

# **Toward Effective Detection of the Bifurcation Locus of Real Polynomial Maps**

Luis Renato G. Dias<sup>1</sup> · Susumu Tanabé<sup>2</sup> · Mihai Tibăr<sup>3</sup>

Received: 18 March 2015 / Revised: 15 September 2015 / Accepted: 28 December 2015 /

Published online: 25 January 2016

© SFoCM 2016

**Abstract** We answer to a problem raised by recent work of Jelonek and Kurdyka: how can one detect by rational arcs the bifurcation locus of a polynomial map  $\mathbb{R}^n \to \mathbb{R}^p$  in case p > 1. We describe an effective estimation of the "non-trivial" part of the bifurcation locus.

**Keywords** Bifurcation locus · Real polynomial maps · Regularity at infinity · Detection

Mathematics Subject Classification  $14D06 \cdot 14Q20 \cdot 58K05 \cdot 57R45 \cdot 14P10 \cdot 32S20 \cdot 58K15$ 

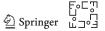
Communicated by Teresa Krick.

Mihai Tibăr tibar@math.univ-lille1.fr

Luis Renato G. Dias lrgdias@famat.ufu.br

Susumu Tanabé tanabe@gsu.edu.tr

- Faculdade de Matemática, Universidade Federal de Uberlândia, Av. João Naves de Ávila 2121, 1F-153, Uberlândia CEP: 38408-100, Brazil
- Département de Mathématiques, Université de Galatasaray, 34357 Istanbul, Turkey
- Mathématiques UMR 8524 Laboratoire Paul Painlevé, Univ. Lille, 59000 Lille, France



#### 1 Introduction

The *bifurcation locus* of a polynomial map  $f: \mathbb{R}^n \to \mathbb{R}^p$ ,  $n \geq p$ , is the smallest subset  $B(f) \subset \mathbb{R}^p$  such that f is a locally trivial  $C^{\infty}$ -fibration over  $\mathbb{R}^p \backslash B(f)$ . It is well known that B(f) is the union of the set of critical values  $f(\operatorname{Sing} f)$  and the set of *bifurcation values at infinity*  $\mathcal{B}_{\infty}(f)$  (see Definition 2.1) which may be non-empty and disjoint from  $f(\operatorname{Sing} f)$  even in very simple examples. Finding the bifurcation locus in the cases p > 1 or p = 1 and n > 2 is yet an unreached ideal. Nevertheless, one can obtain approximations by supersets of  $\mathcal{B}_{\infty}(f)$  from exploiting asymptotical regularity conditions [2,6,9,10,13,15,16,18,19,21,23,24] etc.

Improving the effectivity of the detection of asymptotically non-regular values becomes an important issue; for instance, it leads to applications in optimization problems [11,22]. Along this trend, Jelonek and Kurdyka [14] produced recently an algorithm for finding the set of asymptotically critical values  $\mathcal{K}_{\infty}(f)$  in case p=1. It is known that in this case,  $\mathcal{K}_{\infty}(f)$  is finite and includes  $\mathcal{B}_{\infty}(f)$ . A sharper estimation of  $\mathcal{B}_{\infty}(f)$  has been found in the real setting [7] by approximating the set of asymptotic  $\rho_a$ -non-regular values of f. The later method provides a finite set of values A(f) with the following property:  $\mathcal{B}_{\infty}(f) \subset A(f) \subset \mathcal{K}_{\infty}(f)$ .

In case p > 1, the bifurcation locus  $\mathcal{B}_{\infty}(f)$  may be no more finite. Actually, by the Morse–Sard result proved by Kurdyka et al. [16] for  $\mathcal{K}_{\infty}(f)$ , or by the one obtained in [6] for the sharper estimation  $\mathcal{B}_{\infty}(f) \subset \mathcal{S}_{0}(f) \subset \mathcal{K}_{\infty}(f)$ , one only knows that the sets  $\mathcal{K}_{\infty}(f)$  and  $\mathcal{S}_{0}(f)$  are contained in a 1-codimensional semi-algebraic subsets of  $\mathbb{R}^{p}$ .

