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Learning a mathematical concept from comparing examples: the importance of variation and prior knowledge

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Abstract In experiment 1, novice fourth-grade students (N=92) who compared multiple examples that separately varied each critical aspect and then simultaneously varied all critical aspects developed better conceptual knowledge about the *altitude of a triangle* than students who compared multiple examples that did not separately vary each critical aspect but simultaneously varied all critical aspects. In experiment 2, this pattern was the same for fourth-grade students (N=90) but not for sixth-grade students (N=94) who had greater prior knowledge about the concept. Aspects that are critical for learning should be varied first separately and then simultaneously, and students with different levels of prior knowledge may perceive different aspects as critical for their learning and thus benefit differently from the identical instruction.

Keywords Example variability · Comparison · Conceptual knowledge · Mathematics education · Positive and negative examples

Learning from examples has attracted great attention from educational psychologists and has been a major topic in educational research for at least the past four decades (Atkinson et al. 2000). One of the issues that have been intensively investigated by researchers is the quantity of examples, i.e., how many examples are needed for learning? Researchers consistently demonstrate that multiple examples are better than one example in facilitating learning, because the comparison evoked by comparing multiple examples is generally good for learning (e.g., Catrambone and Holyoak 1989; Cooper and Sweller 1987; Gentner 2005; Gibson and Gibson 1955; Gick and Paterson 1992; Reed 1993; Rittle-Johnson and Star 2009; Schwartz and Bransford 1998; Silver et al. 2005; Sweller and Cooper 1985; Tennyson 1973). As Gentner and Namy (1999) argued, a comparison among examples can

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highlight a common structure which is important for learning. Rittle-Johnson and Star (2007) also claimed that, "comparison is emerging as a fundamental learning mechanism" (p. 561).

Not all comparisons, however, may equally be effective (Gick and Paterson 1992; Quilici and Mayer 1996; Rittle-Johnson et al. 2009). The effectiveness of multiple examples depends on the variability of multiple examples being compared (Gentner and Namy 1999; Quilici and Mayer 1996; Renkl et al. 1998; Rittle-Johnson and Star 2007, 2009; Rittle-Johnson et al. 2009) and the prior knowledge of students who compare the examples (Albro et al. 2007; Clarke et al. 2005; Gentner et al. 2007; Schwartz and Bransford 1998), which are still unsolved questions and need further research.

In the introduction, we briefly overview a literature on learning from comparing multiple examples and identify limitations in existing research as the point of departure for the present study. Next, we elaborate the "critical aspects/features" and "patterns of variation and invariance" from a variation theory that provides guidelines for this study to investigate unsolved issues. Finally, we introduce the geometry concept that is under examination in the study.

Experiment research on learning from comparing multiple examples

Although plenty of studies on learning from examples have shown that multiple examples are better than one in terms of promoting learning, it is still unclear with respect to the role of example variability and students' prior knowledge in learning from comparing examples.

First, it is unclear regarding how similar or different examples should be in order to promote learning (Renkl et al. 1998; Rittle-Johnson and Star 2009). Examples are generally analyzed from two aspects: surface (irrelevant) features and structural (relevant) features (Gick and Holyoak 1980, 1983; Holyoak and Koh 1987; Paas and Van Merrienboer 1994; Quilici and Mayer 1996; Reed 1989; Ross 1989b, 1997; Ross and Kennedy 1990; VanderStoep and Seifert 1993). It is the similarity between the two aspects that determines the comparison from examples. Holyoak and Koh (1987) used the relevance to goal attainment to distinguish the types of features: surface features are irrelevant to the goal attainment, such as names, objects, numbers, and story lines; structural features are relevant to the goal attainment, such as underlying mathematical procedures, rules, solutions, and principles (Quilici and Mayer 1996; Ross and Kilbane 1997). For instance, an apple is an example of the concept of fruit. The shape, size, and color are surface features of the apple, and the attribution that the apple is edible is a structural feature.

Contradictory findings have been reported regarding how similar or different the multiple examples should be in terms of surface and structural features. Examples with different surface features would help the learner focus on structural features and induce a schema; given superficially similar examples the learner might consider surface features as relevant, which might spoil schema induction and future problem solving (e.g., Merrill and Tennyson 1978; Paas and Van Merrienboer 1994; Quilici and Mayer 1996; Ranzijn 1991; Reed 1989; Tennyson 1973; Tennyson et al. 1972). The reverse, however, seems to be true as well. Providing superficially similar examples might help the learner notice and align the structural features and form the schema; high variable examples might make the structural features difficult to be discovered (e.g., Gentner and Namy 1999; Gick and Holyoak 1980, 1983; Namy and Gentner 2002; Richland et al. 2004; Ross 1989a; Ross and Kennedy 1990). Furthermore, most existing studies focus on the effect of variability of surface features rather than structural features on learning from multiple examples. More research is needed to investigate different effects of surface and structural features on learning, and to explore what relevant important dimensions should vary and what dimensions should remain invariant (Gick and Paterson 1992; Rittle-Johnson and Star 2009). As Rittle-Johnson and Star (2009) concluded, it is still unclear from existing literature how similar multiple examples should be and what should be compared.

Second, it is also unclear regarding the role of students' prior knowledge in learning from multiple examples (Rittle-Johnson et al. 2009). Some researchers argued that students with low prior knowledge can hardly benefit from comparing multiple examples, especially the complex and unfamiliar examples (e.g., Albro et al. 2007; Gentner et al. 2007; Holmqvist et al. 2007; Schwartz and Bransford 1998); some researchers found that high able students can benefit from comparing any kind of examples, while less able students can only benefit from comparing very different examples (e.g., Quilici and Mayer 1996); and some researchers did not find any interaction effect between students' prior knowledge and the variability of examples being compared (e.g., Renkl et al. 1998).

Inspired by the two issues, the present study was conducted to investigate how to design multiple examples in terms of the example variability to promote learning on the one hand and to examine the role of students' prior knowledge in learning from comparing multiple examples on the other hand. Furthermore, as most research on learning from multiple examples involves non-academic tasks (Rittle-Johnson and Star 2009; Rittle-Johnson et al. 2009; Star and Rittle-Johnson 2009), varying from preschool-aged children learning perceptual categories (Namy and Gentner 2002) to undergraduate students learning contract negotiation strategies (Gentner et al. 2007) to bank apprentices learning calculation of interest (Renkl et al. 1998), the present study chose mathematics as the subject area for research.

Critical aspects/features and patterns of variation and invariance

Our research is motivated by variation theory which holds variation to be epistemologically fundamental for all learning to occur (see Bowden and Marton 1998; Marton and Booth 1997; Marton and Tsui 2004; Pang 2003 for details). Different from cognitive theories that consider learning as construction of mental representation, variation theory interprets learning from a relational and perceptual perspective. As Marton and Booth (1997) argued, an experience is in its essence non-dualistic; that is, the human and the world are not separated. When we see something, we create individual–world relations through our experiences. For example, if a person experiences an object as a bird, then the meaning of the bird is not in the object, neither is it "in the subject's head." Instead, as Svensson (1984) explained, it is constituted as the relation between the object to which awareness is directed and the person as the subject. According to variation theory, learning means the formation of new individual–world relations and thus a new way of seeing something rather than psychological entities located in individuals. And "learning to see something in a certain way amounts to discerning certain critical features of that phenomenon and focusing on them simultaneously" (Marton 1999). In particular, the notions of "critical aspects/features" and "patterns of variation and invariance" from variation theory provide direct guidelines for the present study to address the unsolved issues mentioned above.

Critical aspects/features of the object of learning

In variation theory, learning and teaching must be learning and teaching of something. The something to be learned, i.e., the object of learning, is a particular insight, skill, or capability that students are supposed to develop after instruction (Marton and Pang 2006). For a particular object of learning to be learned, critical aspects of the object of learning should be first identified. According to variation theory, aspects and features of a phenomenon and its examples are analyzed as critical or uncritical to students' understanding and learning, rather than surface or structural to the objective disciplinary knowledge. Critical aspects are aspects that cause difficulty for students in the process of learning; they might be superficial or structural. For example, if a child believes that fruit can only be round, then the surface aspect (shape) and feature (round) become critical for the child to learn the concept of fruit. In order to help the child realize that fruit could be in different shapes, we should show him/her different examples (e.g., an apple, a banana, and a carambola) that have different shapes. Here, the surface aspect of shape is the critical aspect for learning the concept of fruit. As Marton and Pang (2008) argued, both the disciplinary knowledge and the students' understanding should be taken into account when identifying the critical aspects of an object of learning.

Patterns of variation and invariance

According to variation theory, to learn a phenomenon means to simultaneously discern the critical aspects/features of the phenomenon. To discern, a learner must experience variation. When an aspect of a phenomenon varies while other aspects remain invariant, the varied aspect would be discerned (Pang and Marton 2005). In particular, Marton and his colleagues (e.g., Marton and Pang 2006; Marton and Tsui 2004) defined four patterns of variation and invariance (contrast, separation, fusion, and generalization) to facilitate the discernment of critical aspects.

Contrast occurs when a learner experiences variation of different values or features in an aspect of a phenomenon. To experience something, the learner must experience something else in order to make a comparison. For example, to experience what "black" is, the learner must experience other colors such as red or white. Only after having experienced other values (red, white, etc.) of the aspect (color) can the learner discern the specific color of black. The pattern of contrast focuses on a particular value or feature of an aspect.

Separation happens when a learner focuses on an aspect of a phenomenon. To experience a certain aspect of something separately from other aspects, it must vary while other aspects remain invariant. In this pattern, the varied aspect is discerned by the learner. For example, to discern the aspect of the "color" of an object, other aspects (e.g., size, shape, and height) must be kept invariant while varying the aspect of "color." In this way, the aspect of "color" can be separated from other aspects. Contrast and separation occur when two or more objects have a varying aspect while other aspects remain invariant.

Fusion takes place when a learner wants to discern several aspects of a phenomenon that vary simultaneously. To experience a phenomenon, the learner must discern all critical aspects at the same time when different critical aspects vary simultaneously. For example, if a teacher wants to teach students what a robin is, he/she should expose the students to simultaneous variation in all critical aspects of a robin (e.g., feather, size, color, shape, and sound). The students will grasp a concept if they can simultaneously discern all critical aspects of the concept.

Generalization occurs when a learner wants to apply his/her previous discernment to various contexts. To fully understand an object of learning, the learner must experience many other examples to generalize the meaning. The idea of "blackness" can only be achieved after the learner has experienced various black objects, such as black clothes, a black ball, black hair, and so on.

With respect to the sequence of using the four patterns of variation and invariance, researchers suggested that contrast and separation should be first used to help students discern each critical aspect separately, followed by fusion that simultaneously varies all critical aspects (Ki 2007; Marton and Tsui 2004; Pang 2002). Generalization could be used after students have simultaneously discerned all critical aspects to generalize the discernment to other contexts.

In the current study, we analyzed a geometrical concept and its examples from the critical aspects for learning, and investigated the effectiveness of patterns of variation and invariance in helping students discern these critical aspects. We expected that this way of analyzing a concept and designing its multiple examples would shed some light on the controversial issues found in the area of example comparison. Specifically, we compared the effectiveness of different types of comparison in supporting the learning of the concept, for students with different levels of prior knowledge. These types varied in how the multiple examples were designed. In one type, critical aspects of the object of learning were varied first separately (*contrast* and *separation*) and then simultaneously (*fusion*). In the other type, critical aspects were not separately varied before being simultaneously experienced. We predicted that the first type should be a better method for determining the example variability to promote learning, and critical aspects that vary for students with different prior knowledge should be the focus during example design.

