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Disconjugacy and the rotation number for linear, non-autonomous Hamiltonian systems*

This paper is dedicated to Prof. Roberto Conti on the occasion of his eightieth birthday

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Abstract. The rotation number for non-autonomous Hamiltonian systems is used to characterize the weak disconjugacy property for such systems. The argument functions of Lidskiĭ-Yakubovich are an important tool in the proofs. It is shown that weakly disconjugate systems admit principal solutions.

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Key words. disconjugacy – rotation number – argument function

1. Introduction

There is an extended classical theory concerning linear Hamiltonian systems with the disconjugacy property. The basic facts concerning disconjugate Hamiltonian systems are discussed in [6] and [14]. One of the main results of the classical theory states that, if mild auxiliary hypotheses hold, then a disconjugate Hamiltonian system admits a principal solution.

Recently, disconjugate systems have been studied using the methods of the modern theory of non-autonomous differential systems [19,20]. Many of these methods are drawn from the fields of topological dynamics and ergodic theory. They make it possible to study the dynamical and ergodic properties of principal solutions. In fact, information about these solutions can be extracted from the flow induced by the Hamiltonian system and its time-translates on a certain fiber bundle whose fiber Λ is the set of Lagrange subspaces of $\mathbb{R}^{2n}$.

In this paper, we will prove a result which implies that, for a large class on non-autonomous Hamiltonian systems, the disconjugacy property can be character-

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ized in an ergodic-theoretic way. More generally, we will give an ergodic-theoretic characterization of a property we call weak disconjugacy; this last property is often but not always equivalent to the true disconjugacy. The characterization of weak disconjugacy is stated in terms of the rotation number for linear Hamiltonian systems [15, 10, 11, 28]. This quantity can also be used to discuss the exponential dichotomy property for systems of the form (1) [17], and to study certain issues in control theory revolving around the Frequency Theorem of Yakubovich [35, 37, 12].

Let us describe our results in somewhat more detail. Our starting point is the time-varying linear Hamiltonian system

$$J\,\mathbf{z}' = H(t)\,\mathbf{z} \qquad (\mathbf{z} \in \mathbb{R}^{2n})\,. \tag{1}$$

Here $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$ is the usual antisymmetric matrix; I_n represents the $n \times n$ identity matrix. The coefficient $H(\cdot)$ takes values in the set of real symmetric $2n \times 2n$ matrices. Sometimes we will write $\mathbf{z} = \left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{y} \end{smallmatrix}\right)$ where $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$. It will be convenient to write $H(\cdot)$ in the block form

$$H(t) = \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix} \tag{2}$$

where $H_{11}^* = H_{11}$, $H_{22}^* = H_{22}$ and $H_{12}^* = H_{21}$. Here * denotes the matrix transpose. It will be assumed that $H_{22} \geq 0$ for all $t \in \mathbb{R}$, and that a certain controllability condition is valid (see Hypothesis 2 below). This condition will serve as a substitute for the hypothesis of identical normality which is often imposed when studying disconjugate linear Hamiltonian systems. We will assume throughout the paper that the function $H(\cdot)$ is uniformly bounded and uniformly continuous. (Our results are almost certainly valid under the weaker assumptions of boundedness and Lebesgue measurability, but technical problems prevent us from discussing this more general case.)

When $H(\cdot)$ is bounded and uniformly continuous, its *hull* $\Omega = \Omega_H$ is a compact metric space, and the one-parameter group $\{\tau_t \mid t \in \mathbb{R}\}$ defined by translating the argument of H induces a topological flow on Ω. Deferring the discussion of this and other dynamical/ergodic concepts to Section 2, let us suppose that the flow $(\Omega, \{\tau_t\})$ admits an ergodic measure μ whose topological support is all of Ω. Though this assumption is not satisfied for all uniformly continuous functions $H(\cdot)$, it is quite natural if one is interested in such ergodic-theoretic quantities related to non-autonomous linear systems as Lyapunov exponents and rotation numbers.

Let $\alpha(\mu)$ be the *rotation number* corresponding to μ; again we refer to Section 2 for a discussion of this concept. Then if H_{22} is positive semi-definite, and if the controllability Hypothesis 2 holds, a necessary and sufficient condition that all equations (1) corresponding to functions $\widetilde{H}$ in the hull Ω of H be weakly disconjugate is that $\alpha(\mu) = 0$. In other words, all these equations are weakly disconjugate exactly when the "average rotation" of their solutions is zero. We mention two corollaries of this result. First of all, if $\widetilde{H}_{22}(t) > 0$ for all $t \in \mathbb{R}$ and all $\widetilde{H} \in \Omega$, then weak disconjugacy implies disconjugacy. So in this case we obtain a sufficient condition (namely $\alpha(\mu) = 0$) for the disconjugacy of system (1).

Second, if (1) is weakly disconjugate for each $\widetilde{H} \in \Omega$, then (1) admits a principal solution for each $\widetilde{H} \in \Omega$.

The results presented in this paper are related to those proved in [12], where the non-oscillation concept of Yakubovich [35,37] was discussed in the context of non-periodic, non-autonomous Hamiltonian systems. However, in [12], emphasis was placed on the concept of exponential dichotomy and its relation to the rotation number. Here we investigate the connection between the rotation number and the more subtle notion of disconjugacy.

We finish the Introduction by giving some definitions and notation which will be used without comment throughout the paper.

Definition 1. The system (1) is said to be *disconjugate on* $[0, \infty)$ – or simply disconjugate – if for each non-zero solution $\mathbf{z}(t) = \binom{\mathbf{x}(t)}{\mathbf{y}(t)}$ of (1), $\mathbf{x}(t) = 0$ for at most one $t \in (0, \infty]$.

Definition 2. The system (1) is said to be *weakly disconjugate* (*on* $[0, \infty)$) if there exists $T \geq 0$ such that, whenever $\mathbf{z}(t)$ is a non-trivial solution of (1) such that $\mathbf{x}(0) = 0$, there holds $\mathbf{x}(t) \neq 0$ for all $t > T$.

We introduce the following notational conventions. Let $\mathcal{M}_k$ denote the set of all real $k \times k$ matrices and let $\mathcal{S}_k \subset \mathcal{M}_k$ denote the set al all symmetric real $k \times k$ matrices ($k \geq 1$). The symbol $\langle\ ,\ \rangle$ will indicate the Euclidean inner product on $\mathbb{R}^k$, and $\|\cdot\|$ the corresponding norm ($k \geq 1$). Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$ ($n \geq 1$). If $n \geq 1$, a Lagrange subspace λ of $\mathbb{R}^{2n}$ is a vector subspace $\lambda \subset \mathbb{R}^{2n}$ of dimension n such that $\langle \mathbf{x}, J\mathbf{y} \rangle = 0$ for all $\mathbf{x}, \mathbf{y} \in \lambda$. Let $\Lambda = \{\lambda\}$ denote the set of all real Lagrange subspaces of $\mathbb{R}^{2n}$.

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2. Preliminaries

First we introduce the so-called hull of the coefficient matrix $H(\cdot)$ in equation (1); see [5,23,30]. Let $\mathcal{H}$ denote the set of all bounded uniformly continuous functions $H : \mathbb{R} \to \mathcal{S}_{2n}$. Give $\mathcal{H}$ the topology of uniform convergence on compact sets (compact-open topology). For each $t \in \mathbb{R}$, let $\tau_t : \mathcal{H} \to \mathcal{H}$ be the t-translation, defined by $\tau_t(H)(\cdot) = H(\cdot + t)$. Then $\{\tau_t \mid t \in \mathbb{R}\}$ defines a topological flow on $\mathcal{H}$. That is, the following conditions are satisfied: (i) $\tau_0(H) = H$ for all $H \in \mathcal{H}$; (ii) $\tau_{t+s}(H) = \tau_t(\tau_s(H))$ for all $H \in \mathcal{H}$ and all $t, s \in \mathbb{R}$; (iii) the map $\mathcal{H} \times \mathbb{R} \to \mathcal{H} : (H, t) \to \tau_t(H)$ is continuous. We let $(\mathcal{H}, \{\tau_t\})$ indicate this flow.

Let H be a fixed element of $\mathcal{H}$. Then the *hull* $\Omega = \Omega_H = \mathrm{cls}\{\tau_t(H) \mid t \in \mathbb{R}\}$ is a compact subset of $\mathcal{H}$. It is also *invariant* in the sense that, if $\omega \in \Omega$, then $\tau_t(\omega) \in \Omega$ for all $t \in \mathbb{R}$. Hence $(\Omega, \{\tau_t\})$ is a topological flow. For each $\omega \in \Omega$, set $\widetilde{H}(\omega) = \omega(0)$. Then $\widetilde{H}(\tau_t(\omega)) = \omega(t)$ for all $t \in \mathbb{R}$; that is, $\widetilde{H}(\tau_t(\omega))$ "reproduces" the function $t \to \omega(t)$. Let $\omega_0 = H \in \Omega$; then $\widetilde{H}(\tau_t(\omega_0)) = H(t)$ for all $t \in \mathbb{R}$. Clearly $\widetilde{H} : \Omega \to \mathcal{S}_{2n}$ is continuous.

