{R}.$$

Therefore by Lemmas 3.2 and 3.3 we can construct an interval $[\tau, \tau+\epsilon]$, such that for all $\widetilde{\tau} \in [\tau, \tau+\epsilon]$ we have

$$u(x+\widetilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}.$$

Let us define the quantity

$$\bar{\gamma} = \sup\{\gamma \mid \forall \hat{\tau} \in [\tau, \gamma], \ \ u(x+\hat{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}\}. \tag{3.17}$$

We claim that $\bar{\gamma} = +\infty$. If not, $\bar{\gamma} < +\infty$ and by continuity we have

$$u(x+\bar{\gamma}) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{3.18}$$

Recall that from the definition of $\bar{\gamma}$ we have

$$\forall \hat{\tau} \in [\tau, \bar{\gamma}], \ \ u(x+\hat{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{3.19}$$

Therefore to get a contradiction it is sufficient to construct ϵ_0 such that for all $\epsilon \in [0, \epsilon_0]$ we have

$$u(x+(\bar{\gamma}+\epsilon)) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{3.20}$$

Since $\bar{\gamma} > 0$, we can apply Lemma 3.2 to get

$$u(x+\bar{\gamma}) > u(x) \ \ \forall x \in \mathbb{R}. \tag{3.21}$$

We can now apply Lemma 3.3 to find the desired $\epsilon > 0$. Therefore, from the definition of $\bar{\gamma}$ we get

$$\forall \hat{\tau} \in [\tau, +\infty], \ \ u(x+\hat{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R},$$

which proves Proposition 3.1. □

We now turn our attention to the proofs of the two technical lemmas. We start with the proof of Lemma 3.2.

Proof of Lemma 3.2 To prove that

$$u(x+\tau) > u(x) \ \ \forall x \in \mathbb{R}, \tag{3.22}$$

we argue by contradiction. Assume there exists a point x_0 such that

$$w(x) = u(x+\tau) - u(x) \geq w(x_0) = 0 \ \ \forall \ \ x \in \mathbb{R}.$$

At this point, w satisfies :

$$Lw(x_0) = f(u(x_0)) - f(u(x_0+\tau)) = f(u(x_0)) - f(u(x_0)) = 0.$$

By Lemma 1.1 we get $w \equiv Cte$. Since $w(x_0) = 0$, we have $w \equiv 0$. Therefore we have $u(x+\tau) = u(x)$ for all x in $\mathbb{R}$, which says that u is τ periodic.

Now, since $\tau > 0$, we have for any positive integer N,

$$u(0) = u(N\tau). \tag{3.23}$$

Letting N go to infinity in (3.23), we end up with

$$1 = u(0) < 1,$$

which is a contradiction. Therefore (3.22) holds for every x in $\mathbb{R}$. □

We now turn our attention to the proof of Lemma 3.3.

Proof of Lemma 3.3 Let u be a positive solution of (3.1)–(3.3), which satisfies

$$u(x+\tau) > u(x) \ \ \forall x \in \mathbb{R}, \tag{3.24}$$

for a given $\tau > 0$. Observe that since $0 < u < 1$ satisfies (3.14) and (3.15) we have for all $\epsilon > 0$,

$$u(x+\tau+\epsilon) + \frac{\delta}{2} > u(x) \ \ \forall x \in \mathbb{R} \setminus [-M, M]. \tag{3.25}$$

Since u is continuous and satisfies (3.24), we can find ϵ_0, such that for all $\epsilon \in [0, \epsilon_0]$, we have

$$u(x+\tau+\epsilon) > u(x) \ \text{ for } \ x \in [-M-1, M+1]. \tag{3.26}$$

Therefore for all $\epsilon \in [0, \epsilon_0]$, we have

$$u(x+\tau+\epsilon) + \frac{\delta}{2} > u(x) \ \ \forall x \in \mathbb{R}. \tag{3.27}$$