Critical aspects of the object of learning in the present study

The geometrical concept used in the study

The geometrical concept used in the study was the *altitude of a triangle*. A number of studies have examined this concept and have reported that it is a difficult concept for students at various age levels to master. For example, Fischbein and Nachlieli (1998) found that many students in grades 9–11 could neither correctly define the *altitude of a triangle* nor draw the required altitude in a right-angled or obtuse triangle. Of the students in their study, 61% were unable to complete the latter task. Vinner and Hershkowitz (1983) asked 189 students in grades 6–8 (ages 11–14) to complete two versions of a questionnaire to indicate their understanding of basic geometric concepts that they had encountered previously. The results of the study showed that only about 30% of the subjects could correctly construct the required altitude of a right-angled or obtuse triangle, even when they were provided with the definition of the concept; about 20% could construct the altitude correctly without being provided with the definition. In another study, Hershkowitz (1989) asked students to draw the required *altitude of a triangle* and found that even in grade 8, when the concept had been taught, less than 30% of the students had grasped the concept correctly. Similarly, after examining 190 pre-service primary teachers' understanding of the concept of *altitude of a triangle*, Gutierrez and Jaime (1999) concluded that the concept is not easily grasped by either pupils or teachers.

It appears from these studies that the concept of the *altitude of a triangle* is difficult for both primary and secondary students to understand, regardless of whether they have been taught the concept. Gutierrez and Jaime (1999) described seven kinds of errors that students made on tests of this concept when asked to draw a required *altitude of a triangle*: (a) no answer, (b) the median to the specified base, (c) the perpendicular bisector of a side, (d) the correct altitude to a side different from the specified base, (e) a segment perpendicular to the specified base but with the wrong length, (f) a segment internal to the triangle from the opposite vertex to the specified base but not perpendicular to it or the median to the base, and (g) other incorrect responses. Other researchers (Fischbein and Nachlieli 1998; Hershkowitz 1989; Vinner and Hershkowitz 1983) have also contended that the inappropriate assumption that the altitude should fall inside the triangle is the major factor affecting students' performance in this area.

Six critical aspects for learning altitude of a triangle

The object of learning in this study was to develop students' conceptual knowledge of *altitude of a triangle*; conceptual knowledge is "an integrated and functional grasp of mathematics ideas" (Kilpatrick et al. 2001, p. 118). The conceptual knowledge was measured via the ability to recognize and to explain (a) whether or not a segment was an altitude of a triangle, (b) a specified side of a triangle to which an altitude was perpendicular, and (c) three specified sides of a triangle to which three altitudes were perpendicular (see Fig. 5), in line with past measures of conceptual knowledge (e.g., Bruner et al. 1956; Hiebert and Wearne 1996; Klausmeier 1992; Rittle-Johnson and Star 2007, 2009; Tennyson et al. 1983).

To achieve this object of learning, six aspects were identified as critical for fourth-grade students' learning: vertex, perpendicularity, opposition, orientation, location, and altitude-base-correspondence, as shown in Table 1. The selection of these critical aspects was based on the accepted definition of the concept, research related to the subject matter, and two pilot studies conducted by the authors previously. The critical aspects of vertex, perpendicularity, and opposition were defining aspects that were derived from the definition of the *altitude of a triangle* and were considered to be singly necessary and jointly sufficient for defining the concept. The critical aspects of orientation, location, and altitude-base-correspondence were obtained from students' understanding of the concept and were aspects known to cause difficulty for students in the learning. For instance, many students might incorrectly assume that an altitude is always up-down and vertical to the level (or page), or could only fall inside the triangle. There were other aspects that might affect the learning of this concept; however, they were excluded based on our previous work and judgment.

To achieve adequate variation in the critical aspects, different features/values were identified for each aspect as shown in Table 1. For example, two values were defined for

Table 1 Critical aspects and features/values of the object of learning

Critical aspects	Values	
	Value ①	Value ②
① Vertex	Pass through a vertex	Not pass through a vertex
② Perpendicularity	Perpendicular to the line of one side	Not perpendicular to the line of one side
③ Opposition	The vertex is opposite to the side	The vertex is not opposite to the side
④ Orientation	Up-down and vertical to the level (or page)	Other orientations
⑤ Location	Inside the triangle	Coincident with or outside the triangle
⑥ Altitude-base-correspondence	Altitudes do not correspond with bases	Each altitude corresponds with a specified base

the aspect of vertex: value ① refers to the situation in which a segment passes through a vertex and value ② refers to the situation in which a segment does not pass through a vertex. If what is being measured is the altitude of a triangle, then only value ① should apply for the first three critical aspects. That is, an altitude passes through a vertex and is perpendicular to the line on its (the vertex's) opposite side. In contrast, if what is being measured is not the altitude of a triangle, then value ② may apply for the first three critical aspects. For instance, it might not pass through a vertex or might not be perpendicular to the line of one side. In terms of the last three critical aspects, students often incorrectly consider only value ①, ignoring other values, which is why they might have difficulty learning the concept. For example, many students have the misconception that the altitude of a triangle should be always inside the triangle.

Therefore, students need to simultaneously discern these six critical aspects in order to develop a complete understanding of the *altitude of a triangle*. In other words, they should understand that the altitude of a triangle must pass through a vertex and be perpendicular to its opposite side, could be at one of many possible orientations, is not necessarily always inside the triangle, and corresponds with a specified base.

Present study

We conducted two experiments (experiments 1 and 2) to investigate the effects of example variability and students' prior knowledge on learning the *altitude of a triangle*. Three conditions were created for the experiments; the three conditions differed in how examples were designed. Students were randomly assigned to (a) experience variation of the six critical aspects first separately and then simultaneously (six-separation-simultaneity (SSS)), (b) experience variation of three critical aspects first separately and then experience the variation of all six critical aspects simultaneously (three-separation-simultaneity (TSS)), or (c) experience variation of the six critical aspects simultaneously without having separately experienced any of these aspects (no-separation-simultaneity (NSS)). With the use of the three conditions, the present study could examine whether separate experience of critical aspects facilitated the later simultaneous discernment and its relation with students' prior knowledge.

Experiment 1 recruited fourth-grade students as participants who had not previously learned about *altitude of a triangle*, and evaluated the three types of comparison for supporting fourth-grade students' learning about the concept. It was hypothesized that the SSS condition would lead to greater learning than the TSS condition and the NSS condition. In addition to fourth-grade students, Experiment 2 included sixth-grade students as participants who had partially learned the concept in school previously and gained some knowledge regarding the concept. Thus, experiment 2 examined the possible interaction effects between students' prior knowledge and conditions. We expected that the SSS condition would better improve fourth-grade students' learning than the other two conditions but would lose some benefits on sixth-grade students; and the differences between SSS and TSS, SSS, and NSS would be insignificant.

Critical aspects could be separately experienced via the use of examples designed by *contrast* and *separation*. For example, Fig. 1 shows how students could separately discern the critical aspects of vertex, perpendicular, and opposition by comparing pairs of positive and negative examples. Within the first pair, the only difference between the positive and negative examples was in the critical aspect of vertex: segment *AD* in the positive example passed through a vertex, whereas the corresponding segment *ED* in the negative example

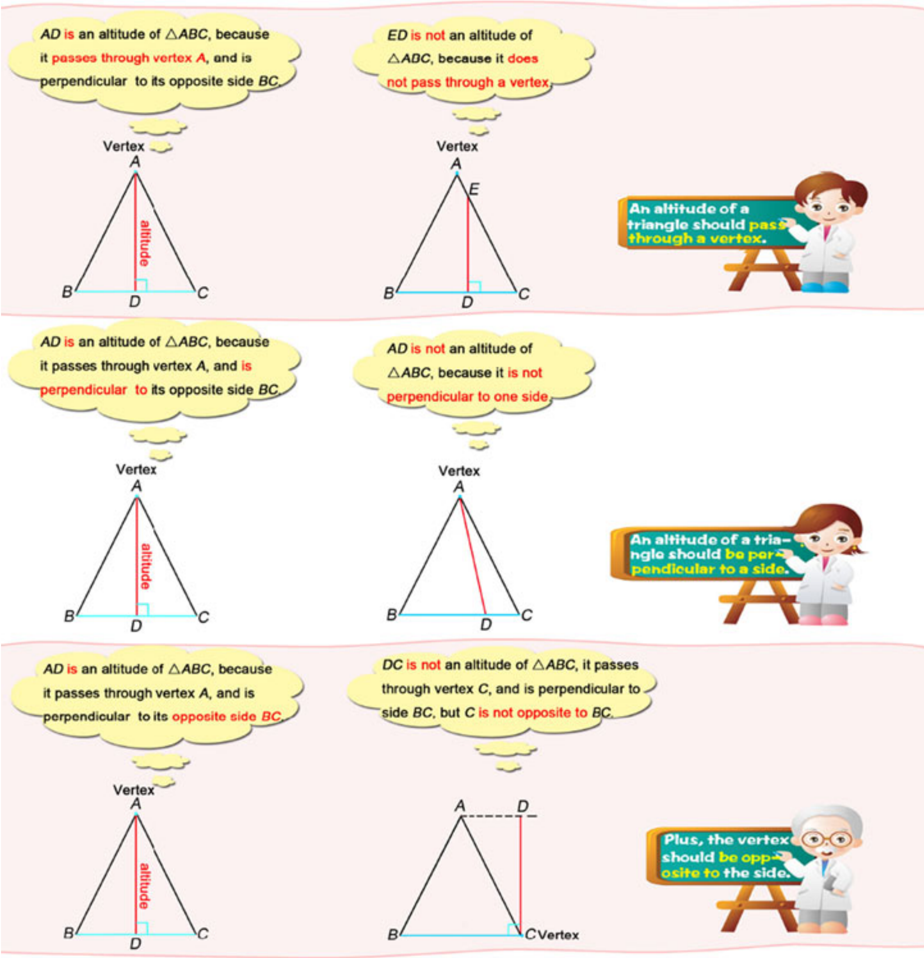


Fig. 1 A sample packet page showing separate variation of critical aspects of vertex, perpendicularity, and opposition (translated from Chinese)

did not pass through a vertex. By keeping all of the other critical aspects the same, the varied aspect of vertex and the value of passing through a vertex should be discerned by students. These techniques are known in variation theory as *separation* and *contrast*. In other words, after comparing the positive and negative examples in the first pair, students should understand that the altitude of a triangle must pass through a vertex.

Similarly, the second pair of positive and negative examples was designed according to patterns of *separation* and *contrast* for students to discern the critical aspect of perpendicularity. After comparing the two examples in this pair, students were expected to understand that an altitude must be perpendicular to a side. For the third pair of examples, students were expected to discern the critical aspect of opposition by comparing the positive and negative examples. Therefore, with the use of *contrast* and *separation*, students were expected to separately discern the critical aspects of vertex, perpendicularity, and opposition; in other words, they would understand that the altitude of a triangle should pass through a vertex and be perpendicular to a side that is opposite to the vertex.

Figure 2 shows how the six critical aspects were simultaneously varied in examples: different altitudes of different types of triangles (acute, right-angled, and obtuse) passed through different vertexes, were perpendicular to different opposite sides, and were at different orientations and locations. Experiencing simultaneous variation of all critical aspects is considered a necessary stage for developing a complete understanding of the object of learning.

