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Quantum Computation and the Localization of Modular Functors *

Michael H. Freedman

Microsoft Research
One Microsoft Way
Redmond, WA 98052-6399, USA
michaelf@microsoft.com

Dedicated to my teachers and collaborators: *Alexei Kitaev, Greg Kuperberg, Kevin Walker, and Zhenghan Wang. Their work has been the inspiration for this lecture.*

Abstract. The mathematical problem of localizing modular functors to neighborhoods of points is shown to be closely related to the physical problem of engineering a local Hamiltonian for a computationally universal quantum medium. For genus $= 0$ surfaces, such a local Hamiltonian is mathematically defined. Braiding defects of this medium implements a representation associated to the Jones polynomial and this representation is known to be universal for quantum computation.

1. The Picture Principle

Reality has the habit of intruding on the prodigies of purest thought and encumbering them with unpleasant embellishments. So it is astonishing when the chthonian hammer of the engineer resonates precisely to the gossamer fluttering of theory. Such a moment may soon be at hand in the practice and theory of quantum computation. The most compelling theoretical question, "localization," is yielding an

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answer which points the way to a solution of Quantum Computing's (QC) most daunting engineering problem: reaching the accuracy threshold for fault tolerant computation.

After Shor's discovery [**21**] of a polynomial time factoring algorithm in the quantum model QC, skeptics properly questioned whether a unitary evolution could ever be induced to process information fault tolerantly. The most obvious tricks, such as making a backup copy, useful in a dissipative system (e.g., pencil and paper) are unavailable in quantum mechanics. To overcome these difficulties, a remarkable theoretical framework based on "stabilizer codes," "transversal gates," "cat-state-ancilli," and nested concatenations of these was erected [**22**], [**23**], [**2**], [**12**], and [**16**]. While the result is a consistent recipe for fault-tolerant quantum computation, the accuracy threshold which would allow this combinatorial behemoth to overcome its own overhead has been estimated as about 10^{-6}, one i.i.d. error per one million physical gate operations and requiring gates accurate also to one part in a million. This places a formidable task before the engineer and physicist. But within the year the beginnings of a new idea on fault tolerance had been generated by Kitaev [**13**].

While the term is not yet present in that paper the idea is to construct (first mathematically) a "quantum medium" and to store quantum states as topological structures within the medium and (eventually) manipulate these states, that is, apply gates to them, by topological transformations of the medium. For our purposes, we define a quantum medium as a collection of many finite level systems coupled together by a Hamiltonian H obeying a strong locality condition: The individual systems are located in a two-dimensional lattice or a more irregular cellulation of a surface Σ. We postulate a constant $d > 0$ so that $H = \Sigma \overline{H}_k$ and each $\overline{H}_k = H_k \otimes \mathrm{id}$, where the identity is on all tensor factors (= subsystem) not located within some ball B_ℓ of diameter d in the lattice. For example, the Heisenberg magnet with $H = -J \Sigma_{a,b=\partial \text{ edge}} \vec{\sigma}_a \otimes \vec{\sigma}_b$ is a quantum medium of diameter $= 1$. (But engineer be warned; localizing H^ℓ within balls of diameter $= d$ implies n-ary interaction for $n \sim d^2$. Controlling effective n-ary terms for $n \geq 2$ will be tricky in the extreme and probably will require enforcing symmetries to cancel lower-order terms.) Kitaev's "toric code" [**13**] in which quantum states are stored as first homology of a torus, can be counted as having $d = 2$; they require 4-ary interactions.

We study here a partial generalization of the toric code which also stores quantum information in a degenerate ground state $V(\Sigma)$ of a quantum medium. The medium is on a disk with pointlike defects which we treat as punctures. The dimension of $V(\Sigma)$, Σ the punctured disk, grows exponentially with the number of punctures. Transformations of Σ, that is, braidings (up to isotopy) of the punctures in space–time, $\Sigma \times R$, operate unitarily on $V(\Sigma)$. Other work ([**13**], [**19**], and [**14**]) also explores the realization of elements of computation by braiding anionic "quasi-particles" or "defects" of a quantum medium.

The vision is that stability of computation, at least sufficient to reach the 10^{-6} threshold for "software" error correction, is to be realized by the discreteness of algebraic topology: two Z_2-homology cycles are never "close," two words in

the braid group are equal or distinct. More exactly, it is geometry not topology which will confer stability. Working in a lattice model one may calculate [**13**] that the perturbation Hamiltonian P must be raised to the length scale L before nonzero terms, $\langle\zeta|P^L|\eta\rangle$, $\zeta, \eta \in$ ground state (H), are encountered and so the splitting of the ground state is estimated to be proportional to $e^{-\Omega(L)}$. The length scale in the previous two examples are: $L =$ (length of shortest essential cycle); and in the anionic context, the closest that two defects are allowed to come to each other during braiding. The "engineering goal" is to construct a *physical* quantum medium on a material disk whose ground state admits many localized excitations ("anions") whose braidings effect computationally universal unitary transformations of the ground state. It is further hoped that actual "errors," the result of unwanted noisy excitations, are to be removed automatically by some relaxation process in which the system is coupled to a cold bath by another much weaker Hamiltonian H'. The mathematicians first cut at the engineering goal is to produce a *mathematical* quantum medium with these properties and this is accomplished by the theorem below. This "first cut" is not yet interesting to experimentalists since the Hamiltonian contains summands which have as many as 30 nontrivial indices, but it represents an exact existence theorem. The question for physicists is whether this phase can also be represented perturbatively with a simple Hamiltonian, perhaps an RVB model [**1**], [**18**]. This would be a major step toward physical realization.

Theorem 1.1. *Consider a rectangle R of Euclidian square lattice consisting of* 15 *boxes by* 30 n *boxes. Associate a* 2*-level spin system* $\mathbb{C}^2$ *with each of the* $e := 960n+36$ *box edges in* R*. The disjoint union of these spin systems has Hilbert space* $(\mathbb{C}^2)^{\otimes e} =: X$*. There is a time-dependent local Hamiltonian* $H_t = (\Sigma_k \overline{H}_{k,t})$ *with fewer than* 2000 n *terms and each* H_k *having* 30 *or fewer indices, supported in at most a* 5×3 *rectangle of boxes, "diameter* $= 5$*." For* $t = 0$*, the ground states of* H_0 *form a sub-Hilbert space* $W \subset X$*, and geometrically determines* $3n$ *exceptional points or "defects" spaced out along the midline of* R*. Within* W *there is a "computational" sub-Hilbert space* $V \cong (\mathbb{C}^2)^{\otimes n}$*,* $V \subset W$*. W may be identified with the* $SU(2)$*-Witten–Chern–Simons modular functor at level* $l = r - 2 = 3$ *of the* $3n$*-punctured disk with the fundamental representation of* $SU(2)$ *labeling each of the* $3n + 1$ *boundary components. The Braid group* $B(3n)$ *of the defects acts unitarily on W according to the Jones' representation at level* $= 5$*. Any quantum algorithm can be efficiently simulated on V by restricting the action of* $B(3n)$ *to a "computational subspace."*

The representation is implemented adiabatically by gradually deforming H_t to H_{t+1} and then to H_{t+2} and so on. The passage from H_t to H_{t+1} involves turning off an exceptional term $\overline{H}_{k,t}$ which defines a defect site, and turning on a new term $\overline{H}_{k,tH}$ which determines an alternative, adjacent, site for the defect at time $t + 1$. Each braid generator can be implemented in $4(r + 1)$ *or time steps. We believe, based on a conjectural energy gap, that the geometry confers stability to this implementation which increases exponentially, error* $= e^{-\Omega(L)}$*, under refinement*

of the lattice on R by a factor of L, while the number of time steps needed for a computation increases only linearly in L.