Observe that for all $\epsilon \in [0, \epsilon_0]$, $b := \tau + \epsilon$ satisfies assumptions (2.11) and (2.12) of Lemma 2.1. Therefore we can apply Lemma 2.1 for each $\epsilon \in [0, \epsilon_0]$ and get

$$u(x+\tau+\epsilon) \geq u(x) \ \ \forall x \in \mathbb{R}. \tag{3.28}$$

Thus, we end up with

$$u(x+\tilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}, \tag{3.29}$$

for all $\tilde{\tau} \in [\tau, \tau+\epsilon_0]$. This ends the proof of Lemma 3.3. □

Third step: By the first step and Proposition 3.1, we can define the quantity

$$\tau^* = \inf\{\tau > 0 | \ \forall \tilde{\tau} > \tau, \ \ u(x+\tilde{\tau}) \geq u(x) \ \ \forall x \in \mathbb{R}\}. \tag{3.30}$$

We end the proof of Theorem 3.1, by proving the following claim

Claim 3.2 $\tau^* \leq 0$.

Proof: We follow the arguments used in the previous subsection on the uniqueness up to translation. We argue by contradiction and assume that $\tau^* > 0$. We will show that for ϵ small enough, we still have,

$$u(x+(\tau^*-\epsilon)) \geq u(x) \ \text{ for all } \ x \in \mathbb{R}. \tag{3.31}$$

Using the previous step, we will have for all $\tilde{\tau} \geq \tau^* - \epsilon$

$$u(x+\tilde{\tau}) \geq u(x) \ \text{ for all } \ x \in \mathbb{R}, \tag{3.32}$$

which will contradict the definition of τ^*.

The construction of ϵ is obtained as follows. By the definition of τ^* and by continuity, we have

$$u(x+\tau^*) \geq u(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.33}$$

Since $\tau^* > 0$, by Lemma 3.2, we have

$$u(x+\tau^*) > u(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.34}$$

Therefore, on the compact [−M−1,M+1], we can find $\epsilon_1 > 0$ such that

$$\forall \epsilon \in [0, \epsilon_1) \;\; u(x+(\tau^*-\epsilon)) > u(x) \;\; \text{for all } x \in [-M-1, M+1]. \tag{3.35}$$

The arguments used in the proof of Claim 3.1 apply to u, therefore there exists $\epsilon \in (0, \epsilon_1]$ such that u satisfies

$$u(x+(\tau^*-\epsilon)) + \frac{\delta}{2} > u(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.36}$$

Fix now $\epsilon \in (0, \epsilon_1)$, such that (3.36) holds. Again, observing that $b := \tau^* - \epsilon$ satisfies assumptions (2.11) and (2.12) of Lemma 2.1 with u as sub and supersolution, we conclude that (3.31) holds.This ends the proof of Claim 3.2 and at the same time proves Theorem 3.1. □

3.3 Non-existence and applications

In this subsection, we obtain non-existence results. More precisely, we have the following non-existence result.

Theorem 3.2 *Let f be as in Theorem 1.2. If there exists a continuous sub or supersolution u of Problem (3.1)–(3.3), such that u is not a solution of (3.1)–(3.3) then there exists no solution of Problem (3.1)–(3.3).*

Theorem 3.2 comes as a consequence of the uniqueness up to translation of the solution. The uniqueness of the speed of a travelling front (Theorem 1.3) is then obtained as a corollary of Theorem 3.2 and the monotonicity of the solution.