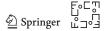
Our approach is based on the set  $\mathcal{S}_{\infty}(f)$  of non-regular values at infinity with respect to the Euclidean distance function from any point as origin, and which includes  $\mathcal{B}_{\infty}(f)$ . Since the set of critical values  $f(\operatorname{Sing} f)$  is the image of an algebraic set and the well-known estimation methods apply, we consider it as the "trivial" part of the job. The most difficult task is to apprehend the complements of  $f(\operatorname{Sing} f)$  to the bifurcation locus  $\mathcal{B}_{\infty}(f)$ .

We shall detect here the "non-trivial" part  $NS_{\infty}(f)$  of the bifurcation locus at infinity (defined at Sect. 2.6) which, roughly speaking, contains the values of  $S_{\infty}(f)$  which are not coming from the branches at infinity of the singular locus Sing f.

This note answers a question raised by the results [14] and [7], as of how can one detect the bifurcation locus by rational arcs in the case p > 1.

More precisely, given a polynomial map  $f = (f_1, \ldots, f_p) \colon \mathbb{R}^n \to \mathbb{R}^p$ , deg  $f_i \leq d$ , we find all the values of the "non-trivial" part  $N\mathcal{S}_{\infty}(f)$  of  $\mathcal{S}_{\infty}(f)$  and hence of non-trivial part  $N\mathcal{B}_{\infty}(f)$  of the bifurcation locus  $\mathcal{B}_{\infty}(f)$ , as follows:

- (1) We consider a set of rational paths:  $(x(t), y(t)) = \left(\sum_{-ds \le i \le s} a_i t^i, \sum_{-ds \le j \le 0} b_j t^j\right) \subset \mathbb{R}^n \times \mathbb{R}^p$ , where  $s = [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}$ . This means a finite number of vectorial coefficients  $a_i \in \mathbb{R}^n$ , for  $-ds \le i \le s$ , and  $b_j \in \mathbb{R}^p$ , for  $-ds \le j \le 0$ .
- (2) The coefficients are subject to several conditions, namely:  $||b_0|| = 1$ ,  $\exists k > 0$ ,  $a_k \neq 0 \in \mathbb{R}^n$ , we ask the annulation of the coefficients of the terms with positive exponents in the expansion of f(x(t)) and the annulation of the coefficients of



the terms with nonnegative exponents in the expressions  $x_i(t)\phi_j(x(t), y(t))$ , for all  $i, j \in \{1, ..., n\}$  (cf (13) for the definition).

We denote by  $\operatorname{Arc}_{\infty}(f)$  the algebraic subset of arcs obtained by this construction [steps (1) and (2) above], and by  $\alpha_0(\operatorname{Arc}_{\infty}(f))$  the set of limits  $\lim_{t\to\infty} f(x(t))$ , i.e., the free coefficient in the expansion of f(x(t)) for f(x(t)) fo

$$NS_{\infty}(f) \subset \alpha_0(Arc_{\infty}(f)) \subset \mathcal{K}_{\infty}(f).$$

#### 2 Regularity Conditions at Infinity and Bifurcation Loci

#### 2.1 Bifurcation Locus

Let  $f = (f_1, ..., f_p) : \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial map,  $n \ge p$ .

**Definition 2.1** We say that  $t_0 \in \mathbb{R}^p$  is a *typical value* of f if there exists a disk  $D \subset \mathbb{R}^p$  centered at  $t_0$  such that the restriction  $f_|: f^{-1}(D) \to D$  is a locally trivial  $C^{\infty}$ -fibration. Otherwise, we say that  $t_0$  is a *bifurcation value* (or atypical value). We denote by B(f) the set of bifurcation values of f.

