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Stability properties of an inverse parabolic problem with unknown boundaries

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Abstract We treat the stability issue for an inverse problem arising from non-destructive evaluation by thermal imaging. We consider the determination of an unknown portion of the boundary of a thermic conducting body by overdetermined boundary data for a parabolic initial-boundary value problem. We obtain that when the unknown part of the boundary is a priori known to be smooth, the data are as regular as possible and all possible measurements are taken into account, the problem is exponentially ill-posed. Then, we prove that a single measurement with some a priori information on the unknown part of the boundary and minimal assumptions on the data, in particular on the thermal conductivity, is enough to have stable determination of the unknown boundary. Given the exponential ill-posedness, the stability estimate obtained is optimal.

Keywords Inverse problems · Stability · Parabolic equations · Bessel functions · Unique continuation

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1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$, with a sufficiently smooth boundary $\partial\Omega$, a closed part of which, say I , is not known and not accessible. For instance,

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I could be some interior component of $\partial\Omega$ or some inaccessible portion of the exterior component of $\partial\Omega$. On the other hand, we assume that the set $A = \partial\Omega \setminus I$ is accessible and known. Let T be a positive number and let $\kappa = \kappa(x, t)$, $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, be a symmetric $N \times N$ matrix whose entries are Lipschitz continuous (real) functions. We assume that κ is also uniformly elliptic. Given a nontrivial function f on $A \times [0, T]$ such that $\text{supp } f$ is compactly contained in $A \times [0, T]$, let us consider the initial-boundary value problem

$$\begin{cases} \partial_t u - \text{div}(\kappa(x, t)\nabla u) = 0 & \text{in } \Omega \times (0, T), \\ u(x, 0) = 0 & x \in \bar{\Omega}, \\ u = 0 & \text{on } I \times [0, T], \\ u = f & \text{on } A \times [0, T]. \end{cases} \quad (1.1)$$

Given an open portion Σ of $\partial\Omega$ such that $\Sigma \subset A$, we consider the *inverse problem* of determining I from the knowledge of $\kappa \nabla u \cdot \nu$ on $\Sigma \times [0, T]$, where ν denotes the exterior normal to Ω .

This problem arises, for instance, from non-destructive testing using thermal imaging. In fact, Ω may represent a thermic conducting body, with thermal conductivity κ , A is a known part of the boundary and I may represent an unknown privileged isothermal surface, such as a solidification front [2], a corroded portion of $\partial\Omega$ or the boundary of a cavity inside Ω [3, 21]. We prescribe the temperature on the accessible part of the boundary, A , and we measure on $\Sigma \times [0, T]$ the heat flux exiting the body, and, through this additional measurement, we want to determine I . Actually in some applications, in particular in the continuous casting of steel, see [2, 8], the conductivity κ depends explicitly on the temperature u , rather than on t . Nevertheless, the case in which κ depends on x and t only is an important starting point to study the stability also for the general non-linear case. We shall return to this point later.

The uniqueness result for the problem discussed earlier has been proved by Isakov, see [12]. In [5] it is shown via a counterexample that uniqueness may fail when $u(\cdot, 0) \neq 0$.

In this paper we approach the stability issue of this problem from two different points of view. First of all, we analyse the instability character of the problem and we find that this inverse problem is exponentially unstable, that is logarithmic stability estimates are best possible. Then, we prove a stability estimate under some a priori information on I and some assumptions on A , κ and the oscillation character of f . The stability estimate we prove is of logarithmic type and it is therefore very weak; however, it is essentially optimal. We wish to remark that to properly analyse the instability of the problem, we study what happens in the most favourable situation, that is we suppose to have strong a priori information on I , A and κ as simple as possible and, more notably, instead of a single measurement with a given f we take into account all possible measurements. Still, the instability is of exponential type, and this means that performing different or more measurements does not substantially improve the stability of the problem. Conversely, for what concerns the stability estimate, the aim is to keep as minimal as possible the a priori information on the unknown I and the assumptions on the data A , κ and f while still keeping the optimality of the estimate.

By the stability estimates and the instability analysis we can therefore characterize in a quite precise manner the modulus of continuity of the map which

associates to the measurement $\kappa \nabla u \cdot \nu$ on $\Sigma \times [0, T]$ the unknown part of the boundary I .

More precisely, in the instability analysis we deal with the following framework. We assume that $\Omega = B_1(0) \setminus D$, where D represents an unknown cavity. Therefore, we identify A with the exterior boundary of Ω , $\partial B_1(0)$, and I with the boundary of the cavity, ∂D . We further assume that κ is identically equal to the identity matrix, that is the body is homogeneous and isotropic. Let us fix an integer m . To each C^m regular cavity D we associate its Dirichlet-to-Neumann map $\mathcal{D}(D)$, that is the operator that maps each prescribed temperature f on $A \times [0, T]$ into the corresponding heat flux $\frac{\partial u}{\partial \nu}|_{A \times [0, T]}$, where u solves (1.1). We establish the instability properties of the function that associates to each operator $\mathcal{D}(D)$ the corresponding cavity D .

The basic idea of the method we use for such a purpose goes back to [13]. More recently, this idea has been applied successfully to the inverse problem of conductivity, see [18]. Later, in [7], the method has been formulated in an abstract framework suitable to be applied in the context of inverse problems. In that paper, in fact, the abstract formulation has been applied to many different inverse elliptic problems. Here we also make use of the abstract setting; however, the parabolic case presents additional difficulties with respect to the elliptic case. A crucial step in order to apply the abstract method relies on the construction of a sequence $\{u_k\}_{k \in \mathbb{N}}$ of solutions to an auxiliary initial-boundary value problem (namely (1.1) with $\Omega = B_1(0)$, that is $D = \emptyset$), satisfying the following properties. First, the linear space generated by $\{u_k\}_{k \in \mathbb{N}}$ should be dense among all possible solutions to the auxiliary problem and, second, u_k should decay exponentially with respect to k on any compact subset of $B_1(0)$. In the elliptic case, such a sequence is provided by the harmonic polynomials, whereas in the parabolic case its construction is much more delicate and is performed in Sect. 3, with the crucial exponential decay estimate given in Proposition 3.5.

We wish to remark that the stability estimate is established when I is assumed to be $C^{1,\beta}$, $0 < \beta \leq 1$, whereas A is just Lipschitz. Furthermore, κ may depend on both time and space variables. We notice that in [5] an analogous logarithmic stability estimate has been proved when A is assumed to be $C^{1,1}$ and κ does not depend on t .

The proof of the stability estimate has the same structure of those given in [1, 4, 5, 19]. As in these papers, the main effort consists of deducing some quantitative estimates of unique continuation, in particular three cylinder inequalities in the interior and at the boundary and stability estimates for Cauchy problems. An optimal three cylinder inequality in the interior has been proved in [10, 22] and it is recalled in Theorem 4.4. We also need an optimal three cylinder inequality at the boundary and this is obtained in Theorem 4.5. In turn, in order to obtain such quantitative estimates of unique continuation, the main effort is to find a suitable Carleman estimate, see Theorem 4.1. The main novelty here is the dependence of the thermal conductivity by time, too. This leads to three cylinder inequalities whose constants depend on the parameters of the cylinders in a different way with respect to the time autonomous case. However, we show that, even with this difference, analogous estimates of unique continuation can be proved, a stability estimate for Cauchy problems, Theorem 4.6, which by the way has been obtained for a Lipschitz boundary, and a smallness propagation estimate, Proposition 4.7.

Let us further remark that treating the case in which κ depends on t is a basic step to approach the non-linear case in which κ depends on u . In fact, we observe that the techniques used for proving quantitative estimates of unique continuation, such as Theorems 4.4–4.6 and Proposition 4.7, can be applied also to the differential inequality

$$|\partial_t u - \operatorname{div}(\kappa(x, t)\nabla u)| \leq C(|\nabla u| + |u|)$$

instead of the corresponding homogeneous equation, we refer for the obvious changes to [10, 22], and this would allow us, provided that κ and the boundary of Ω are regular enough, to extend the results also to the non-linear case.

Finally, let us note that similar results may be obtained also if we consider a homogeneous Neumann condition on I . For what concerns the stability estimate, however, the regularity assumptions on I need to be changed, for instance I should be of $C^{1,1}$ class, see the discussion in [1] and [4].

The plan of the paper is as follows. In Sect. 2 we state the main results of the paper: the exponential instability, Theorem 2.3, and the stability estimate, Theorem 2.5. In Sect. 3, we prove Theorem 2.3. In Sect. 4 we prove Theorem 2.5, in particular we prove the Carleman inequality, Theorem 4.1, and state the quantitative estimates of unique continuation. In the appendix the rather technical proofs of the quantitative estimates are developed.

2 Statement of the main results

We begin by giving some notation and definitions. We shall fix the space dimension N , $N \geq 2$, throughout the paper. We shall use the letter C or K to denote positive constants. The value of the constants may change from line to line, but we shall specify their dependence everywhere they appear (sometimes we emphasize their difference writing C_0, C_1, \dots). We shall always omit the dependence of the constants on N .

For every $x \in \mathbb{R}^N$ we shall set $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, and we shall denote by $B_r(x)$ and $B'_r(x')$, respectively, the open ball in \mathbb{R}^N centred at x of radius r and the open ball in \mathbb{R}^{N-1} centred at x' of radius r . Sometimes we shall write B_r and B'_r instead of $B_r(0)$ and $B'_r(0)$, respectively.

Let r, t_0 be positive numbers. For every $x \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^N$ such that $|\zeta| = 1$ and $\alpha \in (0, \pi)$ we shall denote by $C(x, \zeta, \alpha, r)$ the cone $\{y \in B_r(x) : \frac{(y-x)\cdot\zeta}{|y-x|} > \cos \alpha\}$ and we shall denote by $C^{t_0}(x, \zeta, \alpha, r)$ the set $C(x, \zeta, \alpha, r) \times (0, t_0)$. We shall denote by $D_r^{t_0}(x)$ the cylinder $B_r(x) \times (0, t_0)$. For every function ψ on \mathbb{R}^{N-1} such that $\psi(0) = 0$, we shall denote by $D_{\psi,r}^{t_0}$ the set $\{(x', x_N, t) \in B_r \times (0, t_0) : \psi(x') < x_N\}$. When dealing with $N + 1$ variables (x, t) , with $x = (x_1, \dots, x_N)$, we shall denote $\nabla = \nabla_x$, $\operatorname{div} = \operatorname{div}_x$, $\Delta = \operatorname{div}(\nabla)$, $D^2 = D_x^2$. We shall write, for brevity, $\partial_i f = \frac{\partial f}{\partial x_i}$, $\partial_t f = \frac{\partial f}{\partial t}$. Finally, we shall denote by \mathcal{I} the identity matrix.

Definition 2.1 Let Ω be a bounded domain in \mathbb{R}^N . Given a nonnegative integer m and β , $0 \leq \beta \leq 1$, we shall say that a portion Σ of $\partial\Omega$ is of $C^{m,\beta}$ class with constants $R_0, E > 0$, if for every $P \in \Sigma$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{R_0} = \{(x', x_N) \in B_{R_0} : x_N > \varphi(x')\},$$

where φ is a $C^{m,\beta}$ function on B'_{R_0} satisfying $\varphi(0) = 0$, $\|\varphi\|_{C^{m,\beta}(B'_{R_0})} \leq ER_0$ and, whenever $m \geq 1$, also $|\nabla\varphi(0)| = 0$. We remark that when $m = 0$ and $\beta = 1$ we shall also speak of Lipschitz class and that when $\beta = 0$ we shall speak of C^m class.

Remark 2.2 We have chosen to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter R_0 is equal to one. For instance, we have that

$$\|\varphi\|_{C^{1,\beta}(B'_{R_0})} = \|\varphi\|_{L^\infty(B'_{R_0})} + R_0\|\nabla\varphi\|_{L^\infty(B'_{R_0})} + R_0^{1+\beta}|\nabla\varphi|_{\beta,B'_{R_0}},$$

where

$$|\nabla\varphi|_{\beta,B'_{R_0}} = \sup_{\substack{x',y' \in B'_{R_0} \\ x' \neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x' - y'|^\beta}.$$

In the same fashion, if $D = \Omega \times (0, T)$, $T > 0$, and u is a function belonging to $H^{2,1}(D)$ the norm $\|u\|_{H^{2,1}(D)}$ is meant as follows

$$\|u\|_{H^{2,1}(D)}^2 = \frac{1}{TR_0^N} \int_D (u^2 + R_0^2|\nabla u|^2 + R_0^4|D^2u|^2 + T^2(\partial_t u)^2) dx dt,$$

and so on for any other integral or fractional order Sobolev space defined on D or on $\Sigma \times (0, T)$, Σ being a portion of $\partial\Omega$. For what concerns the definition and main properties of these Sobolev spaces we refer to [17].

2.1 The instability result

Let $\Omega = B_1 \setminus D$, where D is a compact subset of B_1 . Let $A = \partial B_1 = \mathbb{S}^{N-1}$, $I = \partial D$. We also set $\Sigma = A$. For simplicity, throughout this subsection, we also set the dimensional parameter R_0 equal to 1. We set $Q = B_1 \times (\pi/2, 3\pi/2) \subset \mathbb{R}^{N+1}$ and $\Gamma = A \times (\pi/2, 3\pi/2)$.

We wish to remark that we could equivalently consider the time interval $(0, \pi)$ or $(0, 1)$ instead. We have chosen $(\pi/2, 3\pi/2)$ only because it will turn out to be convenient for the purpose of the proof.

The following Hilbert spaces will be used. The space $H = H_0^{3/2,3/4}(\Gamma)$, its dual $H' = H_1 = H^{-3/2,-3/4}(\Gamma)$, and $H_0 = H^{1/2,1/4}(\Gamma)$. We consider now the interpolation spaces between H_0 and H_1 . For what concerns interpolation we refer again to [17]. For any θ , $0 \leq \theta \leq 1$, we define H_θ as $[H_0, H_1]_\theta$, where this denotes the interpolation at level θ between the two spaces H_0 and H_1 . The norm in H_θ will be denoted by $\|\cdot\|_\theta$. First, we notice that for any θ , $0 \leq \theta \leq 1$, there exists a constant C_θ , which depends on θ only, such that the following interpolation inequality holds for any $\psi \in H_0$

$$\|\psi\|_\theta \leq C_\theta \|\psi\|_0^{1-\theta} \|\psi\|_1^\theta. \quad (2.1)$$

By using the interpolation properties of fractional order Sobolev spaces on Γ , see [17], we can characterize H_θ as follows

$$H_\theta = \begin{cases} H^{2(1/4-\theta), 1/4-\theta}(\Gamma) & \text{if } 0 \leq \theta \leq 1/4, \\ H^{-2(\theta-1/4), -(\theta-1/4)}(\Gamma) & \text{if } 1/4 \leq \theta \leq 1, \theta \neq 3/4. \end{cases} \quad (2.2)$$

Let us notice the interesting case of $\theta = 1/4$, where we have $H_\theta = L^2(\Gamma)$, and that $H_{3/4}$ does not coincide with $H^{-1, -1/2}(\Gamma)$.