Indeed, let us assume that Theorem 3.2 holds and assume by contradiction that there exist (u, c) and (v, c') two continuous solutions of (3.1)–(3.3) with different speeds ($c \neq c'$). Recall that

$$Lu = \alpha u'' + \beta \int_{\mathbb{R}} J(x-y)(u(y)-u(x))\mathrm{d}y - cu' - \mathrm{d}u.$$

We note L_c and $L_{c'}$ the operator L with parameter respectively c and c'. By the previous subsection we have $u' > 0$ and $v' > 0$. Note that u satisfies the set of equations

$$L_{c'}u + f(u) = (c-c')u' \quad \text{on} \;\; \mathbb{R} \tag{3.37}$$

$$u(x) \to 0 \quad \text{as} \quad x \to -\infty \tag{3.38}$$

$$u(x) \to 1 \quad \text{as} \quad x \to +\infty. \tag{3.39}$$

Since $u' > 0$, u is not a solution of (3.1)–(3.3) with speed c' and is either a sub or a supersolution of this problem. Theorem 3.2 then provides a contradiction. Thus we must have $c = c'$.

Let us turn our attention to the proof of Theorem 3.2.

Proof of Theorem 3.2 Without loss of generality we can assume that u is a supersolution of (3.1)–(3.3). We argue with a contradiction argument. Let us assume that there exists a continuous solution v to (3.1)–(3.3). Since u and v are respectively a super and a subsolution of (3.1)–(3.3), the argument developed in the proof of the uniqueness up to translation (i.e. Sect. 3.1) holds. We then have $u_\tau = v$ for some real τ, which is a contradiction. □

Remark: When $c' = 0$ and v have finitely many discontinuities, the proof of the uniqueness of the speed still holds. Indeed, assume by contradiction that (u, c) is another solution with $c \neq 0$. Following the previous argumentation, u is continuous and is either a supersolution or a subsolution of (3.1)–(3.3) with speed $c' = 0$. We can assume that u is a supersolution. The proof in the other case is similar. Using the observation in Sect. 2 on the first part of Theorem 2.1, there exists $\tau > 0$ such that $u_\tau > v$. Define as in Sect. 3.1

$$\tau^* = \inf\{\tau \geq 0|\ \ u_\tau(x) \geq v(x)\ \ \forall x \in \mathbb{R}\}.$$

Since a Maximum Principle holds for u_τ and v, working as in Sect. 3.1 yields a contradiction. Thus we must have $c = 0$.

4 Monotonicity of solutions of the integrodifferential equation: the monostable case

In this section, we present a proof of Theorem 1.4. Recall that we are interested in the monotonicity of solutions of problem (3.1)–(3.3), when the nonlinearity f is monostable (i.e. $f \in C^1(\mathbb{R})$ satisfying $f(0) = f(1) = 0$, $f_{|(0,1)} \geq 0$ and $f'(s) \leq 0$ in $1 - \epsilon < s$ for some $\epsilon > 0$).

Using the comparison principle and the translation invariance, without loss of generality, as in the previous section we can restrict our attention on solutions satisfying $0 < u < 1$.

We start as in Sect. 3.2 by breaking down our proof in three steps.

- first step: we prove that for any solution u of (3.1)–(3.3) there exists a positive τ such that
$$u(x) \geq u(x - \tau)\ \ \forall x \in \mathbb{R}.$$
- second step: we show that for any $\widetilde{\tau} \geq \tau$, u satisfies
$$u(x) \geq u(x - \widetilde{\tau})\ \ \forall x \in \mathbb{R}.$$
- third step: we prove that
$$\inf\{\tau > 0|\forall \widetilde{\tau} > \tau,\ \ u(x) \geq u(x - \widetilde{\tau})\ \ \forall x \in \mathbb{R}\} \leq 0.$$

We easily see that the last step provides the conclusion of the theorem. The next three subsections are devoted to each step of the proof.

Proof of the first step We show that most of the technical lemmas developed in the previous section can be adapted to this situation. First we show the following

Lemma 4.1 *Let u be a positive solution of (3.1)–(3.3), such that u is increasing in a neighborhood $(-\infty, -M)$ of $-\infty$. Then there exists a positive τ such that*

$$u(x) \geq u(x-\tau) \ \ \forall x \in \mathbb{R}.$$

Remark 4.1 Since f does not satisfy $f'(s) \leq 0$ when $s < \epsilon$ for some $\epsilon > 0$, Theorem 2.1 does not readily apply. However, in the case of monotonicity, an analogue of Lemma 2.1 can be obtained with minor change to the proof.