Comments 1.2.

- The second paragraph of the theorem should be read as a defensible physical proposition, whereas the first paragraph is mathematics.
- Our Hamiltonian may be too complicated to prove the persistence of an energy gap above the ground state in the thermodynamic limit. But based on an analogy with a simpler system the gap is conjectured and will be discussed at the end of the proof.
- The passage from the Jones' representation to computation on V is the subject of [**6**] and [**7**] where it is proved that universality holds for $r = 5$ and $r \geq 7$. Functorially, V is a tensor summand of a subspace of W but by fixing a reference vector in the complementary tensor factor we regard V simply as a subspace of W.
- The idea of anionic computation is taken from [**13**] and in a more speculative form from [**8**]. The new ingredient is the implementation of a computationally complete modular functor by a local Hamiltonian. Witten's approach [**26**] to CSr was Lagrangian and so nonlocal; it yields an identically zero Hamiltonian under Legendre transform, [**5**] and [**1**]. This lecture, in contrast, supplies a Hamiltonian interpretation for CS5. (We may replace 5 by any $r \geq 7$ in the statement at the expense of scaling the constants in the theorem by $r/5$ or $r^2/25$ according to whether they scale as lengths or areas.).
- We know of two works in progress with a similar objective. Kitaev and Bravyi [**14**] study a local model for the weaker functor CS4 on high genus surfaces, and Kitaev and Kupperberg [**15**] have an approach to construct local Hamiltonians generally for modular functors on surfaces of any genus which (unlike CS5) are quantum doubles [**3**]. Their approach has the advantage that the local contributions to the Hamiltonian can be arranged to commute so that an energy gap will be rigorously established. In contrast, an interesting feature of the present paper is that topologically a combinatorial means yields an exact determination of a ground state defined by *noncommuting* terms. This is not usually possible. Finally, we will see that our local construction for H extends to the higher genus surfaces if CSr is replaced by any modular functors of the form $V \otimes V^*$. The simple topological reason for this may illuminate the analysis of [**15**].
- Shortly, we will give the reader a completely pictorial understanding of CSr on planar surfaces.

So far, we have only discussed the "engineering": the quest is to specify H (which will be described in the proof). Let us take a brief digression from that sulfurous underworld of grinding gears to the Elysian fields of abstract thought. The Witten–Chern–Simons theory descends from the signature (= Pontryagin form)

in dimension 4 and every step of the desent to a lower dimension leads to deeper abstraction until mathematical wit is well nigh exhausted as the point (dimension $= 0$) is reached. To tell this story in its barest outline, we restrict to $G = SU(2)$, and borrow from Atiyah [**1**], Freed [**4**], and Walker [**24**]. The signature of a closed 4-manifold is an integer, as is the Pontryagin class of an $SU(2)$ bundle over a closed 4-manifold. An $SU(2)$-bundle over a closed 3-manifold is topologically trivial, but if endowed with a connection acquires a secondary " Chern–Simons" class in the circle $= R/\mathcal{Z}$. Quantizing [**26**] at level l, leads to the topological Jones–Witten–Chern–Simons invariant $\in \mathbb{C}$ which is morally an average of the classical Chern–Simons invariant over all connections. The invariant for a closed surface Σ (with some additional structure) is a finite-dimensional vector space V; and each 3-manifold bounding Σ determines a vector $v \in V$. Before dividing by gauge symmetry, the vector space $\overline{V}$ is the infinite-dimensional space of sections of the associated complex line bundle to a natural S^1-bundle over the space of $SU(2)$ connections A on $SU(2)$ bundles over Σ. A 3-manifold Y with connection, $\overline{A}$, on a bundle extending the bundle over the boundary, $\partial(Y, \overline{A}) = (\Sigma, A)$, determines a map $f\{(Y', \overline{A'}) \mid \partial(Y', \overline{A'}) = (\Sigma, A)\} \to S^1$ by integrating the Chern–Simons form over $Y \cup -Y'$. The consistent choices for such functionals constitute the total space of this "natural" S^1-bundle. In general, a map f is "consistent" if it obeys the additivity properties of the Chern–Simons integral: $f(Y') - f(Y'') =$ C.S.$(Y' \cup -Y'')$. Symplectic reduction followed by quantization as explained in [**1**] produces a finite-dimensional V from $\overline{V}$ with $v(Y) \in V$ depending only on the topology of Y. The definition of the Witten–Chern–Simons invariant for a surface with boundary is a collection of vector spaces indexed by certain labelings. For a 1-manifold the invariant seems to be a certain type of "2-category" while the correct definition for a point is but dimly perceived and the object of current research. Several authors assert that it is unnecessary to finish the progression, that we can be content with a theory whose smallest building blocks are "pairs of pants" (three punctured spheres). The invariant for these while technically a vector in a 2-vector space is easily understood in terms of sets of vector spaces parameterized by "labelings" of the boundary circles so no unusual categorical abstractions need be mastered. The reason for this assertion is that, using a handle body decomposition, all closed 3-manifold invariants can be calculated from gluing along surfaces with smooth boundary; gluings along faces with corners on the boundary, which one would encounter computing from a cellulation, can be avoided. But the Freed–Walker program rejects this advice on two grounds. First, localizing $V(\Sigma)$ not merely to "pants," but to cells (i.e., neighborhoods of points) may give more natural consistency conditions, to replace the 14 consistency equations of [**24**]; which in turn could eventually lead to classification of modular functors and a conceptual understanding. Second, to paraphrase Edmund Hillary, we should localize to points "because they are there."

The hyperbole of the first paragraph can now be made sound. CS5 is a universal model for quantum computation and for the physicist/engineer to implement it a local Hamiltonian H must be described. For the pure mathematician to be satisfied

with his understanding of CS5 it must be localized to points. The two objectives are certainly similar in spirit and possibly identical. To clarify the connection, we introduce an intermediate concept, undoubtedly plebeian, but dear to a topologist. We would like, when possible, to describe a vector in a modular functor as a linear combination of "admissible" *pictures* up to "equivalence." This, after all, is exactly how we understand homology: $v \in H_1(\Sigma, Z_2)$ is an equivalence class of admissible pictures. To be admissible the picture must be a closed 1-manifold, the equivalence relation is bordism. Both "1-manifold-ness" and "bordism" can be defined by local conditions which are the combinatorial analogs of "closed" and "co-closed" familiar forms from de Rham's theory of differential forms. In Kitaev's toric code these condition are imposed by vertex and face operators A_v and B_f, respectively. There is a subtle shift here from the usual way of thinking of homology as equivalence classes of cycles, to the "harmonic" representative which is merely the equally weighted average of all cycles in the homology class. In this way quotients and equivalence classes are never encountered and homology is located within cycles, within chains, just as a C.S.S. code space is located within the fixed space of stabilizers built from products of σ_z's and further within the fixed space of stabilizers, $\Pi\sigma_x$'s.

To generalize from homology, we should think of a picture as (linear combinations of) anything we can draw on a surface Σ. If helpful, we allow various colors and/or notational labels, framing fields, etc. . . . , and even additional dimensions bundled over Σ. But in the present case no such embellishments are required. What is important is that if we move the surface by a diffeomorphism, the picture should also move and move canonically. Thus if Σ is a torus it would not suit our purposes to draw the picture of $v \in V(\Sigma)$ in a solid torus T, $\partial T = \Sigma$: a meridial Dehn twist on Σ extends over T, twisting the picture, but a longitudinal Dehn twist does not have any obvious way to act on a picture drawn in T. (To anticipate, a modular functor we will have an S-matrix which can transform a picture in one (call it the "inside") solid torus to a picture in the dual ("outside") solid torus, where longitudinally a Dehn twist does act. But resorting to the S-matrix does not solve our problem since its input and output pictures are on a scale of the injectivity radius of the surfaces and hence nonlocal.) We demand that the "admissibility" and "equivalence" of pictures be locally determined, i.e., decided on the basis of restriction to small patches on Σ. To make the connection with lattice models, we consider Σ discretized as a cell complex; the conditions must span only clumps of cells of constant combinatorial diameter. As in the example of harmonic 1-cycles, "equivalence" is a slight misnomer: what we impose instead are invariance conditions on the (linear combinations of) admissible pictures representing any fixed $v \in V$ which ensure that the stabilized vectors are in fact equally weighted superpositions of all admissible pictures representing v.