Let us fix an integer $m \geq 2$ and positive constants δ, b and r .

To any strictly positive function g defined on $r\mathbb{S}^{N-1} = \partial B_r$, we denote its *radial subgraph* as $\text{subgraph}_{\text{rad}}(g) = \{y \in \mathbb{R}^N : y = \rho\omega, 0 \leq \rho \leq g(r\omega), \omega \in \mathbb{S}^{N-1}\}$. Then, $X_{mb\delta}(B_r)$ denotes the set given by $\{\text{subgraph}_{\text{rad}}(g) : g \in C^\infty(r\mathbb{S}^{N-1}), \|g\|_{C^m(r\mathbb{S}^{N-1})} \leq b \text{ and } r \leq g \leq r + \delta\}$.

Let us consider the metric space $(X, d) = (X_{mb\frac{1}{4}}(\bar{B}_{1/2}), d_{\mathcal{H}})$ where $d_{\mathcal{H}}$ denotes the Hausdorff distance, namely

$$d_{\mathcal{H}}(D_1, D_2) = \max \left\{ \sup_{x \in D_1} \text{dist}(x, D_2), \sup_{x \in D_2} \text{dist}(x, D_1) \right\},$$

for any $D_1, D_2 \in X$. Let us notice that every $D \in X$ is closed, is star-shaped with respect to the origin and satisfies $\bar{B}_{1/2} \subset D \subset \bar{B}_{3/4}$.

For any $D \in X$, we set $Q(D) = (B_1 \setminus D) \times (\pi/2, 3\pi/2)$, $\Gamma(D) = \partial D \times (\pi/2, 3\pi/2)$. If $D = \emptyset$, then we set $Q(D) = Q$ and $\Gamma(D) = \emptyset$.

For any $D \in X \cup \{\emptyset\}$, we consider the operator $\mathcal{D}(D) : H \mapsto H_0$ which is defined as follows. For any $\psi \in H$, let $u \in H^{2,1}(Q(D))$ be the solution to

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } Q(D), \\ u(x, \pi/2) = 0 & x \in \Omega \setminus D, \\ u = 0 & \text{on } \Gamma(D), \\ u = \psi & \text{on } \Gamma. \end{cases} \quad (2.3)$$

Then, for any $\psi \in H$, we set

$$\mathcal{D}(D)\psi = \frac{\partial u}{\partial \nu} \Big|_\Gamma, \quad u \text{ solution to (2.3)}. \quad (2.4)$$

We have that Theorems 4.3 and 6.2 in [17, Chapter 4] imply, respectively, existence and uniqueness of a solution $u \in H^{2,1}(Q(D))$ to (2.3) and its continuous dependence from the boundary datum $\psi \in H$. Finally, by the trace theorem [17, Chapter 4, Theorem 2.1], we can conclude that, for any $D \in X \cup \{\emptyset\}$, the operator $\mathcal{D}(D) : H \mapsto H_0$ is linear and bounded. We can also consider $\mathcal{D}(D)$ as a linear and bounded operator between H and $H' = H_1$, by setting

$$\langle \mathcal{D}(D)\psi, \phi \rangle_{H', H} = \left\langle \frac{\partial u}{\partial \nu} \Big|_\Gamma, \phi \right\rangle_{H', H} = \int_\Gamma \frac{\partial u}{\partial \nu} \phi, \quad \text{for any } \psi, \phi \in H, \quad (2.5)$$

where u solves (2.3) and $\langle \cdot, \cdot \rangle_{H', H}$ is the duality pairing between H' and H .

Let us remark that the operator \mathcal{D} is usually referred to as the *Dirichlet-to-Neumann* map. We are in the position of stating the instability result.

Theorem 2.3 *Let us fix an integer $m \geq 2$ and a positive constant b . Let $(X, d) = (X_{mb^{\frac{1}{4}}(\bar{B}_{1/2}, d_{\mathcal{H}})})$. Then there exists a positive constant δ_1 , depending on m and b only, such that for every δ , $0 < \delta < \delta_1$, we can find $D_1, D_2 \in X$ satisfying*

$$d(D_i, \bar{B}_{1/2}) \leq \delta, \quad \text{for any } i = 1, 2; \quad d(D_1, D_2) \geq \delta; \quad (2.6)$$

and, for any θ , $0 \leq \theta \leq 1$,

$$\|\mathcal{D}(D_1) - \mathcal{D}(D_2)\|_{\mathcal{L}(H, H_\theta)} \leq K \exp\left(-\theta \delta^{-\frac{N-1}{2m(2N+1)}}\right), \quad (2.7)$$

where K is a constant depending on m , b and θ only.

2.2 The stability result

Let $M, R_0, E, F, \lambda, \Lambda, T$ and β be given positive numbers with $\lambda, \beta \in (0, 1]$. Theorem 2.5 is based on the following assumptions and a priori information.

Assumptions on the domain and the accessible part of the boundary

We assume that Ω is a bounded domain in \mathbb{R}^N satisfying

$$|\Omega| \leq MR_0^N, \quad (2.8)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω . We also assume that

$$\partial\Omega \text{ is of } C^{0,1} \text{ class with constants } R_0, E. \quad (2.9)$$

We shall distinguish two nonempty parts A, I of $\partial\Omega$ which satisfy

$$I \cup A = \partial\Omega, \quad \overset{\circ}{I} \cap \overset{\circ}{A} = \emptyset, \quad I \cap A = \partial A = \partial I, \quad (2.10)$$

where interiors and boundaries are intended here in the relative topology of $\partial\Omega$.

Moreover we assume that we can select a portion Σ of A satisfying

$$\partial\Omega \cap B_{R_0}(P_1) \subset \Sigma \subset A_{R_0}, \quad (2.11)$$

where P_1 is a point belonging to Σ and $A_{R_0} = \{x \in \partial\Omega : \text{dist}(x, I) \geq R_0\}$.

Remark 2.4 Observe that earlier condition (2.9) implies a lower bound on the diameter of Ω and on the diameter of every connected component of $\partial\Omega$. On the other hand, conditions (2.8) and (2.9) imply an upper bound on the diameter of Ω .

A priori information on the unknown part of the boundary

We suppose that

$$I \text{ is of } C^{1,\beta} \text{ class with constants } R_0, E. \quad (2.12)$$

Assumptions about the prescribed boundary datum

We shall assume that the Dirichlet datum f appearing in problem (1.1) belongs to $H^{1/2,1/4}(A \times (0, T))$, it is nontrivial and satisfies

$$\text{supp } f \subset A_{R_0} \times [0, T], \quad \frac{\|f\|_{H^{1/2,1/4}(A \times (0, T))}}{\|f\|_{L^2(A \times (0, T))}} \leq F. \quad (2.13)$$

Let us notice that f can be trivially extended to $\partial\Omega \times (0, T)$ by setting $f = 0$ on $I \times (0, T)$. In such a way, denoting

$$S_T = \partial\Omega \times (0, T),$$

we shall often consider f as belonging to $H^{1/2,1/4}(S_T)$ and we shall set, unless otherwise specified,

$$\|f\| = \|f\|_{H^{1/2,1/4}(S_T)}.$$

Assumptions about the thermal conductivity

We assume that the thermal conductivity $\kappa = \kappa(x, t)$, $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, is a symmetric $N \times N$ matrix satisfying the following conditions for every (x, t) , $(y, s) \in \mathbb{R}^{N+1}$ and every $\xi \in \mathbb{R}^N$

$$\lambda|\xi|^2 \leq \kappa(x, t)\xi \cdot \xi \leq \lambda^{-1}|\xi|^2, \quad (2.14)$$

$$|\kappa(x, t) - \kappa(y, s)| \leq \Lambda \left(\frac{|x - y|}{R_0} + \frac{|t - s|}{T} \right). \quad (2.15)$$

In the sequel we shall refer to the set of numbers $\{R_0^2/T, M, E, F, \lambda, \Lambda, \beta\}$ as to the *a priori data*. We shall also use the following notation

$$W(\Omega \times (0, T)) = \{v : v \in L^2((0, T); H^1(\Omega)), \partial_t v \in L^2((0, T); H^{-1}(\Omega))\}.$$

Theorem 2.5 *Let Ω_1, Ω_2 be two domains satisfying (2.8), (2.9). For any $i = 1, 2$, let A_i, I_i , satisfying (2.10), be the accessible and inaccessible part of $\partial\Omega_i$, respectively. Assume that $A_1 = A_2 = A$ and that Ω_1 and Ω_2 lie on the same side of A . Let us take $\Sigma \subset A$ satisfying (2.11). Finally, we suppose that, for any $i = 1, 2$, I_i satisfies the a priori information (2.12).*

Let us assume that (2.13)–(2.15) are also satisfied and let $u_i \in W(\Omega_i \times (0, T))$ be the weak solution to (1.1) when $\Omega = \Omega_i$, $i = 1, 2$. If, given $\varepsilon > 0$, we have

$$R_0^{1/2} \|\kappa \nabla u_1 \cdot \nu - \kappa \nabla u_2 \cdot \nu\|_{L^2(\Sigma \times (0, T))} \leq \varepsilon, \quad (2.16)$$

then we have

$$d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq R_0 \omega(\varepsilon/\|f\|), \quad (2.17)$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$\omega(s) \leq C |\log s|^{-C_1}, \quad \text{for every } s < 1, \quad (2.18)$$

and C, C_1 are positive constants depending on the a priori data only.

3 Proof of Theorem 2.3

We begin with the following definition.

Definition 3.1 Let (X, d) be a metric space. For a given positive δ , $Z \subset X$ is δ -discrete if for any two distinct points z_1, z'_1 in Z we have $d(z_1, z'_1) \geq \delta$.

Let (X, d) be as in the hypotheses of Theorem 2.3. Let us set $D_0 = \bar{B}_{1/2}$ and let us call, for any $\delta > 0$, $X_\delta = \{D \in X : d(D, D_0) \leq \delta\}$. We observe that for any δ , $0 < \delta \leq 1/4$, we have $X_\delta = X_{mb\delta}(\bar{B}_{1/2})$. We have that (X, d) satisfies the following proposition, see for instance [7, Proposition 3.1].

Proposition 3.2 *There exists $\delta_0 > 0$, depending on m and b only, such that for any δ , $0 < \delta < \delta_0$, we can find $Z_\delta \subset X_\delta$ such that Z_δ is δ -discrete, with respect to the Hausdorff distance, and Z_δ has at least $\exp(2^{-N} \delta_0^{(N-1)/m} \delta^{-(N-1)/m})$ elements.*

Before proving Theorem 2.3, we need to introduce some further notation. Let $\tilde{Q} = \Omega \times (0, 2\pi)$, $\tilde{\Gamma} = A \times (0, 2\pi)$ and, for any $D \in X$, let $\tilde{Q}(D) = (\Omega \setminus D) \times (0, 2\pi)$ and $\tilde{\Gamma}(D) = \partial D \times (0, 2\pi)$. Clearly, we set $\tilde{Q}(\emptyset) = \tilde{Q}$ and $\tilde{\Gamma}(\emptyset) = \emptyset$.

Let us consider the Hilbert space $L^2(\tilde{\Gamma})$ endowed with the scalar product

$$(\psi, \phi)_0 = \int_{\tilde{\Gamma}} \psi \phi, \quad \text{for any } \psi, \phi \in L^2(\tilde{\Gamma}).$$

We can choose as an orthonormal basis of $L^2(A)$ the following set

$$\{f_{jp} : j \geq 0 \text{ and } 1 \leq p \leq p_j\} \quad (3.1)$$

where each f_{jp} is a real valued *spherical harmonic* of degree j , j being a non-negative integer. Therefore, we have that

$$-\Delta_A f_{jp} = \lambda_j f_{jp} = j(j + N - 2)f_{jp}, \quad (3.2)$$

where Δ_A is the Laplace–Beltrami operator on A . For any $j \geq 0$, the integer p_j is the dimension of the space of spherical harmonics of degree j and we have that, see for instance [20, p. 4],

$$p_j = \begin{cases} 1 & \text{if } j = 0, \\ \frac{(2j + N - 2)(j + N - 3)!}{j!(N - 2)!} & \text{if } j \geq 1, \end{cases}$$

so that

$$p_j \leq 2(j + 1)^{N-2}, \quad j \geq 0.$$

Then we have that the set

$$\left\{ \psi_{njp} = \frac{1}{\sqrt{\pi}} \sin\left(\frac{n}{2}t\right) f_{jp}(\omega) : n \geq 1, j \geq 0 \text{ and } 1 \leq p \leq p_j \right\} \quad (3.3)$$

is an orthonormal basis of $L^2(\tilde{\Gamma})$ with respect to the scalar product $(\cdot, \cdot)_0$.

For any $s, 0 < s \leq 1$, let us consider the Sobolev space $H_0^{2s,s}(\tilde{\Gamma})$. We have the following properties. First, $H_0^{2s,s}(\tilde{\Gamma}) = H^{2s,s}(\tilde{\Gamma})$ if and only if $s \leq 1/2$. Then, for any $s \neq 1/2$, we can endow $H_0^{2s,s}(\tilde{\Gamma})$ with the following scalar product

$$(\psi, \phi)_s = \sum_{\substack{n \geq 1, j \geq 0 \\ 1 \leq p \leq p_j}} \left(1 + \lambda_j^{2s} + \left(\frac{n}{2}\right)^{2s} \right) (\psi, \psi_{njp})_0 (\phi, \psi_{njp})_0,$$

for any $\psi, \phi \in H_0^{2s,s}(\tilde{\Gamma})$, (3.4)

with respect to which the set

$$\left\{ \tilde{\psi}_{njp} = \frac{\psi_{njp}}{\sqrt{1 + \lambda_j^{2s} + \left(\frac{n}{2}\right)^{2s}}} : n \geq 1, j \geq 0 \text{ and } 1 \leq p \leq p_j \right\}$$

(3.5)

is an orthonormal basis of $H_0^{2s,s}(\tilde{\Gamma})$.