Proof of Lemma 4.1 Let u be a positive solution of (3.1)–(3.3).

We start with the definition of quantities that we will use all along the proof. Let δ positive be such that

$$f'(p) \leq 0 \quad \forall p \quad \text{such that} \ \ 1 - p < \delta. \tag{4.1}$$

Choose $M > 0$ such that:

$$|u(x) - 1| < \frac{\delta}{2} \qquad \forall x > M, \tag{4.2}$$

$$u(x) < \frac{\delta}{2} \qquad \forall x < -M, \tag{4.3}$$

$$u(x) > u(\widetilde{x}) \quad \forall \widetilde{x} < x \leq -M. \tag{4.4}$$

Again the proof of Lemma 4.1 is mainly based on the following technical lemma which will be proved later on.

Lemma 4.2 *Let u be a positive solution of (3.1)–(3.3) satisfying (4.2)–(4.4). Assume there exist positive constants a and b such that u satisfies:*

$$u(x) > u(x-b) \ \ \forall x \in (-\infty, M+1] \tag{4.5}$$

$$u(x) + a > u(x-b) \ \ \forall x \in \mathbb{R}. \tag{4.6}$$

Then we have $u(x) \geq u(x-b)$ $\forall x \in \mathbb{R}$.

Proof of Lemma 4.1 Assume for the moment that Lemma 4.2 holds. Then to prove Lemma 4.1 we just have to find appropriate constants a and b which satisfy (4.5) and (4.6).

Since we chose M such that u is increasing on $(-\infty, -M]$, then for every positive b, u satisfies

$$u(x) > u(x-b) \ \ \forall x \in (-\infty, -M-1].$$

Now, since u satisfies (3.2) and (3.3), there exists a positive constant D such that on the compact set $[-M-1, M+1]$ we have for every $b \geq D$,

$$u(x) > u(x-b) \ \ \forall x \in [-M-1, M+1].$$

Therefore, for b greater than D, u satisfies

$$u(x) > u(x-b) \ \ \forall x \in (-\infty, M+1].$$

Now take $a = 1$ and observe that $u(x) + a > u(x-b) \ \ \forall x \in \mathbb{R}$ since $u < 1$. This ends the construction of the constants a and b. □

Now we turn our attention to Lemma 4.2.

Proof of Lemma 4.2 As in the previous section, let us define

$$a^* = \inf\{a > 0 \mid u(x) + a > u(x-b) \ \forall x \in \mathbb{R}\}. \tag{4.7}$$

We claim that

Claim 4.1 $a^* = 0$.

Observe that by Claim 4.1 we end up with $u(x) \geq u(x-b) \ \forall x \in \mathbb{R}$ which is the desired conclusion. □

Proof of Claim 4.1 As in Sect. 2 we argue by contradiction. If not, since $\lim_{x\to\pm\infty} u(x) + a^* - u(x-b) = a^* > 0$, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) + a^* = u(x_0 - b)$.

Let $w(x) := u(x) + a^* - u(x-b)$, then we have

$$0 = w(x_0) = \min_{\mathbb{R}} w(x).$$

Observe that w also satisfies the following equation:

$$\begin{aligned} Lw &= f(u(x-b)) - f(u(x)) \\ w(+\infty) &= a^* \\ w(-\infty) &= a^*. \end{aligned}$$

From (4.5) we deduce that

$$w(x) = u(x) + a^* - u(x-b) \geq u(x) - u(x-b) > 0 \ \ \forall x \in (-\infty, M+1].$$

Thus, $x_0 > M + 1$.

As in Sect. 3.2, by the maximum principle property, at its minimum x_0, w satisfies:

$$f(u(x_0) + a^*) - f(u(x_0)) = Lw(x_0) > 0.$$

Thus

$$Q = f\big(u(x_0) + a^*\big) - f(u(x_0)) > 0 \tag{4.8}$$

$$Q = f'(d)a^* > 0, \tag{4.9}$$

for some $d \in]u(x_0), u(x_0) + a^*[$.