Let us now abuse notation and write H instead of $\widetilde{H}$. We consider the family of equations

$$J\,\mathbf{z}' = H(\tau_t(\omega))\,\mathbf{z} \tag{1_ω}$$

where ω ranges over Ω. If $\omega = \omega_0$, equation (1_{ω_0}) coincides with equation (1). At this point, there is no particular reason to require that Ω be the hull of a fixed function in $\mathcal{H}$. From now on, we will let Ω be a general compact metric space which supports a topological flow $(\Omega, \{\tau_t\})$, and let $H : \Omega \to \mathcal{S}_{2n}$ be a continuous function.

We will use various techniques from topological dynamics and from ergodic theory to study the family of equations (1_ω). We repeat some basic definitions. Let $\omega_0 \in \Omega$; the *orbit* through ω_0 is $\{\tau_t(\omega_0) \mid t \in \mathbb{R}\}$. The *positive semi-orbit* containing ω_0 is $\{\tau_t(\omega_0) \mid t \geq 0\}$, while the *negative semi-orbit* is $\{\tau_t(\omega_0) \mid t \leq 0\}$. The *omega-limit set* of ω_0 is by definition $\{\omega \in \Omega \mid$ there exists a sequence $t_n \to \infty$ such that $\tau_{t_n}(\omega_0) \to \omega$ as $n \to \infty\}$. The *alpha-limit set* is defined analogously, using sequences $t_n \to -\infty$. Both the omega-limit set and the alpha-limit set of ω_0 are compact invariant subsets of Ω. A compact invariant subset $M \subset \Omega$ is said to be *minimal* [8] if for every $\omega \in M$ the orbit through ω is dense in M. It is easy to see that, if M is minimal, then each positive or negative semi-orbit in M is dense in M.

Now let μ be a regular Borel probability measure on Ω. The *topological support* $\operatorname{Supp}\mu$ is by definition the complement in Ω of the largest relatively open subset $W \subset \Omega$ which satisfies $\mu(W) = 0$. The measure μ is said to be $\{\tau_t\}$-*invariant* if for each Borel subset $B \subset \Omega$ and each $t \in \mathbb{R}$ there holds $\mu(\tau_t(B)) = \mu(B)$. The measure μ is said to be $\{\tau_t\}$-*ergodic* if, in addition, it satisfies the following indecomposability condition: whenever $B \subset \Omega$ is a Borel set such that the symmetric difference $\tau_t(B)\triangle B$ has μ-measure zero for all $t \in \mathbb{R}$, then either $\mu(B) = 0$ or $\mu(B) = 1$. Using a classical construction of Krylov and Bogoliubov (see, e.g., [24]), one shows that there exists at least one invariant measure on Ω. The existence of an ergodic measure on Ω can be then proved using the Krein-Mil'man Theorem (e.g. [7]).

Next, let $\omega \in \Omega$, and let $\Phi_\omega(t)$ be the fundamental matrix solution at $t = 0$ of equation (1_ω). Thus $\Phi_\omega(t)$ is the $2n \times 2n$ matrix function satisfying (1_ω) such that $\Phi_\omega(0) = I_{2n} = 2n \times 2n$ identity matrix. Then $\Phi_\omega(t)$ belongs to the symplectic group $\mathrm{Sp}(n, \mathbb{R}) = \left\{\left(\begin{smallmatrix} A & C \\ B & D\end{smallmatrix}\right) \mid AB^* = BA^*,\ CD^* = DC^*, AD^* - BC^* = I_n\right\}$ for each $t \in \mathbb{R}$. The matrix functions $\Phi_\omega(t)$ define a topological flow (linear skew-product flow) on $\Omega \times \mathbb{R}^{2n}$, as follows. If $(\omega, \mathbf{z}) \in \Omega \times \mathbb{R}^{2n}$, set $\widetilde{\tau}_t(\omega, \mathbf{z}) = (\tau_t(\omega), \Phi_\omega(t)\,\mathbf{z})$ $(t \in \mathbb{R})$; then $(\Omega \times \mathbb{R}^{2n}, \{\widetilde{\tau}_t\})$ is a topological flow. They also define a topological flow in $\Omega \times \Lambda$, where $\Lambda = \{\lambda\}$ is the set of all real Lagrange subspaces of $\mathbb{R}^{2n}$. In fact, if $\lambda \in \Lambda$, then the image subspace $\Phi_\omega(t)\,\lambda$ belongs to Λ because $\Phi_\omega(t)$ is symplectic $(\omega \in \Omega,\ t \in \mathbb{R})$. One now checks that, if $\widehat{\tau}_t : \Omega \times \Lambda \to \Omega \times \Lambda$ is defined by $\widehat{\tau}_t(\omega, \lambda) = (\tau_t(\omega), \Phi_\omega(t)\,\lambda)$ for each $t \in \mathbb{R}$, then $(\Omega \times \Lambda, \{\widehat{\tau}_t\})$ is a topological flow.

We will make use of various elementary facts concerning Λ and its elements λ. First, there are several ways of parametrizing Lagrange subspaces of $\mathbb{R}^{2n}$. (a) Let u and v be real $n \times n$ matrices such that $u^* v = v^* u$ and such that, if $u\,\mathbf{x} = v\,\mathbf{x} = 0$ for some $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} = 0$. Write $\left[\begin{smallmatrix} u \\ v\end{smallmatrix}\right] = \operatorname{Span}\left\{\left(\begin{smallmatrix} u\mathbf{e}_1 \\ v\mathbf{e}_1\end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} u\mathbf{e}_n \\ v\mathbf{e}_n\end{smallmatrix}\right)\right\}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the canonical basis in $\mathbb{R}^n$. Then $\lambda = \left[\begin{smallmatrix} u \\ v\end{smallmatrix}\right]$ is a Lagrange subspace of $\mathbb{R}^{2n}$, and each

Lagrange subspace can be (non-uniquely) parametrized in this way. (b) Let φ, ψ be real $n \times n$ matrices such that $\varphi + i\psi$ lies in the group $\mathrm{U}(n)$ of unitary complex $n \times n$ matrices. Then $\begin{bmatrix}\varphi\\ \psi\end{bmatrix}$ is a Lagrange subspace of $\mathbb{R}^{2n}$. If u, v are real $n \times n$ matrices such that $\lambda = \begin{bmatrix}u\\ v\end{bmatrix} \in \Lambda$, then one can write $u = \varphi r$, $v = \psi r$, where r is a non-singular $n \times n$ matrix and $\varphi + i\psi \in \mathrm{U}(n)$, and hence $\lambda = \begin{bmatrix}\varphi\\ \psi\end{bmatrix}$. (c) If $\lambda = \begin{bmatrix}u\\ v\end{bmatrix}$ where $\det u \neq 0$, then $\lambda = \begin{bmatrix}1\\ vu^{-1}\end{bmatrix}$, where 1 is the $n \times n$ identity matrix and vu^{-1} is symmetric.

The space Λ itself can be given the structure of a real-analytic manifold of dimension $n(n+1)/2$. An important subset of Λ is the (vertical) Maslov cycle $\mathcal{C}$ defined as follows. Let $\lambda_v = \begin{bmatrix}0\\ 1\end{bmatrix}$ be the "vertical" Lagrange plane, where again 1 denotes the $n \times n$ identity matrix. Then $\mathcal{C} = \{\lambda \in \Lambda \mid \dim(\lambda \cap \lambda_v) \geq 1\}$. Clearly $\mathcal{C} = \mathcal{C}^{(1)} \cup \cdots \cup \mathcal{C}^{(n)}$ where $\mathcal{C}^{(i)} = \{\lambda \in \Lambda \mid \dim(\lambda \cap \lambda_v) = i\}$. One sees that $\mathcal{C}$ is the complement in Λ of the set $W = \left\{\lambda \in \Lambda \mid \lambda = \begin{bmatrix}u\\ v\end{bmatrix} \text{ with } \det u \neq 0\right\}$. This set W is open and dense in Λ. According to point (c) above, W can be parametrized by the set $\mathcal{S}_n$ of symmetric, real $n \times n$ matrices. In particular, W is simply connected.