Now consider the question, perhaps the first question a geometric topologist should ask about a modular functor $V(\Sigma)$; Can you draw a (local) picture of it on Σ so that the mapping class group of Σ acts on $V(\Sigma)$ by the obvious induced action on pictures?

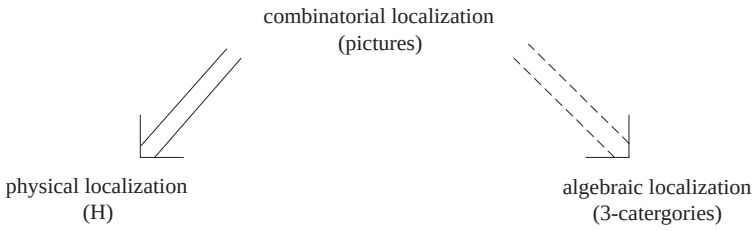


Fig. 1. Picture principle.

We should not expect it to be easy to discover the local rules for the pictures associated to a given modular functor V and in fact they may not exist in much generality. Recall that a 3-manifold Y bounding Σ, $\partial Y = \Sigma$ determines a vector $v(Y) \in V(\Sigma)$, so we might think of our proposed picture $P(v(Y))$ drawn on Σ as some ghostly recollection of Y. The present understanding of modular functors is closely related to surgery formulas on links, but to think in this way we must choose a "base point" 3-manifold Y_0, with $\partial Y_0 = \Sigma$ to hold the links. This choice seems to create an asymmetry which should not be present in $P(v(Y))$. Thus for a pictorial representation of V which is derived from surgery, we expect that only part of the mapping group—that part extending over Y_0—will act locally. To localize V, this problem must be overcome.

Let us propose a meta theorem or "principle" that solving the "picture problem," which we call "combinatorial localization," should imply both the Freed–Walker program, which we call "algebraic localization," and the design problem for the Hamiltonian H, which we call "physical localization."

The solid arrow is asserted with some confidence at least as a mathematical statement; the dotted arrow is speculative. While the solid arrow seems unlikely to have a literal converse: ground states of even simple Hamiltonians in dimension ≥ 2 are too complicated to draw pictures of; conceivably the dotted arrow might be an equivalence constituting a culmination of the Freed–Walker program.

2. Combinatorial Localization of CS5 on Marked Disks, and the Proof of the Theorem

We show how to represent CS5 (and by extension all CSr) on a disk with marked points by local pictures. Since the representation of quantum computing within CS5 [**6**] only used the braid group acting on a disk with marked points, this partial solution to the combinatorial localization problem will suffice to prove the theorem (once we have explained the solid arrow in Figure 1).

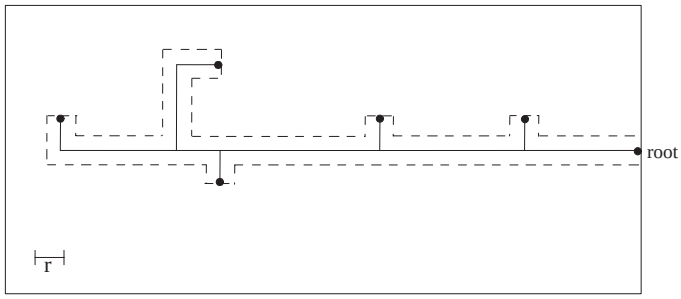


Fig. 2.

For any $r \geq 2$, CSr has a combinatorial localization on any cellulated disk with marked labeled points (labels $\epsilon\{0, 1, \ldots, r-2\}$ lie on the marked points and disk boundary), provided the cellulation has bounded combinatorics and the marked points stay sufficiently far from each other and the boundary. For a concrete statement, let us take the cellulated disk to be a rectangle R with a square Euclidean cellulation. We suppose that all marked points are at least r lattice spacings from the boundary and $9r$ from each other. The marked points and ∂R are all assigned the label 1 (the irreducible two-dimensional representation of $sl(2, \mathbb{C})_q$). In this circumstances it is easy to build a trivalent "r-collared rooted tree" T_r for the disk with marked points, as shown in Figure 2.

All straight segments of the tree are to be more than $3r$ lattice bonds in length; the root is on ∂R and the leaves are the marked points. The r-collard condition is that an $\lceil r/2 \rceil$ relative regular neighborhood $N(T)$ of lattice cells—the region within the dashed line—should be embedded in R.

The box counts in the statement of the theorem are designed to permit a (discontinuous) family of T_r's to be found at all times during braiding. We say that the boxing of R is *roomy* relative to the location of the marked points if it has this property. The key, Lemma 2.1, will show that, for roomy boxing, two discrete pictures, which we regard as smoothly equivalent are in fact combinatorially equivalent. More precisely, the infinity of smooth averaging operators acting on the space of combinatorial pictures has exactly the same joint fixed set as a finite subset of combinatorial operators.

Let us begin with a geometric interpretation of CS5$(\Sigma) =: V(\Sigma)$. For a closed surface Σ it is implicit in [**11**]. Let Σ be bound by a handle body H. A general 3-manifold Y with boundary Σ can now be represented as a "blackboard framed" surgery diagram in H. The special cabling morphism w of the Temperley–Lieb category (see, Chapter 12 of [**11**] or [**20**]), when composed into the surgery diagram, yields a linear combination of 1-manifolds, each labeled by "1". We may write H as a planar surface cross interval, $H \cong \Sigma_- \times I$, so that $\Sigma = \Sigma_- \cup_\partial -\Sigma_-$, where $-\Sigma_-$ denotes Σ_- with its orientation reversed. Now projecting these 1-manifolds to Σ_-, we see a linear combination of immersed 1-labeled 1-manifolds with overcrossings

indicated at double points. This picture determines the vector $v(Y)$. The Kauffman relations at a root of unity, in our case $e^{2\pi i/5}$, allow extensive simplification of these pictures via the recoupling formalism. In fact, each $v \in V$ can be encoded as a labeling of a fixed (framed, embedded, and vertex planar) trivalent graph, which is a spine for a Σ_-.

It is an important observation of Walker's (personal communication) and Gelca's [**9**] that this description can be extended to labeled surfaces with boundary. (Verification follows directly from the gluing axiom.) In the case of a disk with n marked points (D, n)—treating marked points as crushed boundary components—the modular functor with $n+1$ labels $\vec{\ell}$, $V_{\vec{\ell}}(D, n)$ has as its basis q-admissible labelings with boundary conditions on a fixed trivalent tree embedded in D, rooted on ∂D, with leaves on the marked points. The boundary condition is that the label on the root is the label given on ∂D^2 and each leaf has the label associated to its marked point. As in [**6**], we only need consider the case where all labels $= 1$.