For any n, j and p , let us define $\gamma(\tilde{\psi}_{njp}) = \sqrt{n + j}$ and let us call, for any positive integer q , $N_1(q) = \#\{(n, j, p) : \gamma(\tilde{\psi}_{njp}) \leq q\}$, where $\#$ denotes the number of elements. By the previous estimate on p_j , we have that

$$N_1(q) \leq 2(1 + q)^{2N}, \quad \text{for any } q \in \mathbb{N}. \tag{3.6}$$

Let $\tilde{H} = H_0^{3/2,3/4}(\tilde{\Gamma})$, $\tilde{H}' = \tilde{H}_1 = H^{-3/2,-3/4}(\tilde{\Gamma})$ and $\tilde{H}_0 = H^{1/2,1/4}(\tilde{\Gamma})$. For any $D \in X \cup \{\emptyset\}$, let us define in the same fashion as before the Dirichlet-to-Neumann map associated to D , that is the linear and bounded operator $\tilde{\mathcal{D}}(D) : \tilde{H} \mapsto \tilde{H}_0$ such that for any $\tilde{\psi} \in \tilde{H}$ we have

$$\tilde{\mathcal{D}}(D)\tilde{\psi} = \frac{\partial \tilde{u}}{\partial \nu} \Big|_{\tilde{\Gamma}},$$

where \tilde{u} solves

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} = 0 & \text{in } \tilde{Q}(D), \\ \tilde{u}(x, 0) = 0 & x \in \Omega \setminus D, \\ \tilde{u} = 0 & \text{on } \tilde{\Gamma}(D), \\ \tilde{u} = \tilde{\psi} & \text{on } \tilde{\Gamma}. \end{cases} \tag{3.7}$$

Let us also define the following two linear and bounded operators $\tilde{G}, \tilde{G}^* : \tilde{H} \mapsto \tilde{H}$ such that for any $\tilde{\psi} \in \tilde{H}$ we have

$$\begin{aligned} \tilde{G}\tilde{\psi}(\omega, t) &= \tilde{\psi}(\omega, t)e^{-1/t}t^{-3/2}, \\ \tilde{G}^*\tilde{\psi}(\omega, t) &= \tilde{\psi}(\omega, t)e^{-1/(2\pi-t)}(2\pi - t)^{-3/2}. \end{aligned} \tag{3.8}$$

We have that there exists a constant C_1 such that

$$\|\tilde{G}\|_{\mathcal{L}(\tilde{H}, \tilde{H})}, \|\tilde{G}^*\|_{\mathcal{L}(\tilde{H}, \tilde{H})} \leq C_1. \tag{3.9}$$

Now we can define, for any $D \in X \cup \{\emptyset\}$, the following linear and bounded operator $\tilde{\mathcal{D}}_1(D) : \tilde{H} \mapsto \tilde{H}'$ as follows

$$\langle \tilde{\mathcal{D}}_1(D)\tilde{\psi}, \tilde{\phi} \rangle_{\tilde{H}', \tilde{H}} = \langle \tilde{\mathcal{D}}(D)\tilde{G}\tilde{\psi}, \tilde{G}^*\tilde{\phi} \rangle_{\tilde{H}', \tilde{H}}, \quad \text{for any } \tilde{\psi}, \tilde{\phi} \in \tilde{H}. \quad (3.10)$$

Then the proof of Theorem 2.3 is an immediate consequence of the following two propositions.

Proposition 3.3 *There exists a constant C_2 , depending on m and b only, such that for any $D \in X$ and any $\tilde{\psi} \in \tilde{H}$*

$$\|(\tilde{\mathcal{D}}(D) - \tilde{\mathcal{D}}(\emptyset))\tilde{\psi}\|_{\tilde{H}_0} \leq C_2 \|\tilde{u}_0\|_{H^{2,1}(B_{7/8} \times (0, 2\pi))}, \quad (3.11)$$

where \tilde{u}_0 solves (3.7) with $D = \emptyset$.

Proposition 3.4 *There exists a positive constant δ_1 , depending on m and b only, such that for any δ , $0 < \delta < \delta_1$, we can find $D_1, D_2 \in X$ satisfying (2.6) and such that*

$$\|\tilde{\mathcal{D}}_1(D_1) - \tilde{\mathcal{D}}_1(D_2)\|_{\mathcal{L}(\tilde{H}, \tilde{H}')} \leq 2 \exp\left(-\delta^{-\frac{N-1}{2m(2N+1)}}\right). \quad (3.12)$$

Proof of Theorem 2.3 We observe that there exists a constant C_3 such that the following two inequalities are satisfied for any $\tilde{\psi} \in \tilde{H}$

$$\|\tilde{\mathcal{D}}(\emptyset)\tilde{\psi}\|_{\tilde{H}_0} \leq C_3 \|\tilde{u}_0\|_{H^{2,1}(\tilde{Q})}, \quad \|\tilde{u}_0\|_{H^{2,1}(\tilde{Q})} \leq C_3 \|\tilde{\psi}\|_{\tilde{H}}.$$

By Proposition 3.3, we have that there exists a constant C_4 , depending on m and b only, such that

$$\|\tilde{\mathcal{D}}(D)\|_{\mathcal{L}(\tilde{H}, \tilde{H}_0)} \leq C_4, \quad \text{for any } D \in X. \quad (3.13)$$

Then, we consider the following fact. For any $\psi \in H$, let $\tilde{\psi} \in \tilde{H}$ be its extension by 0 outside Γ . We have that $J : H \mapsto \tilde{H}$, where $J(\psi) = \tilde{\psi}$ for any $\psi \in H$, is a linear isometry. Furthermore, we have that the linear operators $G, G^* : H \mapsto H$ such that for any $\psi \in H$

$$G(\psi) = \tilde{G}(\tilde{\psi})|_{\Gamma}, \quad G^*(\psi) = \tilde{G}^*(\tilde{\psi})|_{\Gamma},$$

are invertible and there exists a constant C_5 such that

$$\|G\|_{\mathcal{L}(H, H)}, \|G^*\|_{\mathcal{L}(H, H)}, \|G^{-1}\|_{\mathcal{L}(H, H)}, \|(G^*)^{-1}\|_{\mathcal{L}(H, H)} \leq C_5. \quad (3.14)$$

For any $D \in X$, for any $\psi, \phi \in H$, we have that

$$\mathcal{D}(D)\psi = \tilde{\mathcal{D}}(D)J\psi|_{\Gamma} \quad (3.15)$$

and

$$\begin{aligned} \langle \mathcal{D}(D)\psi, \phi \rangle_{H', H} &= \langle \tilde{\mathcal{D}}(D)J\psi, J\phi \rangle_{\tilde{H}', \tilde{H}} \\ &= \langle \tilde{\mathcal{D}}_1(D)JG^{-1}\psi, J(G^*)^{-1}\phi \rangle_{\tilde{H}', \tilde{H}}. \end{aligned} \quad (3.16)$$

By (3.15) and (3.13), we infer that for any $D_1, D_2 \in X$ we have

$$\|\mathcal{D}(D_1) - \mathcal{D}(D_2)\|_{\mathcal{L}(H, H_0)} \leq 2C_4. \quad (3.17)$$

By (3.16) and (3.14), we infer that for any $D_1, D_2 \in X$ we have

$$\|\mathcal{D}(D_1) - \mathcal{D}(D_2)\|_{\mathcal{L}(H, H')} \leq C_5^2 \|\tilde{\mathcal{D}}_1(D_1) - \tilde{\mathcal{D}}_1(D_2)\|_{\mathcal{L}(\tilde{H}, \tilde{H}')} . \quad (3.18)$$

The proof of Theorem 2.3 follows immediately from Proposition 3.4, (3.17), (3.18) and the interpolation inequality (2.1). \square

Proof of Proposition 3.3 Let $\chi \in C_0^\infty(B_1)$ be a cutoff function such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ in } B_{5/6}, \quad \chi \equiv 0 \text{ outside } B_{7/8}.$$

Let us take the auxiliary function $v = \tilde{u} - (1 - \chi)\tilde{u}_0$, \tilde{u} being the solution to (3.7). We have that $\frac{\partial v}{\partial \nu}|_{\tilde{\Gamma}} = (\tilde{\mathcal{D}}(D) - \tilde{\mathcal{D}}(\emptyset))\tilde{\psi}$ and that v solves

$$\begin{cases} \partial_t v - \Delta v = f & \text{in } \tilde{Q}(D), \\ v(x, 0) = 0 & x \in \Omega \setminus D, \\ v = 0 & \text{on } \tilde{\Gamma}(D), \\ v = 0 & \text{on } \tilde{\Gamma}, \end{cases} \quad (3.19)$$

where $f = -\Delta \chi \tilde{u}_0 - 2\nabla \chi \cdot \nabla \tilde{u}_0$ and hence, by construction of χ , it is supported in $(\bar{B}_{7/8} \setminus B_{5/6}) \times [0, 2\pi]$ and, for a constant C_6 ,

$$\|f\|_{L^2(\tilde{Q}(\bar{B}_{1/2}))} \leq C_6 \|\tilde{u}_0\|_{H^{2,1}(B_{7/8} \times (0, 2\pi))}. \quad (3.20)$$

Then we perform the following change of variables. There exists a constant C_7 , depending on m and b only, such that for any $D \in X$ we can find a bijective function $\varphi : B_1 \mapsto B_1$ satisfying

$$\begin{aligned} \varphi(x) = x, \quad \text{for any } x \in \bar{B}_{1/4} \cup (B_1 \setminus B_{5/6}), \quad \varphi(\bar{B}_{1/2}) = D, \\ \|\varphi\|_{C^m(\bar{B}_1)}, \|\varphi^{-1}\|_{C^m(\bar{B}_1)} \leq C_7. \end{aligned}$$

Let $w(x, t) = v(\varphi(x), t)$. We have that w solves

$$\begin{cases} a \partial_t w - \operatorname{div}(A \nabla w) = f & \text{in } \tilde{Q}(\bar{B}_{1/2}), \\ w(x, 0) = 0 & x \in \Omega \setminus \bar{B}_{1/2}, \\ w = 0 & \text{on } \tilde{\Gamma}(\bar{B}_{1/2}), \\ w = 0 & \text{on } \tilde{\Gamma}, \end{cases} \quad (3.21)$$

where a is a function depending on x only, A is a symmetric $N \times N$ matrix depending on x only, and there exists a constant $c_1 > 0$, depending on m and b only, such that for any $x \in B_1$ we have

$$\begin{aligned} a(x) \geq c_1, \quad A(x) \Xi \cdot \Xi \geq c_1 \|\Xi\|^2, \quad \text{for any } \Xi \in \mathbb{R}^N, \\ \|a\|_{C^{m-1}(\bar{B}_1)}, \|A\|_{C^{m-1}(\bar{B}_1)} \leq 1/c_1. \end{aligned}$$

By standard regularity estimates, see for instance [14, Chapter III, Sect. 6], we obtain that there exists a constant C_8 , depending on c_1 only, so that

$$\|w\|_{H^{2,1}(\bar{Q}(\bar{B}_{1/2}))} \leq C_8 \|f\|_{L^2(\bar{Q}(\bar{B}_{1/2}))}.$$

Therefore, the proof is concluded by noticing that, by the trace theorem [17, Chapter 4, Theorem 2.1], there exists a constant C_9 such that

$$\left\| \frac{\partial w}{\partial \nu} \Big|_{\bar{\Gamma}} \right\|_{\tilde{H}_0} \leq C_9 \|w\|_{H^{2,1}(\bar{Q}(\bar{B}_{1/2}))}$$

and that, by construction, $\frac{\partial w}{\partial \nu} \Big|_{\bar{\Gamma}} = \frac{\partial v}{\partial \nu} \Big|_{\bar{\Gamma}}$. \square

We now turn our attention to Proposition 3.4. Let us fix integers $n \geq 1$, $j \geq 0$, p , $1 \leq p \leq p_j$, and let

$$\Psi_{njp}(\omega, t) = y_n(t) f_{jp}(\omega), \quad (\omega, t) \in A \times \mathbb{R},$$

where

$$y_n(t) = \begin{cases} \frac{1}{\sqrt{\pi}} \sin(\frac{\pi}{2}t) e^{-1/t} t^{-3/2} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

Let us remark that $y_n \in C^\infty(\mathbb{R})$ and $y_n \in H^m(\mathbb{R})$ for any nonnegative integer m . Then we consider the following boundary value problem. Let $U = U_{njp}(x, t)$ be a classical solution to

$$\begin{cases} \partial_t U - \Delta U = 0 & \text{in } \Omega \times (0, +\infty), \\ U(x, t) = 0 & (x, t) \in \Omega \times (-\infty, 0], \\ U = \Psi_{njp} & \text{on } A \times \mathbb{R}. \end{cases} \quad (3.22)$$

The following decay estimate of exponential type will be crucial.

Proposition 3.5 *For any n, j, p , let U_{njp} solve (3.22). Let us fix $\rho_0, 0 < \rho_0 < 1$. Then there exist positive constants K_1 and k_1 , depending on ρ_0 only, such that*

$$\|U_{njp}\|_{L^2(B_{\rho_0} \times \mathbb{R})} \leq K_1 \exp(-k_1 \sqrt{n+j}). \quad (3.23)$$

The proof of Proposition 3.5 is rather technical and, therefore, we postpone it to the end of the section. As a corollary of Proposition 3.5, we obtain the following result.

Lemma 3.6 *For any n, j, p , and any $D \in X$, we have*

$$\|(\tilde{D}_1(D) - \tilde{D}_1(\emptyset)) \tilde{\psi}_{njp}\|_{\tilde{H}'} \leq K_2 \exp(-k_1 \gamma(\tilde{\psi}_{njp})), \quad (3.24)$$

where K_2 depends on m and b only.