Now we discuss the argument functions of Yakubovich [33,34], who used work of Gel'fand-Lidskiĭ [13] and Lidskiĭ [22] as a foundation for his analysis. Let $t \to \Phi(t) = \begin{pmatrix} u_1(t) & u_2(t)\\ v_1(t) & v_2(t)\end{pmatrix}$ be a continuous curve in $\mathrm{Sp}(n, \mathbb{R})$. Set

$$\begin{aligned}
\mathrm{Arg}_1\Phi(t) &= \arg\det(u_1 - iv_1)\,,\\
\mathrm{Arg}_2\Phi(t) &= \arg\det(u_2 - iv_2)\,,\\
\mathrm{Arg}_3\Phi(t) &= \arg\det(u_1 + iu_2)\,,\\
\mathrm{Arg}_4\Phi(t) &= \arg\det(v_1 + iv_2)\,.
\end{aligned}$$

Here arg is the usual argument relation on $\mathbb{C}^* = \mathbb{C} - \{0\}$; one chooses arg det to be continuous in t. It turns out that Arg_1, Arg_2, Arg_3, Arg_4 are induced by multivalued argument relations Arg_i on $\mathrm{Sp}(n, \mathbb{R})$ whose branches differ one from another by integer multiples of 2π for each fixed index $i = 1, 2, 3, 4$. See [33] for the precise definition of an argument on $\mathrm{Sp}(n, \mathbb{R})$ and for a detailed discussion of the above arguments and others as well. We note in particular that, if $\Phi_0 \in \mathrm{Sp}(n, \mathbb{R})$, then $\mathrm{Arg}_i(\Phi(t)\Phi_0)$ also defines an argument on $\mathrm{Sp}(n, \mathbb{R})$ $(1 \leq i \leq 4)$.

The arguments $\mathrm{Arg}_1, \ldots, \mathrm{Arg}_4$ have strong equivalence properties which are discussed in detail in [33]. We give one such property now; a slight generalization will be needed in Section 3 and will be discussed there. There is an uniform constant κ such that, if $1 \leq i \neq j \leq 4$, if $\Phi : [t_0, t_1] \to \mathrm{Sp}(n, \mathbb{R})$ is any continuous curve, and if $\mathrm{Arg}_i\Phi(t_0)$ and $\mathrm{Arg}_j\Phi(t_0)$ are chosen to lie in $[0, 2\pi)$, then

$$|\mathrm{Arg}_i\Phi(t) - \mathrm{Arg}_j\Phi(t)| \leq \kappa \tag{3}$$

for all $t_0 \leq t \leq t_1$. The point is, of course, that κ does not depend on the choice of $\Phi(\cdot)$. We remark that, in relations to be consider in the sequel, the same letter κ will be used to denote other, perhaps larger constants. We will always assume that κ is large enough so that all relations in which it intervenes are valid.

Next we discuss the concept of rotation number for the family (1_ω). The rotation number was introduced in the case $n = 1$ in [16]. When $n \geq 2$, it was defined and its basic properties were worked out in [15]; see also [28]. Its connection with the

Yakubovich argument functions was elucidated in [25]. See [10,11] for a review of these matters. We proceed to summarize some basic facts.

The rotation number is defined in terms of a given ergodic measure μ on Ω. So fix such a measure μ. If $\omega \in \Omega$, set

$$\alpha(\mu) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}_i \Phi_\omega(t) \qquad (i = 1, 2, 3, 4)\,. \tag{4}$$

It follows from the inequalities (3) that the limit (if it exists) does not depend on the choice of i. It is shown in [25] that there is a subset $\Omega_1 \subset \Omega$, with $\mu(\Omega_1) = 1$, such that, if $\omega \in \Omega_1$, then the limit in (4) exists and does not depend on the choice of $\omega \in \Omega_1$. The number $\alpha(\mu)$ is called the *rotation number* (with respect to μ) of the family (1_ω). It has remarkable properties which are discussed in [15,17,25,11].

There are several equivalent ways of defining the rotation number which we now discuss. We first review a definition given in [25]. Let $\varphi,\ \psi$ be real $n \times n$ matrices such that $\varphi + i\psi \in \mathrm{U}(n)$. Define a function $Q : \Omega \times \mathrm{U}(n) \to \mathcal{M}_n$ as follows:

$$Q(\omega, \varphi, \psi) = (\varphi^*, \psi^*)\, H(\omega) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

Let us identify $\mathrm{U}(n)$ with $\left\{ \begin{pmatrix} \varphi & -\psi \\ \psi & \varphi \end{pmatrix} \mid \varphi + i\psi \in \mathrm{U}(n) \right\}$; then $\mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$. We can also identify the orthogonal group $\mathrm{O}(n)$ on $\mathbb{R}^n$ with $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \text{ is a real } n \times n \text{ matrix such that } u^*u = I_n \right\} \subset \mathrm{U}(n)$. Then Λ can be identified with the left coset space $\mathrm{U}(n)/\mathrm{O}(n)$. Explicitly, let λ_h be the "horizontal" Lagrange plane $\lambda_h = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let us identify the identity coset $I_{2n}{\cdot}\mathrm{O}(n)$ with λ_h. In this way, the Lagrange plane $\begin{bmatrix} \varphi \\ \psi \end{bmatrix}$ is identified with the coset $\begin{pmatrix} \varphi & -\psi \\ \psi & \varphi \end{pmatrix}{\cdot}\mathrm{O}(n)$ whenever $\varphi + i\psi \in \mathrm{U}(n)$.

Carrying on with this identification of Λ with $\mathrm{U}(n)/\mathrm{O}(n)$, we see that the trace $\operatorname{Tr} Q$ can be viewed as a real valued function on $\Sigma = \Omega \times \Lambda$. Now, it is proved in [25] that, whenever $\omega \in \Omega_1$,

$$\alpha(\mu) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{Tr} Q(\widehat{\tau}_s(\omega, \lambda))\, ds \tag{5}$$

for every $\lambda \in \Lambda$. In fact, [25] prove the following stronger assertion. Let ν be a $\{\widehat{\tau}_t\}$-ergodic measure on Σ which is a *lift* of μ: this means that, if we write $\pi : \Sigma \to \Omega : (\omega, \lambda) \to \omega$ for the canonical projection, then for each Borel subset $B \subset \Omega$ there holds $\nu(\pi^{-1}(B)) = \mu(B)$. One can prove the existence of ergodic lifts ν of μ using the Choquet theory [26]; see [21] for details. Now, Novo, Núñez and Obaya show that, for each ergodic lift ν of μ, there holds

$$\alpha(\mu) = \int_\Sigma \operatorname{Tr} Q\, d\nu\,. \tag{6}$$

Still another way of defining the rotation number uses the Maslov intersection index of closed curves on Λ. Briefly, let $c : [0, 1] \to \Lambda$ be a continuous closed curve; i.e. $c(0) = c(1)$. The Maslov cycle $\mathcal{C}$ is two-sided in Λ in the sense that there is a continuous, nowhere vanishing vector field on $\mathcal{C}$ which is not tangent to

$\mathcal{C}^{(1)}$ for each $\lambda \in \mathcal{C}^{(1)}$. Using this fact, one can define the Maslov index Ind (c) as "the number of oriented crossings of c with the Maslov cycle $\mathcal{C}$"; see [2,3]. Let $\omega \in \Omega_1$ and $\lambda \in \Lambda - \mathcal{C}$, then write $\lambda(t) = \Phi_\omega(t)\lambda$ for $t \geq 0$. If $\lambda(t) \in \mathcal{C}$ perturb it so that it lies in $\Lambda - \mathcal{C}$, then slide $\lambda(t)$ to $\lambda(0)$ through the simply-connected set $\Lambda - \mathcal{C}$. Let $m(t, \omega, \lambda)$ be the Maslov index of the resulting closed curve. Then

$$\alpha(\mu) = -\lim_{t\to\infty} \frac{\pi}{t}\, m(t, \omega, \lambda)\,; \tag{7}$$

see [15,10].

A final method for defining the rotation number goes as follows. Following Arnold [2], define $\mathrm{Det}^2 : \Lambda = \mathrm{U}(n)/\mathrm{O}(n) \to \mathbb{S}^1$ by $\mathrm{Det}^2(u{\cdot}\mathrm{O}(n)) = -(\det u)^2$. It is easily seen that Det^2 is a well-defined smooth map from Λ to the circle $\mathbb{S}^1$. If $\omega \in \Omega_1,\ \lambda \in \Lambda$, and $\lambda(t) = \Phi_\omega(t)\,\lambda$, one has

$$2\alpha(\mu) = \lim_{t\to\infty} \frac{1}{t} \arg \mathrm{Det}^2 \lambda(t)\,;$$

see again [15,10].

Let us now write $H(\cdot)$ in the form (2):

$$H(\omega) = \begin{pmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{pmatrix} \qquad (\omega \in \Omega)\,,$$

where the H_{ij} are continuous functions with values in $\mathcal{M}_n$, which satisfy the symmetry conditions $H_{11}^* = H_{11},\ H_{22}^* = H_{22},\ H_{12}^* = H_{21}$. We impose the following additional conditions on these matrices. Unless explicitly stated otherwise, they will be in force for the rest of the paper.

Hypothesis 1. For each $\omega \in \Omega$, the matrix $H_{22}(\omega)$ is positive semi-definite: $H_{22}(\omega) \geq 0$.

Hypothesis 2. For each minimal subset $M \subset \Omega$, there exists a point $\omega_0 \in M$ such that the control system

$$\mathbf{x}' = H_{21}(\tau_t(\omega_0))\,\mathbf{x} + H_{22}(\tau_t(\omega_0))\,\mathbf{u} \tag{8}$$

is null-controllable.