The (framed) braid group acts on the labeled tree T via its embedding in the disk. To see the induced action on $V(D, n)$ (we drop labeling subscripts), perturb the embedding of T (rel its endpoints) by pushing it downward into a three ball $D \times [0, -1]$, where we think of D identified with $D \times 0$. Now implement any desired braid b as a diffeormorphism of $D \times [0, -\epsilon]$ where $\epsilon > 0$ is small with respect to the previous push. Viewed from above, $b(T)$ has overcrossings but the recoupling $(6j)$ rules (and isotopies) allow $b(T)$ to be described in the original basis of q-admissible labelings on T (with root and leaves still carrying the label 1). For example, the simplest Kauffman relations, on strands of $b(T)$ labeled by "1" read,

A detailed example: the effect of a single braid generator is given immediately following the statement of Lemma 2.1 to elucidate the recoupling of braids.

There is a topological observation inherent in inducing the braid action on $V(D, n)$. By capping off, any diffeomorphism of a planar surface extends to the two spheres and can be extended further to a diffeomorphism of the 3-ball B^3. The action on V comes from projecting this topological extension acting on labeled trivalent trees back into the original planar surface (after crushing the inner boundary components to points). In fact, it is the correspondence between 3-manifolds and diagrams which proves that we have correctly specified the action on the functor, for we have $v(\overline{f}Y) = f_* v(Y)$ where $\overline{f}|_{\partial Y = \Sigma} = f$. Generally, when a surface Σ has genus > 0 there will be no way of including it in the boundary of a 3-manifold M so that all diffeomorphisms of Σ extend over Y. However, it is a triviality that any diffeomorphism of Σ extends over $\Sigma \times I$ by product with id_I. Now let this extension act on the appropriate equivalence classes of framed q-admissibly labeled trivalent graphs embedded in $\Sigma \times I$ projected back into Σ to define the action on any $SU(N)$-level $= r$ modular functor V. Thus the "doubled" functor $V(\Sigma) \otimes V^*(\Sigma) = V(\Sigma \amalg \overline{\Sigma}) = V(\partial(\Sigma \times I))$ has a combinatorial

localization, i.e., is describable by local pictures. This may have some relation to the unpublished work of Kitaev and Kupperberg (private communication) on local descriptions for Drinfeld doubles.

We set aside for later study the problem of devising combinatorial local rules for the necessary elementary equivalences of such trees T: $6j$-moves, ribbon equivalence, vertex half-twist equivalence, and regular homotopy.

One would hope to define a quantum medium for CS5 of individual systems with levels to record labels 0, 1, 2, 3 (and possible additional levels to store other information) and terms H_k with at most six indices (as in a $6j$-symbol) corresponding to these elementary equivalences. While this count seems correct in the smooth setting, there the crude Hilbert space is infinite dimensional which may create new difficulties. We have not been able to find a discrete setting in which all the equivalences are expressed efficiently. For the purpose of this lecture, we stay with discrete models for quantum media built from 2-level systems, but to do this we accept terms H_k with up to 30 indices.

The fundamental two-dimensional representation of $SU(2)$ generates $SU(2)$'s complex representation ring and, as a result, recoupling theory achieves a very simple result: an element $v \in V(D, n)$ is a linear combination of embedded 1-manifolds each labeled by "1", i.e., the standard two-dimensional representation, and given the boundary condition: each 1-manifold of the linear combination meets each marked point (and ∂D) once. Thus "manifoldness" and the "boundary condition" define admissibility for our picture. This makes good sense combinatorically in the lattice of R, as well as smoothly. We point out that our notion of 1-manifold is strict: at each vertex 0 or 2 edges (not 4) should be occupied.

It is time to define the local equivalence moves between pictures. We are working within the Temperley–Lieb category modulo the relation that the $(r-1)th = 4th$ Jones–Wenzl projector is trivial. This is our most interesting relation. As a smooth equivalence relation this has only one form but, combinatorially, we need to impose two versions of it according to how the output endpoints are grouped. We denote these by  and  . The second picture stands for  in conventional projector notation [**11**].

A second relation says that removing a circle, which bounds a disk free from punctures, multiplies the diagram by the scalar $1/d$, $d = e^{\pi i/5} + e^{-\pi i/5}$. A third relation replaces the undercrossings that arise through braiding with legitimate morphisms in the category. In terms of smooth pictures, the relation replaces the "virtual" uncrossing in the middle diagram with a two-term sum.

The middle picture is "virtual"; it is not actually an admissible picture to be assigned a weight. This relation requires a little care and lattice space to discretize, since we do not want to permit the intermediate picture which would represent the wrong boundary data at the indicated defect. Recall that each defect is labeled by 1 representing the two-dimensional irreducible representation of $sl(2, \mathbb{C})_q$ which is recorded by a single line leaving the defect.

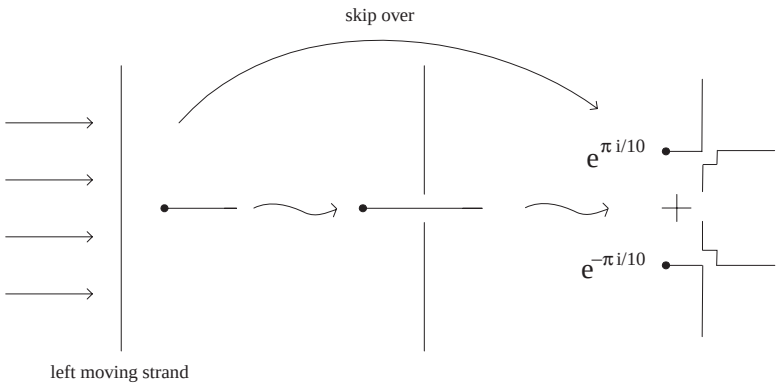


Fig. 3.

Finally, a fourth class of equivalence permits isotopy. Again the reader should note that enough neighboring sites should be observed by the appropriate H_k to preserve embeddedness. For example, cases 1 and 2 are allowable, case 3 is not.

There will be isotopy relations for arc end points as well. For example, in cases 1 and 2 of Figure 3 imagine the open circle filled to become an end point and the shorter of the two line segments meeting it deleted. Morally, we should define operators $\overline{H}_k$ which enforce the average of the initial I and final F configuration of cases 1 and 2. However, there is a detail, to get the overall phase correct, and not settle for merely a projective representation, we must fix a base point direction: say the positive ray emanating from each endpoint at 45° and find positive semidefinite $\overline{H}_k$'s which assign zero norm to $(1/2)(I_1 - e^{-\pi i/10}F_1)$ and $(1/2)[I_2 - F_2]$ in cases 1 and 2, respectively. These operators correspond to asserting equivalences: $I_1 \sim -e^{-\pi i/10}F_1$ and $I_2 \sim F_2$. The general rule is that a state obtained by clockwise (counterclockwise) isotopy through the base point direction must be adjusted by the phase $+(-)e^{i\pi/2r}$ before being averaged. Similarly, there is an isotopy relation for the arc endpoint on the boundary circle of the disk D. Here some point on the boundary is chosen and the phase is adjusted by $-(+)e^{i\pi/2r}$ as this point is crossed clockwise (counterclockwise).

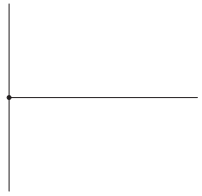


Fig. 4.

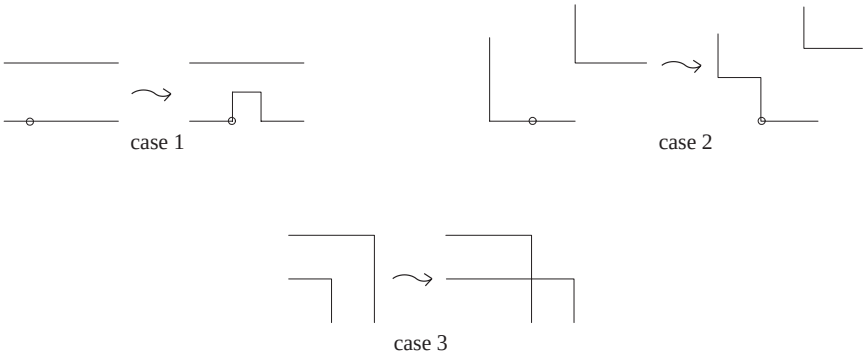


Fig. 5.