We say that f is  $C^{\infty}$ -trivial at infinity at  $t_0 \in \mathbb{R}^p$  if there exists a compact set  $\mathcal{K} \subset \mathbb{R}^n$  and a disk  $D \subset \mathbb{R}^p$  centered at  $t_0$  such that the restriction  $f_{|}: f^{-1}(D) \setminus \mathcal{K} \to D$  is a locally trivial  $C^{\infty}$ -fibration. Otherwise, we say that  $t_0$  is a bifurcation value at infinity of f. We denote by  $\mathcal{B}_{\infty}(f)$  the bifurcation locus at infinity of f.

#### 2.2 The Rho-Regularity

Let  $a=(a_1,\ldots,a_n)\in\mathbb{R}^n$  and let  $\rho_a\colon\mathbb{R}^n\to\mathbb{R}_{\geq 0},$   $\rho_a(x)=(x_1-a_1)^2+\ldots+(x_n-a_n)^2$ , be the Euclidian distance function to a. Let  $f:\mathbb{R}^n\to\mathbb{R}^p$  be a polynomial map, where  $n\geq p$ .

**Definition 2.2** (Milnor set at infinity and the  $\rho_a$ -non-regularity locus) [7] The critical set  $\mathcal{M}_a(f)$  of the map  $(f, \rho_a) \colon \mathbb{R}^n \to \mathbb{R}^{p+1}$  is called the Milnor set of f (with respect to the distance function). The following semi-algebraic set, cf [6, Theorem 5.7] and [7, Theorem 2.5]:

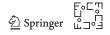
$$S_a(f) := \left\{ t_0 \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset \mathcal{M}_a(f), \lim_{j \to \infty} \|x_j\| = \infty \text{ and } \lim_{j \to \infty} f(x_j) = t_0 \right\}$$

will be called the set of asymptotic  $\rho_a$ -non-regular values. If  $t_0 \notin S_a(f)$ , we say that  $t_0$  is  $\rho_a$ -regular at infinity. Let  $S_{\infty}(f) := \bigcap_{a \in \mathbb{R}^n} S_a(f)$ .

**Lemma 2.3**  $S_{\infty}(f)$  is a semi-algebraic set.

*Proof* Let  $f: \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial mapping and let us consider the following semi-algebraic set:

$$\mathcal{W} := \left\{ (x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \mathcal{M}_a(f) \right\}.$$



By the definition of  $S_{\infty}(f)$ , we have:

$$S_{\infty}(f) := \left\{ y \in \mathbb{R}^p \mid \forall a \in \mathbb{R}^n, \exists \{(x_k, a)\} \subset \mathcal{W} \text{ such that } f(x_k) \to y \right\},\,$$

which tells that  $S_{\infty}(f)$  can be written by using first-order formulas. This means that  $S_{\infty}(f)$  is a semi-algebraic set, see, for instance, [4, pp. 28–29] and [1, Prop. 2.2.4].  $\square$ 

It has been proved in [6,7,24] that one has the inclusion  $\mathcal{B}_{\infty}(f) \subset \mathcal{S}_a(f)$ , for any  $a \in \mathbb{R}^n$ , thus in particular:

$$\mathcal{B}_{\infty}(f) \subset \mathcal{S}_{\infty}(f). \tag{2}$$

It was believed, cf [7, Conjecture 2.11], that (2) was an equality. We show here by an example that this is not the case, at least in the real setting.

### 2.3 Example for $B_{\infty}(f) \neq S_{\infty}(f)$

We consider the two-variable real polynomial<sup>1</sup> constructed in [25],  $f: \mathbb{R}^2 \to \mathbb{R}$ ,  $f(x, y) = y(2x^2y^2 - 9xy + 12)$ . We show that  $S_{\infty}(f) = \{0\}$  and  $B_{\infty}(f) = \emptyset$ .

It was already proved in [25] that f has no singular value, no bifurcation value and that  $S_0(f) \subset \{0\}$ . We shall prove here that this inclusion is an equality. Moreover, we prove here that  $\{0\} \subset S_a(f)$  for any center  $a \in \mathbb{R}^2$ .