We give a consequence of Hypothesis 1; it is proved in ([12], Corollary 3.7).

Proposition 1. *Let μ be an ergodic measure on Ω. If Hypothesis* 1 *holds, then* $\alpha(\mu) \geq 0$.

Hypothesis 2 may seem artificial, but it will become apparent that it is actually a weak version of "identical normality", a condition which is often imposed in studying the disconjugacy phenomenon. Let us explain its significance in a bit more detail. First of all, the control system (8) is called *null controllable* if for each $\mathbf{x}_0 \in \mathbb{R}^n$ there exist a time $T > 0$ and an integrable "control function" $\bar{u} : [0, T] \to \mathbb{R}^n$ such that the solution $\mathbf{x}(t)$ of (8) with $u = \bar{u}(t)$ and $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\mathbf{x}(T) = 0$. It is shown in [18] that Hypothesis 2 is actually equivalent to the a priori stronger condition of *uniform* null controllability of all the control systems (8) obtained by letting ω range over Ω. This last property can be re-expressed in the following way.

Proposition 2. *Suppose that Hypothesis 2 holds. Then there are positive numbers T, δ such that, for all $\omega \in \Omega$ and all $\mathbf{x} \in \mathbb{R}^n$, there holds*

$$\int_0^T \left\| H_{22}(\tau_t(\omega))\, \Psi_\omega(t)^{-1*}\mathbf{x}\right\|^2 dt \geq \delta \|\mathbf{x}\|^2. \tag{9}$$

Here $\Psi_\omega(t)$ is the fundamental matrix solution at $t = 0$ of the system $\mathbf{x}' = H_{21}(\tau_t(\omega))\,\mathbf{x}$.

One can convince oneself that if $H_{22} > 0$, then for each $T > 0$ there is a $\delta = \delta(T)$ such that (9) holds. It follows that each system (1_ω) is identically normal in the sense that, if $\mathbf{z}(t) = \binom{\mathbf{x}(t)}{\mathbf{y}(t)}$ is a non-zero solution of (1_ω), then $\mathbf{x}(t)$ does not vanish identically in any subinterval of $\mathbb{R}$.

3. Analysis

Our main goal in this section is to state and prove theorems which relate the disconjugacy property for a family (1_ω) of time-varying linear Hamiltonian systems to the rotation number for such a family. We begin with a technical result which may be of interest in its own right. Throughout this section we assume (unless otherwise stated) that Ω is a compact metric space, that $(\Omega, \{\tau_t\})$ is a topological flow, that $H : \Omega \to \mathcal{S}_{2n}$ is a continuous function, and that Hypotheses 1 and 2 are satisfied.

Theorem 1. *Let μ be an ergodic measure on Ω with topological support* Supp $\mu = \Omega$. *Assume that the rotation number $\alpha(\mu)$ of the family (1_ω) equals zero. Let $\widetilde{\mu}$ be any other ergodic measure on Ω. Then $\alpha(\widetilde{\mu}) = 0$.*

Proof. We will use various facts about the oscillation of Lagrange subspaces induced by a linear Hamiltonian system; some of them were stated in Section 2.

Let $\mathrm{Tr}\, Q : \Sigma = \Omega \times \Lambda \to \mathbb{R}$ be the function of Novo-Núñez-Obaya discussed in Section 2. We first prove a relation between $\int_0^t \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega, \lambda))\, ds$ and the function Arg_1 of Yakubovich.

Lemma 1. *Let $\lambda_* \in \Lambda$ and let $\omega \in \Omega$. Let $\lambda_h = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ be the horizontal Lagrange plane. Let $\Phi_* \in \mathrm{Sp}(n, \mathbb{R})$ be a matrix such that $\Phi_*\lambda_h = \lambda_*$. Then*

$$\mathrm{Arg}_1(\Phi_\omega(t)\, \Phi_*) = \mathrm{Arg}_1\Phi_* + \int_0^t \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega, \lambda_*))\, ds\,.$$

Proof. Let us write

$$\Phi_\omega(t)\, \Phi_* = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix} \qquad (t \in \mathbb{R})\,.$$

Then $\mathrm{Arg}_1(\Phi_\omega(t)\, \Phi_*) = \arg\det(u_1(t) - i v_1(t))$, where we choose a branch of the argument arg on $\mathbb{C}^* = \mathbb{C} - \{0\}$. On the other hand, let

$$\theta(t) = (v_1(t) + i u_1(t))(v_1(t) - i u_1(t))^{-1}\,;$$

see [22,27,34]. Then $\theta(t)$ takes values in the group $\mathrm{U}(n)$ of complex unitary $n \times n$ matrices. As is shown in [25],

$$\arg\det\theta(t) = 2\arg\left[i^n \det(u_1(t) - i v_1(t))\right] = n\pi + 2\mathrm{Arg}_1(\Phi_\omega(t)\,\Phi_*)\,,$$

where we fix the branch of the complex argument on the left by setting $\arg\det\theta(0) = n\pi + 2\mathrm{Arg}_1\Phi_*$. Now, one also has [25]:

$$\det\theta(t) = \det\theta(0)\exp\int_0^t 2i\mathrm{Tr}\;Q(\widehat{\tau}_s(\omega,\lambda_*))\,ds\,,$$

so

$$\begin{aligned}\arg\det\theta(t) &= \arg\det\theta(0) + 2\int_0^t \mathrm{Tr}\;Q(\widehat{\tau}_s(\omega,\lambda_*))\,ds\\ &= n\pi + 2\mathrm{Arg}_1\Phi_* + 2\int_0^t \mathrm{Tr}\;Q(\widehat{\tau}_s(\omega,\lambda_*))\,ds\,.\end{aligned}$$

Hence $\mathrm{Arg}_1(\Phi_\omega(t)\,\Phi_*) = \mathrm{Arg}_1\Phi_* + \int_0^t \mathrm{Tr}\;Q(\widehat{\tau}_s(\omega,\lambda_*))\,ds$ as was to be proved.

Next let μ be an ergodic measure on Ω such that $\mathrm{Supp}\,\mu = \Omega$. There is a subset $\Omega_1 \subset \Omega$ with $\mu(\Omega_1) = 1$ such that, if $\omega \in \Omega_1$, then the positive semi-orbit $\{\tau_t(\omega) \mid t \geq 0\}$ is dense in Ω. Let ν be an ergodic lift of μ to Σ. By hypothesis $\alpha(\mu) = 0$, and so by the discussion in Section 2 one has $\int_\Sigma \mathrm{Tr}\;Q\,d\nu = 0$; see (6). We apply an important recurrence result of Schnei'berg [31] to the ergodic measure μ, to obtain the following conclusion. There is a subset $\Sigma_* \subset \Sigma$ with $\nu(\Sigma_*) = 1$ such that, if $(\omega_*, \lambda_*) \in \Sigma_*$, then there is a sequence of times $t_k \to \infty$ such that

$$\left|\int_0^{t_k} \mathrm{Tr}\;Q(\widehat{\tau}_s(\omega_*,\lambda_*))\,ds\right| < 1\,. \tag{10}$$

There is no loss of generality in assuming that $\omega_* \in \Omega_1$. Fix such a point (ω_*, λ_*) until the end of the proof of Theorem 1.

Now suppose for contradiction that there is an ergodic measure $\widetilde{\mu}$ in Ω such that $\alpha(\widetilde{\mu}) \neq 0$. Let ν be an ergodic lift of $\widetilde{\mu}$ to Σ. Then by (6) we have

$$\int_\Sigma \mathrm{Tr}\;Q\,d\nu = \alpha(\widetilde{\mu})\,.$$

Using the Birkhoff ergodic theorem [24], we conclude that there is a set $\widetilde{\Omega} \subset \Omega$ with $\widetilde{\mu}(\widetilde{\Omega}) = 1$ such that, if $\widetilde{\omega} \in \widetilde{\Omega}$, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{Tr}\;Q(\widehat{\tau}_s(\widetilde{\omega},\widetilde{\lambda}))\,ds = \alpha(\widetilde{\mu}) > 0$$

for all $\widetilde{\lambda} \in \Lambda$. Here we use the independence of the limit with respect to the element of Λ chosen [15,25]. We see that, if $\widetilde{\omega} \in \widetilde{\Omega}$ and if $\widetilde{\lambda}$ is any element of Λ, then

$$\lim_{t\to\infty}\int_0^t \mathrm{Tr}\;Q(\widehat{\tau}_s(\widetilde{\omega},\widetilde{\lambda}))\,ds = \infty\,. \tag{11}$$

Now fix $\widetilde{\omega} \in \widetilde{\Omega}$. There is a sequence $t_l \to \infty$ such that $\tau_{t_l}(\omega_*) \to \widetilde{\omega}$. Choosing a subsequence if necessary we can assume that $\widehat{\tau}_{t_l}(\omega_*,\lambda_*) \to (\widetilde{\omega},\widetilde{\lambda})$ where $\widetilde{\lambda} \in \Lambda$.

