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Mathematics of Smoothed Particle Hydrodynamics: A Study via Nonlocal Stokes Equations

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Abstract

Smoothed particle hydrodynamics (SPH) is a popular numerical technique developed for simulating complex fluid flows. Among its key ingredients is the use of nonlocal integral relaxations to local differentiations. Mathematical analysis of the corresponding nonlocal models on the continuum level can provide further theoretical understanding of SPH. We present, in this part of a series of works on the mathematics of SPH, a nonlocal relaxation to the conventional linear steady-state Stokes system for incompressible viscous flows. The nonlocal continuum model is characterized by a smoothing length δ which measures the range of nonlocal interactions. It serves as a bridge between the discrete approximation schemes that involve a nonlocal integral relaxation and the local continuum models. We show that for a class of carefully chosen nonlocal operators, the resulting nonlocal Stokes equation is well-posed and recovers the original Stokes equation in the local limit when δ approaches zero. For some other commonly used smooth kernels, there are risks in getting ill-posed continuum models that could lead to computational difficulties in practice. This leads us to discuss the implications of our finding on the design of numerical methods.

Keywords Nonlocal Stokes equation · Nonlocal operators · Smoothed particle hydrodynamics · Peridynamics · Incompressible flows · Stability and convergence

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1 Introduction

There has been much recent interest in nonlocal continuum models. In solid mechanics, the theory of peridynamics [46] was proposed as a possible alternative to conventional models of elasticity and fracture mechanics. It has also been shown to be an integral relaxation to the conventional models when the latter are valid such as the case of linear elasticity. Mathematical and numerical analyses of peridynamics have provided a solid theoretical foundation to nonlocal mechanical models and their numerical approximations [15,37,49]. In this work, we are interested in extending such mathematical studies to problems in fluid mechanics. Indeed, nonlocal integral relaxations are naturally linked to numerical schemes developed for simulating fluid flows such as the smoothed particle hydrodynamics (SPH) [23,33,34,39,43], vortex methods [2,11] and others [3,4,7,8,13,21,28,50]. While developed originally for astrophysical applications, SPH has become a very popular computational technique for simulating complex flows, including both compressible and incompressible flows; meanwhile, it has also encountered issues like the loss of stability and lack of resolution. The mathematical analysis of SPH remains limited today except [5,30]. We present, as part of a series of investigations on the mathematics of SPH, a nonlocal relaxation to the conventional linear steady-state Stokes system for incompressible viscous flows, with the aim of providing further insight into the theoretical foundation of methods like SPH. The proposed nonlocal models serve as bridges linking SPH with the local differential equation models and allow us to delineate effects resulted from the different aspects of the approximation process. This adds new angle to the subject that has not been systematically explored in the literature before. The nonlocal equation studied here is also different from but closely connected to other nonlocal and fractional fluid models such as those considered for geostrophic flows [9,10] and hyper-dissipative models [27,48] as well as hydrodynamic limit of kinetic models [47].

One of the main contributions of this work is to formulate a well-posed nonlocal analog of the linear Stokes system (in two and three space dimensions). In general, a spatially nonlocal model may depend on the laws of nonlocal interactions specified in the bulk spatial region and necessary modifications near the boundary involving possible boundary conditions or nonlocal constraints [15]. As a first step, we will be focusing on the bulk nonlocal interactions in this work, namely finding suitable nonlocal relaxations of the first- and second-order differential operators used in the local Stokes equation such that the resulting nonlocal Stokes equation remains well-posed and retains a consistent local limit. To that end, we only consider periodic boundary conditions to avoid the discussion near physical boundary. Such a simplification allows us to use Fourier analysis to carry out much needed technical derivations. Though the analysis would work in any space dimension, our attention is on two- and three-dimensional spaces for physical relevance and this also complements a similar discussion in the one-dimensional case presented in [18] for the correspondence model of peridynamic materials. The nonlocal analogs of differential operators adopted in this paper are basic elements of nonlocal vector calculus introduced in [16] and have been successfully applied to nonlocal modeling and analysis [14,15,38]. Our main finding in this work is that the choices of the nonlocal gradient and divergence operators are more subtle issues that need more careful treatment. More specifically, we reveal that nonlocal interaction kernels associated with nonlocal gradient and divergence operators should have suitably strengthened interactions as particles (materials points) get close together, which in this work is phrased as *strong nearby interactions* for easy reference. Many popular choices used in practical implementation of SPH, however, do not yield such type of interactions, thus leading to possibly ill-posed problems on the continuum level in the smoothing step. Although suitable numerical discretization may add regularization effect to help alleviating the impact of any intrinsic ill-posedness, it is expected that these numerical regularization effect would be highly dependent on the resolution level (like particle distribution). Such speculations would require further theoretical investigation. Nevertheless, by revealing potential flaws in the key smoothing step for developing a robust and practical SPH methodology, our analysis provides strong links between the popular SPH discretization and a new mathematical foundation of nonlocal operators.

To further highlight the bridging role of the nonlocal models in the analysis of SPH-type methods and to show that the ill-posedness of nonlocal models can be avoided with carefully designed nonlocal operators, we present examples of well-posed nonlocal Stokes models involving nonlocal gradient and divergence operators with suitably strengthened nearby interactions. For these nonlocal models, we actually show that the spaces of nonlocal and local divergence-free vector fields coincide, a nice property-preserving feature of the nonlocal relaxation. We also get, as a byproduct, that using the nonlocal kernels under consideration, the nonlocal Laplacian in terms of the composition of nonlocal gradient and divergence operators is a well-defined and invertible operator which may be used in either the nonlocal Stokes system or a scalar Poisson equation. The latter is often used to do the pressure correction for maintaining incompressibility in practical SPH implementation. With well-posed nonlocal Stokes equation, to make further connections with the second step of discretizing the nonlocal operators in SPH, we deduce the convergence and the asymptotic compatibility of the Fourier spectral approximations. The latter is natural due to the special periodic setting. Moreover, the mathematical findings in this special case demonstrate the possibility that once well-posed nonlocal continuum models can be constructed from the smoothing step, it is possible to develop robust discretization that can maintain the convergence in different parameter regimes, for example, either for a fixed smoothing length $\delta > 0$ or for $\delta \to 0$, as the numerical resolution improves. It also serves as a hint for future analysis of collocation and mesh-free methods like SPH that are originally designed for more complex geometric settings. Indeed, one may design other remedies such as using nonlocal gradient operators with biased nonlocal interactions (similar in spirit to upwind differences [31]) or a formulation involving artificial compressibility [51], The latter two subjects are explored in separate works. In short, our study here further illustrates that the mathematics developed for nonlocal continuum models may offer foundation for a better understanding of related numerical discretizations.

The rest of the paper is organized as follows. We begin by formulating the nonlocal linear Stokes system in Sect. 2, in which the notation and assumptions are introduced. We then establish the well-posedness of the resulting nonlocal model in Sect. 3 and the connection to the usual local Stokes system in Sect. 4. In Sect. 5, we discuss the Fourier spectral methods and related convergence issues. We conclude in Sect. 6 with a summary and a discussion on ongoing and future research.

2 Nonlocal Stokes System

Let us first recall the conventional, local Stokes equation. Let $\boldsymbol{u}$ be the velocity field, p the pressure, $\boldsymbol{f}$ the body force, ν the given viscosity coefficient and a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$, the conventional, local Stokes equation of interests here refers to the system

$$\begin{cases} -\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } \Omega \\ -\nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega. \end{cases} \tag{1}$$

The linear Stokes equation (1) serves as a simplification of the stationary and time-dependent Navier–Stokes equations which are among the most widely studied mathematical models for fluid flows. In recent years, there have been much interests in various generalizations and relaxations of the linear Stokes and nonlinear Navier–Stokes systems, in particular, those involving nonlocal operators. We refer to earlier mentioned examples like geostrophic flows [9,10], hyper-dissipative models [27,48] and hydrodynamic limit of kinetic models [47]. Of particular interests to us is the nonlocal relaxation used in the SPH for the simulation of complex flows. While developed for astrophysical applications, SPH for incompressible viscous flows has also been a subject of ongoing study [1,6,22,25,26,29,41,42,45]. Despite broad applications and much progress in the algorithmic development effort, there has not much rigorous examination on the underlying relaxed nonlocal continuum models which could serve as bridges between local continuum models and their numerical discretizations.