Let us return to the relations $= 0 =$. Combinatorically, the first may be written with the left-hand side a 3×3 lattice square foliated by parallel straight lines (of label $= 1$). Wenzl's [We] recursion formula yields an identity equating four parallel lines with a linear combination of 13 "smaller" terms each containing "turn arounds." The form of the relation is shown in Figure 6.0. In its other incarnation, the fouth Jones–Wenzl's projector relation $= 0$, looks as in Figure 6.1. The coefficients a_i are rational functors of d which can be computed from the Wenzl's recursion relation for projectors (see page 18 of **11**] or [**25**]). Figure 6.0 is merely the lattice counterpart of the more familiar smooth relation, Figure 6.0′, which may be applied within any diagram (at $r = 5$) whenever four $1-$ labeled lines are found running parallel. Obviously, Figure 6.1 also has a smooth counterpart.

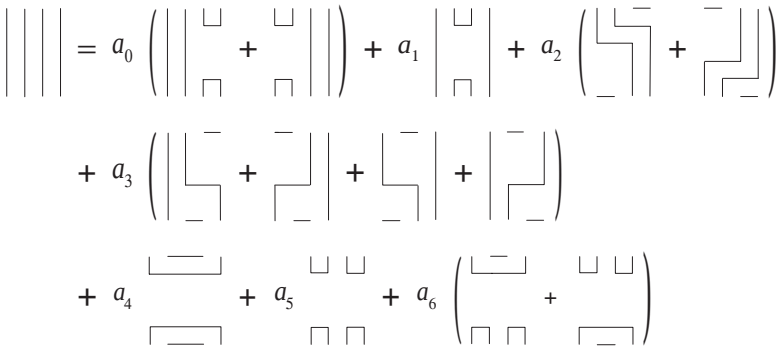


Fig. 6.0.

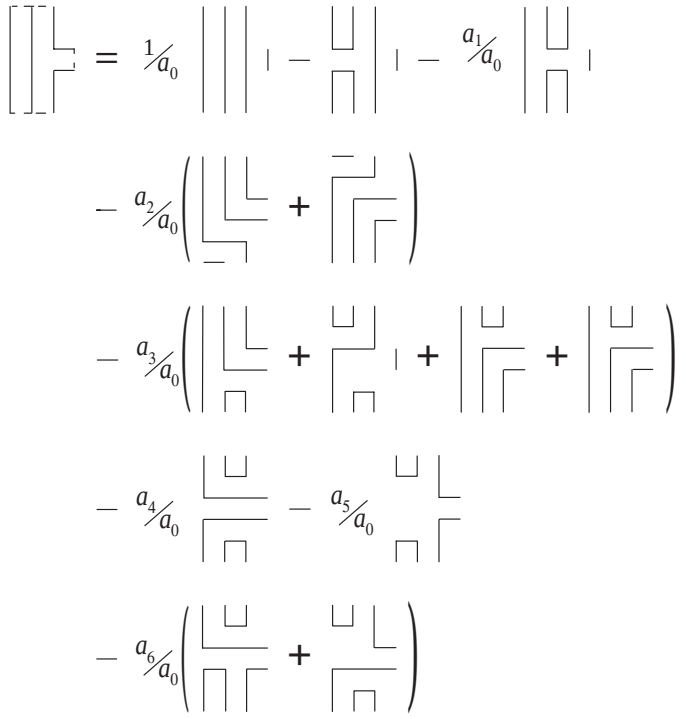


Fig. 6.1.

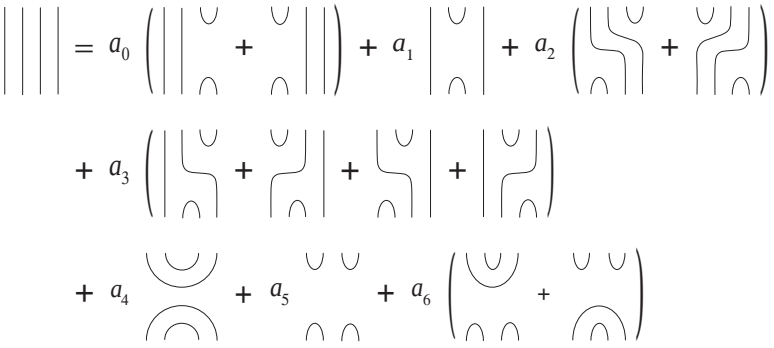


Fig. 6.0′.

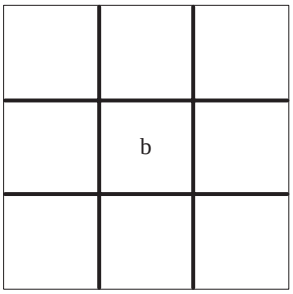


Fig. 7.

The admissibility conditions and the above four classes of "equivalences" must be rewritten as operators A_i and B_j, respectively; collectively denoted $\overline{H}_k$. Let G denote the ground state of the soon-to-be-defined Hamiltonian $H = \Sigma_k \overline{H}_k$. Let V denote the CSr modular functor of the disk with $3n$ marked points and all labels $= 1$. Via recoupling, we may describe V in the fashion of homology. Set $P = \mathbb{C}$ [admissible pictures] and write $V = V_s = P/\sim_s$, where $\sim_s$ is the smooth-category equivalence relation corresponding to our four combinatorical equivalences $\sim_c$. Lemma 2.1 will prove that under the "roomy hypothesis" $\sim_s$ and $\sim_c$ induce identical equivalence classes of admissible pictures (which of course are combinatorial objects). So we may also write $V = V_c = P/\sim_c$. Our goal is to tailor H so that the ground states $g \in G$ correspond bijectively to linear functionals ϕ: $V \to \mathbb{C}$ under the map $\phi \mapsto \Sigma_{p \in \text{ admissible pictures}} \phi(p)(p)$. This will identify G with V^*, but since V has a canonical nonsingular Hermitian inner product ([**26**] and [**11**]) this also gives an isomorphism $G \cong V$.

The inner product $\langle p_1, p_2 \rangle$ is defined on pictures by embedding the disk D into the (x, y)-plane, deforming p_1 upward rel endpoints and p_2 downward rel endpoints. The union of the deformed pictures $\tilde{p}_1 \cup \tilde{p}_2$ is a (vertically framed) link in R^3 and its Kauffman bracket is $\langle p_1, p_2 \rangle$. Note that the vertical framing is singular where p_1 and p_2 share a common lattice bond meeting ∂p_1 and ∂p_2; here the convention is to bend such bonds of p_2 slightly clockwise at the endpoints internal to D and counterclockwise at an endpoint on ∂D.

The definition of the A_i operators is quite obvious. Consider, a vertex v in the interior of R. A Hermitian A_v with four indices, whose ground state is spanned by classical states of valence 0 or 2 at v, is said to enforce "1-manifoldness" at v. Clearly the ground state of A_v has dimension 7. To enforce, instead, a "defect" or marked point labeled by the fundamental representation, "1" of $SU(2)$, we would use instead a Hermitian operator A'_v with a ground state spanned by the four classical states of valence $= 1$ at v.