Proof By (3.9) and (3.10), and the relation between ψ_{njp} and $\tilde{\psi}_{njp}$, (3.5), we obtain that

$$\|(\tilde{\mathcal{D}}_1(D) - \tilde{\mathcal{D}}_1(\emptyset))\tilde{\psi}_{njp}\|_{\tilde{H}'} \leq C_1\|(\tilde{\mathcal{D}}(D) - \tilde{\mathcal{D}}(\emptyset))\tilde{G}\psi_{njp}\|_{\tilde{H}'}$$

Let U_{njp} solve (3.22). Then we have that the restriction of U_{njp} to the time interval $(0, 2\pi)$ solves (3.7) with $D = \emptyset$ and boundary datum $\tilde{G}\psi_{njp}$. By (3.11), we infer that there exists a constant K_3 , depending on m and b only, such that

$$\|(\tilde{\mathcal{D}}_1(D) - \tilde{\mathcal{D}}_1(\emptyset))\tilde{\psi}_{njp}\|_{\tilde{H}'} \leq K_3\|U_{njp}\|_{H^{2,1}(B_{\gamma/8} \times (0, 2\pi))}$$

Then the conclusion follows by Proposition 3.5 and a Caccioppoli-type inequality of the form

$$\|U_{njp}\|_{H^{2,1}(B_{\rho_1} \times (0, 2\pi))} \leq K_4\|U_{njp}\|_{L^2(B_{\rho_0} \times (0, 2\pi))},$$

where $0 < \rho_1 < \rho_0 < 1$ and K_4 depends on ρ_0 and ρ_1 only. For similar Caccioppoli-type inequalities we refer to [14, Chapter III, Sect. 6]. \square

Now we almost have what is needed to prove Proposition 3.4 and, hence, conclude the proof of Theorem 2.3.

Proof of Proposition 3.4 We need to introduce the operator which is adjoint to $\tilde{\mathcal{D}}_1$. For any $D \in X \cup \{\emptyset\}$, let us define the bounded and linear operator $\tilde{\mathcal{D}}^*(D) : \tilde{H} \mapsto \tilde{H}_0$ such that, for any $\tilde{\phi} \in \tilde{H}$, we have

$$\tilde{\mathcal{D}}^*(D)\tilde{\phi} = \frac{\partial \tilde{v}}{\partial \nu} \Big|_{\tilde{\Gamma}},$$

where \tilde{v} solves

$$\begin{cases} \partial_t \tilde{v} + \Delta \tilde{v} = 0 & \text{in } \tilde{Q}(D), \\ \tilde{v}(x, 2\pi) = 0 & x \in \Omega \setminus D, \\ \tilde{v} = 0 & \text{on } \tilde{\Gamma}(D), \\ \tilde{v} = \tilde{\phi} & \text{on } \tilde{\Gamma}. \end{cases} \quad (3.25)$$

Then we define $\tilde{\mathcal{D}}_1^*(D) : \tilde{H} \mapsto \tilde{H}'$ so that

$$\langle \tilde{\mathcal{D}}_1^*(D)\tilde{\phi}, \tilde{\psi} \rangle_{\tilde{H}', \tilde{H}} = \langle \tilde{\mathcal{D}}^*(D)\tilde{G}^*\tilde{\phi}, \tilde{G}\tilde{\psi} \rangle_{\tilde{H}', \tilde{H}}, \quad \text{for any } \tilde{\phi}, \tilde{\psi} \in \tilde{H}.$$

Using the weak formulation of (3.7) and (3.25), it is easy to show that the following adjointness property holds true for any $D \in X \cup \{\emptyset\}$

$$\langle \tilde{\mathcal{D}}_1(D)\tilde{\psi}, \tilde{\phi} \rangle_{\tilde{H}', \tilde{H}} = \langle \tilde{\mathcal{D}}_1^*(D)\tilde{\phi}, \tilde{\psi} \rangle_{\tilde{H}', \tilde{H}}, \quad \text{for any } \tilde{\psi}, \tilde{\phi} \in \tilde{H}.$$

It is not difficult to show, with a simple change of variable in time, that (3.24) holds true if we replace $\tilde{\mathcal{D}}_1$ with $\tilde{\mathcal{D}}_1^*$. Then, through the adjointness, we have that for any n, n', j, j', p, p'

$$|\langle (\tilde{\mathcal{D}}_1(D) - \tilde{\mathcal{D}}_1(\emptyset))\tilde{\psi}_{njp}, \tilde{\psi}_{n'j'p'} \rangle_{\tilde{H}', \tilde{H}}| \leq K_5 \exp(-k_1 \max\{\gamma(\tilde{\psi}_{njp}), \gamma(\tilde{\psi}_{n'j'p'})\}), \quad (3.26)$$

where K_5 depends on m and b only.

Then, recalling Proposition 3.2, the properties of the orthonormal basis $\{\tilde{\psi}_{njp}\}$ and of γ , in particular (3.6), and the decay estimate (3.26), we notice that we are exactly in the position of applying the abstract instability theorem stated as Theorem 2.1 in [7]. Hence, Proposition 3.4 follows. \square

Proof of Proposition 3.5 We suppose that U_{njp} can be written as follows

$$U_{njp}(x, t) = V_{njp}(r, t) f_{jp}(\omega), \quad r = \|x\|, \quad \omega = x/\|x\|.$$

For the time being, let us denote $V = V_{njp}(r, t)$, $V_t = \partial_t V$ and $V_r = \frac{\partial V}{\partial r}$, $V_{rr} = \frac{\partial^2 V}{\partial r^2}$. We have that V satisfies

$$\begin{cases} V_t - V_{rr} - \frac{N-1}{r} V_r + \frac{j(j+N-2)}{r^2} V = 0 & \text{in } (0, 1) \times (0, +\infty), \\ V(r, t) = 0 & (r, t) \in (0, 1) \times (-\infty, 0], \\ V(1, t) = y_n(t) & t \in (0, +\infty), \\ V(0, t) = 0 \quad (V_r(0, t) = 0) & t \in (0, +\infty), \quad j \geq 1 \quad (j = 0). \end{cases} \quad (3.27)$$

Let \hat{V} be the Fourier transform of V with respect to time, that is

$$\hat{V}(r, \xi) = \int_{-\infty}^{+\infty} e^{-i\xi t} V(r, t) dt, \quad \xi \in \mathbb{R}, \quad r \in (0, 1).$$

By using classical tables of integral transforms, see for instance [9], we have that the Fourier transform of y_n is, for any $\xi \in \mathbb{R}$,

$$\begin{aligned} \hat{y}_n(\xi) &= \frac{e^{-\sqrt{2|\xi-n/2|}}}{2i} (\cos(\sqrt{2|\xi-n/2|}) - \text{sign}(\xi-n/2) i \sin(\sqrt{2|\xi-n/2|})) \\ &\quad - \frac{e^{-\sqrt{2|\xi+n/2|}}}{2i} (\cos(\sqrt{2|\xi+n/2|}) - \text{sign}(\xi+n/2) i \sin(\sqrt{2|\xi+n/2|})), \end{aligned}$$

hence the following estimate holds

$$|\hat{y}_n(\xi)| \leq e^{-\sqrt{2|\xi-n/2|}} + e^{-\sqrt{2|\xi-(-n/2)|}}, \quad \text{for any } \xi \in \mathbb{R}. \quad (3.28)$$

We have that \hat{V} solves

$$\begin{cases} \hat{V}_{rr} + \frac{N-1}{r} \hat{V}_r - (i\xi + \frac{j(j+N-2)}{r^2}) \hat{V} = 0 & \text{in } (0, 1) \times \mathbb{R}, \\ \hat{V}(1, \xi) = \hat{y}_n(\xi) & \xi \in \mathbb{R}, \\ \hat{V}(0, \xi) = 0 \quad (\hat{V}_r(0, \xi) = 0) & \xi \in \mathbb{R}, \quad j \geq 1 \quad (j = 0). \end{cases} \quad (3.29)$$

If we choose σ , α and ν so that $\sigma^2 = -i\xi$, $1 - 2\alpha = N - 1$ and $\alpha^2 - \nu^2 = -j(j + N - 2)$, that is

$$\sigma = \frac{\sqrt{2|\xi|}}{2} (i - \text{sign}\xi), \quad \alpha = \frac{2-N}{2}, \quad \nu = \sqrt{\frac{(N-2)^2}{4} + j(j+N-2)}, \quad (3.30)$$

then, using formulae (5.4.11) and (5.4.12) in [15], we infer that for any $\xi \in \mathbb{R}$, any $r \in (0, 1)$,

$$\hat{V}(r, \xi) = r^{\frac{2-N}{2}} (a_1^\xi J_\nu(\sigma r) + a_2^\xi Y_\nu(\sigma r)),$$

where J_ν and Y_ν are Bessel functions of order ν of first and second kind, respectively, and a_1^ξ and a_2^ξ are coefficients depending on ξ . Concerning Bessel functions we refer mainly to [23] and [15]. We always restrict ourselves to the case in which $\nu \geq 0$, and we recall here the following basic properties. The Bessel functions have the following asymptotic behaviour as $z \rightarrow 0$

$$\begin{aligned} J_\nu &\sim z^\nu, & Y_\nu &\sim (1/z)^\nu, & \text{as } z \rightarrow 0, & \text{if } \nu > 0, \\ J_0 &\rightarrow 1, & Y_0 &\sim \log z/2, & \text{as } z \rightarrow 0, \end{aligned} \tag{3.31}$$

and these formulae hold for the derivative

$$\frac{d}{dz}(z^{-\nu} J_\nu(z)) = -z^{-\nu} J_{\nu+1}(z), \quad \frac{d}{dz}(z^{-\nu} Y_\nu(z)) = -z^{-\nu} Y_{\nu+1}(z). \tag{3.32}$$

Then, it is easy to show that the boundary condition at $r = 0$ is satisfied if and only if $a_2^\xi = 0$.

For any $r > 0$ and any $\xi \neq 0$, we have that $J_\nu(\sigma r) \neq 0$, see for instance [15, Theorem 2, p. 127] and [23, Chapter XV]. Therefore, inserting the boundary condition at $r = 1$, we obtain that for any $r \in (0, 1)$,

$$\hat{V}(r, \xi) = r^{\frac{2-N}{2}} \frac{J_\nu(\sigma r)}{J_\nu(\sigma)} \hat{y}_n(\xi), \quad \text{for any } \xi \neq 0. \tag{3.33}$$

We wish to estimate the modulus of $\frac{J_\nu(\sigma r)}{J_\nu(\sigma)}$. We use the following formula, see [23, formula 3, p. 498],

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \prod_{l=1}^{+\infty} \left(1 - \frac{z^2}{\tau_{\nu,l}^2} \right), \tag{3.34}$$

where $0 < \tau_{\nu,1} < \tau_{\nu,2} < \dots < \tau_{\nu,l} < \dots$ are the positive zeroes of J_ν and here $\Gamma(\nu + 1)$ is the value of the Gamma function in $\nu + 1$.

We infer that

$$\left| \frac{J_\nu(\sigma r)}{J_\nu(\sigma)} \right| = r^\nu \prod_{l=1}^{+\infty} \left(\frac{\tau_{\nu,l}^4 + r^4 \xi^2}{\tau_{\nu,l}^4 + \xi^2} \right)^{1/2}.$$

We have that

$$\log \prod_{l=1}^{+\infty} \left(\frac{\tau_{\nu,l}^4 + r^4 \xi^2}{\tau_{\nu,l}^4 + \xi^2} \right)^{1/2} \leq -\frac{1}{2} (1 - r^4) \xi^2 \sum_{l=1}^{+\infty} \frac{1}{\tau_{\nu,l}^4 + \xi^2}.$$

In order to evaluate the right-hand side of the last formula, we make use of the following properties of the zeroes of the Bessel functions. First, for any $l \in \mathbb{N}$, the function $\nu \mapsto \tau_{\nu,l}$ is increasing, see [23, p. 508], therefore

$$\sum_{l=1}^{+\infty} \frac{1}{\tau_{\nu,l}^4 + \xi^2} \geq \sum_{l=1}^{+\infty} \frac{1}{\tau_{[\nu]+1,l}^4 + \xi^2},$$

where $[\nu]$ denotes the integer part of ν . The so-called interlacing property [23, p. 479], implies that

$$\sum_{l=1}^{+\infty} \frac{1}{\tau_{[\nu]+1,l}^4 + \xi^2} \geq \sum_{l=[\nu]+2}^{+\infty} \frac{1}{\tau_{0,l}^4 + \xi^2}.$$

By Bôcher’s Theorem, see [23, p. 494], we obtain that $\tau_{0,l} \leq (2l - 1)\tau_{0,1}$, for any $l \in \mathbb{N}$. Therefore, if we denote $\tau = \tau_{0,1}$, which is a positive absolute constant, we can conclude that

$$\sum_{l=1}^{+\infty} \frac{1}{\tau_{\nu,l}^4 + \xi^2} \geq \sum_{l=[\nu]+2}^{+\infty} \frac{1}{\tau^4(2l - 1)^4 + \xi^2},$$

which in turn is greater than or equal to

$$\int_{[\nu]+2}^{+\infty} \frac{1}{\tau^4(2x - 1)^4 + \xi^2} dx = \frac{1}{2\tau|\xi|^{3/2}} \int_{\frac{(2[\nu]+3)\tau}{\sqrt{|\xi|}}}^{+\infty} \frac{1}{y^4 + 1} dy.$$

If we estimate the last integral, distinguishing whether the first endpoint of the interval of integration is less than 1 or not, we can find a positive absolute constant K_6 such that

$$\frac{1}{2} \sum_{l=1}^{+\infty} \frac{1}{\tau_{\nu,l}^4 + \xi^2} \geq K_6 \min \left\{ \frac{1}{|\xi|^{3/2}}, \frac{1}{(2[\nu] + 3)^3 \tau^3} \right\}.$$

We continue by noticing that there exists a positive constant K_7 such that

$$\frac{1}{(2[\nu] + 3)^3 \tau^3} \geq \frac{K_7}{(j + 1)^3}, \quad \text{for any } j \geq 0.$$

Thus, recalling (3.33), there exists $K_8 > 0$ such that we have, for any $r \in (0, 1)$ and any $\xi \neq 0$, the following crucial estimate

$$|\hat{V}(r, \xi)| \leq r^{\nu - \frac{N-2}{2}} \exp \left(-K_8(1 - r^4)\xi^2 \min \left\{ \frac{1}{|\xi|^{3/2}}, \frac{1}{(j + 1)^3} \right\} \right) |\hat{y}_n(\xi)|. \tag{3.35}$$

We recall that $|\hat{y}_n(\xi)|$ can be estimated as in (3.28).