We begin by focusing on a linear nonlocal system defined on the periodic cell given by $\Omega = (-\pi, \pi)^d \subset \mathbb{R}^d$ in dimension $d = 2$ or 3. Given a small parameter $\delta > 0$ representing the nonlocal interaction length, we define the nonlocal Stokes equation as: for a given periodic function $\boldsymbol{f}$ on Ω, find periodic functions $\boldsymbol{u}_\delta$ and p_δ such that

$$\begin{cases} -\nu \mathcal{L}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) + \mathcal{G}_\delta p_\delta(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ -\mathcal{D}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \end{cases} \tag{2}$$

with normalization conditions on $\boldsymbol{u}_\delta$ and p_δ to eliminate constant shifts, and on $\boldsymbol{f}$ to assure compatibility

$$\int_\Omega \boldsymbol{u}_\delta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0, \qquad \int_\Omega p_\delta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0, \qquad \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0. \tag{3}$$

The nonlocal operators used in (2) are the nonlocal diffusion operator $\mathcal{L}_\delta$, nonlocal gradient operator $\mathcal{G}_\delta$ and nonlocal divergence operator $\mathcal{D}_\delta$ given, respectively, by

$$\mathcal{L}_\delta \boldsymbol{u}(\boldsymbol{x}) = \int_{\mathbb{R}^d} \underline{\omega}_\delta(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}(\boldsymbol{x}))\mathrm{d}\boldsymbol{y}, \tag{4}$$

$$\mathcal{G}_\delta p(\boldsymbol{x}) = \int_{\mathbb{R}^d} \boldsymbol{\omega}_\delta(\boldsymbol{y}-\boldsymbol{x})(p(\boldsymbol{y})-p(\boldsymbol{x}))\mathrm{d}\boldsymbol{y} \tag{5}$$

$$\mathcal{D}_\delta \boldsymbol{u}(\boldsymbol{x}) = \int_{\mathbb{R}^d} \boldsymbol{\omega}_\delta(\boldsymbol{y}-\boldsymbol{x}) \cdot (\boldsymbol{u}(\boldsymbol{y})+\boldsymbol{u}(\boldsymbol{x}))\mathrm{d}\boldsymbol{y}. \tag{6}$$

The above operators are determined by a nonlocal scalar-valued kernel $\underline{\omega}_\delta$ and a vector-valued kernel $\boldsymbol{\omega}_\delta$. In this work, we take a special form $\boldsymbol{\omega}_\delta(\boldsymbol{x}) = \hat{\omega}_\delta(|\boldsymbol{x}|)\boldsymbol{x}/|\boldsymbol{x}|$, and for the moment, we assume that $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ both are nonnegative, radial symmetric and with a compact support in the δ neighborhood $B(\mathbf{0}, \delta)$ of the origin. Here, δ is a parameter that characterizes the range of nonlocal interaction. It is called a *nonlocal horizon parameter* in peridynamics, following [46], but in SPH, it is more commonly called the *smoothing length*. While more specific assumptions on the nonlocal kernels are given later, we note that these operators have been studied extensively in recent years, see for instance, [16,35,36,38] for more discussions and generalizations. The development of these operators, and in particular, their vector and tensor forms, has been motivated by the mathematical analysis of peridynamics and other nonlocal integral equation models, though the connection to SPH has also been alluded to previously [17]. In more general mathematical context, these nonlocal operators serve as the nonlocal analog of the classical diffusion, gradient and divergence operators, upon properly choosing the nonlocal kernels, and they form part of the building blocks of the nonlocal vector calculus, together with relevant integral identities. Indeed, by a nonlocal integration by parts formula (see [38, Theorem 2.7]), we have $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$ to be adjoint operators of each other, in the sense that, for functions in the suitable spaces (and periodic in our context),

$$\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \mathcal{G}_\delta p(\boldsymbol{x}) d\boldsymbol{x} = -\int_\Omega \mathcal{D}_\delta \boldsymbol{u}(\boldsymbol{x}) p(\boldsymbol{x}) d\boldsymbol{x}, \tag{7}$$

just like the conventional gradient and divergence operators. In addition, the expression $\boldsymbol{u}(\boldsymbol{y}) + \boldsymbol{u}(\boldsymbol{x})$ in the definition of $\mathcal{D}_\delta \boldsymbol{u}$ in (4) can be used interchangeably with $\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})$, although the plus sign is preferred in the more general function class over a bounded domain Ω to have (7) satisfied.

We note that integral relaxations to differential operators have also been discussed in many works related to the SPH methods as mentioned earlier, except that they are more often given by approximate quadrature forms in disguise. In the SPH community, δ is called the smoothing length; thus, we refer δ as either horizon parameter or smoothing length interchangeably in this work.

Nonlocal operators such as $\mathcal{L}_\delta$ can also be seen as continuum forms of popular discrete and graph Laplacians (see e.g. [24,32,40]). The study on $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$, as explained in this work, bears even greater significance for the nonlocal Stokes equation. Such a study is also related to the so-called correspondence theory of peridynamics, see a recent study in [18].

Now we specify conditions on the kernels ω_δ and $\hat{\omega}_\delta$ used in the definition of the nonlocal diffusion operator $\mathcal{L}_\delta$, and the nonlocal gradient and divergence operators $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$.

Assumption 1 The nonnegative and radial symmetric kernels $\underline{\omega}_\delta = \underline{\omega}_\delta(|\boldsymbol{x}|)$ and $\hat{\omega}_\delta = \hat{\omega}_\delta(|\boldsymbol{x}|)$ are assumed to satisfy the following assumptions.

1. The kernels $\underline{\omega}_\delta = \underline{\omega}_\delta(|\boldsymbol{x}|)$ and $\hat{\omega}_\delta = \hat{\omega}_\delta(|\boldsymbol{x}|)$ have compact support in the sphere $B(\mathbf{0}, \delta)$ and satisfy the normalization conditions:

$$\frac{1}{2}\int_{\mathbb{R}^n} \underline{\omega}_\delta(|\boldsymbol{x}|)|\boldsymbol{x}|^2 \mathrm{d}\boldsymbol{x} = d. \tag{8}$$

   and

$$\int_{\mathbb{R}^n} \hat{\omega}_\delta(|\boldsymbol{x}|)|\boldsymbol{x}| \mathrm{d}\boldsymbol{x} = d. \tag{9}$$

2. The kernels $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ are rescaled from kernels ω and $\hat{\omega}$ that have compact support in the unit sphere:

$$\begin{cases} \underline{\omega}_\delta(|\boldsymbol{x}|) = \dfrac{1}{\delta^{d+2}}\omega\left(\dfrac{|\boldsymbol{x}|}{\delta}\right), \\ \hat{\omega}_\delta(\boldsymbol{x}) = \dfrac{1}{\delta^{d+1}}\hat{\omega}\left(\dfrac{|\boldsymbol{x}|}{\delta}\right). \end{cases} \tag{10}$$

The above conditions are often made in earlier studies on the nonlocal operators and are also default assumptions throughout this paper. In the literature on SPH and vortex-blob methods, these moment conditions have also been used as well, see, e.g., [11]. It turns out that, as shown in the next section, additional conditions on the kernels, particularly on $\hat{\omega}_\delta$, are needed in order to obtain a well-posed nonlocal Stokes system.

3 Well-Posedness of the Nonlocal Stokes System with Periodic Condition

In this section, we examine the well-posedness of the nonlocal Stokes system given by (2). To begin with, let us define an operator associated with the vector system by

$$\mathcal{A}_\delta = \begin{pmatrix} -\nu\mathcal{L}_\delta & \mathcal{G}_\delta \\ -\mathcal{D}_\delta & 0 \end{pmatrix}. \tag{11}$$

Under periodic conditions and the constraints (3), we write $\boldsymbol{u}$ and p in terms of their Fourier series, namely,

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d, \boldsymbol{\xi}\neq 0} \widehat{\boldsymbol{u}}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \text{ and } p(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d, \boldsymbol{\xi}\neq 0} \widehat{p}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}},$$

where

$$\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) = \int_{\Omega} \boldsymbol{u}(\boldsymbol{x}) e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} d\boldsymbol{x} \text{ and } \widehat{p}(\boldsymbol{\xi}) = \int_{\Omega} p(\boldsymbol{x}) e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} d\boldsymbol{x}.$$

The following lemma provides the associated Fourier symbols of the nonlocal operators.

Lemma 2 *The Fourier symbols of operators $\mathcal{L}_\delta$, $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$ are given by*

$$\widehat{\mathcal{L}_\delta \boldsymbol{u}}(\boldsymbol{\xi}) = -\lambda_\delta(\boldsymbol{\xi})\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) \tag{12}$$

$$\widehat{\mathcal{G}_\delta p}(\boldsymbol{\xi}) = i\boldsymbol{b}_\delta(\boldsymbol{\xi})\widehat{p}(\boldsymbol{\xi}) \tag{13}$$

$$\widehat{\mathcal{D}_\delta \boldsymbol{u}}(\boldsymbol{\xi}) = i(\boldsymbol{b}_\delta(\boldsymbol{\xi}))^T \widehat{\boldsymbol{u}}(\boldsymbol{\xi}), \tag{14}$$

where $\lambda_\delta(\boldsymbol{\xi})$ and $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ are given by

$$\lambda_\delta(\boldsymbol{\xi}) = \int_{|\boldsymbol{s}|\leq\delta} \underline{\omega}_\delta(|\boldsymbol{s}|)(1-\cos(\boldsymbol{\xi}\cdot\boldsymbol{s}))\mathrm{d}\boldsymbol{s} \tag{15}$$

$$\boldsymbol{b}_\delta(\boldsymbol{\xi}) = \int_{|\boldsymbol{s}|\leq\delta} \hat{\omega}_\delta(|\boldsymbol{s}|)\frac{\boldsymbol{s}}{|\boldsymbol{s}|}\sin(\boldsymbol{\xi}\cdot\boldsymbol{s})\mathrm{d}\boldsymbol{s}. \tag{16}$$

Proof The results follow immediately from the definitions of $\mathcal{L}_\delta$, $\mathcal{G}_\delta$ and $\mathcal{D}_\delta$. □

For convenience, in the following lemma, we further express $\lambda_\delta(\boldsymbol{\xi})$ and $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ in (15) and (16) using polar coordinates.