We have

Lemma 2. *There is a number $T > 0$ such that, for all $t \geq T$, there holds*

$$\int_0^t \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds > 1\,.$$

Proof. Let $\lambda_h = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the horizontal Lagrange plane, and let $\Phi_* \in \mathrm{Sp}(n, \mathbb{R})$ be a matrix such that $\Phi_* \lambda_h = \lambda_*$.

Let $\varphi(t) = \mathrm{Arg}_3(\Phi_{\omega_*}(t)\Phi_*)$, and assume that $\varphi(0) \geq 0$. Since $H_{22}(\omega) \geq 0$ for all $\omega \in \Omega$, $\varphi(t)$ is a non-decreasing function of t (see [22,34]). Now, according to [33], there is a constant κ such that

$$\left|\mathrm{Arg}_3(\Phi_{\omega_*}(t)\Phi_*) - \mathrm{Arg}_1(\Phi_{\omega_*}(t)\Phi_*)\right| \leq \kappa \tag{12}$$

for all $t \in \mathbb{R}$. Using Lemma 1 and (12), we see that there is a constant κ such that

$$\left|\varphi(t) - \int_0^t \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds\right| \leq \kappa \tag{13}$$

for all $t \in \mathbb{R}$. Since $\varphi(t)$ is non-decreasing and $\varphi(0) \geq 0$, this certainly implies that

$$\int_0^t \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds \geq -\kappa \tag{14}$$

for all $t \geq 0$.

Let N be a positive number. Using the fact that $\widehat{\tau}_{t_l}(\omega_*, \lambda_*) \to (\widetilde{\omega}, \widetilde{\lambda})$ together with (11), we see that, if l is large enough, then there is a positive number T_l such that

$$\int_{t_l}^{t_l+T_l} \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds \geq N\,. \tag{15}$$

Combining (14) and (15), and writing $T = t_l + T_l$, we see that

$$\int_0^T \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds \geq N - \kappa\,.$$

Then (13) implies that $\varphi(T) \geq N - 2\kappa$, and hence

$$\varphi(t) \geq N - 2\kappa$$

whenever $t \geq T$. Using (13) again, we see that

$$\int_0^t \mathrm{Tr}\, Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds \geq N - 3\kappa$$

if $t \geq T$. Choosing $N > 3\kappa + 1$ we obtain the conclusion of Lemma 2.

It is interesting to note that the equivalence between argument functions stated in [33] assures that inequality (13) holds if the matrix Φ_* appearing in the definition of $\varphi(t)$ is replaced by any other matrix of the symplectic group.

It is now clear how to complete the proof of Theorem 1: if we choose $t_r > T$, we see that (10) is incompatible with Lemma 2. Thus Theorem 1 is finally proved.

Theorem 1 has some interesting consequences; to discuss them we recall some basic facts about non-autonomous linear Hamiltonian systems. Let us remove for the time being the Hypotheses 1 and 2. Consider the case when the coefficient matrix $H(\cdot)$ in (1) is T-periodic: $H(t+T) = H(t)$ for all $t \in \mathbb{R}$. In this case the hull Ω of H is homeomorphic to the circle $\mathbb{S}^1$, and the flow $\{\tau_t\}$ is defined by the rigid rotations on $\mathbb{S}^1$. There is a unique ergodic measure μ on $\Omega = \mathbb{S}^1$; it coincides with the normalized Lebesgue measure on the circle. It follows from ([37], Theorem 2) that the rotation number $\alpha = \alpha(\mu)$ provides a labelling of the so-called instability zones in the space of all T-periodic, continuous Hamiltonian systems (1).

The point we wish to make is that an analogous labelling of instability zones is available in the general non-autonomous case. Let Ω be a compact metric space, let $(\Omega, \{\tau_t\})$ be a topological flow, and let $H : \Omega \to \mathcal{S}_{2n}$ be a continuous function. We say that H lies in a stability zone (relative to $(\Omega, \{\tau_t\})$) if the family of equations (1_ω) admits an *exponential dichotomy* over Ω; see [29]. We repeat the definition of exponential dichotomy. Let $\mathcal{P}$ be the set of linear projectors $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with the usual topology. The family (1_ω) is said to have an exponential dichotomy (ED) over Ω if there are constants $C > 0$, $\gamma > 0$ and a continuous function $P : \Omega \to \mathcal{P} : \omega \to P_\omega$ such that

$$\begin{aligned} \left\| \Phi_\omega(t)\, P_\omega\, \Phi_\omega(s)^{-1} \right\| &\le C e^{-\gamma(t-s)} \qquad && t \ge s, \\ \left\| \Phi_\omega(t)\, (I_{2n} - P_\omega)\, \Phi_\omega(s)^{-1} \right\| &\le C e^{\gamma(t-s)} \qquad && t \le s. \end{aligned}$$

It is well-known that the dimension of the image $\operatorname{Im} P_\omega$ equals n for all $\omega \in \Omega$; see e.g. [25]. In fact, $\operatorname{Im} P_\omega = \lambda_\omega \subset \mathbb{R}^{2n}$ is a Lagrange subspace of $\mathbb{R}^{2n}$ for all $\omega \in \Omega$.

Next let $\check{H}^1(\Omega, \mathbb{Z})$ be the first Čech cohomology group of Ω with integer coefficients. Fix an ergodic measure μ on Ω. Let $h_\mu : \check{H}^1(\Omega, \mathbb{Z}) \to \mathbb{R}$ be the well-known Schwarzmann homomorphism [32]. It is proved in [15,11] that, if the family of equations (1_ω) has an ED over Ω, then

$$2\alpha(\mu) \in \operatorname{Im} h_\mu = \text{image of } h_\mu\,.$$

Since $\operatorname{Im} h_\mu$ is an at most countable subgroup of $\mathbb{R}$, one sees that the set of unstable Hamiltonian systems (relative to $(\Omega, \{\tau_t\})$) divides into at most countably many zones, defined by the various possible values of $\alpha(\mu)$. This division into instability zones may depend on the choice of μ. If $\check{H}^1(\Omega, \mathbb{Z}) = \{0\}$ and if $\operatorname{Supp} \mu = \Omega$, then $\operatorname{Im} h_\mu = \{0\}$, and so there is necessarily one stability zone relative to μ. Certainly these conditions hold if Ω is a single point, which corresponds to the case when the Hamiltonian system (1) has constant coefficients. They hold in other cases as well; e.g., there are vector fields on the 3-sphere $\Omega = \mathbb{S}^3$ which admit ergodic measures μ with $\operatorname{Supp} \mu = \mathbb{S}^3$ [1]. It does not seem to be known if the instability zones are connected or not.

Now we reimpose Hypotheses 1 and 2. Let μ be an ergodic measure on Ω such that $\operatorname{Supp} \mu = \Omega$. Suppose that $\alpha(\mu) = 0$ and that the equations (1_ω) are unstable (admit an exponential dichotomy over Ω). By Theorem 1, $\alpha(\widetilde{\mu}) = 0$ for every ergodic measure $\widetilde{\mu}$ on Ω. We can therefore apply Proposition 3.8 of [12] to conclude that equations (1_ω) satisfy a strong non-oscillation condition of Yakubovich type [35,37]. Namely, for each $\omega \in \Omega$, the Lagrange plane

$\lambda_\omega = \mathrm{Im}\, P_\omega$ lies in the complement $\Lambda - \mathcal{C}$ of the Maslov cycle $\mathcal{C}$. The significance of this fact in the context of certain basic issues of optimal control theory and of absolute stability theory is discussed in [12,9]. The results of these papers generalize to the case of non-autonomous control systems certain theorems of Yakubovich regarding periodic control systems [35–37].

Now we turn to our characterization of the property of weak disconjugacy for non-autonomous linear Hamiltonian systems. To avoid interruption of the discussion, we first state a result from [12] which will be applied several times.

Proposition 3. *Suppose that Hypotheses* 1 *and* 2 *hold. Let* μ *be an ergodic measure on* Ω *such that* $\mathrm{Supp}\,\mu = \Omega$. *Suppose that* $\alpha(\mu) = 0$. *Let* $(\omega, \lambda) \in \Sigma$, *and let* $\lambda(t) = \Phi_\omega(t)\,\lambda$ $(0 \le t < \infty)$. *Then it is not the case that* $\lambda(t) \in \mathcal{C}$ *for all* $t \ge 0$.

This statement is proved in ([12], Proposition 3.5) subject to the additional hypothesis that, for each ergodic measure $\widetilde{\mu}$ on Ω, there holds $\alpha(\widetilde{\mu}) = 0$. However this additional hypothesis is in fact verified because of Theorem 1.

Let us now state and prove our main results.