Let us now fix $\rho_0, 0 < \rho_0 < 1$. We wish to estimate

$$\int_{B_{\rho_0} \times (0, +\infty)} |u(x, t)|^2 dx dt = \frac{1}{2\pi} \int_0^{\rho_0} \int_{-\infty}^{+\infty} r^{N-1} |\hat{V}(r, \xi)|^2 d\xi dr.$$

Here we have used the fact that $\|f_{j\rho}\|_{L^2(A)} = 1$ and Plancherel’s Theorem. Then, by (3.35) and (3.28) we have

$$\int_{B_{\rho_0} \times (0, +\infty)} |u(x, t)|^2 dx dt \leq \frac{1}{2\pi} \frac{\rho_0^{2\nu+2}}{2\nu + 2} E, \tag{3.36}$$

where

$$E \leq 4 \int_{-\infty}^{+\infty} \exp\left(-2K_8(1 - \rho_0^4)\xi^2 \min\left\{\frac{1}{|\xi|^{3/2}}, \frac{1}{(j+1)^3}\right\}\right) e^{-2\sqrt{2}|\xi-n/2|} d\xi.$$

Since $\nu \geq j$ and $\rho_0 < 1$, then

$$\frac{1}{2\pi} \frac{\rho_0^{2\nu+2}}{2\nu+2} \leq \rho_0^{2j}. \quad (3.37)$$

We now turn our attention to the term E . We distinguish between two cases.

First, if $n/4 \geq (j+1)^2$, we have, setting $K_9 = K_8(1 - \rho_0^4)$,

$$E \leq 4 \int_{|\xi| \leq n/4} \exp(-\sqrt{2n}) d\xi + 4 \int_{|\xi| \geq n/4} \exp(-K_9\sqrt{n}) \exp(-2\sqrt{2}|\xi - n/2|) d\xi,$$

therefore, for a positive absolute constant K_{10} ,

$$E \leq 2n \exp(-\sqrt{2n}) + K_{10} \exp(-K_9\sqrt{n}).$$

We conclude that there exist positive constants K_{11} , K_{12} , depending on ρ_0 only, such that

$$E \leq \begin{cases} K_{11} \exp(-K_{12}\sqrt{n}), & \text{if } n/4 \geq (j+1)^2, \\ K_{11}, & \text{if } n/4 < (j+1)^2. \end{cases} \quad (3.38)$$

Then, inserting (3.37) and (3.38) into (3.36), a simple computation yields to (3.23) with constants K_1 and k_1 depending on ρ_0 only.

In order to conclude the proof, we have just to check that if we set

$$U(x, t) = V(\|x\|, t) f_{jp}(x/\|x\|), \quad (x, t) \in \Omega \setminus \{0\} \times \mathbb{R},$$

where

$$V(r, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it\xi} \hat{V}(r, \xi) d\xi, \quad (r, t) \in (0, 1) \times \mathbb{R}$$

and \hat{V} is defined as in (3.33), then U can be extended to a continuous function on $\bar{\Omega} \times \mathbb{R}$ which satisfies (3.22).

For this purpose, the following estimate will be crucial. For any $\eta \leq 0$ and any $\xi \in \mathbb{R}$, let $\zeta = \xi + i\eta$ and $\sigma^2 = -i\zeta$, with $\arg(\sigma)$ satisfying $\pi/4 \leq \arg(\sigma) \leq 3\pi/4$. Then, if we apply (3.34) again we obtain that

$$\left| \frac{J_\nu(\sigma r)}{J_\nu(\sigma)} \right| \leq r^\nu, \quad \text{for any } \eta \leq 0, \xi \neq 0, r \in (0, 1). \quad (3.39)$$

Since the support of y_n is contained in $[0, +\infty)$, (3.39) and the Paley–Wiener theory, see for instance [11, Theorem 7.4.3], imply that, for any $r \in (0, 1)$, the support of $V(r, t)$ is contained in $[0, +\infty)$ as well. Finally, the regularity of y_n , (3.29), that is the equation and the boundary conditions at $r = 0$ and $r = 1$ satisfied by \hat{V} , the asymptotic behaviour of the Bessel functions as $z \rightarrow 0$, and, when $j = 0$, formula (3.32), imply that V satisfies (3.27) and, in turn, that the function U defined previously is indeed the solution to (3.22). \square

4 Quantitative estimates of unique continuation and proof of Theorem 2.5

First, we shall prove some Carleman estimates and we shall apply them to solutions to parabolic equations and we shall obtain an optimal three cylinder inequality at the boundary, a stability estimate for Cauchy problems and an estimate of smallness propagation, which are the main ingredients of the proof of Theorem 2.5. Then, we shall sketch the proof of Theorem 2.5. Since the Carleman estimates are obtained by the technique employed in [10] we shall adopt the notation used in that paper.

Let $\{g^{ij}(x, t)\}$ be a symmetric $N \times N$ matrix whose entries are real functions. When $\xi \in \mathbb{R}^N$ and $(x, t), (y, s) \in \mathbb{R}^{N+1}$, we assume that

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^N g^{ij}(x, t)\xi_i\xi_j \leq \lambda|\xi|^2 \quad (4.1)$$

and

$$\left(\sum_{i,j=1}^N (g^{ij}(x, t) - g^{ij}(y, s))^2 \right)^{1/2} \leq \Lambda(|x - y| + |t - s|). \quad (4.2)$$

Let q_0 be a given positive number and let L be the following parabolic operator

$$Lu = \partial_i(g^{ij}(x, t)\partial_j u) - q_0\partial_t u. \quad (4.3)$$

For any positive numbers r and t_0 we set $Q_r^{t_0} = B_r \times (-t_0, t_0)$, $\tilde{Q}_r^{t_0} = (B_r \setminus \{0\}) \times (-t_0, t_0)$. When h is a $C^{0,1}(\mathbb{R}^{N-1})$ function such that $h(0) = 0$, we shall denote by Γ_h the set

$$\Gamma_h = \{(x, t) \in \mathbb{R}^{N+1} : x_N = h(x')\},$$

we shall denote by $\Gamma_{h,r}^{t_0}$ the set $\Gamma_h \cap Q_r^{t_0}$ and we shall denote by $Q_{h,r}^{t_0}$ the set

$$Q_{h,r}^{t_0} = \{(x, t) \in Q_r^{t_0} : x_N > h(x')\}.$$

If $x = (x', h(x'))$ we denote by $\nu(x)$, or simply by ν , the unit vector of \mathbb{R}^N

$$\nu(x) = \frac{(-\nabla_{x'} h(x'), 1)}{\sqrt{1 + |\nabla_{x'} h(x')|^2}}.$$

To simplify the notation, we shall use some of the standard notation in Riemannian geometry, but we shall always drop the corresponding volume element in the definition of the Laplace–Beltrami operator associated to a Riemannian metric. We do this because it simplifies the formulae appearing in the proofs of the following lemmas, especially when the metric is allowed to depend on the time variable and we make use of partial integration with respect to this variable.

In particular, if $g(x, t) = \{g_{ij}(x, t)\}_{i,j=1}^N$ denotes the inverse matrix of the matrix of coefficients of L , we have $g^{-1}(x, t) = \{g^{ij}(x, t)\}_{i,j=1}^N$, and, for any function f and any two variable vector fields ξ and η , we set

$$\begin{aligned} \xi \cdot \eta &= \sum_{i,j=1}^N g_{ij}(x, t)\xi_i\eta_j, & |\xi|^2 &= \xi \cdot \xi, \\ \nabla f &= g^{-1}\nabla_x f, & \operatorname{div}(\xi) &= \sum_{i=1}^N \partial_i \xi^i, & \Delta f &= \operatorname{div}(\nabla f). \end{aligned}$$

When h is a $C^{0,1}(\mathbb{R}^{N-1})$ function and $x = (x', h(x'))$ we denote by $\mathbf{n}(x, t)$, or simply by \mathbf{n} , the vector

$$\mathbf{n}(x, t) = g^{-1}(x, t)v(x).$$

With this notation we have

$$Lu = \Delta u - q_0 \partial_t u.$$

For the sake of brevity, in the sequel we shall denote, respectively, by \int the integral over Q_1^1 with respect to the Lebesgue measure $dX = dx dt$ and by $\int_{\Gamma_{h,1}^1}$ the integral over $\Gamma_{h,1}^1$ with respect to the N -dimensional surface measure. In Theorem 4.1 we shall adopt the following notation. We denote

$$\rho(x, t) = \left(\sum_{i,j=1}^N g_{ij}(0, t)x_i x_j \right)^{1/2}. \tag{4.4}$$

For positive numbers a and μ to be chosen later we set

$$\sigma(x, t) = \rho(x, t) - a(\rho(x, t))^\beta \left(\sum_{j=1}^N g_{Nj}(0, t)x_j \right) \tag{4.5}$$

and

$$w(x, t) = \varphi(\sigma(x, t)), \quad \text{where } \varphi(s) = s \exp \left(\int_0^s \frac{e^{-\mu\tau^\beta} - 1}{\tau} d\tau \right). \tag{4.6}$$

Observe that ρ is the distance function from $x = 0$ associated to the metric $\sum_{i,j=1}^N g_{ij}(0, t) dx_i dx_j$ and σ is a perturbation of ρ satisfying

$$|\sigma(x, t) - \rho(x, t)| \leq C a (\rho(x, t))^{1+\beta}, \tag{4.7}$$

where C depends on λ only.

Theorem 4.1 *Assume that the parabolic operator L satisfies the conditions (4.2) and (4.3) and let us take $h \in C^{1,\beta}(B'_1)$ such that $h(0) = |\nabla_{x'} h(0)| = 0$ and $\|h\|_{C^{1,\beta}(B'_1)} \leq E$, with $0 < \beta \leq 1$ and $E > 0$. Then, there are constants $0 < d < 1$, $\mu_0 > 0$, $a_0 > 0$ and $C > 1$ depending on λ, Λ, E and β only such that, for any $\alpha \geq C(q_0 + 1)$ and any $u \in C^1(\bar{Q}_{h,1}^1) \cap C^\infty(Q_{h,1}^1)$ such that $u = 0$ on $\Gamma_{h,1}^1$ and $\text{supp } u \subset Q_d^1$, the following inequality holds*

$$C \int w^{2-2\alpha} (Lu)^2 \geq \int (\alpha w^{\beta-2\alpha} |\nabla u|^2 + \alpha^3 w^{\beta-2-2\alpha} u^2), \tag{4.8}$$

where w is defined by (4.5) and (4.6) with $a = a_0$ and $\mu = \mu_0$.

We begin the proof of the previous theorem by setting, for any function $w \in C^{2,1}(Q_1^1)$ such that $w, |\nabla w| > 0$ in $Q_{h,1}^1$, the operator $L_\alpha(f) = w^{-\alpha}L(w^\alpha f)$. We have

$$L_\alpha(f) = \Delta f + \alpha^2 \frac{|\nabla w|^2}{w^2} f - \alpha q_0 (\partial_t \log w) f + 2\alpha \frac{|\nabla w|^2}{w^2} A(f) - q_0 \partial_t f, \quad (4.9)$$

where

$$A(f) = w \frac{\nabla w \cdot \nabla f}{|\nabla w|^2} + \frac{1}{2} F_w^g f, \quad F_w^g = \frac{w \Delta w - |\nabla w|^2}{|\nabla w|^2}. \quad (4.10)$$

We shall denote by \mathcal{M}_w^g the symmetric $N \times N$ matrix

$$\mathcal{M}_w^g = \frac{1}{2} \{M_{ij} + M_{ji}\}_{i,j=1}^N, \quad (4.11)$$

where, using the summation notation of repeated indices,

$$M_{ij} = \frac{1}{2} \operatorname{div} \left(\frac{w \nabla w}{|\nabla w|^2} \right) \delta_{ij} - \partial_j \left(\frac{w g^{ik} \partial_k w}{|\nabla w|^2} \right) + \frac{1}{2} g_{jh} \frac{w g^{kl} \partial_l w}{|\nabla w|^2} \partial_k g^{hi} - \frac{1}{2} F_w^g \delta_{ij}. \quad (4.12)$$

The following lemmas hold true [10].

Lemma 4.2 *Let ρ be defined by (4.4), let φ be a positive nondecreasing function on $(0, +\infty)$. Let w be a function of $C^{2,1}(Q_1^1)$ class such that $w, |\nabla w| > 0$ in $Q_{h,1}^1$. Let*

$$\phi(s) = \varphi(s)/(s\varphi'(s)).$$

Then the symmetric matrix \mathcal{M}_w^g satisfies $\mathcal{M}_w^g \nabla w = 0$ and the following facts hold

$$\begin{aligned} F_{\varphi(w)}^g &= \phi(w) F_w^g - w \phi'(w), \\ \mathcal{M}_{\varphi(w)}^g &= \phi(w) \mathcal{M}_w^g + w \phi'(w) \left(\mathcal{I} - \frac{\nabla w \otimes \nabla w}{|\nabla w|^2} g \right), \\ F_\rho^{g(0,t)} &= N - 2, \mathcal{M}_\rho^{g(0,t)} = 0. \end{aligned}$$

Lemma 4.3 *Let w be a function of $C^{2,1}(Q_1^1)$ class such that $w, |\nabla w| > 0$ in $Q_{h,1}^1$. Then, for any $\alpha \geq 1$ and any $u \in C^1(\bar{Q}_1^1) \cap C^\infty(Q_1^1)$ such that $u = 0$ on $\Gamma_{h,1}^1$ and $\operatorname{supp} u \subset Q_1^1$, the following inequality holds*

$$\begin{aligned} & \int \frac{w^2}{|\nabla w|^2} (L_\alpha f)^2 - \mathcal{B}_w(f) \\ & \geq 4\alpha \int \mathcal{M}_w^g \nabla f \cdot \nabla f + \alpha \int F_w^g \Delta(f^2) - 2q_0 \int F_w^g f \partial_t f \\ & \quad + 2\alpha \int \frac{|\nabla w|^2}{w^2} A(f)^2 + \frac{q_0^2}{\alpha} \int \frac{w^2}{|\nabla w|^2} (\partial_t f)^2 - q_0 \int |\nabla f|^2 \partial_t \frac{w^2}{|\nabla w|^2} \\ & \quad - q_0 \int \frac{w^2}{|\nabla w|^2} \partial_t g^{ij} \partial_i f \partial_j f - 2q_0 \int \frac{w^2 \nabla |\nabla w|^2 \cdot \nabla f}{|\nabla w|^4} \partial_t f \\ & \quad - 4\alpha^2 q_0 \int (\partial_t \log w) A(f) f + 2\alpha q_0^2 \int (\partial_t \log w) \frac{w^2}{|\nabla w|^2} f \partial_t f, \quad (4.13) \end{aligned}$$

where

$$\mathcal{B}_w(f) = 2\alpha \int_{\Gamma_{h,1}^1} \frac{w \nabla w \cdot \mathbf{n}}{|\nabla w|^2 |\mathbf{n}|^2} (\nabla f \cdot \mathbf{n})^2. \quad (4.14)$$