Lemma 3 *The Fourier symbols $\lambda_\delta(\boldsymbol{\xi})$ and $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ in (15) and (16) can be equivalently expressed as*

$$\lambda_\delta(\boldsymbol{\xi}) = \begin{cases} 4\displaystyle\int_0^{\pi/2}\int_0^\delta r\underline{\omega}_\delta(r)\big(1-\cos(r\cos(\phi)|\boldsymbol{\xi}|)\big)\mathrm{d}r\mathrm{d}\phi & \textit{for } d=2, \\ 4\pi\displaystyle\int_0^{\pi/2}\sin(\phi)\int_0^\delta r^2\underline{\omega}_\delta(r)\big(1-\cos(r\cos(\phi)|\boldsymbol{\xi}|)\big)\mathrm{d}r\mathrm{d}\phi & \textit{for } d=3, \end{cases} \tag{17}$$

and

$$\boldsymbol{b}_\delta(\boldsymbol{\xi}) = b_\delta(|\boldsymbol{\xi}|)\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}, \tag{18}$$

where the scalar coefficient $b_\delta(|\boldsymbol{\xi}|)$ *is given by*

$$b_\delta(|\boldsymbol{\xi}|) = \begin{cases} 4\int_0^{\pi/2} \cos(\phi) \int_0^{\delta} r\hat{\omega}_\delta(r) \sin(r\cos(\phi)|\boldsymbol{\xi}|)\mathrm{d}r\mathrm{d}\phi & \textit{for } d = 2, \\ 4\pi \int_0^{\pi/2} \cos(\phi)\sin(\phi) \int_0^{\delta} r^2\hat{\omega}_\delta(r) \sin(r\cos(\phi)|\boldsymbol{\xi}|)\mathrm{d}r\mathrm{d}\phi & \textit{for } d = 3. \end{cases} \tag{19}$$

Proof Let us show (18) with $d = 3$. The case with $d = 2$ is similar and omitted. First we observe that for any orthogonal matrix R, we have

$$\boldsymbol{b}_\delta(\boldsymbol{\xi}) = R^T \boldsymbol{b}_\delta(R\boldsymbol{\xi}).$$

Now denote $\boldsymbol{e} = (0, 0, 1)^T$. Let R be the rotation matrix which rotates $\boldsymbol{\xi}$ to be aligned with $\boldsymbol{e}$, namely

$$R\boldsymbol{\xi} = |\boldsymbol{\xi}|\boldsymbol{e}.$$

Then $R\boldsymbol{\xi} \cdot \boldsymbol{s} = |\boldsymbol{\xi}|s_3$ and

$$\boldsymbol{b}_\delta(\boldsymbol{\xi}) = \int_{|\boldsymbol{s}|\le\delta} \hat{\omega}_\delta(|\boldsymbol{s}|)\frac{R^T\boldsymbol{s}}{|\boldsymbol{s}|} \sin(|\boldsymbol{\xi}|s_3)\mathrm{d}\boldsymbol{s}.$$

So each component of $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ is given by

$$\begin{aligned} {}_i &= \int_{|\boldsymbol{s}|\le\delta} \frac{\hat{\omega}_\delta(|\boldsymbol{s}|)}{|\boldsymbol{s}|} \sum_{j=1}^{3} R_{ji}s_j \sin(|\boldsymbol{\xi}|s_3)\mathrm{d}\boldsymbol{s} \\ &= \int_{|\boldsymbol{s}|\le\delta} \frac{\hat{\omega}_\delta(|\boldsymbol{s}|)}{|\boldsymbol{s}|} R_{3i}s_3 \sin(|\boldsymbol{\xi}|s_3)\mathrm{d}\boldsymbol{s} \quad \text{for } i = 1, 2, 3. \end{aligned}$$

The next task at hand is to find $\{R_{3i}, i = 1, 2, 3\}$. We know that R is the matrix obtained by rotating $\boldsymbol{\xi}$ by an angle of $\arccos(\boldsymbol{e} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}) = \arccos(\xi_3/|\boldsymbol{\xi}|)$ around the axis in the direction of

$$\frac{\boldsymbol{\xi} \times \boldsymbol{e}}{|\boldsymbol{\xi} \times \boldsymbol{e}|} = \frac{1}{\sqrt{\xi_1^2 + \xi_2^2}}(\xi_2, -\xi_1, 0).$$

Such a rotation matrix can be explicitly constructed. In particular, R_{3i} is given by

$$R_{3i} = \frac{\xi_i}{|\boldsymbol{\xi}|}, \quad \text{for } i = 1, 2, 3.$$

Combing the above arguments, we obtain

$$\begin{aligned} \boldsymbol{b}_\delta(\boldsymbol{\xi}) &= \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\int_{|\boldsymbol{s}|\le\delta} \hat{\omega}_\delta(|\boldsymbol{s}|)\frac{s_3}{|\boldsymbol{s}|}\sin(|\boldsymbol{\xi}|s_3)\mathrm{d}\boldsymbol{s} \\ &= \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\left(2\pi\int_0^\pi \cos(\phi)\sin(\phi)\int_0^\delta r^2\hat{\omega}_\delta(r)\sin(r\cos(\phi)|\boldsymbol{\xi}|)\mathrm{d}r\mathrm{d}\phi\right) \\ &= \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}b_\delta(\boldsymbol{\xi}), \end{aligned}$$

where $b_\delta(\boldsymbol{\xi})$ is given by (19).

Now (17) can be obtained similarly by noticing that $\lambda_\delta(\boldsymbol{\xi}) = \lambda_\delta(R\boldsymbol{\xi})$ for any orthogonal matrix R. □

Via Fourier analysis, we now get a $(d+1)\times(d+1)$ matrix system:

$$A_\delta(\boldsymbol{\xi})\begin{pmatrix}\widehat{\boldsymbol{u}}(\boldsymbol{\xi})\\ \widehat{p}(\boldsymbol{\xi})\end{pmatrix} = \begin{pmatrix}\widehat{\boldsymbol{f}}(\boldsymbol{\xi})\\ 0\end{pmatrix}$$

where

$$A_\delta(\boldsymbol{\xi}) = \begin{pmatrix}\lambda_\delta(\boldsymbol{\xi})I_d & i\boldsymbol{b}_\delta(\boldsymbol{\xi})\\ -i(\boldsymbol{b}_\delta(\boldsymbol{\xi}))^T & 0\end{pmatrix}.$$

For each fixed nonzero $\boldsymbol{\xi}$, in order for the matrix $A_\delta(\boldsymbol{\xi})$ to be invertible, we need $\lambda_\delta(\boldsymbol{\xi})$ and $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ to be nonzero. Under such assumptions, the inverse of $A_\delta(\boldsymbol{\xi})$ is given by

$$(A_\delta(\boldsymbol{\xi}))^{-1} = \begin{pmatrix}\frac{1}{\lambda_\delta(\boldsymbol{\xi})}\left(I_d - \frac{\boldsymbol{b}_\delta(\boldsymbol{\xi})\otimes\boldsymbol{b}_\delta(\boldsymbol{\xi})}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2}\right) & i\frac{\boldsymbol{b}_\delta(\boldsymbol{\xi})}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2}\\ -i\frac{(\boldsymbol{b}_\delta(\boldsymbol{\xi}))^T}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2} & \frac{-\lambda_\delta(\boldsymbol{\xi})}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2}\end{pmatrix}. \tag{20}$$

Now we can easily observe from the expression in Eq. (17) that $\lambda_\delta(\boldsymbol{\xi})$ is positive for any nonzero $\boldsymbol{\xi}$. It is however more delicate to check the non-degeneracy of $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ since the latter is an integral of a product of a positive function with a sign-changing oscillatory function. The following lemma (also utilized in [18]) gives a simple observation on the sine Fourier coefficient that is useful to our discussion on the positivity of $\boldsymbol{b}_\delta(\boldsymbol{\xi})$.

Lemma 4 *Given a measurable, nonnegative and nonincreasing function $g = g(x)$ with $xg(x)$ integrable, we have*

$$\int_0^{2\pi} g(x)\sin(x)\mathrm{d}x \ge 0 \tag{21}$$

with the equality holds only for g being a constant function. Consequently, for any $h > 0$ and $a > 0$, we have

$$\int_0^h g(x)\sin(ax)\mathrm{d}x \ge 0, \tag{22}$$

with the equality holds only for g being a constant function (and with value zero if the product ha is not an integer multiple of 2π).

Proof The inequality (21) follows immediately from the observation that

$$\int_0^{2\pi} g(x)\sin(x)\mathrm{d}x = \int_0^{\pi} [g(x) - g(x+\pi)]\sin(x)\mathrm{d}x \geq 0.$$

By the nonincreasing property, we see that the equality holds only for g being a constant function. The more general case follows by applying a change of variable and taking a zero extension of g outside $(0, h)$ to cover complete periods of the scaled sine function. □

A simple consequence of Lemma 4 and (19) is that $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ is positive for any fixed $\boldsymbol{\xi}$ if $r^{d-1}\hat{\omega}(r)$ is a nonincreasing function. This simple fact gives us a hint that in order for the nonlocal Stokes system to be well-posed, one should expect the nonlocal interaction in the gradient and divergence operators be suitably strengthened for physical points (particles) in closer proximity. Now to offer a precise energy estimate, in the rest of this section, we assume that the kernel $\hat{\omega}_\delta(r)$ satisfies the following additional conditions.

Assumption 5 The kernel $\hat{\omega}_\delta(r)$ is the rescaling of $\hat{\omega}(r)$ given by (10) with $\hat{\omega}(r)$ satisfying the following conditions.

1. $r^{d-1}\hat{\omega}(r)$ is nonincreasing for $r \in (0, 1)$;
2. $\hat{\omega}(r)$ is of fractional type in at least a small neighborhood of origin, namely there exists some $\epsilon > 0$ such that for $s \in (0, \epsilon)$ we have

$$\hat{\omega}(r) = \frac{c}{r^{d+\beta}}, \tag{23}$$

   for some constant $c > 0$ and $\beta \in (-1, 1)$.

Remark 1 The condition that $r^{d-1}\hat{\omega}(r)$ is nonincreasing gives us a sufficient condition for $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ to stay positive for finite $|\boldsymbol{\xi}|$. It is not a necessary condition, but it is shown later by some examples that $\boldsymbol{b}_\delta(\boldsymbol{\xi})$ can be zero for finite $|\boldsymbol{\xi}|$ if the nonincreasing condition is violated; thus, the nonlocal Stokes system is ill-posed in such cases. Consequently, SPH schemes based on the nonlocal gradients with bounded and smooth kernels obtained from the smoothing step may contain intrinsic instabilities that are difficult to eliminate on the discretization level, especially when the particle distribution becomes uneven or highly disordered.