Theorem 2. *Suppose that Hypotheses* 1 *and* 2 *hold. Let* μ *be an ergodic measure on* Ω *such that* $\mathrm{Supp}\,\mu = \Omega$.

a) The equations (1_ω) *are all weakly disconjugate if and only if* $\alpha(\mu) = 0$.
b) If $\alpha(\mu) = 0$, *then each equation* (1_ω) *admits n linearly independent solutions* $\mathbf{z}_i(t) = \begin{pmatrix} \mathbf{x}_i(t) \\ \mathbf{y}_i(t) \end{pmatrix}$ $(1 \le i \le n)$ *such that, if* $u(t)$ *is the* $n \times n$ *matrix with columns* $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$, *then* $\det u(t) \neq 0$ $(t \in \mathbb{R})$.
c) If $\alpha(\mu) = 0$, *then each equation* (1_ω) *admits a principal solution.*

Proof. We will prove statements a) and b) together. Suppose first that the equations (1_ω) are all weakly disconjugate. Let $\lambda_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be the vertical Lagrange plane, and let $\omega \in \Omega$. It follows from the Definition 2 of weak disconjugacy that there exists $T > 0$ such that, if $t > T$, then $\Phi_\omega(t)\,\lambda_v$ does not lie in $\mathcal{C}$. Using formula (7) for the rotation number $\alpha(\mu)$, we see that $\alpha(\mu) = 0$ because $m(t, \omega, \lambda_v)$ is constant for $t > T$. It is worth pointing out that the above argument can be carried out if it is assumed only that equation (1_ω) is weakly disconjugate for μ-almost all $\omega \in \Omega$.

Next we prove part b). This is the main step in the proof of Theorem 2. First of all, we use Proposition 3 to conclude that, if $(\omega, \lambda) \in \Sigma$, then the positive semi-orbit $\{\widehat{\tau}_t(\omega, \lambda) \mid t \ge 0\}$ cannot be entirely contained in $\Omega \times \mathcal{C}$. Here, as always, $\mathcal{C}$ is the vertical Maslov cycle in Λ.

We will prove that there is a compact, $\{\widehat{\tau}_t\}$-invariant subset $\Sigma_* \subset \Sigma$ which is disjoint from $\Omega \times \mathcal{C}$ and which projects to all of Ω: $\pi(\Sigma_*) = \Omega$. Note that this statement implies part b). For, if $\omega \in \Omega$, and in $\lambda \in \Lambda$ is a point such that $(\omega, \lambda) \in \Sigma_*$, then $\Phi_\omega(t)\,\lambda = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ lies in $\Lambda - \mathcal{C}$ for all $t \in \mathbb{R}$. But this is equivalent to the condition that $\det u(t) \neq 0$ for all $t \in \mathbb{R}$.

We will need the following lemma to prove the existence of Σ_*.

Lemma 3. *Let* $\omega \in \Omega$, $\lambda \in \Lambda$, *and set* $\lambda(t) = \Phi_\omega(t)\,\lambda$ $(-\infty < t < \infty)$. *Suppose that there are points* $t_1 < t_0 < t_2$ *such that* $\lambda(t_1) \notin \mathcal{C}$, $\lambda(t_0) \in \mathcal{C}$, *and* $\lambda(t_2) \notin \mathcal{C}$.

Let c be a closed curve in Λ obtained by sliding $\lambda(t_2)$ through the simply-connected set $\Lambda - \mathcal{C}$ to $\lambda(t_1)$. Let Ind *be the Maslov index (see Section 2). Then* Ind $(c) > 0$.

Proof. This result is certainly well-known but for completeness we sketch a proof. For each $\varepsilon > 0$, consider the function $H_\varepsilon(\omega) = H(\omega) + \varepsilon \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_n \end{smallmatrix}\right)$. Let $\Phi^\varepsilon_\omega(t)$ be the fundamental matrix solution at $t = 0$ of $J\,\mathbf{z}' = H_\varepsilon(\tau_t(\omega))\,\mathbf{z}$, and let $\lambda_\varepsilon(t) = \Phi^\varepsilon_\omega(t)\,\lambda$. Note that $H_{\varepsilon,22}$ is strictly positive definite for all $\omega \in \Omega$. Note also that, if ε is small, then $\lambda_\varepsilon(t_1) \notin \mathcal{C}$ and $\lambda_\varepsilon(t_2) \notin \mathcal{C}$.

We claim that, if ε is small, then there exists $t_\varepsilon \in (t_1, t_2)$ such that $\lambda_\varepsilon(t_\varepsilon) \in \mathcal{C}$. To see this, let $\lambda_v = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ be the vertical Lagrange subspace, and let $\Phi_0 \in \mathrm{Sp}(n, \mathbb{R})$ satisfy $\Phi_0\lambda_v = \lambda$. Write $\Phi_\omega(t)\,\Phi_0 = \left(\begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix}\right)$, and let $\eta(t) = (u_1(t) - iu_2(t))^{-1}(u_1(t) + iu_2(t))$. Then $\eta(t)$ is unitary for all $t \in \mathbb{R}$. As shown in [22,34], there are continuous, non-decreasing functions ("angles") $\varphi_1(t), \ldots, \varphi_n(t)$ such that $\exp i\varphi_j(t)$ is an eigenvalue of $\eta(t)$ for each $t \in \mathbb{R}$ and each $j = 1, \ldots, n$. Moreover, $\lambda(t)$ lies in $\mathcal{C}$ if and only if at least one of these angles equals zero (mod 2π).

We can choose the φ_js so that $\varphi_j(t_1) \in (-2\pi, 0)$ for $1 \leq j \leq n$. Let $t_\alpha = \inf\{t \in [t_1, t_2] \mid \varphi_j(t) = 0$ for at least one $j\}$ and $t_\beta = \inf\{t \in [t_\alpha, t_2] \mid \varphi_j(t) > 0$ for at least one $j\}$. Relabelling the φ_js if necessary, we can assume that $\varphi_1(t), \ldots, \varphi_r(t) = 0$ for at least one $t \in [t_\alpha, t_\beta]$, while $\varphi_{r+1}(t), \ldots, \varphi_n(t) \in (-2\pi, 0)$ for all $t \in [t_1, t_\beta]$. Clearly $\varphi_1(t_\beta) = \ldots = \varphi_r(t_\beta) = 0$.

Now write $\Phi^\varepsilon_\omega(t) = \begin{pmatrix} u^\varepsilon_1 & u^\varepsilon_2 \\ v^\varepsilon_1 & v^\varepsilon_2 \end{pmatrix}$ and $\eta_\varepsilon(t) = (u^\varepsilon_1(t) - iu^\varepsilon_2(t))^{-1}(u^\varepsilon_1(t) + iu^\varepsilon_2(t))$. For each $\varepsilon > 0$, there are continuous, non-decreasing functions $\varphi^\varepsilon_1(t), \ldots, \varphi^\varepsilon_n(t)$ such that $\eta_\varepsilon(t)$ has eigenvalues $\exp i\varphi^\varepsilon_j(t)$ $(1 \leq j \leq n)$. Moreover, $\lambda_\varepsilon(t)$ lies on $\mathcal{C}$ if and only if one of the values $\varphi^\varepsilon_j(t)$ is zero (mod 2π). Assume that $\varphi^\varepsilon_j(t_1) \in (-2\pi, 0)$ $(1 \leq j \leq n,\ \varepsilon > 0)$.

By standard eigenvalue perturbation theory, we have the following. Let ε be sufficiently small, and let t be sufficiently close to t_β. Then there are integers $1 \leq j_1 \leq \cdots \leq j_r \leq n$ (which may a priori depend on ε) such that the *unordered* r-tuple $\{\exp i\varphi^\varepsilon_{j_1}(t), \ldots, \exp i\varphi^\varepsilon_{j_r}(t)\}$ is close to the unordered r-tuple $\{\exp i\varphi_1(t), \ldots, \exp i\varphi_r(t)\}$. The integers $j_1, \ldots, j_r$ do not depend on t, if ε is fixed and t is close to t_β.

Now, if $t > t_\beta$ is close to t_β, then at least one function $\varphi_j(t)$ is strictly positive. This certainly implies that, if ε is small, then there exist $j = j(\varepsilon)$ and $t > t_\beta$ such that $\varphi^\varepsilon_j(t) > 0$. This shows that the desired t_ε exists.

Let us now close $\lambda_\varepsilon(t)$ by sliding $\lambda_\varepsilon(t_2)$ through $\Lambda - \mathcal{C}$ to $\lambda_\varepsilon(t_1)$. Let the resulting closed curve be called c_ε. If ε is small enough then c_ε is homotopic to c. Hence Ind $c_\varepsilon =$ Ind c for small ε. Now, by ([3], Section 7, Lemma 2), the curve c_ε has positive Maslov index (because $H_{\varepsilon,22} > 0$). Hence Ind $c > 0$.

Returning to the proof of Theorem 2b), let ν be an ergodic lift of μ to Σ. Then $\int_\Sigma \mathrm{Tr}\, Q\, d\nu = 0$. Let Σ_* be the topological support of ν. Using the $\{\widehat{\tau}_t\}$-invariance of Σ_* and Proposition 3, we see that Σ_* cannot be contained in $\Omega \times \mathcal{C}$. Let $W = \Sigma_* - (\Omega \times \mathcal{C})$. Then W is relatively open in Σ_*, hence it has positive ν-measure. The projection $\pi(\Sigma_*)$ equals Ω because ν is a lift of μ and Supp $\mu = \Omega$.