Proof of Theorem 4.1. Let w be defined by (4.6). First of all we choose a in such a way that $\nabla w \cdot \mathbf{n} \geq 0$ in a neighbourhood of 0. Denoting by $(\cdot|\cdot)$ the Euclidean scalar product we have

$$\nabla w \cdot \mathbf{n} = \varphi'(\sigma) ((g^{-1}(x, t) - g^{-1}(0, t)) \nabla \sigma |v) + (g^{-1}(0, t) \nabla \sigma |v) \quad (4.15)$$

and

$$\begin{aligned} (g^{-1}(0, t) \nabla \sigma |v) &= a \frac{(\rho(x, t))^\beta}{\sqrt{1 + |\nabla_{x'} h|^2}} + \frac{(x' \nabla_{x'} h - h(x'))}{\rho(x, t) \sqrt{1 + |\nabla_{x'} h|^2}} \\ &\quad - \frac{a\beta (x' \nabla_{x'} h - h(x'))}{\sqrt{1 + |\nabla_{x'} h|^2}} \left(\sum_{j=1}^N g_{Nj}(0, t) x_j \right) (\rho(x, t))^{\beta-2}. \end{aligned} \quad (4.16)$$

Let us now choose $a = 2E\lambda^{-(\beta+1)/2}$. By (4.16) we obtain

$$(g^{-1}(0, t) \nabla \sigma |v) \geq C^{-1} |x'|^\beta, \quad \text{for every } x' \in B'_{1/C},$$

where $C > 1$ depends on λ , E and β only. With this value of a we also have

$$|((g^{-1}(x, t) - g^{-1}(0, t)) \nabla \sigma |v)| \leq C|x|, \quad \text{on } \Gamma_{h,1}^1,$$

where C depends on λ , Λ , E and β only.

By (4.15), (4.14) and the previous inequalities, we obtain

$$\mathcal{B}_w(f) \geq 0, \quad (4.17)$$

for all $f \in C^1(\bar{Q}_{h,1}^1) \cap C^\infty(Q_{h,1}^1)$ such that $f = 0$ on $\Gamma_{h,1}^1$ and $\text{supp } f \subset Q_{1/C}^1$, where $C > 1$ depends on λ , Λ , E and β only.

With the previous choice of a and for a fixed number $\mu \geq 1$, we have $\phi(s) = e^{\mu s^\beta}$, where ϕ is defined in Lemma 4.2. Moreover, the following properties can be easily checked on $Q_{1/C}^1$ for some constant $C > 1$ depending on λ , Λ , E , β and μ only

$$\begin{aligned} |x|/C \leq \sigma \leq C|x|, \quad \sigma/C \leq w \leq C\sigma, \quad 1/C \leq |\nabla w| \leq C, \\ |\partial_t(w^2/|\nabla w|^2)| \leq Cw^2, \quad |\partial_t \phi| \leq Cw^\beta, \quad |\nabla|\nabla w|^2| \leq Cw^{\beta-1}, \\ |\Delta \phi| \leq Cw^{\beta-2}, \quad |F_w^s| \leq C, \quad |\partial_t \log w| \leq C. \end{aligned} \quad (4.18)$$

In order to estimate from below the first integral on the right-hand side of (4.13), we observe that if we denote

$$\tilde{\nabla} f = \nabla f - \frac{(\nabla \sigma \cdot \nabla f)}{|\nabla \sigma|^2} \nabla \sigma = \nabla f - \frac{(\nabla w \cdot \nabla f)}{|\nabla w|^2} \nabla w,$$

then from Lemma 4.2 we have

$$\mathcal{M}_w^g \nabla f \cdot \nabla f = \sigma \phi' |\tilde{\nabla} f|^2 + \phi \mathcal{M}_\sigma^g \tilde{\nabla} f \cdot \tilde{\nabla} f. \tag{4.19}$$

Now, from Lemma 4.2, $\mathcal{M}_\rho^{g(0,t)} = 0$, and by straightforward calculations we have

$$|\mathcal{M}_\sigma^g| = |\mathcal{M}_\sigma^g - \mathcal{M}_\sigma^{g(0,t)}| + |\mathcal{M}_\sigma^{g(0,t)} - \mathcal{M}_\rho^{g(0,t)}| \leq C_0 \sigma^\beta, \tag{4.20}$$

for any $\sigma \leq 1/C_0$,

where $C_0 > 1$ depends on λ, Λ, E and β only (note that C_0 does not depend on μ).

By (4.19) and (4.20) we have, when $\text{supp } f \subset Q_{1/C_0}^1$,

$$\int \mathcal{M}_w^g \nabla f \cdot \nabla f \geq \int (\sigma \phi' - C_0 \sigma^\beta \phi) |\tilde{\nabla} f|^2. \tag{4.21}$$

Now we estimate from below the second and third integrals on the right-hand side of (4.13). From Lemma 4.2 we have

$$F_w^g = (N - 2)\phi + (B\phi - \sigma\phi'), \tag{4.22}$$

where $B = F_\sigma^g - (N - 2)$.

From Lemma 4.2, $F_\rho^{g(0,t)} = N - 2$, hence

$$|B(x, t)| \leq |F_\sigma^g - F_\sigma^{g(0,t)}| + |F_\sigma^{g(0,t)} - F_\rho^{g(0,t)}| \leq C_0 \sigma^\beta, \quad \text{for any } \sigma \leq 1/C_0, \tag{4.23}$$

where $C_0 > 1$ depends on λ, Λ, E and β only.

The identity

$$|\nabla f|^2 = |\tilde{\nabla} f|^2 + \frac{(\nabla w \cdot \nabla f)^2}{|\nabla w|^2}, \tag{4.24}$$

(4.22) and the divergence theorem imply that

$$\begin{aligned} \int F_w^g \Delta(f^2) &= (N - 2) \int f^2 \Delta(\phi) + 2 \int (B\phi - \sigma\phi') f \Delta f \\ &\quad + 2 \int (B\phi - \sigma\phi') |\tilde{\nabla} f|^2 \\ &\quad + 2 \int (B\phi - \sigma\phi') \frac{(\nabla w \cdot \nabla f)^2}{|\nabla w|^2}, \end{aligned} \tag{4.25}$$

and

$$\int F_w^g f \partial_t f = -\frac{(N - 2)}{2} \int f^2 \partial_t \phi + \int (B\phi - \sigma\phi') f \partial_t f. \tag{4.26}$$

By the identity (4.9), we obtain the following formula for the second integral on the right-hand side of (4.25)

$$\begin{aligned} &2 \int (B\phi - \sigma\phi') f \Delta f \\ &= 2 \int (B\phi - \sigma\phi') w^{-\alpha} f L_\alpha f + 2\alpha^2 \int (\sigma\phi' - B\phi) \frac{|\nabla w|^2}{w^2} f^2 \\ &\quad + 2 \int (\sigma\phi' - B\phi) f \left(2\alpha \frac{|\nabla w|^2}{w^2} A(f) - q_0 \partial_t f - q_0 \alpha (\partial_t \log w) f \right). \end{aligned} \tag{4.27}$$

By (4.21), and (4.25)–(4.27) we obtain, for any $\alpha \geq 1$,

$$\begin{aligned} & 4\alpha \int \mathcal{M}_w^g \nabla f \cdot \nabla f + \alpha \int F_w^g \Delta(f^2) - 2q_0 \int F_w^g f \partial_t f \\ & \geq 2\alpha \int (\sigma\phi' - 2C_0\sigma^\beta\phi + B\phi) |\tilde{\nabla} f|^2 + 2\alpha^3 \int (\sigma\phi' - B\phi) \frac{|\nabla w|^2}{w^2} f^2 - \mathcal{R}_1, \end{aligned} \quad (4.28)$$

where

$$\begin{aligned} \mathcal{R}_1 &= 2\alpha \int (B\phi - \sigma\phi') w^{-\alpha} f L_\alpha f + (N-2)\alpha \int f^2 \Delta(\phi) \\ & \quad + 2\alpha \int (\sigma\phi' - B\phi) f \left(2\alpha \frac{|\nabla w|^2}{w^2} A(f) - q_0 \partial_t f - q_0 \alpha (\partial_t \log w) f \right) \\ & \quad + (N-2)q_0 \int f^2 \partial_t \phi - 2q_0 \int (B\phi - \sigma\phi') f \partial_t f. \end{aligned}$$

By (4.18) and Young inequality we obtain

$$\begin{aligned} \mathcal{R}_1 &\leq \frac{1}{2} \int \frac{w^2}{|\nabla w|^2} (L_\alpha f)^2 + C \int (\alpha^3 w^{-2+3\beta/2} + \alpha w^{-2+\beta} + \alpha^2 (w^{-2+2\beta} + q_0)) f^2 \\ & \quad + C\alpha \int w^{-2+\beta/2} (A(f))^2 + C\alpha \int w^\beta \frac{(\nabla w \cdot \nabla f)^2}{|\nabla w|^2} + \frac{q_0^2}{2\alpha} \int \frac{w^2}{|\nabla w|^2} (\partial_t f)^2, \end{aligned} \quad (4.29)$$

where C depends on λ , Λ , E , β and μ only.

Now, let us choose $\mu = 4C_0$. With this choice of μ and by (4.18) we obtain the following estimate from below for the first term on the right-hand side of (4.28)

$$2\alpha \int (\sigma\phi' - 2C_0\sigma^\beta\phi + B\phi) |\tilde{\nabla} f|^2 \geq C_0\alpha \int w^\beta |\tilde{\nabla} f|^2, \quad \text{for any } \alpha \geq 1, \quad (4.30)$$

where C_0 depends on λ , Λ , E and β only.

On the other hand, by (4.10) we obtain

$$\begin{aligned} \int w^\beta \frac{(\nabla w \cdot \nabla f)^2}{|\nabla w|^2} &= \int \frac{|\nabla w|^2}{w^{2-\beta}} (A(f) - F_w^g f)^2 \\ &\leq 2 \int \frac{|\nabla w|^2}{w^{2-\beta}} (A(f))^2 + \frac{1}{2} \int \frac{|\nabla w|^2}{w^{2-\beta}} (F_w^g)^2 f^2. \end{aligned}$$

This inequality and the second formula in (4.24) yield to

$$\int w^\beta |\tilde{\nabla} f|^2 \geq C^{-1} \int w^\beta |\nabla f|^2 - C \int \frac{|\nabla w|^2}{w^{2-\beta}} (A(f))^2 - C \int \frac{|\nabla w|^2}{w^{2-\beta}} (F_w^g)^2 f^2,$$

where $C > 1$ depends on λ , Λ , E and β only.

The previous inequality, (4.13), (4.17), (4.28) and (4.30) yield to

$$\begin{aligned} \int \frac{w^2}{|\nabla w|^2} (L_\alpha f)^2 &\geq C^{-1} \alpha \int w^\beta |\nabla f|^2 + \alpha^3 \int (C^{-1} - C/\alpha^2) \frac{|\nabla w|^2}{w^{2-\beta}} f^2 \\ &\quad + \alpha \int (2 - Cw^\beta) \frac{|\nabla w|^2}{w^2} (A(f))^2 \\ &\quad + \frac{q_0^2}{\alpha} \int \frac{w^2}{|\nabla w|^2} (\partial_t f)^2 - \mathcal{R}_2 \end{aligned} \quad (4.31)$$

where

$$\begin{aligned} \mathcal{R}_2 &= \mathcal{R}_1 - q_0 \int \frac{w^2}{|\nabla w|^2} \partial_t g^{ij} \partial_i f \partial_j f - 2q_0 \int \frac{w^2 \nabla |\nabla w|^2 \cdot \nabla f}{|\nabla w|^4} \partial_t f \\ &\quad - 4\alpha^2 q_0 \int (\partial_t \log w) A(f) f + 2\alpha q_0^2 \int (\partial_t \log w) \frac{w^2}{|\nabla w|^2} f \partial_t f. \end{aligned}$$

Using (4.29) and Young inequality to estimate \mathcal{R}_2 from above, by (4.18) and (4.31) we obtain, for any $\alpha \geq C$ and $\text{supp} f \subset Q_{1/C}^1$,

$$\begin{aligned} \frac{1}{2} \int \frac{w^2}{|\nabla w|^2} (L_\alpha f)^2 &\geq C^{-1} \alpha \int w^\beta |\nabla f|^2 + C^{-1} \alpha^3 \int w^{-2+\beta} f^2 \\ &\quad + C^{-1} \alpha \int w^2 (A(f))^2 + \frac{q_0^2}{2\alpha} \int \frac{w^2}{|\nabla w|^2} (\partial_t f)^2, \end{aligned}$$

where $C > 1$ depends on λ , Λ , E and β only.

Finally, recalling that $f = w^{-\alpha} u$ and (4.9), we easily obtain (4.8). \square

By the Carleman estimates, we can obtain the following three cylinder inequalities. In Theorem 4.4 we state the three cylinder inequality in the interior, proved in [10], in Theorem 4.5 we state the three cylinder inequality at the boundary. We shall omit its proof because it is analogous to the one of Theorem 15 in [22].