Since $x_0 > M + 1$, (4.2) implies that $1 - d < \delta$.

Thus, Q would verify :

$$Q = f'(d)a^* \leq 0,$$

which contradicts (4.9). Hence $a^* = 0$, which ends the proof of Claim 4.1. □

Now, we turn our attention to the second step in the proof of Theorem 1.4.

Proof of the second step As in Sect. 3.2 we achieve the second step with the following proposition.

Proposition 4.1 *Let u be a positive solution of (3.1)–(3.3) satisfying (4.2)–(4.4). If there exists $\tau > 0$ such that*

$$u(x) \geq u(x - \tau) \ \ \forall x \in \mathbb{R}. \tag{4.10}$$

Then, for all $\widetilde{\tau}$ we have, $u(x) \geq u(x - \widetilde{\tau}) \ \ \forall x \in \mathbb{R}$.

As in Sect. 3.2, the proof of the proposition is based on the two following technical lemmas.

Lemma 4.3 *Let u be a positive solution of (3.1)–(3.3) and $\tau > 0$ be such that $u(x) \geq u(x - \tau) \ \ \forall x \in \mathbb{R}$.*

Then, we have $u(x) > u(x - \tau) \ \ \forall x \in \mathbb{R}$.

Lemma 4.4 *Let u be a positive solution of (3.1)–(3.3) satisfying (4.2)–(4.4) and $\tau > 0$ be such that*
$u(x) > u(x - \tau) \ \ \forall x \in \mathbb{R}$.
Then, there exists $\epsilon_0(\tau) > 0$ such that for all $\widetilde{\tau} \in [\tau, \tau + \epsilon_0]$, we have

$$u(x) > u(x - \widetilde{\tau}) \ \ \forall x \in \mathbb{R}. \tag{4.11}$$

We omit the details of the proofs since essentially all the arguments developed in the previous section work. We proceed to the last step.

Proof of the third step By Lemma 4.1 and Proposition 4.1, we can define the quantity

$$\tau^* = \inf\{\tau > 0| \ \forall \widetilde{\tau} > \tau, \ \ u(x) \geq u(x - \widetilde{\tau}) \ \forall x \in \mathbb{R}\}. \tag{4.12}$$

We end the proof of Theorem 1.4 with the following lemma

Lemma 4.5 *Let u be a positive solution of (3.1)–(3.3) satisfying (4.2)–(4.4). Then, we have $\tau^* \leq 0$.*

Proof of Lemma 4.5 Again, we argue by contradiction, suppose that $\tau^* > 0$. We will show that for ϵ small enough, we still have

$$u(x) \geq u(x - (\tau^* - \epsilon)) \ \text{ for all } \ \ x \in \mathbb{R}. \tag{4.13}$$

Then by the previous step, we will have for all $\widetilde{\tau} \geq \tau^* - \epsilon$,

$$u(x) \geq u(x - \widetilde{\tau}) \ \text{ for all } \ \ x \in \mathbb{R}, \tag{4.14}$$

which contradicts the definition of τ^*.

Now, we start the construction. Using the definition of τ^*, the continuity of u and Lemma 4, we end up with

$$u(x) > u(x - \tau^*) \ \text{ for all } \ \ x \in \mathbb{R}.$$

Thus, on the compact $[M, M]$, we can find $\epsilon_1 > 0$ such that,

$$\forall \epsilon \in [0, \epsilon_1) \ \ u(x) > u(x - (\tau^* - \epsilon)) \ \ \forall x \in [-M - 1, M + 1].$$

Since u is increasing on $(-\infty, -M]$, we indeed have

$$\forall \epsilon \in [0, \epsilon_1) \;\; u(x) > u(x - (\tau^* - \epsilon)) \;\; \text{on}(-\infty, M+1].$$