Let $W_1 = \{(\omega, \lambda) \in W \mid$ the positive semi-orbit of (ω, λ) is dense in $\Sigma_*\}$. Then $\nu(W - W_1) = 0$. Also, let W_2 be the set of those points $(\omega, \lambda) \in W$ for which there is a sequence $t_k \to \infty$ such that

$$\left| \int_0^{t_k} \operatorname{Tr} Q(\widehat{\tau}_s(\omega, \lambda))\, ds \right| < 1\,. \tag{16}$$

Then $\nu(W - W_2) = 0$ as well, because of the Schnei'berg recurrence result seen above [31]. Fix a point $(\omega_*, \lambda_*) \in W_1 \cap W_2$.

We claim that there is a time $t_0 \geq 0$ such that, if $t > t_0$, and if $\Phi_{\omega_*}(t)\, \lambda_* = \lambda_*(t)$, then $\lambda_*(t) \notin \mathcal{C}$. To see this, let $\Phi_* \in \mathrm{Sp}(n, \mathbb{R})$ be a matrix such that $\Phi_* \lambda_v = \lambda_*$ and write $\varphi(t) = \mathrm{Arg}_3(\Phi_{\omega_*}(t)\, \Phi_*)$. Put $\Phi_{\omega_*}(t)\, \Phi_* = \left(\begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix}\right)$, and let $\eta(t) = (u_1(t) - iu_2(t))^{-1}(u_1(t) + iu_2(t))$. As in the proof of Lemma 3, there are continuous non-decreasing functions $\varphi_1(t), \ldots, \varphi_n(t)$ such that $\exp i\varphi_1(t), \ldots, \exp i\varphi_n(t)$ are the eigenvalues of $\eta(t)$. Moreover [22,34], one has that $\lambda_*(t) = \Phi_{\omega_*}(t)\, \lambda_*$ lies on $\mathcal{C}$ if and only if at least one of the "angles" $\varphi_j(t)$ equals zero (mod 2π). Still more, one has $\varphi(t) = 2\sum_{j=1}^{n} \varphi_j(t)$.

Now, φ is non-decreasing, so by (13) and (16), φ must be bounded on $[0, \infty)$. Hence each of the angles φ_j is bounded in $[0, \infty)$. This implies that, for each $j = 1, \ldots, n$, the limit $\varphi_j(\infty) = \lim_{t\to\infty} \varphi_j(t)$ exists. If for some j, $\varphi_j(t) = 0 \mod 2\pi$ for all large t, then $\lambda_*(t)$ lies in $\mathcal{C}$ for all large t. However, Proposition 3 ensures that this cannot happen. Hence for each $j = 1, \ldots, n$, the condition $\varphi_j(\infty) = 0 \mod 2\pi$ implies that $\varphi_j(t) \neq 0 \mod 2\pi$ for all large t. Clearly, if $\varphi_j(\infty) \neq 0 \mod 2\pi$, then $\varphi_j(t) \neq 0 \mod 2\pi$ for all large t. We conclude that indeed there exists $t_0 \geq 0$ such that, if $t \geq t_0$, then $\lambda_*(t) \notin \mathcal{C}$.

Next, suppose for contradiction that there exists $(\widetilde{\omega}, \widetilde{\lambda}) \in \Sigma_*$ such that $\widetilde{\lambda} \in \mathcal{C}$. Let $\widetilde{\lambda}(t) = \Phi_{\widetilde{\omega}}(t)\, \widetilde{\lambda}$. There is a time $t_1 < 0$ such that $\widetilde{\lambda}(t_1) \notin \mathcal{C}$, for if not the alpha-limit set of $(\widetilde{\omega}, \widetilde{\lambda})$ would lie entirely on $\Omega \times \mathcal{C}$, contradicting Proposition 3. There is also a time $t_2 > 0$ such that $\widetilde{\lambda}(t_2) \notin \mathcal{C}$.

There is a sequence $t_r \to \infty$ such that $\widehat{\tau}_{t_r}(\omega_*, \lambda_*) \to (\widetilde{\omega}, \widetilde{\lambda})$. Consider the curve $t \to \widetilde{\lambda}(t)$ $(t_1 \leq t \leq t_2)$ in Λ: we define a closed curve $\widetilde{c}$ in Λ by sliding together the endpoints $\widetilde{\lambda}(t_1)$ and $\widetilde{\lambda}(t_2)$ in $\Lambda - \mathcal{C}$. We also consider the curve $t \to \lambda_*(t)$ $(t_s + t_1 \leq t \leq t_s + t_2)$. If s is large enough, we have $\lambda_*(t_s + t_1) \notin \mathcal{C}$, $\lambda_*(t_s + t_2) \notin \mathcal{C}$. Define a closed curve c_* in Λ by sliding together the endpoints of λ_* in $\Lambda - \mathcal{C}$. It s is large enough, then $\widetilde{c}$ and c_* are homotopic.

Now, Lemma 3 implies that $\mathrm{Ind}\,(\widetilde{c}) > 0$. On the other hand, the curve c_* lies entirely in $\Lambda - \mathcal{C}$ is s is large enough. But then $\mathrm{Ind}\,(c_*) = 0$. This contradicts the fact that c_* and $\widetilde{c}$ are homotopic: homotopic curves in Λ have the same Maslov index.

We conclude that $\Sigma_* \subset \Omega \times (\Lambda - \mathcal{C})$. As noted earlier, this implies that part b) is true.

We continue to assume that $\alpha(\mu) = 0$, and prove that each equation (1_ω) is weakly disconjugate. This will complete the proof of part a). We could use classical arguments like those given in ([14], pp. 384–395) or ([6], pp. 34–42). However, we give another proof which illustrates a property of the function $\operatorname{Tr} Q$ of Novo, Núñez and Obaya.

Let (ω_*, λ_*) be the point introduced in the proof of part b). According to that proof, the corresponding function $\mathrm{Arg}_3(\Phi_{\omega_*}(t)\,\Phi_*)$ is bounded above and below for $t \geq 0$. Using (13), one concludes that $\left|\int_0^t \mathrm{Tr}\; Q(\widehat{\tau}_s(\omega_*, \lambda_*))\, ds\right|$ is bounded for $t \geq 0$. It follows in a well-known way that there is a constant κ such that, for all $(\bar{\omega}, \bar{\lambda})$ in the omega-limit set of (ω_*, λ_*), there holds

$$\left|\int_0^t \mathrm{Tr}\; Q(\widehat{\tau}_s(\bar{\omega}, \bar{\lambda}))\, ds\right| \leq \kappa \tag{17}$$

for all $t \in \mathbb{R}$. Since the omega-limit set of (ω_*, λ_*) coincides with Σ_*, we conclude that (17) holds for all $(\bar{\omega}, \bar{\lambda}) \in \Sigma_*$.

Next let $\omega \in \Omega$, and let λ_v be the vertical Lagrange plane. We need only to show that there exists $T > 0$ such that, if $\lambda(t) = \Phi_\omega(t)\,\lambda_v$, then $\lambda(t) \notin \mathcal{C}$ for all $t \geq T$. Let $\bar{\lambda} \in \Lambda$ be a point such that $(\omega, \bar{\lambda}) \in \Sigma_*$. There is a matrix $\bar{\Phi} \in \mathrm{Sp}(n, \mathbb{R})$ such that $\bar{\Phi}\lambda_v = \bar{\lambda}$. Let us write $\varphi(t) = \mathrm{Arg}_3(\Phi_\omega(t))$. According to [33], there is a constant κ such that

$$\left|\varphi(t) - \mathrm{Arg}_3(\Phi_\omega(t)\,\bar{\Phi})\right| \leq \kappa \tag{18}$$

for all $t \geq 0$.

Now, $\varphi(t) = \sum_{j=1}^n \varphi_j(t)$, where the "angles" $\varphi_j(t)$ are continuous and non-decreasing functions of t. Using (12), (17) and (18), we see that these functions must have finite limits as $t \to \infty$ $(1 \leq j \leq n)$. Arguing just as we did in the proof of part b), we see that, for some $T > 0$, we have that $\lambda(t) \notin \mathcal{C}$ for all $t > T$. This completes the proof of part a).

There remains to prove part c) of Theorem 2. To do this, it is sufficient to modify the arguments given in ([6], pp. 38–42). We sketch the details. Let $\omega \in \Omega$, and let $\mathbf{z}_1(t), \dots, \mathbf{z}_n(t)$ be linearly independent solutions of (1_ω) such that the matrix $\begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}$ with columns $\mathbf{z}_1(t), \dots, \mathbf{z}_n(t)$ satisfies $\det u_0(t) \neq 0$ $(-\infty < t < \infty)$. Set

$$S_0(t) = \int_0^t u_0(s)^{-1} H_{22}(\tau_s(\omega))\, u_0(s)^{-1*} ds\,.$$

Clearly $S_0(t)$ is non-decreasing in t (with respect to the natural order on the set of symmetric $n \times n$ real matrices).