Turning now to "relations" B_j consider a box b of R centered in a 3×3 square of boxes (Figure 7); there are 12 nonboundary edges $\{e\}$ (shown in bold). If the $\{c$'s$\}$ are the nonempty (classical) manifold configuration of these edges, i.e.,

valence $\in \{0, 2\}$ at each of the four internal vertices, and iff c_0 and $c_1 = c_0$ or $\partial b \in \{c\text{'s}\}$, set $d = (1/\sqrt{2})(c_0 - c_1)$ and let $\{d\}$ be the set of such vectors. Let $B_b = \Sigma_{d\epsilon\{d\}}|d\rangle\langle d|$ be the Hermitian operator with 12 indices on $(\mathbb{C}^2)^{\otimes\{e\}}$ whose ground state is orthogonal to span $\{d\}$. B_b is the operator which "allows isotopy across b."

To remove circles which bound disks we need, in the presence of isotopy, only introduce operators which delete a box. This operator may be written as $|\theta\rangle\langle\theta|$ where θ is a unit vector proportional to $|\text{box}\rangle + (e^{\pi i/5} + e^{-\pi i/5})|\phi\rangle$.

We postpone the definition of the operator corresponding to Figures 3 and 4 since this must involve the dynamics "t" of H_t. Some trick is needed to avoid adding new levels to our system to encode "crossings."

The projector corresponding to  , Figure 6.0, requires a 24-index operator acting on a 3×3 grid of edges or "box" B whose one-dimensional excited state is spanned by the vector obtained by putting all 14 terms in Figure 6.0 on the left-hand side of the equation. Similarly, the projector corresponding to  , Figure 6.1 is a 30 index operator acting on the bonds of a region the shape of the left-hand side in Figure 6.1. This "nobby box" B' is a 2×5 rectangle union, an additional small box in the middle of one of the long sides.

Now we turn to the dynamics. Almost all conditions H_k that combine to yield H are permanent, only the endpoint operators A'_v should change as we execute braiding. Because of the technical problem illustrated in Figure 4; any lattice resolution into a superposition of two 1-manifolds, as in Figure 3, may cause collision with other strands. One way to deal with this problem is to locate the marked points on a second lattice $\mathbf{L}'$ consisting of the midpoints of the edges in the original lattice $\mathbf{L}$ of boxes in R. This means that we have to add additional 2-index A operators, holding equal the two classical states on both halves of the original edges, i.e., ground state $(A) = (|00\rangle, |11\rangle)$, and that the endpoint operators A'_w actually occur (with two-dimensional ground states) on the finer lattice $\mathbf{L}'$, $w\epsilon\mathbf{L}'$. The dynamics consists of moving an endpoint diagonally on $\mathbf{L}'$, i.e., translating one unit horizontally or vertically in the structure of $\mathbf{L}$. In Figure 8 the endpoint w is moved horizontally to w' by replacing $\{A'_w, A_{w'}\}$ with $\{A_w, A'_{w'}\}$. If w and w' are immediately adjacent in $\mathbf{L}$ the operator swap will cause the endpoint to travel around a corner.

This operator swap can be performed gradually by slowly turning the appropriate terms on or off. If the adiabatic theory is applicable, and following the proof of Lemma 2.1, we discuss the heuristics for an energy gap (in the therom-dynamic limit) for the family H_t, ψ_t will be carried to a unique ground state ψ_t of H_{t+1}. This ground state, as a functional on pictures, is identical to ψ_t provided pictures are identified according to the obvious isotopy rules (and phase rules at endpoints). If the lattice is refined by a linear factor L, tunneling to an undesired orthogonal ground state ψ'_{t+1} should, by arguments analogous to those for the stability of homology classes [**13**], have amplitudes scaling like $\epsilon(L) = e^{-\Omega(L)}$. The

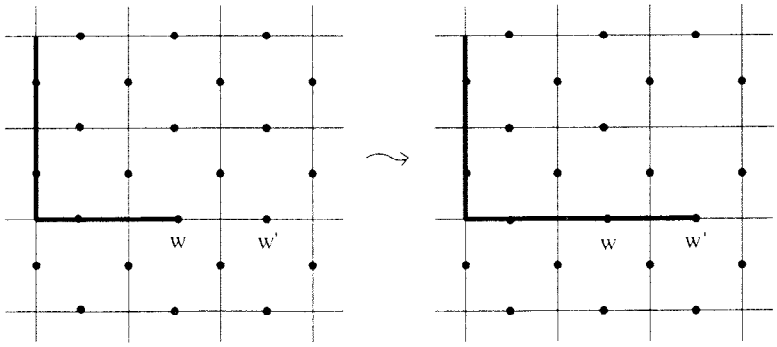


Fig. 8.

mathematical description for adiabatic evolution of the system is via the natural connection A on the tautological bundle over the complex Grassmannian X: The time evolution of $G := \{\text{ground states } (H_t)\}$ defines a path in X, and A-transport covers this motion with a unitary (i.e., isometric) identification $G_0 \equiv G_t$, for all $t \geq 0$. After a braiding b is completed at time $t = T$, the self-identification $G_0 \equiv G_T$ is the representation of the braid b.

A ground state $g \in G \subset P$ defines a functional g^* on P via orthogonal projection. The $\{B_j\}$ have been chosen to correspond to $\sim_c$ precisely so that a unique extension ϕ exists:

$$\begin{array}{ccc} P & \xrightarrow{g^*} & \mathbb{C} \\ \downarrow & \nearrow_{\phi} & \\ V_c & & \end{array}$$

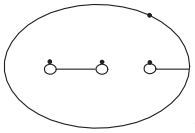
and ϕ satisfies $g = \Sigma\phi(p)(p)$. Conversely, given a functional ϕ on V_c, the g associated to ϕ by the formula above lies in the null space of each B_j, so in fact $G = V_c^*$.

The important remaining point is to see that after braiding, when the marked points have been returned to there original sites set-wise, that the induced transformation on the ground state is precisely, up to error $\approx \epsilon(L)$, the unitary CS5 representation originally introduced by Jones [**10**] and studied in [**6**] and [**7**]. But this follows from the recoupling theory as presented in [**11**] provided we show that the combinatorial relations, that we have imposed through Hermitian operators $\{B_j\}$, in fact are sufficient to span all the relations implied by the infinitely many smooth relations between pictures, that is, $V_s = V_c$. For this the following lemma suffices:

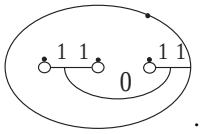
Lemma 2.1. *Let $\rho = \Sigma_i a_i p_i$ be a linear relation between admissible combinatorial pictures in $(R, \{3n\})$ which holds under $\sim_s$, the smooth recoupling theory*

associated to CSr. *Provided that the configuration* $\{3n\} \subset R$ *is roomy in the rectangle* R, *the same relation already holds under* $\sim_c$.

Before proving the lemma let us carry out a simple calculation to get a feel for how the action of braiding is computed via recoupling theory. If the reader wishes to try more complicated examples, the formulas on pages 93–100 of [**11**] are helpful. Here we compute the effect of a braid generator on a vector $\psi_\circ \in V :=$ CS5 (3-punctured disk) where each boundary component has label $= 1$ (the two-dimensional representation of $sl(2, \mathbb{C})_q$), and to account for phase each boundary has a marked base point

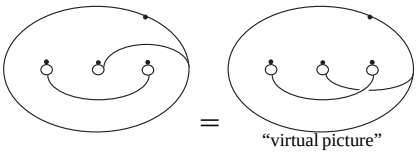
$\psi'_\circ$ is the diagram:  ,

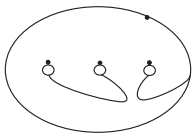
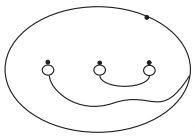
which as a labeled tree is

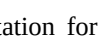
$\psi'_\circ =$ 

.