Now fix $\epsilon \in (0, \epsilon_1)$. We can easily find a positive constant a such that

$$u(x) + a > u(x - (\tau^* - \epsilon)) \;\; \text{for all} \quad x \in \mathbb{R}.$$

We can then apply Lemma 4.2 to obtain the desired result. □

5 The multidimensional case

In this section, we study the extension of the uniqueness results to multidimensional problems. Let us consider the following integrodifferential problem:

$$\epsilon \Delta u + \theta \int_{\Sigma} J(x - t, y, s)(u(t, s) - u(x, y)) \mathrm{d}t \mathrm{d}s \\ + \beta(y) u_x + f(u) = 0 \;\; \text{on} \;\; \Sigma \tag{5.1}$$

$$\frac{\partial u}{\partial \nu} = 0 \;\; \text{on} \;\; \partial \Sigma \tag{5.2}$$

$$u(x, y) \to 0 \;\; \text{uniformly in } y \text{ as} \quad x \to -\infty \tag{5.3}$$

$$u(x, y) \to 1 \;\; \text{uniformly in } y \text{ as} \quad x \to +\infty. \tag{5.4}$$

As we briefly mentioned in the introduction, for general operator $Lu := \theta \int_{\Sigma} J(x - t, y, s)(u(t, s) - u(x, y)) \mathrm{d}s \mathrm{d}t + \beta(y) u_x$ the Strong Maximum Principle as state in Theorem 1.5 no longer holds. However, for operator of the form $Lu = \theta \int_{\Sigma} J(x - t, y - s)(u(t, s) - u(x, y)) \mathrm{d}s \mathrm{d}t + \beta(y) u_x$ with a nonnegative kernel $J(x, y)$ such that the support of J contains a set of the form $([-b, -a] \cup [a, b]) \times \omega$ for some $0 \leq a < b$, where ω is an open subset of Ω containing 0, we can show the following,

Theorem 5.1 *Multidimensional Maximum Principle*
Let u be a continuous function such that $L[u](x, y) \geq 0$ (resp. $L[u](x, y) \leq 0$) on Σ. Assume that u achieves at a global maximum (resp. a global minimum) at some point $(x_0, y_0) \in \Sigma$, then there exists $y \in \bar{\Omega}$ such that $u(x, y) = u(x_0, y_0)$ on $\mathbb{R} \times \{y\}$.

We obtain as a consequence of this maximum principle the following characterization of such operators,

Lemma 5.1 *Let u be a smooth function on $\bar{\Sigma}$. If u achieves a global minimum (resp. a global maximum) at some point $(x_0, y_0) \in \bar{\Sigma}$ then the following holds :*

- *Either $L[u](x_0, y_0) > 0$ $(resp.\; L[u](x_0, y_0) < 0)$*
- *Or $L[u](x_0, y_0) = 0$ and $u(x, y) = u(x_0, y_0)$ on $\mathbb{R} \times \{y\}$ for some $y \in \bar{\Omega}$.*

Remark 5.1 The Multidimensional Maximum Principle also holds for operators defined in (5.1) provided that we assume further that $\frac{\partial u}{\partial \nu} = 0$ on $\partial \Sigma$.

Lemma 5.1 can be proved just as Theorem 5.1.

Proof of Theorem 5.1 Recall that

$$Lu = \epsilon \Delta u + \theta \int_{\mathbb{R}\times\Omega} J(x-t, y-s)(u(t,s) - u(x,y))\, \mathrm{d}s\mathrm{d}t + \beta(y)u_x.$$

First assume that $\epsilon = 0$. Observe that since we only consider the derivatives of u in the direction x, and that u is continuous, $L[u](x, y)$ is well defined on $\bar{\Sigma}$. Assume that $L[u](x, y) \geq 0$ and achieves a global maximum at $(x_0, y_0) \in \bar{\Sigma}$. Then at this point we have

$$\int_{\mathbb{R}\times\Omega} J(x_0 - t, y_0 - s)(u(t,s) - u(x_0, y_0))\, \mathrm{d}t\mathrm{d}s \leq 0 \quad \text{and} \quad u_x(x_0, y_0) = 0.$$

which implies that

$$\int_{\mathbb{R}\times\Omega} J(x_0 - t, y_0 - s)(u(t,s) - u(x_0, y_0))\, \mathrm{d}t\mathrm{d}s = 0.$$