Experiment 1

The primary goal of experiment 1 was to compare the effects of three types of comparison (SSS, TSS, and NSS) on fourth-grade students' learning about the *altitude of a triangle*. It was hypothesized that SSS would be superior to TSS and NSS because students in the SSS condition had the opportunity to separately discern each critical aspect which was necessary when they needed to simultaneously discern all critical aspects. In contrast, students in the

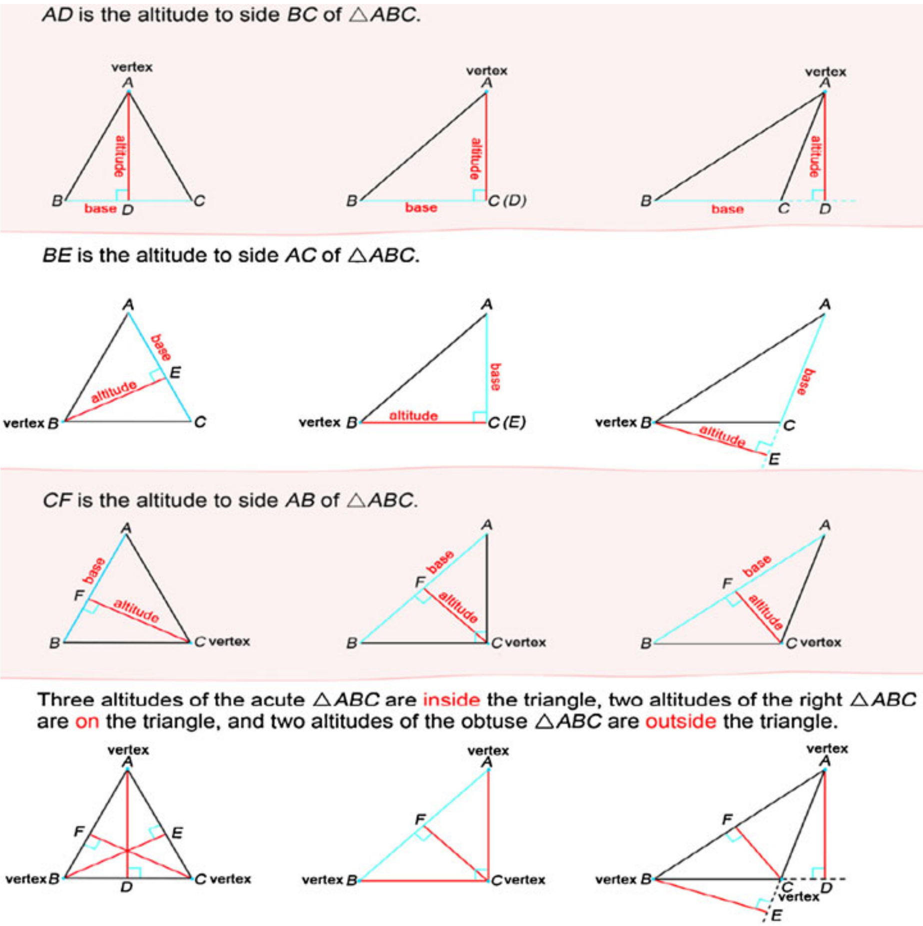


Fig. 2 A sample packet page showing simultaneous variation of all six critical aspects (translated from Chinese)

TSS and NSS conditions would encounter difficulty when comparing examples varying all critical aspects because they had not separately discerned these critical aspects.

Methods

Participants Ninety-two fourth-grade students from a top urban primary school in Beijing, China, were chosen as the subjects. Parental and students' permissions were obtained before the study. None of the participants had previously learned about the *altitude of a triangle* in school. They, however, had gained some relevant knowledge including different types of triangle, vertexes and sides of a triangle, perpendicularity, altitude of a trapezoid, and area of a rectangle. According to the school's curriculum, the concept of *altitude of a triangle* was scheduled to be taught to students about 1 week after experiment 1 was completed; the participants were thus considered to be ready for learning this concept.

Design and procedure We employed a pretest-intervention-posttest design. A pretest and a posttest were used to evaluate students' conceptual knowledge of the *altitude of a triangle*. First, all of the students were asked to take a 40-min pretest to determine their prior knowledge of the concept. One week after the pretest, students were randomly assigned to SSS (n=31), TSS (n=31), or NSS (n=30) and were given a corresponding intervention packet to study for 40 min. An instructor gave a brief (10 min) scripted introduction to students before the intervention. All instructors were our research assistants and followed a script. The instructor gave students a reminder when the time left was 10 min. The packets were collected back after the intervention finished. Then students took a 5-min break before completing a posttest, which was equivalent to the pretest. Finally, 22 students were randomly selected for interviews.

Before the pretest and posttest, the instructor also gave a brief scripted introduction of the tests, explaining the task to the students. They were told to justify their answers in terms of the attributes of the concept and to feel free to elaborate on their ideas as much as possible. The students were told that their answer sheets would not be rated or reviewed by their teachers. They were also reminded when the time left was 10 min. To ensure fidelity of condition, we did not administer any test or intervention, but gave direction and provided assistance to instructors when necessary, as well as observed classrooms.

Intervention The design of the three intervention packets is summarized in Fig. 3. First, a question about how to calculate area of a triangular red scarf was raised to motivate students' interest in learning the concept of the *altitude of a triangle*. Then, the instruction was delivered to students via a definition and examples of and practice with the concept. Finally, answers for the practice exercises were provided for students to evaluate their own understanding. The differences among the three packets occurred in how examples were designed for learning, whereas other parts (e.g., definition, practice, answers, etc.) were kept consistent to control for extraneous factors within the conditions. The remainder of this section illustrates how the patterns of variation and invariance were used to design examples in SSS, and the differences of example design among the three conditions.

The three intervention packets (SSS, TSS, and NSS) all contained three altitudes of three types of triangles in Fig. 2. The primary difference among the packets was how these examples were designed and sequenced. In the SSS packet, the six critical aspects were separately varied via the use of examples designed based on *contrast* and *separation*. First, three pairs of positive and negative examples in an acute triangle were used on one page

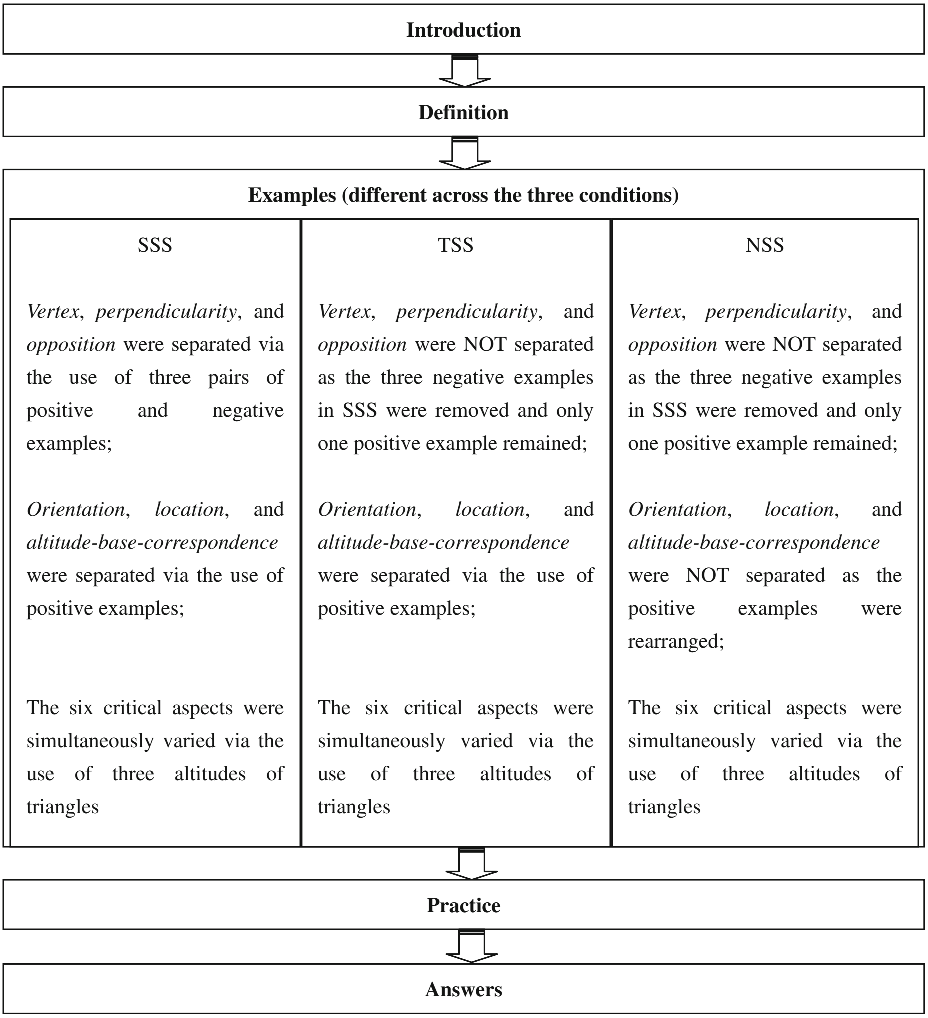


Fig. 3 Design of intervention packets for SSS, TSS, and NSS

(see Fig. 1). These examples were designed based on patterns of *contrast* and *separation* to help students discern the critical aspects of vertex, perpendicularity, and opposition, as described above.

Then, the critical aspects of orientation and location were expected to be separately discerned on two following pages, with the use of *contrast* and *separation*. One page contained the previous positive example in Fig. 1 and two other positive examples. The three examples were the same except for the orientation; the second and third examples were obtained as a result of rotating the first example. In this way, the critical aspect of orientation was varied while other critical aspects remained invariant, which made use of the patterns of *contrast* and *separation*. After comparing the three positive examples that only differed in the aspect of orientation, students were expected to discern this aspect and understand that an altitude could be at any orientation, as long as it satisfies the critical aspects of vertex, perpendicularity, and opposition. The other page included an altitude of an acute triangle (the previous one), an altitude of a right-angled triangle, and an altitude of an obtuse triangle. The three altitudes differed only in the aspect of location: the altitude of the acute triangle was inside the triangle, the altitude of the right-angled triangle was coincident with one side of the triangle, and the altitude the obtuse triangle was outside the triangle. By comparing these three examples which were designed based on *contrast* and *separation*, students were expected to discern the critical aspect of location and realize that the location of an altitude is not necessarily always inside a triangle.

Therefore, via the use of *contrast* and *separation*, students were expected to have separately discerned the five critical aspects of vertex, perpendicularity, opposition, orientation, and location. In order to discern the sixth critical aspect of altitude-base-correspondence (i.e., each altitude corresponds with one specified base), students had to realize that there are three different altitudes for every triangle. Consequently, three altitudes of the same acute triangle were then presented to students on one page. The three altitudes varied in the aspects of vertex, perpendicularity, opposition, and orientation; they passed through different vertexes, were perpendicular to different opposite sides, and were at different orientations. Students comparing the three altitudes would experience the simultaneous variation of the four critical aspects that were expected to have been separately discerned, which made use of the pattern of *fusion*. In other words, after having separately discerned the critical aspects of vertex, perpendicularity, opposition, and orientation, students were expected to be able to simultaneously discern the four aspects and recognize the three different altitudes of the acute triangle.