As an illustration, in Fig. 1 we plot the values of $b_\delta(|\boldsymbol{\xi}|)$ against $|\boldsymbol{\xi}|$ for the kernel $\hat{\omega}(r) = \frac{1}{r^{d+\beta}}$ with $\beta < -1$, using the expression in (19) with $d = 2$ and $d = 3$, respectively. We observe from the plots the tendency for $b_\delta(|\boldsymbol{\xi}|)$ to stay positive with the kernel $\hat{\omega}(r)$ being more singular at zero. On the other hand, clear numerical evidence from the plots shows that $b_\delta(|\boldsymbol{\xi}|)$ may become zero at some finite frequencies for $\beta < -1.5$ in the two-dimensional case and for $\beta < -2$ in the three-dimensional case.

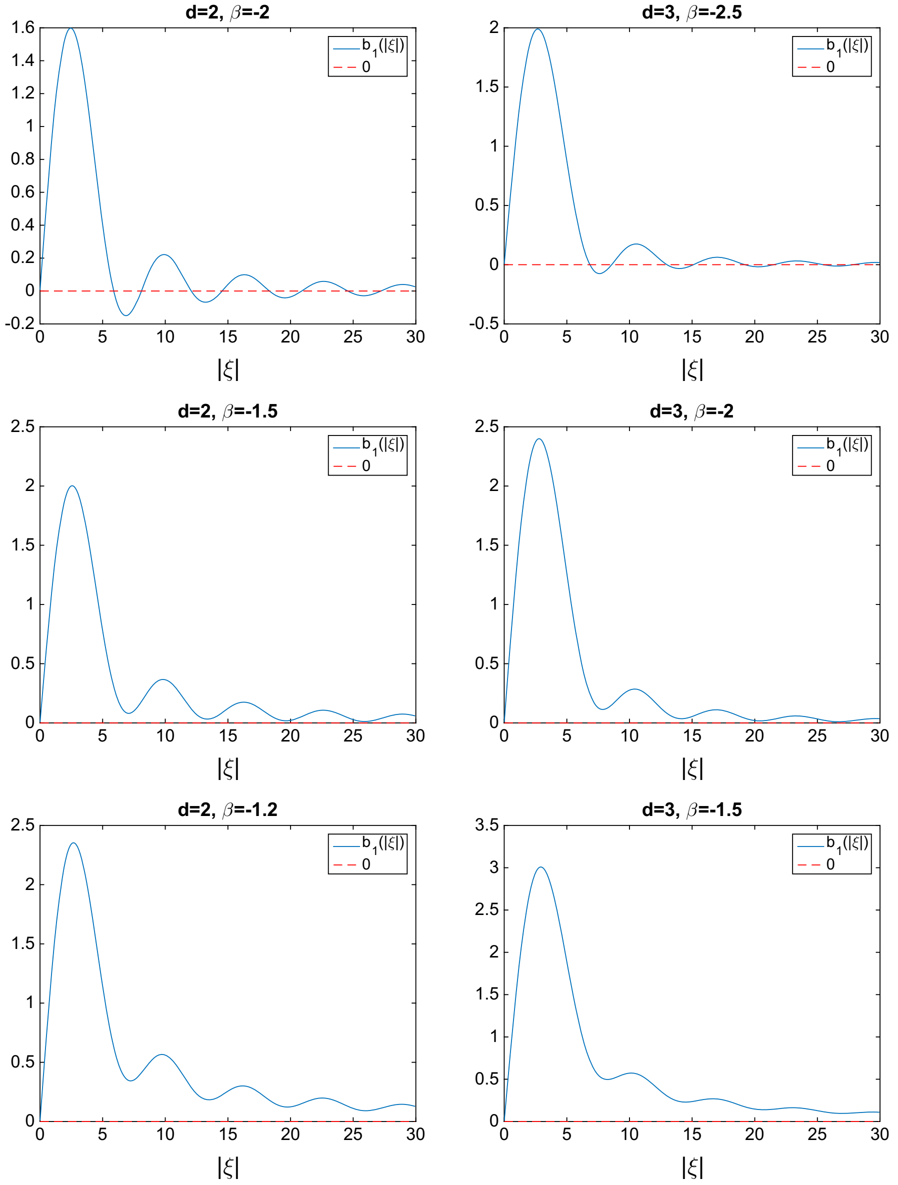

Fig. 1 Values of $b_\delta(|\boldsymbol{\xi}|)$ against $|\boldsymbol{\xi}|$ for $\delta = 1$, $d = 2$ (left column from top to bottom) with $\beta = -2$ (top), $\beta = -1.5$ (middle), $\beta = -1.2$ (bottom), and $d = 3$ (right column from top to bottom) with $\beta = -2.5$ (top), $\beta = -2$ (middle), $\beta = -1.5$ (bottom)

The following theorem establishes the existence of a unique solution to the nonlocal Stokes equation.

Theorem 6 *Assume that the kernels $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ satisfy Assumptions 1 and 5. Given $\delta > 0$, there exists a unique solution $(\boldsymbol{u}_\delta, p_\delta)$ to the nonlocal Stokes system (2)* *with periodic boundary condition given in the form of their Fourier series with* $(\widehat{\boldsymbol{u}}_\delta(\boldsymbol{\xi}), \widehat{p}_\delta(\boldsymbol{\xi}))$ *computed through*

$$\begin{pmatrix} \widehat{\boldsymbol{u}}_\delta(\boldsymbol{\xi}) \\ \widehat{p}_\delta(\boldsymbol{\xi}) \end{pmatrix} = (A_\delta(\boldsymbol{\xi}))^{-1} \begin{pmatrix} \widehat{\boldsymbol{f}}(\boldsymbol{\xi}) \\ 0 \end{pmatrix} \tag{24}$$

where $(A_\delta(\boldsymbol{\xi}))^{-1}$ *are defined by* (20) *for nonzero* $\boldsymbol{\xi}$. *In addition, with* C *independent of* δ *and* $\boldsymbol{f}$ *we have*

$$\|\boldsymbol{u}_\delta\|_{[\mathcal{S}_\delta(\Omega)]^d} \leq C\|\boldsymbol{f}\|_{[\mathcal{S}_\delta^*(\Omega)]^d} \tag{25}$$

$$\|p_\delta\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{[H^{-\beta}(\Omega)]^d}, \tag{26}$$

where $\mathcal{S}_\delta(\Omega)$ *is the energy space with its norm associated with the Fourier symbol* $(\lambda_\delta(\boldsymbol{\xi}))^{1/2}$ *and* $\mathcal{S}_\delta^*(\Omega)$ *is its dual space, and* $\beta \in (-1, 1)$ *is the exponent defined through* (23).

Proof From Lemma 4 and Assumption 5, we know that $A_\delta(\boldsymbol{\xi})$ is invertible for nonzero $\boldsymbol{\xi}$ and the inverse is given by (20). This gives us

$$\widehat{\boldsymbol{u}}_\delta(\boldsymbol{\xi}) = \frac{1}{\lambda_\delta(\boldsymbol{\xi})}\left(I_d - \frac{\boldsymbol{b}_\delta(\boldsymbol{\xi}) \otimes \boldsymbol{b}_\delta(\boldsymbol{\xi})}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2}\right)\widehat{\boldsymbol{f}}(\boldsymbol{\xi})$$

and

$$\widehat{p}_\delta(\boldsymbol{\xi}) = -i\frac{(\boldsymbol{b}_\delta(\boldsymbol{\xi}))^T}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2}\widehat{\boldsymbol{f}}(\boldsymbol{\xi}).$$

So we have

$$|\widehat{\boldsymbol{u}}_\delta(\boldsymbol{\xi})| \leq C\left|\frac{1}{\lambda_\delta(\boldsymbol{\xi})}\right||\widehat{\boldsymbol{f}}(\boldsymbol{\xi})|, \tag{27}$$

and

$$|\widehat{p}_\delta(\boldsymbol{\xi})| \leq \left|\frac{1}{b_\delta(\boldsymbol{\xi})}\right||\widehat{\boldsymbol{f}}(\boldsymbol{\xi})|. \tag{28}$$

From (27), we know immediately that (25) is true.

Now we are left to show (26). From Eq. (28) we only need to estimate $|1/b_\delta(|\boldsymbol{\xi}|)|$. We again address the case $d = 3$. Under the assumption that $r^2\hat{\omega}(r)$ is nonincreasing, we can use Lemma 3 to write

$$\begin{aligned} b_\delta(|\boldsymbol{\xi}|) &= 4\pi\int_0^{\pi/2}\cos(\phi)\sin(\phi)\int_0^\delta r^2\hat{\omega}_\delta(r)\sin(r\cos(\phi)|\boldsymbol{\xi}|)\mathrm{d}r\mathrm{d}\phi \\ &= \frac{2\pi}{\delta}\int_0^{\pi/2}\sin(2\phi)\int_0^1 r^2\hat{\omega}(r)\sin(r\cos(\phi)\delta|\boldsymbol{\xi}|)\mathrm{d}r\mathrm{d}\phi. \end{aligned}$$

Notice that the integral in the above quantity is positive for any finite $\delta|\boldsymbol{\xi}|$ under the assumption that $r^2\hat{\omega}(r)$ is nonincreasing. Indeed, from Lemma 4, we know that the integrand is always nonnegative and it is only possibly zero when $\cos(\phi)\delta|\boldsymbol{\xi}|$ is a multiple of 2π, which is a set of measure zero. Thus the integral above is positive for any fixed numbers $\delta > 0$ and $|\boldsymbol{\xi}| > 0$.