Let b be the counterclockwise braided of the right-most pair of punctures. Then $b\psi'_\circ$ is represented by



$= A$  $+ A^{-1}$  , where $A = e^{2\pi i/10}$. $(*)$

Now  is our notation for the Jones–Wenzl idempotent  $= 1/\sqrt{d^2-1}($  $- 1/d$  $)$ where $d = -A^2 - A^{-2}$ and, as usual, the open ends in the above diagrams can be interpreted as permitting arbitrary (but constant) extension to the outside. *Note*: The orthogonality relations

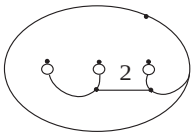
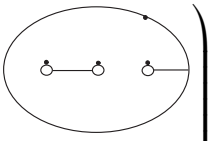
- $\left\langle \frac{1}{d} \supset\subset, \frac{1}{d} \supset\subset \right\rangle = \frac{1}{d^2}$  $= 1$;

- $\left\langle \frac{1}{\sqrt{d^2-1}}\left(\smile - \frac{1}{d} \supset\subset \right), \frac{1}{\sqrt{d^2-1}}\left(\smile - \frac{1}{d} \supset\subset \right) \right\rangle$
  $= \frac{1}{d^2-1}\left(\text{⊜} - \frac{2}{d}\text{◎} + \frac{1}{d^2}\text{⊂⊃} \right) = \frac{1}{d^2-1}\left(d^2 - \frac{2}{d}d + \frac{1}{d^2}d^2 \right) =$ 1; and
- $\left\langle \frac{1}{\sqrt{d^2-1}}\left(\smile - \frac{1}{d} \supset\subset \right), \frac{1}{d} \supset \subset \right\rangle = \frac{1}{d\sqrt{d^2-1}}\left(\text{◎} - \frac{1}{d}\text{⊂⊃} \right)$
  $= \frac{1}{d\sqrt{d^2-1}}\left(d - \frac{1}{d}d^2 \right) = 0.$

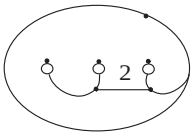
Normalizing, $\psi_\circ = (1/d)\psi_t'$ is a unit vector, $\langle \psi_\circ, \psi_\circ \rangle = 1$.

From the definition of  we have  So we use this to expand the two parallel lines in the second term of $(*)$ to get

$b\psi_\circ = A\psi_\circ + \frac{A^{-1}}{d}$

$= A\psi_\circ + \frac{A^{-1}}{d}\left(\sqrt{d^2-1} \right.$  $+ \frac{1}{d}$  $\left.\vphantom{\sqrt{d^2-1}}\right)$

$= A\psi_\circ + \frac{\sqrt{d^2-1}}{d}A^{-1}\psi_2 + \frac{A^{-1}}{d}\psi_\circ,$ (**)

where $\psi_2 :=$  ,

$= \left(A + \frac{A^{-1}}{d} \right)\psi_\circ + \frac{\sqrt{d^2-1}}{d}A^{-1}\psi_2.$

As a check on unitarity, note that under the sesquelinear pairing

$$\langle b\psi_\circ, b\psi_\circ \rangle = \left(A + \frac{A^{-1}}{d} \right)\left(A^{-1} + \frac{A}{d} \right) + \left(\frac{\sqrt{d^2-1}}{d}A^{-1}\frac{\sqrt{d^2-1}}{d}A \right)$$

$$= 1 + \frac{A^2 + A^{-2}}{d} + \frac{1}{d^2} + \frac{d^2 - 1}{d^2} = 1.$$

Proof of Lemma 2.1. The argument is based on the Birkhoff curve shortening principle whereby a family of embedded arcs and circles can be "pulled tight" to a shorter geodesic position without crossings developing. We work combinatorically. By the "roomy hypothesis" there is an r-collared tree $T := T_r \subset R$. Assign a positive weight $w(\beta)$ to each bond β (or 1-cell) of the cellutation of R so that w grows rapidly with distance to T: as a good first approximation, we may take $w(\beta) = 10^{\#(\beta)}$ where $\#(\beta) =$ minimum number of bonds joining β to T. Now for any (classical) picture p_i define its length $L(p_i) = \Sigma_{\beta \epsilon p_i} w(\beta)$. Permitting combinatorial isotopy (rel the marked points) and the removal of small circles, but not the undercrossing  or  relations, we may pull p_i tight by local moves to equivalent pictures (up to a scalar) which steadily reduce $L(p_i)$ until a local minimum is reached. Call this step "pull tight." Because of our weight function w, the new p_i will try to lie mainly in a small neighborhood of T, and only occupancy of the bonds close to T will force parts of the picture to lie farther away. Also the picture seeking to occupy the bonds near T efficiently will have its strands running parallel to T in $(r-2) \times (r-2)$ lattice blocks β athwart the middle r-bonds of each of the distinguished length $3r$ segments of T. Also at the trivalent vertices of T near which sufficiently many strands pass, we would like to see copies of the left-hand side of Figure 6.1. This will be true up to a small isotopy (across a few boxes), and can be made true on the nose by modifying the weight function w, by adding a small term proportional to the distance from each trivalent vertex out to a distance r from that vertex. Now apply  at some site in B, or  at some site B' if the opportunity presents. This breaks p_i into $\Sigma_j b_{ij} q_{ij}$ and for all j, $L(q_{ij}) < L(p_i)$. Pull tight again to remove the slack created by the "turn arounds" in Figure 6.0 or 6.1. Alternate the pulling tight with applications of  or  until no further reductions in length can be made in this way. Call this cycle "pull and cut." With a slight abuse of notation let q_{ij} denote one of the terminal classical states of this process. Now allow a single "under crossing" move (Figure 3) to further reduce $L(q_{ij})$ if such a move is available. Now alternate the "pull and cut" cycle with single under-crossing moves until no daughter picture (still denote by q_{ij}) can have its length reduced by further iteration of this process.

Manifestly, all the daughter pictures q_{ij} now lie in $N(T)$ and pass through all boxes B parallel to T and with three or fewer strands and pass through each B' in a standard way according to some admissible triple as explained below. Note that $(3, 3, 2)$ is not admissible. At this point it is simple to formally reorganize the term of this sum $\Sigma_{ij} b_{ij}, q_{ij}$ as $\Sigma c_\ell T_\ell$ where T_ℓ is an admissible labeling of T. As explained in [**11**], the leaves and root T of t are always labeled by 1 (this is our choice) and the admissibility condition says that other edges (i.e., components of the intrinsic 1-skeleton of T) are labeled by $a, b, c, d, \ldots$ taken from

$\{0, 1, 2, \ldots, r-2\}$ so that at each trivalent vertex of T the following relations hold on the triple of incident labels $a, b,$ and c:

$$\begin{aligned} a &\leq b + c, \\ b &\leq c + a, \\ c &\leq a + b, \\ a + b + c &\equiv 0 \,(\mathrm{mod}\, 2), \text{ and} \\ a + b + c &< 2(r-1) = 8. \end{aligned}$$

An admissible labeling T_ℓ is interpreted as a linear combination of pictures by replacing each edge with the Jones–Wenzl projector corresponding to its label. The set of admissible labeled trees $\{T_\ell\}$ is an orthogonal basis for the modular functor $V_s(R, \{3n\})$, defined topologically using the smooth equivalence relation. (The subscript s is to emphase that the smooth relations are used in this definition; of course, $V_s = V$.)