After that, the same three altitudes were used again on another page and the critical aspect of altitude-base-correspondence was mentioned to students (see Fig. 4). When students were comparing the three altitudes, the four discerned critical aspects (vertex, perpendicularity, opposition, and orientation) would recede to the background and the aspect of altitude-base-correspondence (i.e., each altitude corresponds with one specified base) would come to the fore of students' attention. In this case, students were expected to discern the varied aspect of altitude-base-correspondence against other aspects that had been discerned. The three altitudes were then integrated in the acute triangle at the bottom of this page to help students simultaneously experience variation of five critical aspects (i.e., vertex, perpendicularity, opposition, orientation, and altitude-base-correspondence) that should have been separately discerned, which made use of the pattern of *fusion*. When seeing this acute triangle, students were expected to discern that the three different altitudes in the triangle were at different orientations, passed through different vertexes, and were perpendicular to their different opposite bases.

Afterward, three altitudes of a right-angled triangle and an obtuse triangle were presented to students on two separate pages (same as the second and third columns in Fig. 2). Because two altitudes of a right-angled triangle were coincident with two legs of the right-angled triangle, all six critical aspects (vertex, perpendicularity, opposition, orientation, location, and altitude-base-correspondence) were varied simultaneously when the three altitudes of the right-angled triangle were presented. Likewise, all six critical aspects were varied simultaneously when all three altitudes of an obtuse triangle were presented. In this way, students were expected to simultaneously discern critical aspects by comparing examples in an acute triangle and generalize the discernment of the right-angled triangle and the obtuse triangle.

Finally, the previous used examples were combined on one page, as shown in Fig. 2. Three altitudes of each type of triangle were shown on this page at the same time to expose students to variation of all critical aspects. This provided an opportunity for students to simultaneously experience variation of all the critical aspects that they should have

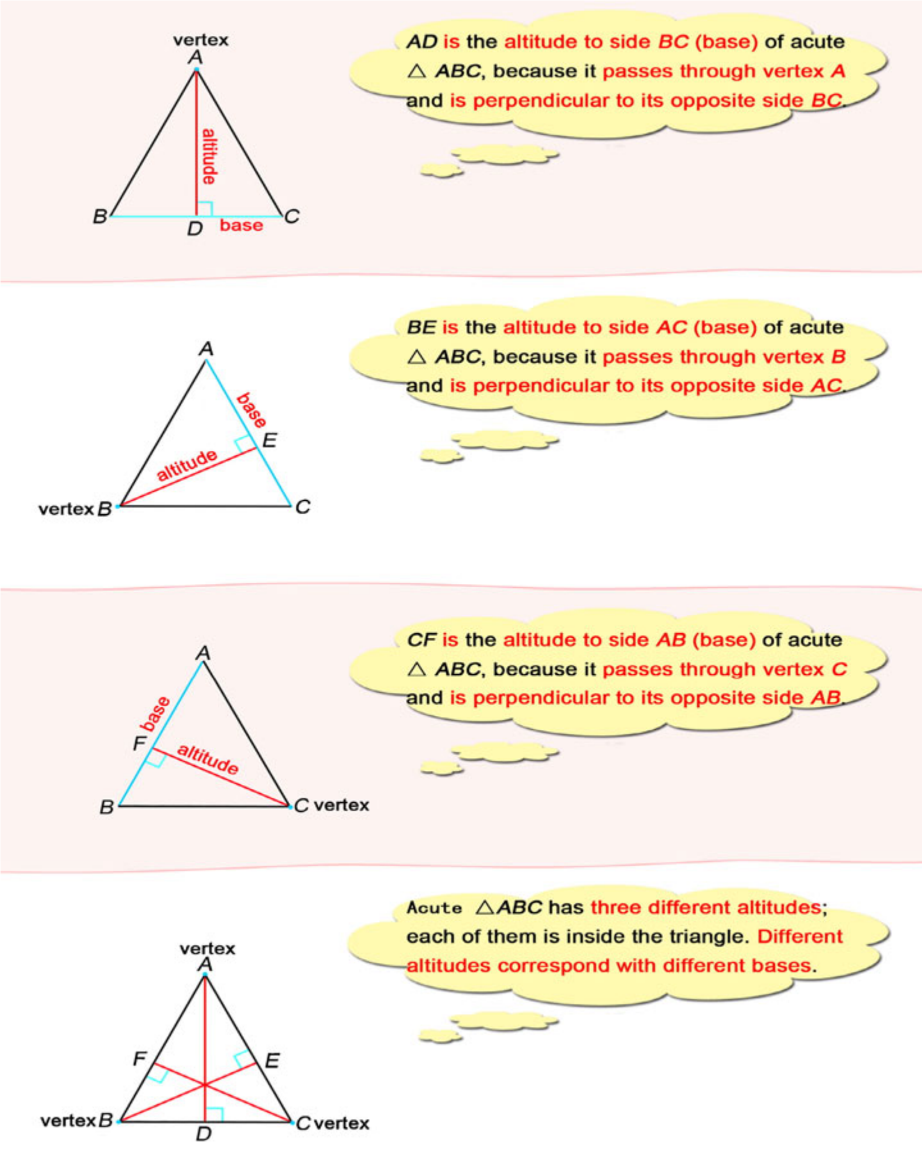


Fig. 4 A sample packet page in SSS showing simultaneous variation of five critical aspects in an acute triangle (translated from Chinese)

previously discerned, and thus to completely understand the concept of *altitude of a triangle*.

The TSS packet was similar to the SSS packet except for one page: negative examples in Fig. 1 were excluded, and only the positive example remained with the instructional prompts. Therefore, students in the TSS condition did not have the opportunity to separately discern the critical aspects of vertex, perpendicularity, and opposition as only one positive example was presented on this page. However, as other pages in TSS were the same as those in SSS, students still had the opportunity to separately discern the other three critical aspects, and to then simultaneously experience variation of all critical aspects, like students in the SSS condition did.

Unlike the SSS packet, examples in the NSS packet were not designed to help students discern critical aspects first separately and then simultaneously. Instead, students in the NSS condition experienced simultaneous variation of critical aspects at the start of comparing examples, without having separately discerned each critical aspect.

In NSS, the three negative examples in Fig. 1 were also excluded as TSS did; only the positive example remained on this page. Then, NSS presented three altitudes of the acute triangle on one page (see Fig. 4) and clearly explained the following: that there were three altitudes for every triangle and that each altitude corresponded with a specified base. Students would experience the simultaneous variation of the critical aspects of vertex, perpendicularity, opposition, orientation, and altitude-base-correspondence by comparing the three altitudes on this page; they, however, had not previously discerned these critical aspects.

After introducing one altitude and three altitudes of the acute triangle, NSS presented one altitude and three altitudes of a right-angled triangle and an obtuse triangle in a similar way on separate pages. The three altitudes of the right-angled triangle and the obtuse triangle varied all six critical aspects, including the aspect of location. Finally, all preciously used examples were summarized on one page to present three altitudes of each type of triangle at the same time as the other two conditions did, which is also shown in Fig. 2. The method of designing examples in NSS focused on different types of triangles and actually represented the traditional way of teaching the concept in the classroom, which could increase the ecological validity of the present study. This was ascertained after we talked with teachers, analyzed their lesson plans, observed their lessons, and consulted subject specialists and mathematical textbook editors.

Across all packets, instructional prompts that elaborate attributes and underlying relationships of the concept were provided to promote learning, which was suggested by literature (e.g., Kurtz et al. 2001; Renkl et al. 1998; Richland et al. 2007). For instance, promptings were provided with positive examples to elaborate attributes and to explain why they were positive example. This could help students "establish the conceptual knowledge by focusing their attention in a given example on the specific and unique characteristics of that concept" (Tennyson and Cocchiarella 1986, p. 62).

Although the total number of examples involved in SSS, TSS, and NSS was different, the differences among the three conditions were deliberate choices rather than design oversights because other design alternatives would have qualitatively changed the research questions of interest. The only difference between SSS and TSS was that the three negative examples in SSS (see Fig. 1) were removed in TSS. The purpose of designing the difference was to investigate whether or not having the opportunity to separately discern the three critical aspects (vertex, perpendicularity, and opposition) facilitated students' subsequent learning. The difference between SSS and NSS was that, in SSS the six critical aspects were varied first separately and then simultaneously while in NSS the six critical aspects were simultaneously varied without having been separately varied before. The purpose of designing the difference was to explore whether or not critical aspects should be separately varied before being simultaneously varied.

To equate the number of examples involved in the three conditions, an alternative of TSS could be using positive examples to replace the negative examples in SSS rather than removing them. Alternatives of NSS could be varying the critical aspects first simultaneously and then separately or simply adding more examples to NSS. However, either of these alternatives would increase the opportunity of experiencing other critical aspects and/or lead to other patterns of variation and invariance. For example, in two pilot studies previously conducted, we designed a condition which replaced the negative examples in SSS by positive examples that were at different orientations. This made the total number of examples absolutely equivalent in the two conditions. However, students in this new condition were found to have more opportunities to experience the critical aspect of orientation although they did not separately experience critical aspects of vertex, perpendicularity, and opposition. As a result, the experiment would examine the effectiveness of different patterns of variation and invariance; this was not the research question in which we were interested.

Therefore, given the same instructional time for SSS, TSS, and NSS, we consider the three conditions suitable for examining the research question of Experiment 1, that is, whether separate discernment of the critical aspects facilitated the later simultaneous discernment. Any possible different effects of the interventions on learning should be attributed to the variables under examination.

The design of TSS also merits some justifications. Three negative examples were removed so that students reading TSS did not have the opportunity to separately discern the three critical aspects of vertex, perpendicularity, and opposition. The choice of the three critical aspects was deliberate as the three aspects were defining and structural aspects of the *altitude of a triangle* (see Table 1); lack of them would result in a negative example. Therefore, we designed three pairs of positive and negative examples in SSS for students to separately discern the three defining aspects, and removed the three negative examples in TSS to eliminate the separate discernment. By this way, we could investigate whether separately discerning the three defining aspects affected learning the concept. In addition, as sixth-grade students in experiment 2 were supposed to have previously discerned the three defining aspects, we could further examine the possible interaction effect between condition and students' prior knowledge. Despite this, we are aware that there are three other aspects (i.e., orientation, location, and altitude-base-correspondence) that are not defining aspects but also critical to students' understanding of the concept. Future research could be conducted to examine whether separate discernment of these aspects facilitates students' subsequent learning.

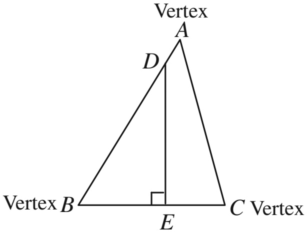
Assessment Both quantitative and qualitative methods were used to assess the students' learning of the concept in question. A test of 12 items was developed to measure the students' conceptual knowledge about the *altitude of a triangle*. The test had two equivalent forms, which were adopted as the pretest and posttest, respectively. The Pearson correlation coefficient between the two forms was $r=0.767$, $p<0.001$. Cronbach's alpha was 0.789 on the pretest and 0.810 on the posttest. Sample items are included in Fig. 5.

The test consisted of three parts to evaluate students' conceptual knowledge: "True/false judgment" (questions 1 to 6), "Completion I: one altitude" (questions 7 to 9), and "Completion II: three altitudes" (questions 10 to 12). "True/false judgment" measured whether students could correctly distinguish positive examples from negative examples of the concept and provide justification. "Completion I" measured whether students were able to correctly identify an altitude with a corresponding side of a triangle and provide justification. "Completion II" measured whether students could correctly identify three altitudes with the corresponding sides of a triangle at the same time and provide justification for the first altitude.