Theorem 4.4 (Three cylinder inequality in the interior) *Let T_1 and R be positive numbers such that $T_1 \in (0, T]$, $R \in (0, R_0]$. Let κ be a symmetric $N \times N$ real matrix satisfying (2.14) and (2.15). Let $u \in H^{2,1}(D_R^{T_1})$ satisfy*

$$\partial_t u - \text{div}(\kappa(x, t) \nabla u) = 0 \text{ in } D_R^{T_1}, \quad u(x, 0) = 0.$$

Then, there exist constants $s_1 \in (0, 1)$ and $C > 1$ depending on λ and Λ only such that, for any ρ_1, ρ_2 and τ satisfying $0 < \rho_1 < \rho_2 < s_1^2 R$, $\tau \in (0, T_1)$, the following inequality holds

$$\|u\|_{L^2(D_{\rho_2}^{T_1-\tau})} \leq \gamma \left(\frac{CR}{\rho_2} \right)^\gamma \|u\|_{L^2(D_{\rho_1}^{T_1})}^\vartheta \|u\|_{L^2(D_R^{T_1})}^{1-\vartheta}, \quad (4.32)$$

where

$$\vartheta = \frac{\log \frac{s_1 R}{\rho_2}}{C \log \frac{CR}{\rho_1}} \quad \text{and} \quad \gamma = C \left(\frac{R^2}{T_1} + \frac{T_1}{\tau} \right)^C. \quad (4.33)$$

Theorem 4.5 (Three cylinder inequality at the boundary) *Let T_1 and R be positive numbers such that $T_1 \in (0, T]$, $R \in (0, R_0]$. Let κ be a symmetric $N \times N$ real matrix satisfying (2.14) and (2.15) and ψ be a function of $C^{1,\beta}$ class with constants E, R_0 such that $\psi(0) = 0$. Let $u \in H^{1,1}(D_{\psi,R}^{T_1}) \cap H_{\text{loc}}^{2,1}(D_{\psi,R}^{T_1})$ be a solution to*

$$\partial_t u - \operatorname{div}(\kappa(x, t)\nabla u) = 0 \text{ in } D_{\psi,R}^{T_1}, \quad u = 0 \text{ on } \Gamma_\psi \cap D_R^{T_1}, \quad u(x, 0) = 0.$$

Then, there exist constants $s_1 \in (0, 1)$ and $C > 1$ depending on λ, Λ, E and β only such that, for any ρ_1, ρ_2 and τ satisfying $0 < \rho_1 < \rho_2 < s_1^2 R$, $\tau \in (0, T_1)$, the following inequality holds

$$\|u\|_{L^2(D_{\psi,\rho_2}^{T_1-\tau})} \leq \gamma \left(\frac{CR}{\rho_2} \right)^\gamma \|u\|_{L^2(D_{\psi,\rho_1}^{T_1})}^\vartheta \|u\|_{L^2(D_{\psi,R}^{T_1})}^{1-\vartheta}, \quad (4.34)$$

where ϑ and γ are as in (4.33).

We now turn our attention to the stability estimates for Cauchy problems and to smallness propagation estimates, whose proofs are postponed to the appendix.

Theorem 4.6 (Stability estimate for Cauchy problems) *Let δ, T_1 and R be positive numbers such that $T_1 \in (0, T]$, $R \in (0, R_0]$. Let κ be a symmetric $N \times N$ real matrix satisfying (2.14) and (2.15) and ψ be a function of $C^{0,1}$ class with constants E, R_0 such that $\psi(0) = 0$. Let $u \in H^{1,1}(D_{\psi,R}^{T_1}) \cap H_{\text{loc}}^{2,1}(D_{\psi,R}^{T_1})$ be a solution to*

$$\partial_t u - \operatorname{div}(\kappa(x, t)\nabla u) = 0 \text{ in } D_{\psi,R}^{T_1}, \quad u = 0 \text{ on } \Gamma_\psi \cap D_R^{T_1}, \quad u(x, 0) = 0$$

and satisfy

$$\frac{1}{R_0^{N-1}T} \int_{\Gamma_\psi \cap D_R^{T_1}} (\kappa \nabla u | v)^2 \leq \delta^2. \quad (4.35)$$

Then, there exists a constant $C > 1$ depending on λ, Λ and E only such that, for any ρ_1, ρ_2 and τ satisfying $0 < \rho_2 \leq C^{-1}R$, $0 < \rho_1 < C^{-1}\rho_2$, $\tau \in (0, T_1)$, the following inequality holds

$$\|u\|_{L^2(D_{\psi,\rho_1}^{T_1-\tau})} \leq K \delta^{\vartheta_1} \|u\|_{L^2(D_{\psi,\rho_2}^{T_1})}^{1-\vartheta_1}, \quad (4.36)$$

where

$$\vartheta_1 = \frac{\log \frac{\rho_2}{\rho_1}}{C \log \frac{C\rho_2}{\rho_1}}, \quad K = \gamma_1 \left(\frac{C\rho_2}{\rho_1} \right)^{\gamma_1}, \quad \gamma_1 = C \left(\frac{R^2}{T_1} + \frac{T_1}{\tau} \right)^C.$$

In Proposition 4.7 we shall use the following notation. Let α and R be positive numbers such that $\alpha < \pi$ and let $s_1 \in (0, 1)$ be defined as in Theorem 4.4. Given $x_0, \zeta \in \mathbb{R}^N$, with $|\zeta| = 1$, we denote

$$\lambda_1 = \frac{R}{1 + \sin \alpha}, \quad w_1 = x_0 + \lambda_1 \zeta, \quad \rho_1 = (1/4)\lambda_1 s_1^2 \sin \alpha. \quad (4.37)$$

Furthermore, given a bounded domain D in \mathbb{R}^{N+1} , a number β , $0 < \beta \leq 1$, and a function u defined on D , we shall denote

$$|u|_{\beta, \beta/2, D} = \sup_{\substack{(x,t), (y,s) \in D \\ (x,t) \neq (y,s)}} \frac{|u(x,t) - u(y,s)|}{(|x - y|^2 + |t - s|)^{\beta/2}}.$$

Proposition 4.7 (Smallness propagation estimate) *Let us take positive numbers α, β, H, T_1 and R such that $\alpha < \pi, T_1 \in (0, T], R \in (0, R_0]$. Let κ be a symmetric $N \times N$ matrix such that (2.14) and (2.15) are satisfied. Let $u \in H_{\text{loc}}^{2,1}(C^{T_1}(x_0, \zeta, \alpha, R))$ be a solution to*

$$\partial_t u - \text{div}(\kappa(x, t)\nabla u) = 0 \text{ in } C^{T_1}(x_0, \zeta, \alpha, R), \quad u(x, 0) = 0 \text{ in } C(x_0, \zeta, \alpha, R),$$

such that

$$\|u\|_{L^2(C^{T_1}(x_0, \zeta, \alpha, R))} + R^\beta |u|_{\beta, \beta/2, C^{T_1}(x_0, \zeta, \alpha, R)} \leq H. \tag{4.38}$$

Then, for every $t \in (0, T_1/2)$,

$$|u(x_0, t)| \leq CH |\log((eH)^{-1} \|u\|_{L^2(D_{\rho_1}^{T_1}(w_1))})|^{-B}, \tag{4.39}$$

where C depends on $\alpha, \beta, \lambda, \Lambda$ and R^2/T_1 only and B depends on α, β, λ and Λ only.

Given these results of unique continuation, the proof of Theorem 2.5 can be concluded with a procedure which is analogous to the one used to prove the main theorems of [4, 5]. However, for the convenience of the reader, we point out here the most important steps of the proof. We begin by stating the following propositions. We shall denote by G the connected component of $\Omega_1 \cap \Omega_2$ such that $\Sigma \subset \bar{G}$.

Proposition 4.8 *Let the hypotheses of Theorem 2.5 be satisfied. We have*

$$\max_{(\bar{\Omega}_i \setminus G) \times [0, T/2]} |u_i| \leq \|f\| \omega(\varepsilon/\|f\|), \tag{4.40}$$

where ω is an increasing continuous function on $[0, \infty)$ satisfying

$$\omega(t) \leq C (\log |\log t|)^{-1/N}, \quad \text{for any } 0 < t < e^{-1}, \tag{4.41}$$

and C depends on the a priori data only.

Furthermore, if in addition we assume that there exist $L > 0$ and $r_0 \in (0, R_0]$ such that ∂G is of Lipschitz class with constants r_0, L , then (4.40) holds true with ω satisfying

$$\omega(t) \leq C_1 |\log t|^{-C_2}, \quad \text{for any } 0 < t < 1, \tag{4.42}$$

where C_1, C_2 depend on R_0/r_0 and the a priori data only.

Proposition 4.9 *Let Ω be a bounded domain in \mathbb{R}^N satisfying (2.8) and (2.9). Let $u \in W(\Omega \times (0, T))$ be the solution to (1.1), where f satisfies (2.13) and κ satisfies (2.14), (2.15). For every $\rho > 0$ and every $x_0 \in \Omega_\rho = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \rho\}$, we have*

$$\|u\|_{L^2(D_\rho^{T/4}(x_0))} \geq C \|f\|, \tag{4.43}$$

where C depends on R_0/ρ and the a priori data only.

Up to obvious changes, Proposition 4.8 can be proved following the lines of the proof of Proposition 5.2 in [5] and using standard regularity estimate for parabolic equations, see for instance [16, Chapter VI, Sect. 11], whereas the proof of Proposition 4.9 is similar to that of Proposition 5.5 in [4]. Here we recall the definition of modified distance introduced in [1].

Definition 4.10 We call modified distance between Ω_1 and Ω_2 the number

$$d_m(\Omega_1, \Omega_2) = \max \left\{ \sup_{x \in \partial\Omega_1} \text{dist}(x, \bar{\Omega}_2), \sup_{x \in \partial\Omega_2} \text{dist}(x, \bar{\Omega}_1) \right\}. \quad (4.44)$$

Note that

$$d_m(\Omega_1, \Omega_2) \leq d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2), \quad (4.45)$$

but, in general, the reverse inequality does not hold. However, the following proposition holds true [1].

Proposition 4.11 *Let Ω_1 and Ω_2 be bounded domains satisfying (2.8) and (2.9). For any $i = 1, 2$, let A_i, I_i satisfy (2.10). Let us also assume that $A_1 = A_2 = A$ and that Ω_1 and Ω_2 lie on the same side of A . There exist numbers $d_0 > 0, r_0 \in (0, R_0]$, such that $\frac{d_0}{R_0}$ and $\frac{r_0}{R_0}$ depend on E only and the following facts hold true. If*

$$d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq d_0, \quad (4.46)$$

then there exists an absolute constant $C > 0$ such that

$$d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2) \leq C d_m(\Omega_1, \Omega_2), \quad (4.47)$$

and any connected component of $\Omega_1 \cap \Omega_2$ has boundary of Lipschitz class with constants r_0, L where r_0 is as defined previously and $L > 0$ depends on E only.

Proof of Theorem 2.5 For the sake of simplicity we denote $d = d_{\mathcal{H}}(\bar{\Omega}_1, \bar{\Omega}_2)$ and $d_m = d_m(\Omega_1, \Omega_2)$. Let us prove that if $\sigma > 0$ is such that

$$\|u_i\|_{L^2((\Omega_i \setminus G) \times (0, T/2))} \leq \sigma, \quad i = 1, 2, \quad (4.48)$$

then

$$d, d_m \leq C_1 R_0 (\sigma / \|f\|)^{C_2}, \quad (4.49)$$

where C_1, C_2 depend on the a priori data only.

We begin by proving (4.49) for d_m . We may assume, without loss of generality, that there exists $x_0 \in I_1 \subset \partial\Omega_1$ such that $\text{dist}(x_0, \Omega_2) = d_m$. By (4.48) we have

$$\|u_1\|_{L^2((\Omega_1 \cap B_{d_m}(x_0)) \times (0, T/2))} \leq \sigma. \quad (4.50)$$

Let s_1 be defined as in Theorem 4.5. We distinguish two cases. If $d_m \geq s_1^2 R_0 / 2$, let $\bar{d} = \frac{s_1^2 R_0}{2(1+\sqrt{1+E^2})}$, $\bar{x} = x_0 - \nu \bar{d} \sqrt{1+E^2}$, where ν denotes the outer unit normal to Ω_1 at x_0 . We have

$$B_{\bar{d}}(\bar{x}) \subset \Omega_1 \cap B_{(s_1^2 R_0 / 2)}(x_0).$$

By Proposition 4.9, (4.50) and the previous inclusion we have

$$\sigma \geq \|u_1\|_{L^2((\Omega_1 \cap B_{s_1^2 R_0 / 2}(x_0)) \times (0, T/2))} \geq C \|f\|, \quad (4.51)$$

where C depends on the a priori data only. Since it is evident that $d_m \leq CR_0$, where C depends on E and M only, and $d_m \geq s_1^2 R_0/2$, by (4.51) we have that, in this case, d_m satisfies (4.49) with C_1 depending on the a priori data only and $C_2 = 1$.

Otherwise, if $d_m < s_1^2 R_0/2$, let us apply Theorem 4.5 with $\rho_1 = d_m$, $\rho_2 = s_1^2 R_0/2$, $R = R_0$, $T_1 = T/2$, $\tau = T/4$ and, by (4.50), we obtain

$$\|u_1\|_{L^2((\Omega_1 \cap B_{s_1^2 R_0/2}(x_0)) \times (0, T/4))} \leq C \|f\| (\sigma/\|f\|)^{\frac{1}{C \log \frac{CR_0}{d_m}}},$$

where C depends on the a priori data only. Then, by Proposition 4.9 and the previous inequality, we can conclude that d_m satisfies (4.49) also in this case.

We have proved that (4.49) holds for d_m and, now, we use this result to show that (4.49) holds for d , too. Without loss of generality, we may assume that there exists $y_0 \in \bar{\Omega}_1$ such that $\text{dist}(y_0, \bar{\Omega}_2) = d$. Let us denote $\delta = \text{dist}(y_0, \partial\Omega_1)$ and let us distinguish three cases, depending on the value of δ with respect to d and d_0 , d_0 as in Proposition 4.11.

First, if $\delta \leq d/2$, we take $z_0 \in \partial\Omega_1$ such that $|y_0 - z_0| = \delta$ and we have

$$d_m \geq \text{dist}(z_0, \bar{\Omega}_2) \geq d - \delta \geq \delta/2,$$

hence $\delta \leq 2d_m$ and, since (4.49) holds for d_m , we have that it also holds for d .

Second, if $d/2 < \delta \leq d_0/2$, then $d < d_0$ and Proposition 4.11 applies, hence by (4.47) d can be controlled by d_m and therefore d satisfies (4.49) as well.