For any fixed  $a = (a_1, a_2) \in \mathbb{R}^2$ , we have:

$$\mathcal{M}_a(f) = \left\{ (x, y) \in \mathbb{R}^2 \mid y^2 (4xy - 9)(y - a_2) = 6(x - a_1)(xy - 1)(xy - 2). \right\}$$

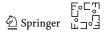
For x = 0, we eventually get solutions of the above equation but which have no influence on the set  $S_a(f)$ . By removing these solutions from  $\mathcal{M}_a(f)$ , we pursue with the resulting set, which we denote by  $\mathcal{M}'_a(f)$ . Thus, assuming that  $x \neq 0$  and multiply the equation by  $x^3$ , we obtain:

$$\mathcal{M}'_{a}(f) = \left\{ (x, y) \in \mathbb{R}^{2} \mid x^{2}y^{2}(4xy - 9)(xy - xa_{2}) = 6x^{3}(x - a_{1})(xy - 1)(xy - 2) \right\}. \tag{3}$$

We show that we can find solutions  $(x_k, y_k)_{k \in \mathbb{N}}$  of the equality in (3) such that  $\|(x_k, y_k)\| \to \infty$  and  $f(x_k, y_k) \to 0$ . Indeed, setting z := xy our equation (3) becomes  $z^2(4z - 9)(z - a_2x) = 6x^3(x - a_1)(z - 1)(z - 2)$ . We then consider each side as a curve of variable z with x as parameter. We consider the graphs of these two curves and observe that for each sign of  $a_2$ , the two graphs intersect at least once for any fixed and large enough |x| and that this happens at some value of z in the interval [0, 1[ (and in the interval [1, 2[ in case  $a_2 = 0$ , respectively). This shows that we can find solutions  $(x_k, y_k) \in \mathcal{M}_a(f)$  with modulus tending to infinity, and since  $z_k = x_k y_k$  is bounded and  $y_k$  tends to 0, we get that  $f(x_k, y_k) \to 0$ .

In conclusion, we have shown that  $S_{\infty}(f) = \{0\}$ , which implies  $B_{\infty}(f) \neq S_{\infty}(f)$ .

<sup>&</sup>lt;sup>1</sup> We thank Y. Chen for suggesting us to test this example.



#### 2.4 Generic Dimension of the Non-singular Part of the Milnor Set

The following statement has been noticed in case p=1 in [10] (see also [8, Lemma 2.2] or [7]). We outline the proof in case p>1, some details of which will be used in Sect. 3.

**Lemma 2.4** Let  $f = (f_1, ..., f_p) \colon \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial map, where n > p and deg  $f_i \leq d$ ,  $\forall i$ . There exists an open dense subset  $\Omega_f \subset \mathbb{R}^n$  such that, for every  $a \in \Omega_f$ , the set  $\mathcal{M}_a(f) \setminus \mathrm{Sing} f$  is either a smooth manifold of dimension p, or it is empty.

*Proof* We denote by  $M_I[D(f)(x)]$  (respectively,  $M_I[D(f, \rho_a)(x)]$ ) the minor of the Jacobian matrix D(f)(x) (respectively  $D(f, \rho_a)(x)$ ) indexed by the multi-index I. We set

$$Z := \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \mathcal{M}_a(f) \backslash \operatorname{Sing} f \}.$$
 (4)

If  $Z = \emptyset$ , then  $\mathcal{M}_a(f) \backslash \operatorname{Sing} f = \emptyset$ ,  $\forall a \in \mathbb{R}^n$ . From now on, let us consider the case that  $Z \neq \emptyset$ . Let  $(x_0, a_0) \in Z$ . Since  $\operatorname{Sing} f$  is closed, there is a neighborhood  $U \subset \mathbb{R}^n$  of  $x_0$  such that  $U \cap \operatorname{Sing} f = \emptyset$ . This means that there exists a multi-index  $I = (i_1, \ldots, i_p)$  of size  $p, 1 \leq i_1 < \ldots < i_p \leq n$ , such that  $M_I[\operatorname{D} f(x)] \neq 0$ ,  $\forall x \in U$ .