Because of the assumed $p = \Sigma_{i,j} b_{ij}, q_{ij} = 0 \in V_s(R, \{3n\})$, $c_\ell = 0$ for all admissible ℓ. But each q_{ij} is an embedded arc pairing x of the {leaves $\cup$ root} in $N(T)$ satisfying the additional admissibility restriction at each trivalent vertex of T. Such pairings are an alternative (though not orthogonal) basis for the modular functor $V(R, \{3n\})$ so, collected in this basis we have for each pairing type x, we have $\Sigma b_{ij}^x, q_{ij}^x = 0$ where

$$\begin{aligned} b_{ij}^x &= b_{ij} \quad \text{if } q_{ij} \text{ has type } = x, \\ &= 0 \quad \text{if } q_{ij} \text{ has type } \neq x. \end{aligned}$$

But all q_{ij} of a fixed type are clearly combinatorally equivalent ($\sim_c$). Thus we have found a combinatorial path through applications of ($\sim_c$) from p to the empty picture or, more precisely, to a sum of zero times various pictures. □

Unlike [**13**] the individual summands of H do not commute. The ground state of H has been computed topologically, however the spectrum spec(H) is less accessible. The most important question is the existence of an energy gap above the ground state which is constant under lattice refinement, $L \to \infty$, i.e., in the thermodynamic limit. The following heuristics motivate the conjectured energy gap.

In finite classical systems, such as random walk on a graph, diffusion time is well known to scale inversely with the spectral gap of the Laplacian. Similarly, in some simple quantum mechanical systems where exact calculation is possible, the energy gap scales inversely to the diffusion time between classical states. In [**13**], where direct calculation yields an energy gap above the ground state, the classical states are cycles and the "diffusion" is through elementary bordisms. Since we have set up our ground state to be analogous to homology: $G \cong P/\sim_c$ with pictures playing the role of cycles and our $\{B_j\}$ playing the role of bordisms, we expect similar diffusion properties and hence an energy gap. In Lemma 2.1 the proof shows that equivalent pictures p_1 and p_2 are connected by a "path" γ of deformations

("down" from p_1 to a neighborhood of T_r and then back up to p_2). Rapid diffusion corresponds to observing that there are a plethora of such paths and, in fact, the procedure for finding γ is highly under determined. More difficult would be a rigorous implication between diffusion and spec(H). Extending the analogy with [**13**], in both cases when the lattice is refined by a factor of L, a sequence of $O(L)$ local operators is required to transform between a pair of orthogonal ground states. So, given the existence of an energy gap, the Hamiltonian H will be stable to order $O(L)$ in perturbation theory, and formally corresponds to tunneling amplitudes between orthogonal ground states which scale like $e^{-\Omega(L)}$.

There are several important open questions. The first is a rigorous treatment of the energy gap, but this is probably too difficult in the present model. Another is how to deal with errors in the form of actual, rather than "virtual," excitation which have already been discussed in the context of tunneling. Can a coupling to a could bath repair such errors or are more active measures required? For example, can broken endpoint pairs of a 1-manifold find each other and cancel through some imposed attraction (as suggested by Dan Gottesman in conversations) or merely through a random walk? Nearby error pairs may be more serious in CS5 than in the toric codes, since isotopy class not just homology needs to be preserved; the wrong reconnection pairing would result in an unrecoverable error. To make this unlikely, should additional terms be included into our Hamiltonian H which could force distinct strands to be widely separated? This would put more weight on the simpler pictures, which are the ones that the quantum medium can most easily correct if damaged.

Kitaev's very general notion of quantum media, with its several antecedents in the study of quantum statistical mechanics, looks likely to become a central object of study shared between theoretical physics, solid state physics, and topology. The main disappointment of the present investigation is the complexity of the local Hamiltonian H used to construct stable universal *topological* quantum computation. One sees no easy road to radically simplifying it and still obtaining an exact description of CS5. However another path may be open. In our discussions, Kitaev has suggested (also see page 46 of [**19**]) that simpler lattice Hamiltonians may renormalize in the scaling limit to topological modular functors. Perhaps the most interesting topological theories, such as CS5, because of their simplicity, will have large "basins of attraction" under renormalization, and that identifiable universality classes of quantum media may not only exist mathematically but may even lie within the reach of engineers.

References

[1] M. Atiyah, The geometry and physics of knots, Lezioni Lincee [Lincei Lectures], Cambridge University Press, Cambridge, 1990, x+78 pp.

[2] D. Aharonov and M. Ben-Or, Fault-tolerant quantum computation with constant error rate, LANL ArXiv: quant-ph/9611025.

[3] V. G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, Vols. 1, 2 (Berkeley, CA., 1986), American Mathematical Society, Providence, RI, 1987, pp. 798–820.
[4] D. S. Freed, Higher algebraic structures and quantization, *Comm. Math. Phys.* **159** (1994), no. 2, 343–398.
[5] M. Freedman, A. Kitaev, and Z. Wang, Simulation of topological field theories by quantum computers, LANL ArXiv: quant-ph/0001071.
[6] M. Freedman, M. Larsen, and Z. Wang, A modular functor which is universal for quantum computation, LANL ArXiv: quant-ph/0001108.
[7] M. Freedman, M. Larsen, and Z. Wang, Density of representations of braid groups and distribution of values of Jones invariants, Preprint.
[8] M. Freedman, P/NP, and the quantum field computer, *Proc. Nat. Acad. Sci. USA* **95** (1998), no. 1, 98–101.
[9] R. Gelca, Topological quantum field theory with corners based on the Kauffman bracket. *Comment. Math. Helv.* **72** (1997), no. 2, 216–243.
[10] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (2) **126** (1987), no. 2, 335–388.
[11] L. H. Kauffman and S. L. Lins, Temperley–Lieb recoupling theory and invariants of 3-manifolds, *Annals of Mathematics Studies*, Vol. 134. Princeton University Press, Princeton, NJ, 1994, x+296 pp.
[12] A. Yu. Kitaev, Quantum computations: Algorithms and error correction, translation in *Russian Math. Surveys* **52** (1997), no. 6, 1191–1249.
[13] A. Yu. Kitaev, Fault-tolerant quantum computation by anyons, LANL ArXiv: quant-ph/9707021.
[14] A Yu. Kitaev and S. Bravyi, Lectures on CS4, Microsoft Research, Redmond, WA, Feb. 2000.
[15] A Yu. Kitaev and G. Kupperberg, Work in progress, private communication.
[16] E. Knill, R. Laflamme, and W. Zurek, Threshold accuracy for quantum computation, LANL arXiv: quant-ph/9610011.
[17] W. B. R Lickorish, Three-manifolds and the Temperley–Lieb algebra. *Math. Ann.* **290** (1991), no. 4, 657–670.
[18] C. Nayak and K. Shtengel, Soluble models of 2D magnets with fractionalized excitations, Preprint, 2000.
[19] J. Preskill, Fault-tolerant quantum computation, LANL ArXiv quant-ph/9712048.
[20] N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), no. 3, 547–597.
[21] P. W. Shor, Algorithms for quantum computation: Discrete logarithms and factoring, *Proceedings of the* 35*th Annual Symposium on Foundations of Computer Science* (Santa Fe, NM, 1994), IEEE Computer Society Press, Los Alamitos, CA, 1994, pp. 124–134.
[22] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, *Phys. Rev. A* (3), **52** No. 4, pp. R2493–R2496.
[23] P. W. Shor, Fault-tolerant quantum computation, *Proceedings of the* 37*th Annual Symposium on Foundations of Computer Science*, IEEE Computer Society Press, Los Alamitos, CA, 1996, quant-ph/9605011.
[24] K. Walker, On Witten's 3-manifold invariants, UCSD preprint, 1991.
[25] H. Wenzl, On sequences of projections, *C. R. Math. Rep. Acad. Sci. Canada* **9** (1987), no. 1, 5–9.
[26] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), no. 3, 351–399.