Therefore $u(t, s) = u(x_0, y_0) = M$ for all $(t, s) \in \Sigma$ such that $(x_0 - t, y_0 - s) \in \text{supp}(J)$. In particular since $([-b, -a] \cup [a, b]) \times \{0\} \subset J$ we have $u(t, y_0) = M$ for all $t \in x_0 + [-b, -a] \cup [a, b]$. Next we show that $u(x, y_0) = M$ for $x \in [x_0, +\infty)$. Let $z \in x_0 + [a, b]$, observe that at the point (z, y_0), u achieves a positive maximum since $u(z, y_0) = u(x_0, y_0)$. We may thus argue as above and conclude that

$$\begin{aligned} u(x, y_0) = M \quad &\text{for all}\, x \in x_0 \\ &+ [-b, -a] \cup [-(b-a), b-a] \cup [a, b] \cup [a+b, 2b]. \end{aligned} \tag{5.5}$$

Thus we have $u(x, y_0) = u(x_0, y_0)$ for all $x \in x_0 + [0, b - a]$. Now repeat all the computations with $z = x_0 + b - a$ instead of x_0 to obtain that $u(x, y_0) = u(x_0, y_0)$ for all $x \in x_0 + [0, 2(b - a)]$. Therefore by repeating infinitely many times this process we obtain $u(x, y_0) = M$ for $x \in [x_0, +\infty)$. By using $z = x_0 - (b - a)$ in the previous computation, we obtain $u(x, y_0) = M$ for $x \in (-\infty, x_0]$. Therefore $u(x, y_0) = M$ on $\mathbb{R} \times \{y_0\}$.

If $\epsilon > 0$ we argue as follows. As in the above proof, assume that $L[u](x, y) \geq 0$ and achieves a global maximum at $(x_0, y_0) \in \bar{\Sigma}$. If $(x_0, y_0) \in \mathbb{R} \times \Omega$, then the previous argument holds and u is constant on $\mathbb{R} \times \{y_0\}$.
If $(x_0, y_0) \in \mathbb{R} \times \partial\Omega$, then we have the following alternative

- Either $\int_{\mathbb{R}\times\Omega} J(x_0 - t, y_0 - s)(u(t,s) - u(x_0, y_0))\, \mathrm{d}t\mathrm{d}s = 0$ and then the previous argument holds.
- Or $\int_{\mathbb{R}\times\Omega} J(x_0 - t, y_0 - s)(u(t,s) - u(x_0, y_0))\, \mathrm{d}t\mathrm{d}s < 0$.
  In that case since $\int_{\mathbb{R}\times\Omega} J(x - t, y_0 - s)(u(t,s) - u(x, y))\, \mathrm{d}t\mathrm{d}s$ is a continuous function on $\bar{\Sigma}$, in a small neighborhood $V(x_0, y_0) = B_r(x_0, y_0) \cap \bar{\Sigma}$, we have

$$\epsilon \Delta u + \beta(y) u_x \geq - \int_{\mathbb{R}\times\Omega} J(x-t, y-s)(u(t,s) - u(x,y))\, \mathrm{d}t\mathrm{d}s \geq 0.$$

  Applying the Hopf Lemma to $Mu = \epsilon \Delta u + \beta(y)u_x$, we obtain a contradiction since $\frac{\partial u}{\partial \nu} = 0$. Therefore $u = u(x_0, y_0)$ on $\mathbb{R} \times \{y_0\}$. □

Remark 5.2 In the case $\epsilon = 0$, the assumption on the normal derivative is not required. However, in that case there is no Hopf Lemma available.