The interviews were conducted individually in a less stressful environment to obtain the most complete responses from selected students. Twenty-two students who were randomly chosen from the three conditions (eight from SSS, nine from TSS, and five from NSS) were

1. True/false judgment: Use your triangular rule, read carefully and judge whether the following segment is an altitude of the triangle or not. **Circle "○" your answer and justify.**

Question 1.

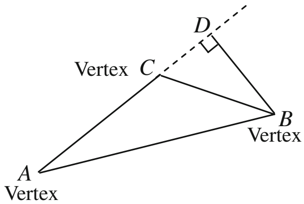


DE **is / is not** an altitude of acute △*ABC*, because____

____

2. Completion Ⅰ: Read carefully and point out **the corresponding side** of the following altitude, **and justify.**

Question 9.

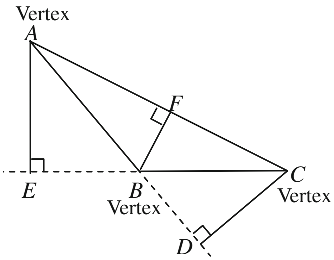


BD is the altitude to side_____of obtuse △*ABC*, because___

____

3. Completion Ⅱ: Read carefully and point out **the corresponding sides** of the following altitudes in △*ABC*, **and justify.**

Question 12.



CD is the altitude to side_____of obtuse △*ABC*, because___

____

BF is the altitude to side_____of obtuse △*ABC*;

Fig. 5 A sample items of different parts on the posttest (translated from Chinese)

asked to elaborate on their answers, especially those that were incorrect. The interviews were intended to reveal the students' understanding of the concept and thus help to understand and code the students' answers at both the pretest and the posttest.

The intervention packets, the tests, and the interview data are available from Jian-peng Guo upon request.

Coding Students' performance was coded according to their answers on the pretest and posttest, including their judgments and defining aspects (vertex, perpendicular, and opposition) discerned in the justifications. The interview data also helped to analyze students' answers and establish the coding scheme. For questions 1, 4, and 6 which involved negative examples, students needed only to identify the missing defining aspect in order to get full credits. Specifically, if a student's judgment was wrong, 0 points were given, indicating that students had no understanding of the question. If the judgment was correct but no justification or unrelated justification was given, 1 point was given. If the judgment was correct and incomplete justification was given, 2 points were given. If the judgment was correct and justification was complete, 3 points were given, indicating that students had complete understanding of the questions.

For other questions which involved positive examples, students needed to identify all three defining aspects in order to get full credits. If a student's judgment was wrong, 0 points were given. If the judgment was correct but no justification or unrelated justification was given, 1 point was given. If the judgment was correct and one defining aspect was indicated in the justification, 2 points were given. If the judgment was correct and two defining aspects were indicated in the justification, 3 points were given. If the judgment was correct and all three defining aspects were indicated in the justification, 4 points were given. Because students needed to indicate two more sides in Questions 10, 11, and 12, the highest score for the three questions was 6 points. As a result, the maximum score for the pretest and the posttest was 51 points. The coding scheme for all questions is shown in Table 2, and sample answers for questions 1 and 9 are shown in Tables 3 and 4.

Data analysis As shown in Table 2, students' answers on each question were assigned a score. The more critical aspects to be taken into consideration simultaneously, the higher score would be assigned to an answer. Because questions were different, the possible points for each question were also different. Then, students' overall performance was analyzed to determine their improvement and more importantly, to compare the effects of the three conditions.

The tests were coded by two independent coders; disagreements were resolved through consensus. The SPSS program was used to conduct the quantitative analyses. To estimate the practical significance of differences between conditions, effect sizes (Cohen's d) were computed as the degree of the mean difference between conditions divided by the pooled standard deviation.

Results

In this section, students' knowledge on the pretest is first presented. Next, students' knowledge gained from the pretest to the posttest is introduced. This is followed by the report of the effect of conditions on students' learning performance.

Pretest knowledge Table 5 summarizes students' overall pretest and posttest performance. With respect to students' pretest knowledge, no significant differences between conditions were found, $F\,(2, 89)=0.066, p>0.05$. The results also showed that students had some prior knowledge about the concept of the *altitude of a triangle*; they scored about 23.6 points on the pretest. Some students even got full credits on some questions. This might be because the interventions occurred after students had completed some relevant classroom lessons including altitude of a trapezoid, area of a rectangle, types of triangles, and so on. Most students, however, had a partial understanding of the concept.

Table 2 Coding scheme for the pretest and the posttest

Questions	Criteria			Scoring (pts)
Questions 1, 4, and 6	Wrong judgment			0
	Right judgment+no/unrelated justification			1
	Right judgment+incomplete justification			2
	Right judgment+complete justification			3
Questions 2, 3, 5, 7, 8, and 9	Wrong judgment			0
	Right judgment+no/unrelated justification			1
	Right judgment+justification of one defining aspect			2
	Right judgment+justification of two defining aspects			3
	Right judgment+justification of three defining aspects			4
Questions 10, 11, and 12	Wrong judgment for 1st altitude (0 pts)	Wrong judgment for 2nd altitude (0 pts)	Wrong judgment for 3rd altitude (0 pts)	0 to 6
	Right judgment for 1st altitude+ no/unrelated justification (1 pt)			
	Right judgment for 1st altitude+ justification of one defining aspect (2 pts)			
	Right judgment for 1st altitude+ justification of two defining aspects (3 pts)	Right judgment for 2nd altitude (1 pt)	Right judgment for 3rd altitude (1 pt)	
	Right judgment for 1st altitude+ justification of three defining aspects (4 pts)			
Total				51 pts

Pt point

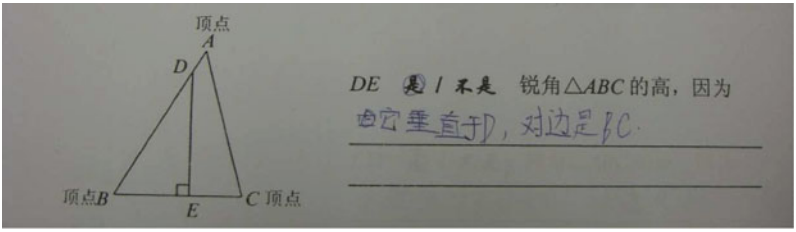
Knowledge gained from the pretest to the posttest Table 5 also shows that students improved from the pretest to the posttest. The mean pretest score was about 23.6 points while the mean posttest score was about 38.1 points.

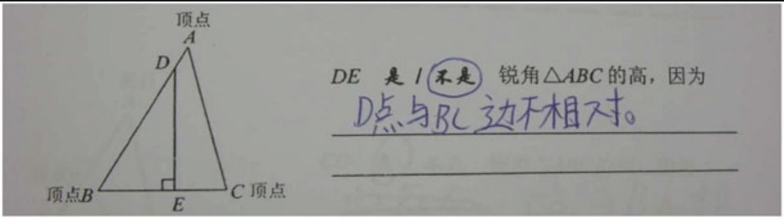
Improvement was also found within each condition, as shown in Table 5. For example, the mean pretest score in the NSS condition was 23.2 points while the mean posttest score in this condition was 35.5 points. Significant differences between the pretest and the posttest were reported for all the conditions, t (30)=−9.107, t (30)=−7.401, and t (29)=−6.554, respectively, $p<0.001$. The results showed that all three intervention packets were effective in promoting student learning of the concept.

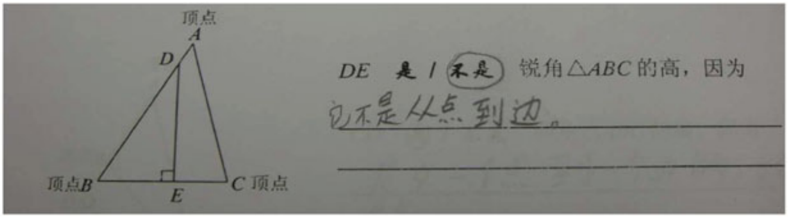
Effect of condition on students' performance Students' overall performance was analyzed to compare the effectiveness of the three conditions, which was the main research aim of experiment 1. A one-way analysis of covariance (ANCOVA) was performed to determine the effectiveness of SSS, TSS, and NSS. The posttest scores were compared among the conditions using the corresponding pretest scores as covariates.

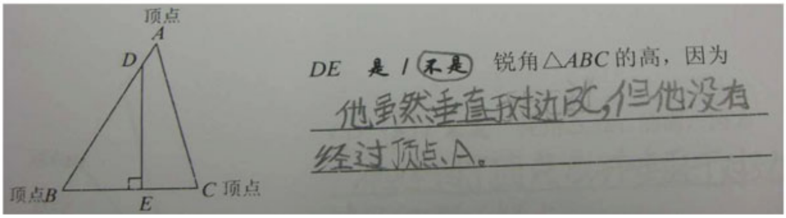
The results indicated that, after partialling out the effects of the covariate on the dependent variable, the differences of the posttest performance among the three conditions were statistically significant, F (2, 88)=3.384, MSE=65.407, $p<0.05$. Post hoc comparisons were subsequently performed and revealed that students assigned to the SSS condition (adjust mean score=40.9) significantly outperformed those assigned to the NSS

Table 3 Sample answers for question 1 on the posttest

<table>
<tr><th>Sample answers</th><th>Scoring</th></tr>
<tr><td>

Answer translated: DE <u>is</u> an altitude of the acute triangle ABC, because <u>it is perpendicular to D, and the opposite side is BC.</u></td><td>0 pts</td></tr>
<tr><td>

Answer translated: DE <u>is not</u> an altitude of the acute triangle ABC, because <u>D is not opposite to Side BC.</u></td><td>1 pt</td></tr>
<tr><td>

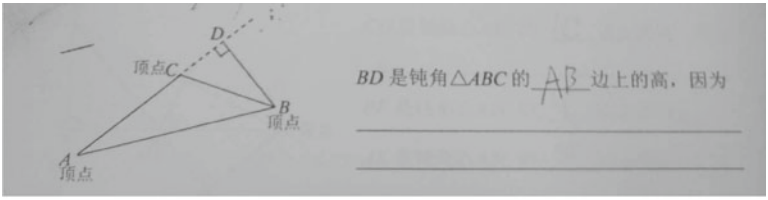
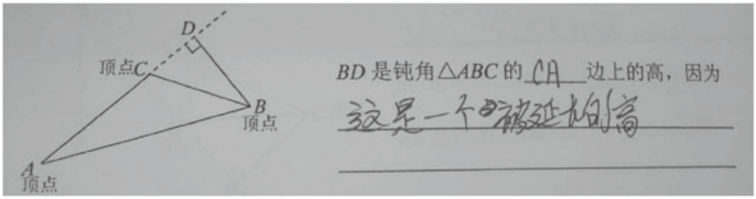
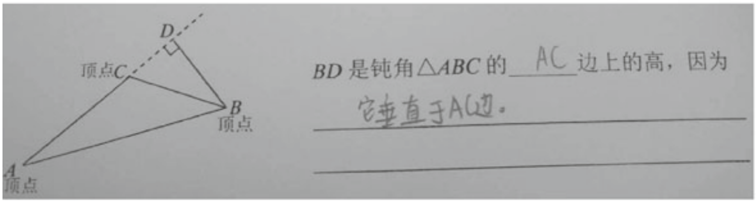
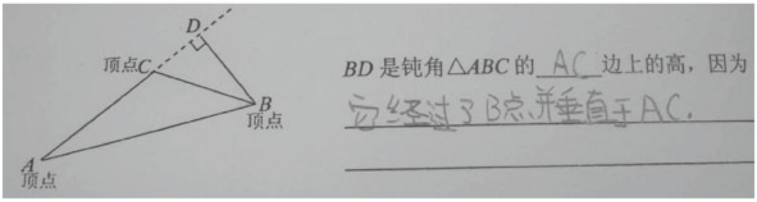
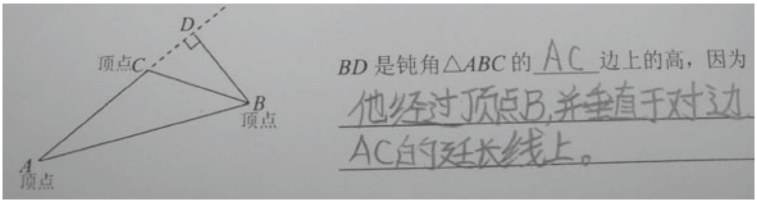
Answer translated: DE <u>is not</u> an altitude of the acute triangle ABC, because <u>it is not from a point to a side.</u></td><td>2 pts</td></tr>
<tr><td>