Let  $S_I := \{J = (j_1, \ldots, j_{p+1}) \mid I \subset J\}$  be the set of multi-indices of size p+1 such that  $1 \leq j_1 < \ldots < j_{p+1} \leq n$  and  $i_1, \ldots, i_p \in \{j_1, \ldots, j_{p+1}\}$ . There are (n-p) multi-indices  $J \in S_I$ ; we set

$$m_J(x,a) := M_J[D(f,\rho_a)(x)], (x,a) \in U \times \mathbb{R}^n.$$
 (5)

From the definitions of Z, U and the functions  $m_J$ , we have:

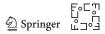
$$Z \cap (U \times \mathbb{R}^n) = \{(x, a) \in U \times \mathbb{R}^n \mid m_J(x, a) = 0; \forall J \in S_I \}.$$
 (6)

Let  $\varphi: U \times \mathbb{R}^n \to \mathbb{R}^{n-p}$  be the map consisting of the functions  $m_J$  for  $J \in S_I$ . Then,  $\varphi^{-1}(0) = Z \cap (U \times \mathbb{R}^n)$  and we notice that  $D\varphi(x, a)$  has rank (n - p) at any  $(x, a) \in U \times \mathbb{R}^n$ . Indeed, let

$$\left(\frac{\partial \varphi}{\partial a_k}(x,a)\right)_{(n-p)\times (n-p)},\ k\notin I, (x,a)\in U\times \mathbb{R}^n.$$

This is a minor of  $D\varphi(x, a)$  of size (n - p). Interchanging if necessary the order of its lines, it is a diagonal matrix with all the entries on the diagonal equal to  $-2M_I[Df(x)]$  and hence nonzero. This and (6) show that Z is a manifold of dimension n + p.

We next consider the projection  $\tau: Z \to \mathbb{R}^n$ ,  $\tau(x, a) = a$ . Thus,  $\tau^{-1}(a) = (\mathcal{M}_a(f) \backslash \operatorname{Sing} f) \times \{a\}$ . By Sard's Theorem, we conclude that, for almost all  $a \in \mathbb{R}^n$ ,  $\tau^{-1}(a) = (\mathcal{M}_a(f) \backslash \operatorname{Sing} f) \times \{a\} \cong (\mathcal{M}_a(f) \backslash \operatorname{Sing} f)$  is either a smooth manifold of dimension p or an empty set.



#### 2.5 The Relation to the Malgrange-Rabier Condition

**Definition 2.5** ([21]) Let  $f: \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial map,  $n \ge p$ . Denote by Df(x) the Jacobian matrix of f at x. We consider

$$\mathcal{K}_{\infty}(f) := \left\{ t \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n, \lim_{j \to \infty} \|x_j\| = \infty, \\ \lim_{j \to \infty} f(x_j) = t \text{ and } \lim_{j \to \infty} \|x_j\| \nu(\mathbf{D}f(x_j)) = 0 \right\},$$
 (7)

where

$$\nu(A) := \inf_{\|y\|=1} \|A^*(y)\|,\tag{8}$$

for a linear map A and its adjoint  $A^*$ .

We call the set  $\mathcal{K}_{\infty}(f)$  of asymptotic critical values of f. If  $t_0 \notin \mathcal{K}_{\infty}(f)$ , we say that f verifies the Malgrange–Rabier condition at  $t_0$ .