Again we denote $a = \delta|\boldsymbol{\xi}|$. For $a < 1$, we use

$$\sin(x) \geq x - \frac{x^3}{6}$$

to get

$$\begin{aligned} b_\delta(|\boldsymbol{\xi}|) \geq & \frac{2\pi a}{\delta} \int_0^{\pi/2} \cos(\phi)\sin(2\phi)\mathrm{d}\phi \int_0^1 r^3\hat{\omega}(r)\mathrm{d}r \\ & - \frac{2\pi a^3}{6\delta} \int_0^{\pi/2} \cos^3(\phi)\sin(2\phi)\mathrm{d}\phi \int_0^1 r^5\hat{\omega}(r)\mathrm{d}r \\ \geq & \frac{a}{\delta} C = C|\boldsymbol{\xi}|, \end{aligned}$$

where C is a constant independent of δ. For $a \in [1, 4\pi/\epsilon]$ where ϵ is the parameter defined in Assumption 5, since the integral defined above is positive, it then has a lower bound, namely, we have

$$b_\delta(|\boldsymbol{\xi}|) \geq \frac{\tilde{C}}{\delta} \geq \frac{\tilde{C}\epsilon}{4\pi}|\boldsymbol{\xi}|.$$

Now for $a > 4\pi/\epsilon$, we have $\cos(\phi)a > 2\pi/\epsilon$ for $\phi \in (0, \pi/3)$. We then write

$$\begin{aligned} b_\delta(|\boldsymbol{\xi}|) &\geq \frac{2\pi}{\delta} \int_0^{\pi/3} \sin(2\phi) \int_0^1 r^2\hat{\omega}(r)\sin(r\cos(\phi)a)\mathrm{d}r\mathrm{d}\phi \\ &\geq \frac{2\pi}{\delta} \int_0^{\pi/3} \sin(2\phi) \left(\int_0^{\frac{2\pi}{\cos(\phi)a}} + \int_{\frac{2\pi}{\cos(\phi)a}}^1 \right) r^2\hat{\omega}(r)\sin(r\cos(\phi)a)\mathrm{d}r\mathrm{d}\phi. \end{aligned}$$

Using Lemma 4 and the nonincreasing assumption, we observe that

$$\int_{\frac{2\pi}{\cos(\phi)a}}^1 r^2\hat{\omega}(r)\sin(r\cos(\phi)a)\mathrm{d}r = \int_0^{1-\frac{2\pi}{\cos(\phi)a}} h(r)\sin(r\cos(\phi)a)\mathrm{d}r \geq 0,$$

where

$$h(r) := (r + \frac{2\pi}{\cos(\phi)a})^2\hat{\omega}(r + \frac{2\pi}{\cos(\phi)a})$$

is a nonincreasing function. Then one can show by using (23) that

$$
\begin{aligned}
b_\delta(|\boldsymbol{\xi}|) &\geq \frac{2\pi}{\delta}\int_0^{\pi/3}\sin(2\phi)\int_0^{\frac{2\pi}{\cos(\phi)a}} r^2\hat{\omega}(r)\sin(r\cos(\phi)a)\mathrm{d}r\mathrm{d}\phi \\
&\geq \frac{2\pi a^\beta}{\delta}\int_0^{\pi/3}\cos^\beta(\phi)\sin(2\phi)\mathrm{d}\phi\int_0^{2\pi}\frac{1}{r^{1+\beta}}\sin(r)\mathrm{d}r \\
&\geq \frac{C|\boldsymbol{\xi}|^\beta}{\delta^{1-\beta}}
\end{aligned}
$$

Thus we obtain (26). □

An interesting consequence is that, under the specific choices of the kernel, we see the equivalence of a vector field being either locally divergence-free or nonlocally divergence-free.

Corollary 7 *Assume that the kernels $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ satisfy Assumptions 1 and 5. Then, in the distribution sense over the periodic cell, a periodic and square integrable vector field $\boldsymbol{u}$ satisfies $\nabla\cdot\boldsymbol{u}\equiv 0$ if and only if $\mathcal{D}_\delta\boldsymbol{u}\equiv 0$. In other words, we have the following equivalent function spaces:*

$$
\{\boldsymbol{u}\in L^2(\Omega)\ :\ \nabla\cdot\boldsymbol{u}\equiv 0,\ in\ \Omega.\}\equiv\{\boldsymbol{u}\in L^2(\Omega)\ :\ \mathcal{D}_\delta\boldsymbol{u}\equiv 0,\ in\ \Omega.\}
$$

The above result follows immediately from the established positivity of the scalar coefficient $b_\delta(\boldsymbol{\xi})$ for any $\boldsymbol{\xi}\neq\mathbf{0}$.

In addition, we may also use the composition of nonlocal divergence and nonlocal gradient to get a nonlocal Laplacian to replace the operator $\mathcal{L}_\delta$ in the nonlocal model (2). Similar argument can be adopted to show the well-posedness of the resulting system. We state the conclusion below without proof.

Theorem 8 *Assume that the kernel $\hat{\omega}_\delta$ satisfies Assumptions 1 and 5. Given $\delta>0$, there exists a unique solution $(\mathbf{u}_\delta, p_\delta)$ to the following modified nonlocal Stokes system*

$$
\begin{cases}
-\nu\mathcal{D}_\delta\mathcal{G}_\delta\boldsymbol{u}_\delta(\boldsymbol{x})+\mathcal{G}_\delta p_\delta(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x}), & \boldsymbol{x}\in\Omega,\\
-\mathcal{D}_\delta\boldsymbol{u}_\delta(\boldsymbol{x})=0, & \boldsymbol{x}\in\Omega,
\end{cases}
\tag{29}
$$

with periodic boundary condition and normalization conditions of the type given in (3). In addition, with C independent of δ and $\boldsymbol{f}$ we have

$$
\|\mathbf{u}_\delta\|_{[\mathcal{V}_\delta(\Omega)]^d}\leq C\|\boldsymbol{f}\|_{[\mathcal{V}_\delta^*(\Omega)]^d} \tag{30}
$$

$$
\|p_\delta\|_{L^2(\Omega)}\leq C\|\boldsymbol{f}\|_{[H^{-\beta}(\Omega)]^d}, \tag{31}
$$

where $\mathcal{V}_\delta(\Omega)$ is the Hilbert space associated with the norm $\|\boldsymbol{u}\|=\{\|\boldsymbol{u}\|^2_{L^2(\Omega)}+\|\mathcal{G}_\delta\boldsymbol{u}\|^2_{L^2(\Omega)}\}^{1/2}$ and $\mathcal{V}_\delta^*(\Omega)$ is its dual space, and $\beta\in(-1,1)$ is the exponent defined through (23).

Remark 2 Naturally, we can also establish the well-posedness of the Poisson equation corresponding to the nonlocal Laplacian $\mathcal{D}_\delta \mathcal{G}_\delta$, just like their local counterparts, under the same conditions given in the above theorem. In fact, this is the usual practice, in the context of solid mechanics, of the correspondence model of peridynamic materials. The study of well-posedness of the latter formulation [18] is similar to that carried out there. For the scalar equation, the resulting nonlocal interactions encoded in $\mathcal{L}_\delta = \mathcal{D}_\delta \mathcal{G}_\delta$ involve both repulsive and attractive types which is different from the $\mathcal{L}_\delta$ operator used in (4) that features only repulsive interactions. We note that this is also relevant to practical incompressible SPH as the pressure correction often relies on a well-posed Poisson equation. Thus, in case that the kernels for $\mathcal{D}_\delta$ and $\mathcal{G}_\delta$ do not have strengthened nearby interactions, the pressure correction step using $\mathcal{D}_\delta \mathcal{G}_\delta$ might become ill-posed which would also impact the convergence and robustness of the numerical solution. Indeed, it has been noted that the composition of SPH divergence and SPH gradient leads to a discretization that are sensitive to particle distributions [12,45]. From our analysis, we can see that it is no surprise that such phenomenon does occur as the kernels in the SPH derivatives usually do not exhibit strong nearby interactions.

Unlike the case of local elliptic systems, the solutions to nonlocal Stokes equation may or may not be more regular than the data ($\boldsymbol{f}$ in our case), depending on the specific forms of the nonlocal operators. For $\omega = \omega(|\boldsymbol{x}|)$ integrable, we can only show that the velocity $\boldsymbol{u}_\delta$ remains in L^2 if the data $\boldsymbol{f}$ is also in L^2. On the other hand, in some special cases where $\omega = \omega(|\boldsymbol{x}|)$ exhibits sufficient singular behavior, we can expect some fractional order regularity pick up. For later references, these results are stated below.

Proposition 9 *Assume that the kernels $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ satisfy Assumptions 1 and 5 with ω_δ being a rescaling of ω. Let $\boldsymbol{u}_\delta$ be the velocity component of the solution to the nonlocal Stokes equation (2). Without loss of generality, we also only consider $\delta \in (0, 1)$. If $\omega(|\boldsymbol{x}|)$ is integrable in $\boldsymbol{x}$, then we have*

$$\|\boldsymbol{u}_\delta\|_{[L^2(\Omega)]^d} \leq C \|\boldsymbol{f}\|_{[L^2(\Omega)]^d}. \tag{32}$$

If instead,

$$\omega(r) \geq \frac{m}{r^{d+2\alpha}} \quad \forall r \in (0, 1), \tag{33}$$

for some $\alpha \in (0, 1)$ and a constant $m \in \mathbb{R}^+$, then we have

$$\|\boldsymbol{u}_\delta\|_{[H^\alpha(\Omega)]^d} \leq C \|\boldsymbol{f}\|_{[H^{-\alpha}(\Omega)]^d}, \tag{34}$$

where C is independent of δ and $\boldsymbol{f}$.