Answer translated: DE <u>is not</u> an altitude of the acute triangle ABC, because <u>it does not pass through Vertex A, although it is perpendicular to its opposite side BC.</u></td><td>3pts</td></tr>
</table>

Pt point

condition (adjust mean score=35.6), $p<0.05$. The effect size (Cohen's $d=0.757$) was medium to large. There was no other significant differences between the performance of students assigned to the SSS and TSS (adjust mean score=37.6) conditions ($p>0.05$, Cohen's $d=0.388$) or between the performance of students assigned to the TSS and NSS

Table 4 Sample answers for question 9 on the posttest

Sample answers	Scoring
 Answer translated: *BD is the altitude to Side* *AB* *of the obtuse triangle ABC, because*____________________	0 pts
 Answer translated: *BD is the altitude to Side* *CA* *of the obtuse triangle ABC, because* *this is an extended altitude.*	1 pt
 Answer translated: *BD is the altitude to Side* *AC* *of the obtuse triangle ABC, because* *it is perpendicular to Side AC.*	2 pts
 Answer translated: *BD is the altitude to Side* *AC* *of the obtuse triangle ABC, because* *it passes through Vertex B and is perpendicular to Side AC.*	3pts
 Answer translated: *BD is the altitude to Side* *AC* *of the obtuse triangle ABC, because* *it passes through Vertex B and is perpendicular to the extension of its opposite side AC.*	4pts

Pt point

Table 5 Students' overall pretest and posttest performance

Test	Condition	M	SD
Pretest	SSS	23.774	8.543
	TSS	23.903	7.880
	NSS	23.200	7.618
	Total	23.630	7.944
Posttest	SSS	40.936	6.811
	TSS	37.645	9.854
	NSS	35.500	7.533
	Total	38.054	8.392

conditions ($p>0.05$, Cohen's $d=0.245$). The sequence of performance was: SSS>TSS>NSS, as shown in Table 5.

The comparison of students' performance supported the research hypotheses: students in the SSS condition outperformed those in the TSS and NSS conditions. This confirmed the effectiveness of the method of example design in SSS on learning the concept of *altitude of a triangle*.

Discussion

The results of experiment 1 clearly supported the research hypotheses that students in the SSS condition would outperform students in the TSS condition and the NSS condition. Students in the SSS condition gained more scores than their counterparts in the other two conditions. The effectiveness of SSS is due to its method of example design, in which the critical aspects were separated and contrasted first, and then simultaneously varied via the use of positive and negative examples.

In contrast, TSS did not separate the discernment of critical aspects of vertex, perpendicularity, and opposition. Students in the TSS condition might have difficulty in discerning the three critical aspects, which should cause difficulty when they were experiencing the simultaneous variation of all critical aspects afterwards and thus inhibited their learning. NSS did not use examples to help students separately discern any critical aspect; critical aspects were simultaneously varied without having been separately discerned. As a result, too much variation of several critical aspects was presented to students at the start of example comparison. Without having discerned these critical aspects previously, students in NSS might get lost when facing the dissimilar examples. Consequently, when required to answer questions that needed simultaneous discernment of several critical aspects, they would make mistakes.

Although all students in the three conditions had the opportunity to experience the simultaneous variation of the six critical aspects, it seemed that whether these critical aspects had been separately contrasted and discerned mattered. Having separately discerned each critical aspect by comparing examples seemed to facilitate the later simultaneously experiencing variation of all critical aspects and thus forming a more advanced way of learning.

Experiment 2

The results of experiment 1 indicated that examples should be designed to help students discern their critical aspects first separately and then simultaneously. There is another research question of the present study regarding the role of students' prior knowledge in learning from examples: why a particular method of example design has different effects for students with different levels of prior knowledge, or what kinds of examples should be provided to students with greater or less prior knowledge? The purpose of experiment 2 is to answer this research question.

Recall that the six critical aspects of the object of learning used in Experiment 1 were identified for fourth-grade students who had not previously learned about *altitude of a triangle*. In addition to fourth graders, experiment 2 also included sixth graders who had partially learned the concept and should have greater prior knowledge than the fourth graders. By doing this, experiment 2 can investigate the possible interaction effect between grade levels and conditions to examine the role of students' prior knowledge in learning from examples. It was hypothesized in experiment 2 that SSS would still be better than TSS and NSS in promoting learning of fourth-grade students. SSS, however, would lose some benefits as students' prior knowledge increases from fourth to sixth grades; the performance of sixth-grade students in the three conditions would not be significantly different. Experiment 2 thus not only replicates experiment 1, but more importantly also extends findings of experiment 1.

Methods

Participant One hundred and eighty-four students were drawn from an urban primary school in Beijing, China, including 90 fourth-grade students and 94 sixth-grade students. Compared with the school in experiment 1which is at top level, this school is considered at above average level in Beijing. None of the fourth-grade students had previously learned about the concept of *altitude of a triangle* in school. They, however, had learned different types of triangle, vertexes and sides of a triangle, perpendicularity, altitude of a trapezoid, and area of a rectangle. According to the school's curriculum, the concept of *altitude of a triangle* was scheduled to be taught to these fourth graders soon after experiment 2 was completed; the participants were thus considered to be ready for learning the concept.

However, the concept is not completely taught in fourth grade according to the curriculum. Fourth-grade students are required to master only an altitude in an equilateral triangle, which is up-down, vertical to the level (or page), and inside the triangle. Altitudes of a right-angled triangle and an obtuse triangle will not be taught to students until seventh grade. On the basis of these considerations, we chose sixth-grade students to represent participants who had gained some knowledge but did not have completely learned the concept. Furthermore, in China, fourth-grade students and sixth-grade students are within a same primary school; their learning experience should not be different.

Materials and procedure The three intervention packets (SSS, TSS, and NSS), the assessment, as well as the procedures of collecting and analyzing data in experiment 2 were the same as those in experiment 1 previously illustrated. One week after the pretest, the fourth-grade students and the sixth-grade students were randomly assigned to one of the three conditions and were given a corresponding intervention packet to study: 30 fourth graders in the SSS condition, 30 fourth graders in the TSS condition, and 30 fourth graders in the NSS condition; 31 sixth graders in the SSS condition, 31 sixth graders in the TSS condition, and 32 sixth graders in the NSS condition. A posttest, which was equivalent to the pretest, was administered immediately after the students finished the intervention.

Results

This section first overviews students' knowledge on the pretest and the knowledge gained from the pretest to the posttest, and then reports the effects of conditions and grade levels on students' overall performance.

Pretest knowledge Table 6 shows the students' overall pretest and posttest performance for each condition. As shown in the table, students' pretest performance in the three conditions was rather close. For example, the mean pretest scores for the SSS, TSS, and NSS conditions were 21.197, 21.262, and 21.871, respectively. Not significant differences in pretest knowledge between conditions were found regardless of the grade level or within each grade level, $ps>0.05$.

In addition, on the whole, sixth-grade students were found to have more prior knowledge than fourth-grade students; for the fourth graders and the sixth graders, the mean pretest scores were 15.633 and 27.011 respectively. Statistically significant differences were found for their pretest performance, $t\ (170.436)=-9.615$, $p<0.001$.

This supported the selection of participants in experiment 2. Although fourth-grade students already had some knowledge regarding the concept of the *altitude of a triangle*, sixth-grade students had greater prior knowledge. It should also be noted that the fourth-grade students in experiment 2 had less knowledge of the concept than did students in experiment 1, though they were all in fourth grade. For example, the mean pretest score for the fourth-grade students in experiment 2 was 15.633 while the mean pretest score for the fourth-grade students in experiment 1 was 23.630. This might be because the students in experiments 1 and 2 came from two different achieving schools, and the school in experiment 1 was better than the school in experiment 2.

Knowledge gained from the pretest to the posttest Table 6 also shows that students in all three conditions improved from the pretest to the posttest. For example, for the fourth graders in the SSS condition the mean scores increased from 15.067 to 31.633; for the sixth graders in the NSS condition the mean scores increased from 26.781 to 42.938.

Significant differences between the pretest and the posttest were found for the three conditions regardless of the grade level and within each grade level, $p<0.001$. The results

Table 6 Students' overall pretest and posttest performance by scores

Test	Condition	Fourth grade		Sixth grade		Total	
		M	SD	M	SD	M	SD
Pretest	SSS	15.067	8.296	27.129	7.107	21.197	9.772
	TSS	15.200	8.438	27.129	6.260	21.262	9.495
	NSS	16.633	9.894	26.781	7.967	21.871	10.244
	Total	15.633	8.834	27.011	7.079		
Posttest	SSS	31.633	12.001	43.677	6.514	37.754	11.300
	TSS	29.167	13.357	43.194	7.931	36.295	12.949
	NSS	24.767	14.132	42.938	7.435	34.145	14.382
	Total	28.522	13.353	43.266	7.246		

showed that the three interventions were effective in promoting students' learning of the concept no matter whether they were fourth graders or sixth graders.

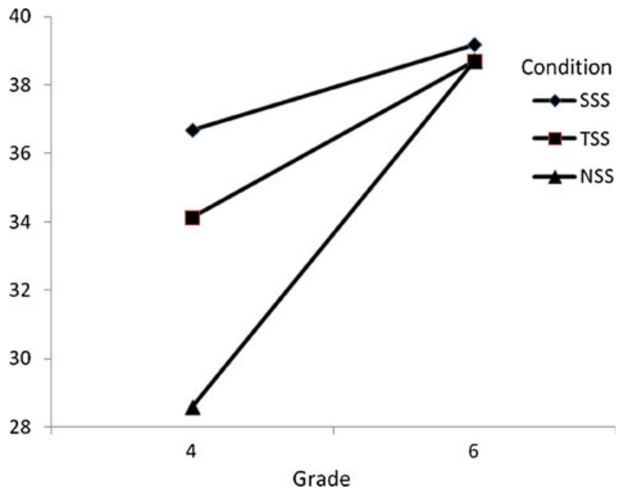
Effects of condition and prior knowledge on students' performance In order to examine the effects of the three conditions and students' prior knowledge on learning the concept, a 3×2 factorial analysis of variance (three levels of conditions: SSS, TSS, and NSS; two levels of grade: fourth and sixth grades) was conducted. The posttest scores were compared among the conditions and grade levels using the corresponding pretest scores as covariates.