Remark 5.3 The multidimensional Maximum Principle holds for Kernel of the form $J(x, y, s) = k(x)\widetilde{k}(y, s)$ with

- $k \in L^1(\mathbb{R})$ is a positive continuous kernel such that $[-b, -a] \cup [a, b] \subset \text{supp}(k)$ for some $0 \leq a < b$.
- $\widetilde{k}(y, s)$ is a positive continuous kernel, which satisfy the following properties:

$$\forall\, y \in \bar{\Omega}\ \exists\, s_y \in \bar{\Omega} \ \text{ such that } \ \widetilde{k}(y, s_y) \neq 0$$

Remark 5.4 Whether generalizations of our Maximum Principle to operators such as $L := \int_{\mathbb{R}\times\Omega} J(x-t, y-s)(u(t, s) - u(x, y))\mathrm{d}t\mathrm{d}s + d(y)u_y$ hold, is still open. An equivalent of the Hopf Lemma for that case must be established in order to treat the cases of extrema achieved on the boundary of the cylinder.

Using the multidimensional Maximum Principle, Lemma 5.1 and the ideas developed in Sect. 2 we have

Theorem 5.2 *Multidimensional Nonlinear Comparison Principle*
Let f satisfy the assumptions of Theorem 1.2. Let u and v be two smooth (C^1) functions on Σ, such that

$$Lu + f(u) \leq 0 \ \text{ on } \ \Sigma \tag{5.6}$$

$$Lv + f(v) \geq 0 \ \text{ on } \ \Sigma \tag{5.7}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \ \mathbb{R} \times \partial\Omega \tag{5.8}$$

$$\lim_{x\to-\infty} u(x, y) \geq 0, \ \ \lim_{x\to-\infty} v(x, y) \leq 0 \ \textit{uniformly in } y \tag{5.9}$$

$$\lim_{x\to+\infty} u(x, y) \geq 1, \ \ \lim_{x\to+\infty} v(x, y) \leq 1 \ \textit{uniformly in } y\ . \tag{5.10}$$

Then there exists a positive real τ such that $u_\tau \geq v$. Moreover, either $u_\tau > v$ on $\bar{\Sigma}$ or $u_\tau \equiv v$ on $\mathbb{R} \times \{y\}$ for some $y \in \bar{\Omega}$.

As in Sect. 2, Theorem 5.2 is proved using the following construction.
Let $0 < \delta \leq \frac{\epsilon}{2}$ such that

$$f'(p) \leq 0 \quad \text{for } p < \delta \ \text{ and } \ 1 - p < \delta. \tag{5.11}$$

Choose $M > 0$ so that

$$1 - u(x, y) < \frac{\delta}{2} \quad \forall (x, y) \in (M, +\infty) \times \bar{\Omega} \tag{5.12}$$

$$\text{and} \quad v(x, y) < \frac{\delta}{2} \quad \forall (x, y) \in (-\infty, -M) \times \bar{\Omega}. \tag{5.13}$$

Lemma 5.2 *Let u and v be as in Theorem 5.2 and satisfy Conditions (5.12) and (5.13). If there exists a positive constant b such that u and v satisfy:*

$$u(x+b, y) > v(x, y) \ \forall (x, y) \in [-M-1, M+1] \times \bar{\Omega} \tag{5.14}$$

$$\text{and} \quad u(x+b, y) + \frac{\delta}{2} > v(x, y) \ \forall (x, y) \in \bar{\Sigma}, \tag{5.15}$$

then we have $u(x+b, y) \geq v(x, y)\ \forall (x, y) \in \bar{\Sigma}$.

As we have already observed in the previous analysis, the proofs of Theorems 1.2–1.3 only rely on a nonlinear comparison principle, a technical lemma such as Lemma 2.1 and a good characterization of $L[u](x)$ at a global extrema of u. The generalization of these two theorems will therefore be straightforward using their multidimensional analog.

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