Proof We follow the proof of Theorem 6. Without loss of generality, we take the case $d = 3$ subject to the additional assumptions of the kernel $\omega(r)$. From the definition of $\lambda_\delta(\boldsymbol{\xi})$, we have

$$\begin{aligned}\lambda_\delta(\boldsymbol{\xi}) &\geq 4\pi \int_0^{\pi/3} \sin(\phi) \int_0^\delta r^2 \underline{\omega}_\delta(r)\big(1-\cos(r\cos(\phi)|\boldsymbol{\xi}|)\big)\mathrm{d}r\mathrm{d}\phi \\ &= \frac{4\pi}{\delta^2}\int_0^{\pi/3} \sin(\phi) \int_0^1 r^2 \omega(r)\big(1-\cos(r\cos(\phi)\delta|\boldsymbol{\xi}|)\big)\mathrm{d}r\mathrm{d}\phi\end{aligned}$$

Now define $a = \delta|\boldsymbol{\xi}|$. For $a < 1$, we use

$$\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

to get

$$\begin{aligned}\lambda_\delta(\boldsymbol{\xi}) \geq\ & \frac{4\pi a^2}{\delta^2}\int_0^{\pi/3} \cos^2(\phi)\sin(\phi)\mathrm{d}\phi \int_0^1 r^4\omega(r)\mathrm{d}r \\ & - \frac{4\pi a^4}{24\delta^2}\int_0^{\pi/3} \cos^4(\phi)\sin(\phi)\mathrm{d}\phi \int_0^1 r^6\omega(r)\mathrm{d}r \\ \geq\ & C\frac{a^2}{\delta^2} = C|\boldsymbol{\xi}|^2,\end{aligned}$$

where C is a constant independent of δ. Now we consider the case $a = \delta|\boldsymbol{\xi}| \geq 1$. Since it is true that for any finite a, $\lambda_\delta(\boldsymbol{\xi})$ is a positive number in the form of $C(a)/\delta^2$, where $C(a)$ depends only on a, then $\lambda_\delta(\boldsymbol{\xi})$ has a lower bound $\tilde{C}/\delta^2$ for a belongs to a finite interval with $\tilde{C}$ being a constant independent of δ.

For the case that $\omega(|\boldsymbol{x}|)$ to be integrable in $\boldsymbol{x}$, we have $r^2\omega(r)$ to be integrable in r. By using the Riemann-Lebesgue Lemma, we can see that $\lambda_\delta(\boldsymbol{\xi})$ goes to C/δ^2 for some constant C as $a \to \infty$. Thus we have shown (32).

As for the case that $\omega(r)$ satisfies (33), we have

$$\begin{aligned}\lambda_\delta(\boldsymbol{\xi}) &\geq \frac{4\pi}{\delta^2}\int_0^{\pi/3}\sin(\phi)\int_0^1 r^2\omega(r)\big(1-\cos(r\cos(\phi)\delta|\boldsymbol{\xi}|)\big)\mathrm{d}r\mathrm{d}\phi \\ &= \frac{4\pi a^{2\alpha}}{\delta^2}\int_0^{\pi/3}\sin(\phi)\cos^{2\alpha}(\phi)\int_0^{\cos(\phi)a}\frac{1}{r^{1+2\alpha}}\big(1-\cos(r)\big)\mathrm{d}r\mathrm{d}\phi \\ &\geq \frac{4\pi a^{2\alpha}}{\delta^2}\int_0^{\pi/3}\sin(\phi)\cos^{2\alpha}(\phi)\mathrm{d}\phi\int_0^{a/2}\frac{1}{r^{1+2\alpha}}\big(1-\cos(r)\big)\mathrm{d}r \\ &= \frac{C}{\delta^{2-2\alpha}}|\boldsymbol{\xi}|^{2\alpha},\end{aligned}$$

for $a \geq 1$. Thus we obtain (34). □

Remark 3 With the conditions in Assumption 5 on the kernel $\hat{\omega}_\delta(r)$, and the more singular behavior imposed in (23), we do get some regularity pickup on the pressure given in (26).

Before ending this section, we present some additional regularity estimates on the nonlocal solutions for smoother data by observing that the nonlocal operators commute with any local differential operators (and their fractional powers, defined via the spectrum decomposition) in the periodic setting. Moreover, we notice that the constants in the estimates given in Theorem 6 and Proposition 9 are independent of δ and $\boldsymbol{f}$, so we can get uniform regularity estimates stated in the following corollary.

Corollary 10 *Assume that the kernels $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ satisfy Assumptions 1 and 5. The solution $(\boldsymbol{u}_\delta, p_\delta)$ to the nonlocal Stokes system (2) with periodic boundary condition satisfies that for any partial (and possibly fractional) differential operators ∂ on the spatial variables of any nonnegative order,*

$$\|\partial \boldsymbol{u}_\delta\|_{[\mathcal{S}_\delta(\Omega)]^d} \leq C\|\partial \boldsymbol{f}\|_{[\mathcal{S}^*_\delta(\Omega)]^d}, \tag{35}$$

$$\|\partial p_\delta\|_{L^2(\Omega)} \leq C\|\partial \boldsymbol{f}\|_{[H^{-\beta}(\Omega)]^d}, \tag{36}$$

where $C > 0$ is a generic constant independent of δ, $\boldsymbol{f}$ and ∂. Moreover, if $\omega(r)$ satisfies (33) for some $\alpha \in (0, 1)$ and constants $m, M \in \mathbb{R}^+$, then

$$\|\partial \boldsymbol{u}_\delta\|_{[H^\alpha(\Omega)]^d} \leq C\|\partial \boldsymbol{f}\|_{[H^{-\alpha}(\Omega)]^d}, \tag{37}$$

$$\|\partial p_\delta\|_{L^2(\Omega)} \leq C\|\partial \boldsymbol{f}\|_{[H^{-\beta}(\Omega)]^d}, \tag{38}$$

for a generic constant $C > 0$ that is independent of δ, $\boldsymbol{f}$ and ∂.

4 The Local Limit

Since the nonlocal operators, as defined here, are constructed to have the corresponding differential operators as the local limits as the horizon (smoothing length) δ shrinks to zero, it is reasonable to expect that the limit of the nonlocal Stokes system (2) recovers the conventional local Stokes system as nonlocal effects vanish. With the energy estimates shown earlier, it is possible to derive rigorously the zero δ limit of (2). Moreover, we can establish the convergence rate of the nonlocal solutions to its local counterpart as $\delta \to 0$ using Fourier analysis.

Theorem 11 *Assume that the kernels $\underline{\omega}_\delta$ and $\hat{\omega}_\delta$ satisfy Assumptions 1 and 5. Let $(\boldsymbol{u}, p)$ be the solution of Stokes system (1) and $(\boldsymbol{u}_\delta, p_\delta)$ be the solution of nonlocal Stokes system (2). with all of them being subject to the condition (3), there is a constant C independent of δ and $\boldsymbol{f}$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{[L^2(\Omega)]^d} \leq C\delta^2\|\boldsymbol{f}\|_{[L^2(\Omega)]^d}. \tag{39}$$

and

$$\|p - p_\delta\|_{L^2(\Omega)} \leq C\delta^{\min\{2,1+\eta\}}\|\boldsymbol{f}\|_{[H^\eta(\Omega)]^d} \quad \textit{for any } \eta \geq -\beta, \tag{40}$$

where $\beta \in (-1, 1)$ is the exponent defined through (23).

Proof Let us work on the case $d = 3$ as illustration. We may again obtain from (20) the estimate of the Fourier coefficients:

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) - \widehat{\boldsymbol{u}}_\delta(\boldsymbol{\xi})| \le \left(\left| \frac{1}{\lambda_\delta(\boldsymbol{\xi})} - \frac{1}{|\boldsymbol{\xi}|^2} \right| + \left| \frac{\boldsymbol{b}_\delta(\boldsymbol{\xi}) \otimes \boldsymbol{b}_\delta(\boldsymbol{\xi})}{\lambda_\delta(\boldsymbol{\xi})|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2} - \frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^4} \right| \right) |\widehat{\boldsymbol{f}}(\boldsymbol{\xi})|$$
$$\le C \left| \frac{1}{\lambda_\delta(\boldsymbol{\xi})} - \frac{1}{|\boldsymbol{\xi}|^2} \right| |\widehat{\boldsymbol{f}}(\boldsymbol{\xi})|$$

and

$$|\widehat{p}(\boldsymbol{\xi}) - \widehat{p}_\delta(\boldsymbol{\xi})| \le C \left| \frac{\boldsymbol{b}_\delta(\boldsymbol{\xi})}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|} - \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right| |\widehat{\boldsymbol{f}}(\boldsymbol{\xi})|.$$

Then (39) is just a consequence of the following estimate of the difference between $1/\lambda_\delta(\boldsymbol{\xi})$ and $1/|\boldsymbol{\xi}|^2$, which we can draw similar arguments from [19, Lemma 1] to obtain:

$$\left| \frac{1}{\lambda_\delta(\boldsymbol{\xi})} - \frac{1}{|\boldsymbol{\xi}|^2} \right| \le C\delta^2,$$

where C is a constant independent of δ and $\boldsymbol{\xi}$.