The analysis of the overall group main effects indicated significant differences for the experimental conditions and grade levels. For the variable of conditions, F (2, 177)=4.026, MSE=72.051, $p<0.05$, indicating that students in different conditions performed significantly different among each other. For the variable of grade level, F (1, 177)= 13.810, MSE=72.051, $p<0.001$, indicating that the sixth-grade students performed significantly better than the fourth-grade students.

Because the main purpose of the experiment was to study differences in grade levels, the primarily interest was the interaction between the conditions and grade levels. It was hypothesized that the effect of the conditions would change with students' prior knowledge increases: SSS would be the most effective condition for fourth-grade students among the three conditions, but its advantage would disappear to some extent for sixth-grade students. In accordance with this prediction, the interaction data for the 3×2 ANCOVA was of major interest. The results showed a significant interaction effect between the two independent variables, F (2, 177)=3.325, $p<0.05$. The significant interactions suggested that which condition was most efficient depended on students' grade level (see Fig. 6).

After the significant interactions had been determined, tests of the simple effects were carried out. A significant effect was found for the fourth-grade students, F (2, 177)=7.123, MSE=72.051, p=0.001. It was noted that the SSS condition (adjust mean score=36.690) performed significantly better than the NSS condition (adjust mean score=28.581), $p<0.001$; the sequence of performance was: SSS>TSS>NSS. For the sixth-grade students, no significant differences were found among conditions, F (2, 177)=0.032, MSE=72.051, $p>0.05$; the sequence of performance was: SSS>NSS>TSS.

Fig. 6 Interaction effect between conditions and grade levels



Discussion

As predicted, students' prior knowledge of the concept of the *altitude of a triangle* altered the effectiveness of example design in the three conditions. Although SSS was consistently effective than TSS and NSS regardless of whether students had previously learned about the concept, the interaction effect revealed in experiment 2 was of more importance. In particular, the fourth-grade students who did not have previously learned about the concept benefited most from the SSS condition, but the sixth-grade students who had partially learned the concept did not show any preference for either condition.

A plausible explanation may lie in the different prior knowledge that students possessed. As noted previously, the sixth-grade students had partially learned the concept in school before Experiment 2 carried out, while the fourth-grade students were completely intact to the concept. As a result, the sixth-grade and fourth-grade students might have different critical aspects for learning the concept. For example, the aspect of vertex might be critical for fourth-grade students, and thus positive and negative examples with only one varied aspect of vertex should be necessary for them to discern this aspect, as SSS did (see Fig. 1). However, sixth-grade students who might have previously discerned the aspect of vertex would not need to compare the positive and negative examples, as this aspect was not critical to them.

Therefore, the six aspects identified as critical for the novice fourth-grade students might not be all critical for sixth-grade students. SSS that was designed to help students discern these six critical aspects first separately and then simultaneously might be particularly beneficial for the fourth-grade students. For the sixth-grade students who might have separately discerned some aspects but still could not simultaneously discern all the aspects, experiencing simultaneous variation of all critical aspects might be particularly important for them, which was presented in all three conditions. This might explain why SSS, TSS, and NSS were equally effective for the sixth-grade students.

General discussion

As expected, multiple examples should be designed to help students discern their critical aspects first separately and then simultaneously. Students would experience too much variation if they compare examples simultaneously varying several critical aspects that have not been previously discerned. In this section, we first discuss the three research issues that have not been solved in literature: "How similar or different should multiple examples be?", "What aspect of examples should be focused on?", and "When examples are most effective (the role of prior knowledge in learning from examples)?" Then, we consider the implications and limitation of the study.

How similar or different examples should be: patterns of variation and invariance

The results of the present study offered a solution for the question regarding how similar or different multiple examples should be. In contrast to previous studies that argued whether to present students with high or low variable examples, the study indicated that certain patterns of variation and invariance according to variation theory could provide a systematic way in determining the similarity or difference among multiple examples: critical aspects of the object of learning should be separately discerned before being simultaneously varied.

In the present study, a complete understanding of the *altitude of a triangle* requires the simultaneous discernment of the six critical aspects which should first be discerned separately. Specifically, in SSS, positive and negative examples that were designed according to the patterns of *separation* and *contrast* were highly similar in all aspects except for one critical aspect. By keeping other aspects invariant, the only varied aspect would be discerned by students as critical. After having separately discerned each critical aspect, students were ready and well prepared to compare examples that were different in all critical aspects; experiencing simultaneous variation of all critical aspects is important for completely understanding the concept. In this way, students would neither consider surface features as relevant when comparing highly similar examples, nor encounter difficulty in discovering structural features when comparing very different examples. Instead, they would discern the critical aspects first separately and then simultaneously.

What aspects should be focused on: critical aspects of the object of learning

As previously discussed, most research adopts the structural/surface distinction and investigates examples that have same structural features but different surface features (i.e., isomorphic examples); this, however, was found to be problematic. The present study suggested that critical aspects and features of the object of learning should be the focus when designing examples. In the study, the object of learning was to develop conceptual knowledge of the *altitude of a triangle* and the complete understanding of the concept was characterized by the six critical aspects identified. The six critical aspects were aspects that could cause difficulty for students in the process of learning. In order to master the concept, students must simultaneously discern the six critical aspects. Therefore, examples should be designed with the focus of the six critical aspects and patterns of variation and invariance should be constituted to enhance the discernment of these aspects. This way of designing examples was indicated in SSS and supported by the effectiveness of SSS.

When comparing examples is effective: the role of prior knowledge in learning from examples

Researchers have reported the different effectiveness of examples for students with different levels of prior knowledge. The example design that is most effective for novices might not be most effective for more experienced students. According to the results of Experiment 2 of the study, this is because students with different levels of prior knowledge might have different critical aspects for learning.

In the present study the fourth-grade students and the sixth-grade students in experiment 2 should have different ways of experiencing the same concept of the *altitude of a triangle*: the fourth-grade students were novices who were intact to the concept but the sixth-grade students had partially learned about the concept previously. Therefore, the critical aspects for learning the concept should be different for the fourth graders and the sixth graders. The six aspects were identified as critical for fourth-grade students; SSS which helped the fourth graders discern these aspects first separately and then simultaneously was thus particularly beneficial. The sixth-grade students, however, might have discerned some of the six aspects beforehand and did not need the stage of separate discernment in SSS. This explains the insignificant differences among conditions in promoting the learning of the sixth-grade students. For instance, if the sixth-grade students had discerned the three critical aspects of vertex, perpendicularity, and opposition before experiment 2 began, the positive and negative examples in SSS (see Fig. 1) that were designed for separately discerning the three critical aspects would become unnecessary; SSS and TSS would thus have no difference in facilitating learning.

Taken together, students who have different levels of prior knowledge may perceive different aspects as critical for their learning and thus benefit differently from the same instruction. For a particular set of examples to be effective, critical aspects should be first identified for specific learners and certain patterns of variation and invariance should be used for discerning these critical aspects.

Instructional implications

The study presented the whole process of applying variation theory in designing multiple examples, which is particularly helpful for teachers who are constructing examples in their instruction. Although using multiple examples to make comparison is a common instructional technique in the classroom, teachers are not doing so in ways that seem most conducive to student learning (Chazan and Ball 1999; Richland et al. 2004; Richland et al. 2007). Teachers usually designed examples with considerations other than variation on critical aspects of the object of learning. In the present study, NSS represented the traditional method of teaching the *altitude of a triangle*. Examples in NSS were presented, from the acute triangle through the right-angled triangle to the obtuse triangle. This way of designing examples in NSS followed the principle of easy-to-difficult; altitudes of an acute triangle are easiest and should be presented first while altitudes of an obtuse triangle are most difficult and should be presented last. However, students in the NSS condition did not learn well by reading the text on their own.

According to the findings of the study, the first step of instruction should be to ascertain the students' critical aspects for learning. Critical aspects are those that have not yet been learned and are still critical to the understanding; they should be identified according to the students' prior knowledge and only aspects that are crucial to students' learning should be deemed critical. Effective and efficient instruction should be tailored to individual students' critical aspects for learning, even though making decisions about the critical aspects of an object of learning for different students is never an easy task.

On the one hand, if the instructions do not provide sufficient information on the necessary critical aspects, then students with lower prior knowledge may have difficulty with learning. For instance, in Experiment 1 students in the TSS condition did not have the opportunity to separately compare the critical aspects of vertex, perpendicularity, and opposition; they were found to perform worse on the posttest than students in the SSS condition who separately compared each critical aspect. On the other hand, instructions focusing on aspects that are uncritical to learning may narrow the space of variation for students with greater prior knowledge, and thus are inefficient. For example, the sixth-grade students in experiment 2 who had partially learned about the concept might have discerned some of the six critical aspects (e.g., the aspect of vertex, perpendicularity, and opposition) before the experiment. They thus did not benefit from the separate discernment of these aspects in SSS; the three conditions equally affected their learning.

In short, the effectiveness of instructions that do not consider the students' individual differences in terms of critical aspects is likely to be random. This is in line with research about treatment-aptitude interaction (e.g., Cronbach 1967; Cronbach and Snow 1977; Snow and Lohman 1984) and more recent research about expertise-reverse effect (see Kalyuga 2007; Kalyuga et al. 2003 for review), which found the interaction effect between the instruction and students' aptitude.

After the critical aspects are identified for individuals, in order to help individuals discern these aspects, examples should be designed with controlled variation to ensure that critical aspects would be separately discerned before being simultaneously varied. For example, in the present study if a student does not recognize that the altitude of a triangle must be perpendicular to one side of the triangle (i.e., a critical aspect), examples should be presented for him/her to compare and discern this critical aspect. It is important to note, however, that examples are not limited to discriminating visual features. For example, Pang and Marton (2005) tried to help students learn the relationship between the change in the price of a commodity and the change in its supply and demand. For students who did not relate the change in price to the change in demand for the good (i.e., a critical aspect), examples were presented for them to experience the simultaneous change of the demand and price while the supply was kept invariant. Via the use of examples focusing on this critical aspect, students would discern that the demand for a good could influence its price.

Theoretical implications

This present study contributes to the development of variation theory, as well as to the literature on learning from examples. The latest development of variation theory focuses on how to apply this theory in promoting learning. In particular, "learning study" has been adopted as an effective approach by researchers (e.g., Linder et al. 2006; Lo et al. 2004; Pang et al. 2006; Pang and Marton 2005). The present study corroborates previous findings and expands the experimental ground for the effectiveness of variation theory to a new field of text reading, In addition, the study shows the whole process of identifying six critical aspects of the object of learning and how to design examples to facilitate the discernment of these aspects. In variation theory, researchers assert that critical aspects of the object of learning should be determined based on disciplinary knowledge and student understanding. The present study might provide the first empirical evidence for this argument by showing the interaction effect between students' prior knowledge and the instruction they receive. Students who have different levels of prior knowledge perceive different aspects as critical for learning; this moderates the effectiveness of certain patterns of variation and invariance. In other words, the effectiveness of patterns of variation and invariance for certain students would decrease or even disappear if students' prior knowledge changes.

The findings of the study could also be generalized to individualized learning. Traditional variation theory research focuses on identifying students' collective critical aspects for learning and facilitating the discernment of these critical aspects. The study shows that students have individual critical aspects for learning, which could be applied to designing individualized learner-tailored learning program. This might be another potential area to apply variation theory.