Let $T > 0$ be a number satisfying the condition (9). We claim that, if $0 \leq t_1 < t_2$ and if $t_2 - t_1 \geq T$, then $S_0(t_2) - S_0(t_1)$ is strictly positive definite. To see this, we proceed as in the proof of Proposition 2, p. 38 of [6]. Suppose for contradiction that there is a vector $0 \neq \mathbf{c} \in \mathbb{R}^n$ such that $S_0(t_2)\,\mathbf{c} - S_0(t_1)\,\mathbf{c} = 0$. Then $S_0(t)\,\mathbf{c} - S_0(t_1)\,\mathbf{c} = 0$ for all $t \in [t_1, t_2]$. One verifies that

$$\begin{aligned} \mathbf{x}(t) &= u_0(t)[-S_0(t_1) + S_0(t)]\,\mathbf{c} \\ \mathbf{y}(t) &= v_0(t)[-S_0(t_1) + S_0(t)]\,\mathbf{c} + u_0(t)^{-1*}\mathbf{c} \end{aligned}$$

is a solution of (1_ω); clearly $\mathbf{x}(t) = 0$ for all $t \in [t_1, t_2]$.

Now let $\omega_1 = \tau_{t_1}(\omega),\ \mathbf{x}_1(t) = \mathbf{x}(t+t_1),\ \mathbf{y}_1(t) = \mathbf{y}(t+t_1)$. Then

$$\begin{aligned}\mathbf{x}_1' &= H_{21}(\tau_t(\omega_1))\,\mathbf{x}_1 + H_{22}(\tau_t(\omega_1))\,\mathbf{y}_1\\ \mathbf{y}_1' &= -H_{21}^*(\tau_t(\omega_1))\,\mathbf{y}_1\end{aligned}$$

if $0 \le t \le t_2 - t_1$. Note that $\bar{\mathbf{y}}_1 = \mathbf{y}(t_1) \neq 0$ because $\mathbf{c} \neq 0$. Clearly $\mathbf{y}_1(t) = \Psi_{\omega_1}(t)^{-1*}\bar{\mathbf{y}}_1$, where $\Psi_{\omega_1}(t)$ is the fundamental matrix solution of the system $\mathbf{x}' = H_{21}(\tau_t(\omega_1))\,\mathbf{x}$. Since $\mathbf{x}_1(t) = 0$ for $0 \le t \le t_2 - t_1$, we have

$$0 = \int_0^T \left\| H_{22}(\tau_s(\omega_1))\,\Psi_{\omega_1}^{-1*}(s)\,\bar{\mathbf{y}}_1 \right\|^2 ds\,.$$

This violates Hypothesis 2 and proves our claim.

Next write

$$L_0 = \lim_{t\to\infty} S_0(t)^{-1}.$$

The limit exists and is positive semidefinite because $S_0(t)$ is positive definite for $t \ge T$ and is nonincreasing in t. Moreover, $S_0(t)^{-1} - L_0$ exists and is invertible for $t \ge T$ because of our claim.

We continue to follow [6], and write

$$\begin{aligned}\widehat{u}(t) &= u_0(t)\,[I_n - S_0(t)\,L_0]\\ \widehat{v}(t) &= v_0(t)\,[I_n - S_0(t)\,L_0] - u_0(t)^{-1*}L_0\,.\end{aligned}$$

One verifies that $\begin{pmatrix}\widehat{u}(t)\\ \widehat{v}(t)\end{pmatrix}$ is a $2n \times n$-matrix solution of (1_ω), that $\det\widehat{u}(t) \neq 0$ for all $t \ge 0$, and that $\widehat{\lambda}(t) = \begin{bmatrix}\widehat{u}(t)\\ \widehat{v}(t)\end{bmatrix}$ is a real Lagrange plane for all $t \ge 0$. Arguing as on p. 42 of [6], we see that, if

$$\widehat{S}(t) = \int_0^t \widehat{u}(s)^{-1} H_{22}(\tau_s(\omega))\,\widehat{u}(s)^{-1*} ds\,,$$

then $\widehat{S}(t) \to 0$ as $t \to \infty$.

Next let $\widetilde{\lambda} \in \Lambda$ and write $\widetilde{\lambda}(t) = \Phi_\omega(t)\,\widetilde{\lambda} = \begin{bmatrix}\widetilde{u}(t)\\ \widetilde{v}(t)\end{bmatrix}$. Suppose that $\det\widetilde{u}(t) \neq 0$ for all $t \ge 0$ and that, if $\widetilde{S}(t) = \int_0^t \widetilde{u}(s)^{-1} H_{22}(\tau_s(\omega))\,\widetilde{u}(s)^{-1*} ds$, then $\lim_{t\to\infty}\widetilde{S}(t) = 0$. Arguing as in p. 42 of [6], we see that $\widetilde{\lambda}(t) = \widehat{\lambda}(t)$ for all $t \ge 0$.

Now, a $2n \times n$-matrix solution $\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}$ of (1_ω) is called a principal solution exactly when $\lambda(t) = \begin{bmatrix}u(t)\\ v(t)\end{bmatrix}$ is a real Lagrange plane for all $t \ge 0$, and when (i) $\det u(t) \neq 0$ for all $t \ge 0$; (ii) $\lim_{t\to\infty} S(t)^{-1} = 0$ where $S(t) = \int_0^t u(s)^{-1} H_{22}(\tau_s(\omega))\,u(s)^{-1*} ds$. So we have proved that (1_ω) admits a unique principal solution $(\omega \in \Omega)$. This completes the proof of Theorem 2.

Remark 1. It is easy to see that, if $H_{22}(\omega) > 0$ for each $\omega \in \Omega$, then weak disconjugacy is equivalent to true disconjugacy. In this case, Theorem 2 provides a characterization of disconjugacy: if Supp $\mu = \Omega$, then all equations (1_ω) are disconjugate if and only if $\alpha(\mu) = 0$.

Using an observation of S. Novo, we can say more. Suppose that $H_{22}(\omega) \geq 0$ for each $\omega \in \Omega$, and suppose that, for each $\omega \in \Omega$ and each non-zero solution $(\mathbf{x}(t), \mathbf{y}(t))$ of (1_ω), $\mathbf{x}(t)$ does not vanish identically on $[0, \infty)$. (This last condition holds in particular if each equation (1_ω) is identically normal.) One can then prove that Hypothesis 2 holds, and so the conclusions of Theorem 2 are valid. One can then go on to show that, if $\operatorname{Supp} \mu = \Omega$, if $H_{22}(\omega) \geq 0$ for all $\omega \in \Omega$, and if each equation (1_ω) is identically normal, then all equations (1_ω) are disconjugate if and only if $\alpha(\mu) = 0$.

We finish this paper by considering the possibility of approximating a weakly disconjugate family (1_ω) by families of Hamiltonian systems which admit an exponential dichotomy. Let us introduce an Atkinson-type condition [4]:

Hypothesis 3. Each minimal subset $M \subset \Omega$ contains a point ω_0 such that, for some $\delta > 0$,

$$\int_0^\infty \left\| \begin{pmatrix} 0 & 0 \\ 0 & H_{22}(\tau_t(\omega_0)) \end{pmatrix} \Phi_{\omega_0}(t)\, \mathbf{z} \right\|^2 dt \geq \delta \|\mathbf{z}\|^2$$

for all $\mathbf{z} \in \mathbb{R}^{2n}$.

Theorem 3. *Suppose that Hypotheses* 1, 2, *and* 3 *all hold. Let* μ *be an ergodic measure on* Ω *such that* $\operatorname{Supp} \mu = \Omega$ *and such that* $\alpha(\mu) = 0$. *Then for each* $\lambda \in (0, 1)$, *the family*

$$J\,\mathbf{z}' = \begin{pmatrix} H_{11}(\tau_t(\omega)) & H_{12}(\tau_t(\omega)) \\ H_{21}(\tau_t(\omega)) & \lambda H_{22}(\tau_t(\omega)) \end{pmatrix} \mathbf{z} \tag{19}$$

admits an exponential dichotomy over Ω

Note that the hypotheses imply that all equations (1_ω) are weakly disconjugate.

Proof. First of all, $\lambda H_{22}(\omega) \geq 0$ for each $\omega \in \Omega$. Hence the rotation number $\alpha(\mu; \lambda)$ of the family (19) is non-negative (Proposition 1). On the other hand, it is easily seen that the function $\lambda \to \alpha(\mu; \lambda)$ is non-decreasing. Hence $\alpha(\mu; \lambda) = 0$ for $0 < \lambda \leq 1$.

Now, Hypothesis 3 is just what is needed to apply the main Theorem of [17]: we conclude that, if $0 < \lambda < 1$, then equations (19) admit an exponential dichotomy. This completes the proof of Theorem 3.

We note that, in the above situation, if equations (1_ω) do not admit an ED over Ω, then $\alpha(\mu; \lambda)$ must be strictly positive for $\lambda > 1$.

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