To get the proof of (40), we notice that for $|\boldsymbol{\xi}| \ge 1$,

$$\left| \frac{\boldsymbol{b}_\delta(\boldsymbol{\xi})}{|\boldsymbol{b}_\delta(\boldsymbol{\xi})|^2} - \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right| = \left| \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \left(\frac{1}{b_\delta(|\boldsymbol{\xi}|)} - \frac{1}{|\boldsymbol{\xi}|} \right) \right| \le \left| \frac{1}{b_\delta(|\boldsymbol{\xi}|)} - \frac{1}{|\boldsymbol{\xi}|} \right|,$$

where $b_\delta(|\boldsymbol{\xi}|)$ is given by (19). Let $a = |\boldsymbol{\xi}|\delta$, then

$$\left| \frac{1}{b_\delta(|\boldsymbol{\xi}|)} - \frac{1}{|\boldsymbol{\xi}|} \right| = \delta \left| \frac{1}{2\pi \int_0^{\pi/2} \int_0^1 r^2 \hat{\omega}(r) \sin(2\phi) \sin(r\cos(\phi)a) \mathrm{d}r \mathrm{d}\phi} - \frac{1}{a} \right|.$$

For $a < 1$, since

$$r\cos(\phi)a - \frac{(r\cos(\phi)a)^3}{3!} \le \sin(r\cos(\phi)a) \le r\cos(\phi)a,$$

we obtain

$$a - \frac{a^3}{3!} \le 2\pi \int_0^{\pi/2} \sin(2\phi) \int_0^1 r^2 \hat{\omega}(r) \sin(r\cos(\phi)a) \mathrm{d}r \mathrm{d}\phi \le a.$$

So

$$\frac{1}{\delta} \left| \frac{1}{b_\delta(|\boldsymbol{\xi}|)} - \frac{1}{|\boldsymbol{\xi}|} \right| \le \frac{1}{a - \frac{a^3}{3!}} - \frac{1}{a} = \frac{a}{6 - a^2} \le a.$$

So we have

$$\left|\frac{1}{b_\delta(|\boldsymbol{\xi}|)}-\frac{1}{|\boldsymbol{\xi}|}\right|\le \delta a=\delta^2|\boldsymbol{\xi}|, \tag{41}$$

which implies that

$$\left|\frac{1}{b_\delta(|\boldsymbol{\xi}|)}-\frac{1}{|\boldsymbol{\xi}|}\right|\frac{1}{|\boldsymbol{\xi}|^\eta}\le C\delta^{\min\{2,1+\eta\}}|\boldsymbol{\xi}|^{\min\{0,1-\eta\}}\le C\delta^{\min\{2,1+\eta\}}, \tag{42}$$

for the case $a=\delta|\boldsymbol{\xi}|<1$. For $a\ge 1$, we proceed the same way as in the proof of Theorem 6 to obtain

$$b_\delta(|\boldsymbol{\xi}|)\ge\begin{cases} C|\boldsymbol{\xi}| & \text{for } \ \delta|\boldsymbol{\xi}|\in[1,4\pi/\epsilon] \\ C\frac{|\boldsymbol{\xi}|^\beta}{\delta^{1-\beta}} & \text{for } \ \delta|\boldsymbol{\xi}|\in(4\pi/\epsilon,\infty). \end{cases}$$

Then we have for $a=\delta|\boldsymbol{\xi}|\in[1,4\pi/\epsilon]$,

$$\left|\frac{1}{b_\delta(|\boldsymbol{\xi}|)}-\frac{1}{|\boldsymbol{\xi}|}\right|\frac{1}{|\boldsymbol{\xi}|^\eta}\le C\frac{1}{|\boldsymbol{\xi}|^{1+\eta}}\le C\delta^{1+\eta}.$$

And for the case $a=\delta|\boldsymbol{\xi}|\ge 4\pi/\epsilon$, we use the assumption that $\eta\ge-\beta$ to obtain

$$\left|\frac{1}{b_\delta(|\boldsymbol{\xi}|)}-\frac{1}{|\boldsymbol{\xi}|}\right|\frac{1}{|\boldsymbol{\xi}|^\eta}\le C\frac{\delta^{1-\beta}}{|\boldsymbol{\xi}|^{\beta+\eta}}+\frac{1}{|\boldsymbol{\xi}|^{1+\eta}}\le C\delta^{1-\beta+\beta+\eta}+\delta^{1+\eta}\le\tilde{C}\delta^{1+\eta}.$$

Combine the above arguments we arrive at (40). □

5 Numerical Discretization

With a well-posed nonlocal Stokes system (2), one may readily consider its numerical discretization. We leave the discussion on its particle approximation and the connection to the incompressible SPH to future works due to the need for more lengthy derivations. Instead, under periodic conditions, it is natural to consider Fourier spectral method for numerical approximation whose convergence can be subject to the similar Fourier analysis.

Let $(\boldsymbol{u}_\delta^N, p_\delta^N)$ stands for the Fourier spectral approximation of $(\boldsymbol{u}_\delta, p_\delta)$. It is easy to see that $(\boldsymbol{u}_\delta^N, p_\delta^N)$ are simply the truncation (projection) of $(\boldsymbol{u}_\delta, p_\delta)$ over all Fourier modes with wave numbers no larger than N. Hence, we get the following convergence result for a fixed $\delta>0$ as $N\to\infty$ for $\boldsymbol{f}\in[L^2(\Omega)]^3$ and also convergence rates for smoother data.

Theorem 12 *Let* $(\boldsymbol{u}_\delta^N(\boldsymbol{x}),\ \ p_\delta^N(\boldsymbol{x}))$ *be Fourier spectral approximation to the* (2). *We assume also that Assumptions 1 and 5 hold true for the kernels. Then for* $\boldsymbol{f}\in[L^2(\Omega)]^d\cap[H^{-\beta}(\Omega)]^d$, *we have as* $N\to\infty$,

$$\|\boldsymbol{u}_\delta^N - \boldsymbol{u}_\delta\|_{[L^2(\Omega)]^d} \to 0 \quad and \quad \|p_\delta^N - p_\delta\|_{L^2(\Omega)} \to 0. \tag{43}$$

Moreover, if $\omega(r)$ satisfies (33) for some $\alpha \in (0, 1)$ and constants $m, M \in \mathbb{R}^+$, then for any $s \geq 0$, with C independent of δ, N, $\boldsymbol{f}$ and s, we have as $N \to \infty$,

$$\|\boldsymbol{u}_\delta^N - \boldsymbol{u}_\delta\|_{[H^{\gamma+\alpha}(\Omega)]^d} \leq \frac{C}{N^s}\|\partial \boldsymbol{f}\|_{[H^{\gamma-\alpha}(\Omega)]^d} \tag{44}$$

and

$$\|p_\delta^N - p_\delta\|_{H^\gamma(\Omega)} \leq \frac{C}{N^s}\|\partial \boldsymbol{f}\|_{[H^{\gamma-\beta}(\Omega)]^d}, \tag{45}$$

where $s > 0$ denote the total order of differentiations of a partial differential operator ∂.

Proof The results follow from standard Fourier analysis and the regularity estimates in Corollary 10. □

Remark 4 By Corollary 7, we can also see that the discrete Fourier spectral approximation automatically leads to a divergence-free vector fields under Assumption 5 and (10) on the kernels, simultaneously in the local and nonlocal sense.

Although nonlocal models may be of interests in their own right, given that they have been used as integral relaxations to the local models, we would like to study the convergence properties of nonlocal discrete solutions to solutions of the corresponding local continuum models. Along this direction, we would like to emphasize the fact that the Fourier spectral approximation for (2) is asymptotically compatible (a notion developed in [49]), in the sense that it is not only a convergent numerical method for the nonlocal problem with fixed δ, but also preserves the compatibility to the asymptotic local limit as δ shrinks to zero. The asymptotic compatibility of Fourier spectral approximation to nonlocal problems has been studied before, see, for example [19] and the references cited therein. Having the asymptotic compatibility provides robustness to the numerical discretization since the numerical solution in various parameter regimes (that involving both the smoothing length and the spatial discretization parameter) is expected to converge to the desired continuum limit with increased numerical resolution. In mathematical terms, one expects the following to be true:

$$\begin{cases} \|\boldsymbol{u}_\delta^N - \boldsymbol{u}\| \to 0, \\ \quad\text{and} \\ \|p_\delta^N - p\| \to 0, \end{cases} \quad \text{as} \quad \begin{cases} \delta \to 0, \\ \quad\text{and} \\ N \to \infty. \end{cases} \tag{46}$$

By the projection property. we have

$$\|\boldsymbol{u}_\delta^N - \boldsymbol{u}^N\| \leq \|\boldsymbol{u}_\delta - \boldsymbol{u}\| \tag{47}$$

and

$$\|p_\delta^N - p^N\| \leq \|p_\delta - p\|. \tag{48}$$

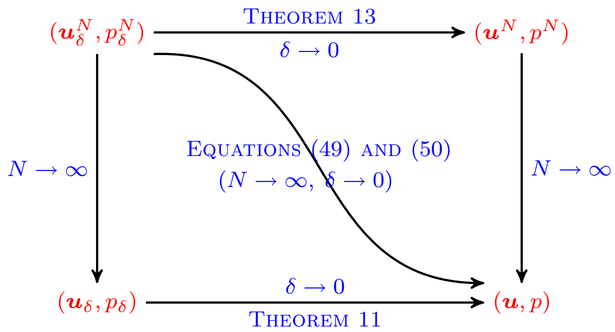

Fig. 2 A diagram for asymptotic compatibility and convergence results

Thus, one way to derive the convergence to the local limit is through the triangle inequalities:

$$\begin{aligned} \|\boldsymbol{u}_\delta^N - \boldsymbol{u}\| &\leq \|\boldsymbol{u}_\delta^N - \boldsymbol{u}^N\| + \|\boldsymbol{u}^N - \boldsymbol{u}\| \\ &\leq \|\boldsymbol{u}_\delta - \boldsymbol{u}\| + \|\boldsymbol{u}^N - \boldsymbol{u}\| \end{aligned} \tag{49}$$

and

$$\begin{aligned} \|p_\delta^N - p\| &\leq \|p_\delta^N - p^N\| + \|p^N - p\| \\ &\leq \|p_\delta - p\| + \|p^N - p\| \end{aligned} \tag{50}$$

where $(\boldsymbol{u}^N, p^N)$ denotes the Fourier approximation for the standard Stokes equation, which converges to $(\boldsymbol{u}, p)$ as $N \to \infty$.