The study furthermore contributes to the literature on learning from examples; it not only gives systematic guidance on how to design multiple examples to enhance student learning, but also provides theoretical justifications to account for the results. Findings of the study shed light on the unclear issues about example design in literature: in order to achieve effective comparison for students with different levels of prior knowledge, multiple examples should be designed to help discerning their critical aspects first separately and then simultaneously. Before confirming and advocating the effectiveness of the example design in the SSS condition in promoting learning from examples, it is important to replicate the present study with a variety of mathematical topics or other subject areas and with a wider range of ages and mathematical prior knowledge. The study is an important initial step in investigating the method of designing multiple examples and more follow-up studies are needed.

References

Albro, E., Uttal, D., De Loache, J., Kaminski, J. A., Sloutsky, V. M., Heckler, A. F., et al. (2007). Fostering transfer of knowledge in education settings. In D. S. McNamara & G. Trafton (Eds.), *Proceedings of the 29th meeting of the Cognitive Science Society* (pp. 21–22). Austin: Cognitive Science Society.

Atkinson, R. K., Derry, S. J., Renkl, A., & Wortham, D. (2000). Learning from examples: instructional principles from the worked examples research. *Review of Educational Research, 70*(2), 181–214.

Bowden, J., & Marton, F. (1998). *The University of learning: beyond quality and competence in higher education*. London: Kogan Page.

Bruner, J. S., Goodnow, J. J., & Austin, G. A. (1956). *A study of thinking*. New York: Wiley.

Catrambone, R., & Holyoak, K. J. (1989). Overcoming contextual limitations on problem-solving transfer. *Journal of Experimental Psychology. Learning, Memory, and Cognition, 15*(6), 1147–1156.

Chazan, D., & Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*, 2–10.

Clarke, T., Ayres, P., & Sweller, J. (2005). The impact of sequencing and prior knowledge on learning mathematics through spreadsheet applications. *Educational Technology Research and Development, 53* (3), 15–24.

Cooper, G., & Sweller, J. (1987). Effects of schema acquisition and rule automation on mathematical problem-solving transfer. *Journal of Educational Psychology, 79*, 347–362.

Cronbach, L. (1967). How can instruction be adapted to individual differences. In R. Gagne (Ed.), *Learning and individual differences* (pp. 23–39). Columbus: Merrill.

Cronbach, L., & Snow, R. (1977). *Aptitudes and instructional methods: A handbook for research on interactions*. New York: Irvington.

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education, 20*(10), 1193–1211.

Gentner, D. (2005). The development of relational category knowledge. In D. H. Rakison & L. Gershkoff-Stowe (Eds.), *Building object categories in developmental time* (pp. 245–275). Mahwah: Erlbaum.

Gentner, D., Loewenstein, J., & Hung, B. (2007). Comparison facilitates children's learning of names for parts. *Journal of Cognition and Development, 8*(3), 285–307.

Gentner, D., & Namy, L. L. (1999). Comparison in the development of categories. *Cognitive Development, 14*(4), 487–513.

Gibson, J. J., & Gibson, E. J. (1955). Perceptual learning: differentiation or enrichment? *Psychological Review, 62*, 32–41.

Gick, M. L., & Holyoak, K. J. (1980). Analogical problem solving. *Cognitive Psychology, 12*, 306–355.

Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology, 15*, 1–38.

Gick, M. L., & Paterson, K. (1992). Do contrasting examples facilitate schema acquisition and analogical transfer? *Canadian Journal of Psychology, 46*, 539–550.

Gutierrez, A., & Jaime, A. (1999). Pre-service primary teachers' understanding of the concept of altitude of a triangle. *Journal of Mathematics Teacher Education, 2*, 253–275.

Hershkowitz, R. (1989). Visualization in geometry: two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61–76.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*, 251–283.

Holmqvist, M., Gustavsson, L., & Wernberg, A. (2007). Generative learning: learning beyond the learning situation. *Educational Action Research, 15*(2), 181–208.

Holyoak, K. J., & Koh, K. (1987). Surface and structural similarity in analogical transfer. *Memory & Cognition, 15*(4), 332–340.

Kalyuga, S. (2007). Expertise reversal effect and its implications for learner-tailored instruction. *Educational Psychology Review, 19*(4), 509–539.

Kalyuga, S., Ayres, P., Chandler, P., & Sweller, J. (2003). The expertise reversal effect. *Educational Psychologist, 38*(1), 23–31.

Ki, W. W. (2007). *The enigma of Cantonese tones: How intonation language speakers can be assisted to discern them.* Unpublished Ph.D. dissertation, University of Hong Kong, Hong Kong.

Kilpatrick, J., Swafford, J. O., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington: National Academy Press.

Klausmeier, H. J. (1992). Concept learning and concept teaching. *Educational Psychologist, 27*(3), 267–286.

Kurtz, K. J., Miao, C., & Gentner, D. (2001). Learning by analogical bootstrapping. *Journal of the Learning Sciences, 10*, 417–446.

Linder, C., Fraser, D., & Pang, M. F. (2006). Using a variation approach to enhance physics learning in a college classroom. *Physics Teacher*, *44*(9), 589–592.

Lo, M. L., Marton, F., Pang, M. F., & Pong, W. Y. (2004). Toward a pedagogy of learning. In F. Marton & A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 189–225). Mahwah: Lawrence Erlbaum.

Marton, F. (1999). *Variatio est mater Studiorum*. In: Opening address delivered to the 8th European Association for Research on Learning and Instruction Biennial Conference, Goteborg, Sweden, August 24–28.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah: Lawrence Erlbaum Associates.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences, 15*(2), 193–220.

Marton, F., & Pang, M. F. (2008). The idea of phenomenography and the pedagogy for conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 553–559). London: Routledge.

Marton, F., & Tsui, A. B. M. (2004). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum Associates.

Merrill, M. D., & Tennyson, R. D. (1978). Concept classification and classification errors as a function of relationships between examples and nonexamples. *Improving Human Performance Quarterly, 7*, 351–364.

Namy, L. L., & Gentner, D. (2002). Making a silk purse out of two sow's ears: young children's use of comparison in category learning. *Journal of Experimental Psychology: General, 131*(1), 5–15.

Paas, F., & van Merrienboer, J. (1994). Variability of worked examples and transfer of geometrical problem-solving skills: a cognitive-load approach. *Journal of Educational Psychology, 86*, 122–133.

Pang, M. F. (2002). *Making learning possible: The use of variation in the teaching of school economics*. Unpublished Ph.D. dissertation, University of Hong Kong, Hong Kong.

Pang, M. F. (2003). Two faces of variation: on continuity in the phenomenographic movement. *Scandinavian Journal of Educational Research, 47*(2), 145–156.

Pang, M. F., & Marton, F. (2005). Learning theory as teaching resource: enhancing students' understanding of economic concepts. *Instructional Science, 33*(2), 159–191.

Pang, M. F., Linder, C., & Fraser, D. (2006). Beyond lesson studies and design experiments – Using theoretical tools in practice and finding out how they work. *International Review of Economics Education*, *5*(1), 28–45.

Quilici, J. L., & Mayer, R. E. (1996). Role of examples in how students learn to categorize statistics word problems. *Journal of Educational Psychology, 88*, 144–161.

Ranzijn, F. J. A. (1991). The number of video examples and the dispersion of examples as instructional design variables in teaching concepts. *The Journal of Experimental Education, 59*(4), 320–330.

Reed, S. K. (1989). Constraints on the abstraction of solutions. *Journal of Educational Psychology, 81*, 532–540.

Reed, S. K. (1993). A schema-based theory of transfer. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 39–67). Norwood: Ablex.

Renkl, A., Stark, R., Gruber, H., & Mandl, H. (1998). Learning from worked-out examples: the effects of example variability and elicited self-explanations. *Cotemporary Educational Psychology, 23*, 90–108.

Richland, L. E., Holyoak, K. J., & Stigler, J. W. (2004). Analogy use in eighth-grade mathematics classrooms. *Cognition and Instruction, 22*, 37–60.

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science, 316*, 1128–1129.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574.

Rittle-Johnson, B., & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*(3), 529–544.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The importance of prior knowledge when comparing examples: influences on conceptual and procedural knowledge of equation solving. *Journal of Educational Psychology, 101*(4), 836–852.

Ross, B. H. (1989a). Remindings in learning and instruction. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (pp. 438–469). Cambridge: Cambridge University Press.

Ross, B. H. (1989b). Distinguishing types of superficial similarities: different effects on the access and use of earlier problems. *Journal of Experimental Psychology. Learning, Memory, and Cognition, 15*, 456–468.

Ross, B. H. (1997). The use of categories affects classification. *Journal of Memory and Language, 37*(2), 240–267.

Ross, B., & Kennedy, P. (1990). Generalizing from the use of earlier examples in problem solving. *Journal of Experimental Psychology. Learning, Memory, and Cognition, 16*(1), 42–55.

Ross, B. H., & Kilbane, M. C. (1997). Effects of principle explanation and superficial similarity on analogical mapping in problem solving. *Journal of Experimental Psychology. Learning, Memory, and Cognition, 23*(2), 427–440.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction, 16*, 475–522.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. (2005). Moving from rhetoric to praxis: issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24*, 287–301.

Snow, R., & Lohman, D. (1984). Toward a theory of cognitive aptitude for learning from instruction. *Journal of Educational Psychology, 76*, 347–376.

Star, J. R., & Rittle-Johnson, B. (2009). It pays to compare: an experimental study on computational estimation. *Journal of Experimental Child Psychology, 102*(4), 408–426.

Svensson, L. (1984). *Människobilden i INOM-gruppens forskning: Den lärande människan.* [The view of man in the research of the INOM-group: The learning man]. Goteborg: Institutionen för pedagogik, Göteborgs universitet.

Sweller, J., & Cooper, G. A. (1985). The use of worked examples as a substitute for problem solving in learning algebra. *Cognition and Instruction, 2*, 59–89.

Tennyson, R. D. (1973). Effect of negative instances in concept acquisition using a verbal learning task. *Journal of Educational Psychology, 64*, 247–260.

Tennyson, R. D., & Cocchiarella, M. J. (1986). An empirically based instructional design theory for teaching concepts. *Review of Educational Research, 56*, 40–71.

Tennyson, R. D., Wooley, F. R., & Merrill, M. D. (1972). Exemplar and nonexemplar variables which produce correct concept classification behavior and specified classification errors. *Journal of Educational Psychology, 63*, 144–152.

Tennyson, R. D., Youngers, J., & Suebsonthi, P. (1983). Acquisition of mathematical concepts by children using prototype and skill development presentation forms. *Journal of Educational Psychology, 75*, 280–291.

VanderStoep, S. W., & Seifert, C. M. (1993). Learning “how” versus learning “when”: improving transfer of problem-solving principles. *Journal of the Learning Sciences, 3*, 93–111.

Vinner, S., & Hershkowitz, R. (1983). On concept formation in geometry. *Zentralbl. Didakt. Math., 1*, 20–25.

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Pang, M. F. (2003). Two faces of variation: On continuity in the phenomenographic movement. *Scandinavian Journal of Educational Research, 47*(2), 145–156.

Pang, M. F. & F. Marton (2003). “Beyond Lesson Study: Comparing Two Ways of Facilitating the Grasp of Some Economic Concepts.” *Instructional Science 31*(3), p. 175–94.

Pang, M. F., & Marton, F. (2005). Learning theory as teaching resource: Enhancing students’ understanding of economic concepts. *Instructional Science, 33*(2), 159–191.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences, 15*(2), 193–220.

Marton, F., & Pang, M. F. (2008). The idea of phenomenography and the pedagogy for conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 553–559). London: Routledge.