Third, if $\delta > \max\{d/2, d_0/2\}$, let us set $d_1 = \min\{d/2, s_1^2 d_0/4\}$ where $s_1 \in (0, 1)$ has been introduced in Theorem 4.4. We have

$$B_{d_1}(y_0) \subset \Omega_1 \setminus \Omega_2, \quad B_{s_1^2 d_0/2}(y_0) \subset \Omega_1. \quad (4.52)$$

Now, let us apply Theorem 4.4 with $\rho_1 = d_1$, $\rho_2 = s_1^2 d_0/2$, $R = d_0$, $T_1 = T/2$, $\tau = T/4$ and, by (4.48) and (4.52), we have

$$\|u_1\|_{L^2((\Omega_1 \cap B_{s_1^2 d_0/2}(y_0)) \times (0, T/4))} \leq C \|f\| (\sigma/\|f\|)^{\frac{1}{C \log \frac{CR_0}{d_m}}},$$

where C depends on the a priori data only. Then, by Proposition 4.9, we obtain that

$$d_1 \leq C_1 R_0 (\sigma/\|f\|)^{C_2}, \quad (4.53)$$

with C_1 and C_2 depending on the a priori data only.

Now, let $\bar{\sigma} = \|f\| (\frac{s_1^2 d_0}{4C_1 R_0})^{C_2}$. If $\sigma < \bar{\sigma}$, then $d_1 < (s_1^2 d_0)/4$, hence $d = 2d_1$ and (4.49) follows by (4.53). If $\sigma \geq \bar{\sigma}$ then (4.49) immediately follows.

By Proposition 4.8 and (4.49) we have

$$d \leq C_1 R_0 |\log |\log(\varepsilon/\|f\|)||^{-C_2},$$

where C_1, C_2 depend on the a priori data only. Then, there exists $\varepsilon_0 > 0$, depending on the a priori data only, such that if $\varepsilon \leq \varepsilon_0$ then $d \leq d_0$. Thus, by Proposition 4.11, G satisfies the hypotheses of Proposition 4.9. Hence in (4.49) we may replace σ with $\|f\| \omega(\varepsilon/\|f\|)$ where $\omega(t) \leq C_1 |\log t|^{-C_2}$, C_1, C_2 depending on the a priori data only, and thus (2.17) follows. \square

Appendix

Proof of Theorem 4.6 We present here only the most remarkable steps of the proof.

First of all, we extend the function u to the cylinder $Q_{\psi,R}^{T_1}$ by setting it equal to zero outside $D_{\psi,R}^{T_1}$ and we continue to denote by u such an extension. Then, let us introduce the change of variables $y = R^{-1}x, s = T_1^{-1}t$. Denoting by $v(y, s) = u(Ry, T_1s)$ and $h(y') = \psi(Ry')$, we have that $v \in H^{1,1}(Q_{h,1}^1) \cap H_{loc}^{2,1}(Q_{h,1}^1)$ is a solution to

$$Lv = \partial_i(g^{ij}(y, s)\partial_j v) - q_0\partial_s v = 0 \text{ in } Q_{h,1}^1, \quad v = 0 \text{ on } \Gamma_{h,1}^1, \tag{A.1}$$

where $g^{-1}(y, s) = \kappa(Ry, T_1s)$ and $q_0 = R^2T_1^{-1}$, and v satisfies also

$$\int_{\Gamma_{h,1}^1} (g^{-1}\nabla v|v)^2 \leq \delta^2. \tag{A.2}$$

Moreover, the matrix g^{-1} satisfies (4.1) and (4.2) and h is a function of Lipschitz class with constants $E, 1$ and such that $h(0) = 0$.

Let ε be a positive number to be chosen later. Let us denote

$$w(y, s) = \varphi(\rho_\varepsilon(y, s)), \quad \text{where } \varphi(\eta) = \eta \exp\left(\int_0^\eta \frac{e^{-\mu\tau} - 1}{\tau} d\tau\right) \tag{A.3}$$

and

$$\rho_\varepsilon(y, s) = \left(\sum_{i,j=1}^N g_{ij}(0, t)(y + \varepsilon e_N)_i(y + \varepsilon e_N)_j\right)^{1/2}. \tag{A.4}$$

Proceeding in the same manner as in the proof of Theorem 4.5, see [22], it is not difficult to show that there exist constants $C > 1, d \in (0, 1)$, depending on λ, Λ and E only, such that, putting $\mu = C$ in (A.3), for any $\alpha \geq C(q_0 + 1)$ and any $\tilde{v} \in H^{1,1}(Q_{h,1}^1) \cap H_{loc}^{2,1}(Q_{h,1}^1)$ such that $\tilde{v} = 0$ on $\Gamma_{h,1}^1$ and $\text{supp } \tilde{v} \subset Q_d^{1,\rho_\varepsilon}$, the following inequality holds

$$\begin{aligned} C \int w^{2-2\alpha} (L\tilde{v})^2 - 2\alpha \int_{\Gamma_{h,1}^1} \frac{w^{1-2\alpha}(\nabla w \cdot \mathbf{n})}{|\nabla w|^2|\mathbf{n}|^2} (\nabla \tilde{v} \cdot \mathbf{n})^2 \\ \geq \int (\alpha w^{1-2\alpha} |\nabla \tilde{v}|^2 + \alpha^3 w^{-1-2\alpha} \tilde{v}^2). \end{aligned} \tag{A.5}$$

Let us apply the previously stated inequality to the function $\tilde{v} = v\zeta$, where v satisfies (A.1) and (A.2) and ζ is a function of class $C_0^2(Q_d^1)$ which is defined as follows. Let us fix $d_1 \in (0, \lambda d)$, $r \in (\varepsilon, d_1/2)$ and $s_0 \in (0, 1)$. Let us denote $t_1 = 1 - s_0/2, t_2 = 1 - s_0$. Let ψ be the even function such that $\psi \in C_0^2(-1, 1)$, ψ is equal to 1 in $[-t_2, t_2]$, it is equal to 0 in $[-1, -t_1]$ and

$$\psi(s) = \exp\left(-\left(\frac{|t_2 + s|}{t_1 + s}\right)^3\right), \quad \text{for any } s \in (-t_1, -t_2].$$

Let f be a function in $C^2([0, d_1])$ that is equal to 1 in $[0, d_1/2]$ and is equal to 0 in $[3d_1/4, d_1]$. Moreover, assume that $|f'| \leq c/d_1, |f''| \leq c/d_1^2$ in $[d_1/2, 3d_1/4]$, where c is an absolute constant.

Then, ζ is defined by

$$\zeta(y, s) = f(\rho_\varepsilon(y, s))\psi(s),$$

and we denote $Q_{h,r}^{2,\rho_\varepsilon} = \{(y, s) \in \mathbb{R}^{N+1} : \rho_\varepsilon(y, s) < r, y_N > h(y'), s \in (-t_2, t_2)\}$. Let us choose $\varepsilon = r\sqrt{\lambda}/2$ in (A.3), (A.4). By (A.2) and (A.5) we have that there exists a constant $C > 1$

depending on λ , Λ and E only such that for all $\alpha \geq C(q_0 s_0^{-1} + 1)^{12}$ the following inequality holds

$$\|v\|_{L^2(Q_{h,r}^1)} \leq C(d_1^{-2} + q_0 s_0^{-1}) \left(\left(\frac{\varphi(r)}{\varphi(d_1/2)} \right)^\alpha \|u\|_{L^2(Q_{h,d_1}^1)} + \left(\frac{\varphi(r)}{\varphi(r_0)} \right)^\alpha \delta \right) \quad (\text{A.6})$$

where $r_0 = \frac{r\sqrt{\lambda}}{2\sqrt{1+E^2}}$. Now, set

$$\alpha_1 = \frac{\log(\|u\|_{L^2(Q_{h,d_1}^1)}/\delta)}{\log(\varphi(d_1/2)/\varphi(r_0))}.$$

If $\alpha_1 \geq C(q_0 s_0^{-1} + 1)^{12}$ then we choose $\alpha = \alpha_1$ in (A.6) and we obtain

$$\|v\|_{L^2(Q_{h,r}^1)} \leq C(d_1^{-2} + q_0 s_0^{-1}) \delta^{\vartheta_0} \|u\|_{L^2(Q_{h,d_1}^1)}^{1-\vartheta_0}, \quad (\text{A.7})$$

where

$$\vartheta_0 = \frac{\log(\varphi(d_1/2)/\varphi(r))}{\log(\varphi(d_1/2)/\varphi(r_0))}.$$

If $\alpha_1 \leq C(q_0 s_0^{-1} + 1)^{12}$ then, by an easy estimate from the previous equation of the left-hand side of (A.6), we obtain

$$\|v\|_{L^2(Q_{h,r}^1)} \leq C(d_1^{-2} + q_0 s_0^{-1} + (Cd_1/r)^{\alpha_1}) \delta^{\vartheta_0} \|u\|_{L^2(Q_{h,d_1}^1)}^{1-\vartheta_0}. \quad (\text{A.8})$$

By (A.7) and (A.8), returning to the original variables, it is easy to obtain the inequality (4.36). \square

Proof of Proposition 4.7 For the sake of simplicity we assume that $x_0 = 0$, $\zeta = e_N$. Let

$$a = \frac{1 - (1/4)s_1^2 \sin \alpha}{1 + (1/4)s_1^2 \sin \alpha}$$

and, for every $k \geq 2$,

$$\lambda_k = a^{k-1} \lambda_1, \quad w_k = \lambda_k e_N, \quad \rho_k = a^{k-1} \rho_1.$$

It is simple to check that, for every $k \geq 1$, the following inclusions hold true

$$B_{\rho_{k+1}}(w_{k+1}) \subset B_{3\rho_k}(w_k) \subset B_{4s_1^{-2}\rho_k}(w_k) \subset C(0, e_N, \alpha, R). \quad (\text{A.9})$$

Denote by

$$d_k = \lambda_k - \rho_k = a^{k-1} \lambda_1 (1 - (1/4)s_1^2 \sin \alpha), \quad k \geq 1.$$

For a number r belonging to $(0, d_1]$, to be chosen later, let us set $\bar{k} = \min\{k \in \mathbb{N} : d_k \leq r\}$. We have

$$\frac{|\log(r/d_1)|}{|\log a|} \leq \bar{k} - 1 \leq \frac{|\log(r/d_1)|}{|\log a|} + 1. \quad (\text{A.10})$$

Moreover, for $j = 0, 1, \dots, \bar{k}$, we set $t_j = T_1(1 - \frac{j}{2\bar{k}})$ and $\sigma_j = \|u\|_{L^2(D_{\rho_j}^j(w_j))}$.

By Theorem 4.4 and (A.9), and since obviously $\sigma_{j+1} \leq \|u\|_{L^2(D_{3\rho_{j+1}}^{t_{j+1}}(w_{j+1}))}$, we obtain

$$\sigma_{j+1} \leq \bar{K} \sigma_j^{\theta_*} (\|u\|_{L^2(C^{T_1}(0, e_N, \alpha, R))})^{1-\theta_*}, \quad \text{for } j = 0, 1, \dots, \bar{k}, \quad (\text{A.11})$$

where

$$\theta_* = \frac{\log(4/3s_1)}{C_1 \log(4C_1/s_1^2)}, \quad \bar{K} = e^{C_2(1+\bar{k})C_1},$$

and here C_1 depends on λ and Λ only and C_2 depends on λ , Λ and R^2/T_1 only. By iterating (A.11), we obtain that

$$\sigma_{\bar{k}} \leq \bar{K}^{1/(1-\theta_*)} \sigma_1^{\theta_{\bar{k}}} (\|u\|_{L^2(C^{T_1}(0, e_N, \alpha, R))})^{1-\theta_{\bar{k}}}. \tag{A.12}$$

Let us recall the following interpolation inequality

$$\begin{aligned} \|v\|_{L^\infty(D_\rho^{T_0})} &\leq C \left(\rho_0^{-(N+2)} \int_{D_\rho^{T_0}} v^2 \right)^{\frac{\beta}{N+2+2\beta}} (|v|_{\beta, \beta/2, D_\rho^{T_0}})^{\frac{N+2}{N+2+2\beta}} \\ &\quad + C \left(\rho_0^{-(N+2)} \int_{D_\rho^{T_0}} v^2 \right)^{1/2}, \end{aligned} \tag{A.13}$$

where $\rho_0 = \min\{\rho, \sqrt{T_0}\}$ and C is an absolute constant. By (4.38), (A.12) and (A.13), we obtain

$$\|u\|_{L^\infty(D_{\rho_{\bar{k}}}^{T_{\bar{k}}}(w_{\bar{k}}))} \leq C (a^{1-\bar{k}})^{(N+2)/2} H^{\frac{N+2}{N+2+2\beta}} (\sigma_{\bar{k}})^{\frac{2\beta}{N+2+2\beta}}, \tag{A.14}$$

where C depends on λ , Λ and R^2/T_1 only. Let us consider the point $x_r = r e_N$. We have that $(x_r, t) \in D_{\rho_{\bar{k}}}^{T_{\bar{k}}}(w_{\bar{k}})$ for every $t \in (0, T_1/2)$. By (4.38) and (A.14), we have

$$\begin{aligned} |u(0, t)| &\leq |u(x_r, t) - u(0, t)| + |u(x_r, t)| \\ &\leq CH \left(\left(\frac{r}{d_1} \right)^\beta + (a^{1-\bar{k}})^{(N+2)/2} \left(\frac{\sigma_{\bar{k}}}{H} \right)^{\frac{2\beta}{N+2+2\beta}} \right), \end{aligned}$$

where C depends on λ , Λ and R^2/T_1 only. From this last inequality and (A.12) we obtain

$$|u(0, t)| \leq CH \left(\left(\frac{r}{d_1} \right)^\beta + (a^{1-\bar{k}})^{(N+2)/2} e^{C(1+\bar{k})C_1} \left(\frac{\sigma_1}{H} \right)^{\frac{2\beta\theta_{\bar{k}}}{N+2+2\beta}} \right), \tag{A.15}$$

where C depends on α , β , λ , Λ and R^2/T_1 only and C_1 depends on λ and Λ only. Let us choose

$$r = d_1 \left| \log \left(\frac{1}{e} \left(\frac{\sigma_1}{H} \right)^{\frac{2\beta}{N+2+2\beta}} \right) \right|^{-\frac{|\log \alpha|}{2|\log \theta_*|}}.$$

From (A.15), taking into account (A.10), we obtain

$$|u(0, t)| \leq CH \left| \log \left(\frac{\sigma_1}{eH} \right) \right|^{-\frac{|\log \alpha|}{2\beta|\log \theta_*|}}$$

and (4.39) follows. □

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