Now to visualize the asymptotic compatibility of the Fourier spectral approximation, we present a diagram in Fig. 2 showing the different paths of convergence, following the work of [49].

Theorem 13 *Suppose $(\boldsymbol{u}_\delta^N(\boldsymbol{x}),\ p_\delta^N(\boldsymbol{x}))$ and $(\boldsymbol{u}^N(\boldsymbol{x}),\ p^N(\boldsymbol{x}))$ are Fourier spectral approximations to the (2) and (1), respectively. We assume also that Assumptions 1 and 5 hold true for the kernels. Then with C independent of δ, N and $\boldsymbol{f}$, we have*

$$\|\boldsymbol{u}_\delta^N - \boldsymbol{u}^N\|_{[L^2(\Omega)]^3} \leq C\delta^2 \|\boldsymbol{f}^N\|_{[L^2(\Omega)]^d}, \tag{51}$$

and

$$\|p_\delta^N - p^N\|_{L^2(\Omega)} \leq C\delta^{\min\{2,1+\eta\}} \|\boldsymbol{f}^N\|_{[H^\eta(\Omega)]^d} \quad \text{for } \eta \geq -\beta, \tag{52}$$

where $\beta \in (-1, 1)$ is the exponent defined through (23).

Proof The result is immediate from (47) and (48) and Theorem 11. □

While the focus of this work is mainly on theoretical analysis, the asymptotic compatibility given in Theorem 13 reveals interesting possibilities to design numerical discretization of the local Stokes equation via nonlocal integral relaxations without imposing the smoothing length δ to be proportional to $h = 1/N$ with N representing the discretization parameter (or h representing the scale of numerical resolution). The latter, with h being the parameter for typical particle spacing, is a common practice in methods like SPH. Relaxing such constraints can potentially lead to more effective and robust approximations especially when simulating complex flow patterns that require more adaptive choices of smoothing and discretization. Finding convergent approximation for particle like approximation to the integral formulations for more general h and δ has been systematically studied and successfully explored computationally [21,44], though the focus there were corrections on the discrete level to assure reproducing conditions, which has been a popular approach developed in mesh-free approximation literature. Further theoretical investigations on the connections of these related ideas will be carried out in future works.

6 Conclusion and Discussions

Recent development of nonlocal continuum models and nonlocal calculus has provided us useful tools to better understand models and numerical methods that may involve nonlocality, either on the physical level or for convenience of numerical computation [14]. A number of studies have been carried out for solid mechanics in the context of peridynamics. This work is an attempt to extend the mathematical study of nonlocal models to fluid mechanics. It is mainly aimed at providing new theoretical insight to popular methods for simulating fluid flows such as SPH and vortex-blob methods. The latter two approaches have the notion of nonlocality and integral relaxations explicitly built in their formulations. In addition to being a computational technique, the introduction of nonlocality can also arise due to physical considerations such as the nonlocal memory effect in viscoelastic fluids and the nonlocal spatial effect in quasi-geostrophic flows. Indeed, the message we want to deliver in this paper is that one should perhaps first study the continuum relaxation of the PDEs more systematically before designing consistent, stable and robust numerical discretization. It is thus a meaningful mathematical exercise to consider a well-posed nonlocal analog of the local continuum equations as one foundation block for understanding and improving the relevant numerical methods. The setting presented here is simplified to the case of nonlocal Stokes equation with periodic boundary conditions so that we can first probe the choices of appropriate nonlocal operators without delving into other technical difficulties. When replacing the local differential operators by their nonlocal counterparts such as those introduced in previous works [16,38], we see that one should be particularly careful about the interaction kernels used in nonlocal gradient and divergence operators, in the sense that the interaction should be sufficiently strengthened to make the system stay solvable and stable. The condition imposed, as elucidated in the work, is a natural one that makes the spaces of local and nonlocal divergence-free vector fields equivalent. The resulting models with proper nonlocal interaction kernels are not only well-posed for any finite horizon and smoothing length, but their solutions also converge to the solution of classical Stokes equation in the local limit, a fact established in this work along with precise convergence rates.

The nonlocal Stokes equation, as a continuum model, serves as a bridge between the local Stokes equation and its discretizations like SPH. In building such a linkage, the notion of asymptotically compatible schemes shown in Fig. 2 can become important for practical applications due to the implied robustness of the underlying numerical methods so that one does not necessarily need the discretization to be refined faster than the reduction of smoothing length. In particular, the Fourier spectral method is shown to enjoy asymptotic compatibility. It is certainly more interesting to look into other numerical methods, in particular, particle discretizations like SPH, which is a main objective of our ongoing series of works. Although no extended investigations are made here on either mesh or particle-based discretization, some preliminary speculations can be offered. For example, while we leave more detailed studies to subsequent works, it is no surprising that the well-posedness of the continuum nonlocal Stokes equation does not guarantee that simple minded discretization is automatically stable. As a comparison, it is well known that to solve the conventional local Stokes equation based on standard centered finite differences on a Cartesian mesh, check-board-type instabilities may arise when the unknown velocity and pressure are placed at the same set of mesh points. Instead, the so-called MAC (Marker and Cell)-type schemes are developed to place the unknown variables at staggered locations. Such issues can be studied in the nonlocal setting, very much in the same spirit of the work presented here. For an brief illustration, consider the one-dimensional nonlocal gradient operator given by

$$\mathcal{G}_\delta p(x) = \int_0^\delta \hat{\omega}_\delta(s)(p(x+s) - p(x-s))\mathrm{d}s,$$

we have naturally two types of finite difference approximation, one with a regular uniform grid and another with a staggered grid that are given in the following, respectively:

$$\text{Regular:} \quad \mathcal{G}_\delta^h p(x_j) = \sum_{k=0}^{r} d_k \left(p(x_{j+k}) - p(x_{j-k})\right)$$

$$\text{Staggered:} \quad \mathcal{G}_\delta^h p(x_j) = \sum_{k=0}^{r} d_k \left(p(x_{j+k+\frac{1}{2}}) - p(x_{j-k-\frac{1}{2}})\right)$$

for some nonnegative coefficients $\{d_k\}$. Again by Fourier analysis, we may find that the eigenvalues of the two discrete operators are:

$$ib_{\delta,h}(n) = \begin{cases} i2\sum_{k=0}^{r} d_k \sin(nkh) & \text{for regular grid} \\ i2\sum_{k=0}^{r} d_k \sin(nkh + nh/2) & \text{for staggered grid} \end{cases}$$

where $h = 2\pi/N$ and $n = -N+1, \dots, N$. One can observe that for the regular grid the discrete eigenvalue is zero if $n = N$ while it is generically not the case for the staggered case. The story of numerical stability is then different for the two discrete operators, see [31] for additional discussions on multidimensional cases.

Finally, there are various possible extensions of the work here. Our nonlocal formulation here is based on a centered nonlocal relaxation to local differential operators. For example, for $\mathcal{G}_\delta$ in (4) , it is determined by a vector $\boldsymbol{\omega}_\delta(\boldsymbol{y}-\boldsymbol{x})$ which can be viewed as an odd and rank-one tensor. A more general form of such a nonlocal gradient operator acting on a vector field $\boldsymbol{u}$, by the Schwartz kernel theorem, can be written as a second-order tensor [38] given by

$$\mathcal{G}_\delta \boldsymbol{u}(\boldsymbol{x}) := \int \rho_\delta(|\boldsymbol{y}-\boldsymbol{x}|)\, \mathsf{M}_\delta(\boldsymbol{y}-\boldsymbol{x}) \frac{\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})}{|\boldsymbol{y}-\boldsymbol{x}|}\, \mathrm{d}\boldsymbol{y},$$

where M_δ is a third-order odd tensor and the integral is interpreted in the principal sense. There are also other forms of nonlocal operators such as those based on non-symmetric or one-sided interaction kernels. For example, the one-dimensional forward/backward nonlocal differentiation operators

$$\mathcal{G}_\delta^+ p(x) = \int_0^\delta \rho_\delta(s)(p(x+s) - p(x)) ds,$$
$$\mathcal{G}_\delta^- p(x) = \int_0^\delta \rho_\delta(s)(p(x) - p(x-s)) \mathrm{d}s,$$

have been studied in [20] that can lead to invertible operators for a wider class of nonlocal interaction kernels $\rho_\delta = \rho_\delta(s)$ with orientation bias [31]. Moreover, one may consider formulations involving stabilization terms by introducing some nonlocal analog of artificial compressibility similar to ideas used for conventional local Stokes models [51]. Studying the nonlocal analog of time-dependent nonlinear Navier–Stokes system is surely another step to take [31]. Extensions can also be considered for compressible flows, interfacial and multiphase flows, magneto-hydrodynamics (MHD) and stochastically driven flows. The extension in the case of a scalar hyperbolic conservation laws can be found in [30]. For SPH, much effort has also been devoted to the accurate treatment of boundary conditions, even though particle like mesh-free methods are thought to be able to handle complex geometry and boundary conditions more effectively. This is perhaps not surprising, given the intrinsic nonlocal continuum formulations explicitly formulated here that demand generically the notion of nonlocal boundary conditions or volumetric constraints (see [15,38]). It leads to another important topic of future research. Naturally, once the model and discretization are in place, there are still many practical issues ranging from quadratures to linear and nonlinear solvers that must also be investigated along with more careful theoretical analysis.

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