We have the following relation between  $\rho_a$ -regularity and Malgrange–Rabier condition:

**Theorem 2.6** ([7, Th. 2.8]) Let  $f = (f_1, ..., f_p) : \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial map, where n > p. Let  $\phi : ]0, \varepsilon[ \to \mathcal{M}_a(f) \subset \mathbb{R}^n$  be an analytic path such that  $\lim_{t\to 0} \|\phi(t)\| = \infty$  and  $\lim_{t\to 0} f(\phi(t)) = c$ . Then,  $\lim_{t\to 0} \|\phi(t)\| v(\mathrm{D}f(\phi(t))) = 0$ . In particular,  $\mathcal{S}_a(f) \subset \mathcal{K}_{\infty}(f)$  for any  $a \in \mathbb{R}^n$ , and  $\mathcal{S}_{\infty}(f) \subset \mathcal{K}_{\infty}(f)$ .

Remark 2.7 See [6] and more precisely [7, Theorem 2.5] for a structure result and a fibration result on  $\mathcal{S}_{\infty}(f)$ . The inclusion  $\mathcal{S}_{\infty}(f) \subset \mathcal{K}_{\infty}(f)$  may be strict (e.g., [20] and [7, Example 2.9]). The inclusion  $B_{\infty}(f) \subset \mathcal{S}_{\infty}(f)$  may be strict, see the above Example Sect. 2.3. One may also have  $\mathcal{S}_a(f) \neq \mathcal{S}_b(f)$  for some  $a \neq b$ , see [7, Example 2.10].

#### 2.6 The Non-trivial Bifurcation Locus at Infinity

We have discussed up to now three types of bifurcation loci:  $\mathcal{B}_{\infty}(f)$ ,  $\mathcal{S}_{\infty}(f)$  and  $\mathcal{K}_{\infty}(f)$ . All of them may contain points of the critical locus  $f(\operatorname{Sing} f)$ . This locus can be estimated separately since it is the image by f of an algebraic set and the known estimation methods apply. What is more difficult to apprehend are the respective complements of  $f(\operatorname{Sing} f)$ . We define here the "non-trivial parts" of the bifurcation loci and next describe a procedure to estimate the one of  $\mathcal{S}_{\infty}(f)$ .

From the definitions of  $\mathcal{M}_a(f)$  and  $\mathcal{S}_a(f)$ , we have the equality  $\mathcal{S}_a(f) = J(f_{|\mathcal{M}_a(f)})$ , where  $J(f_{|\mathcal{M}_a(f)})$  is the *non-properness set* of  $f_{|\mathcal{M}_a(f)}$ . Jelonek defined this set in general:

**Definition 2.8** ([12, Definition 3.3], [14]). Let  $g: M \to N$  be a continuous map, where M, N are topological spaces. One says that g is proper at the value  $t \in N$  if there exists an open neighborhood  $U \subset N$  of t such that the restriction  $g_{|p^{-1}(U)|}$ :



 $g^{-1}(U) \to U$  is a proper map. We denote by J(g) the set of points at which g is not proper.

In our setting  $f: \mathbb{R}^n \to \mathbb{R}^p$ , let us define the *non-trivial*  $\rho$ -bifurcation set at infinity  $NS_{\infty}(f) := \bigcap_{g \in \mathbb{R}^n} NS_a(f)$ , where:

$$NS_a(f)$$

$$:= \left\{ t \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset \mathcal{M}_a(f) \backslash \operatorname{Sing} f, \lim_{j \to \infty} \|x_j\| = \infty, \text{ and } \lim_{j \to \infty} f(x_j) = t \right\}$$

and note that  $S_{\infty}(f) = NS_{\infty}(f) \cup J(f|Sing_f)$  and that  $NS_{\infty}(f)$  is a *closed set* since each set  $NS_a(f)$  is closed, which fact follows from the arguments of [6, Theorem 5.7(a)].

Similarly, we introduce the following notation for the *non-trivial bifurcation set at infinity* which is the object of our main result, Theorem 3.5:

$$N\mathcal{B}_{\infty}(f) := \mathcal{B}_{\infty}(f) \setminus J(f|_{\operatorname{Sing} f}). \tag{9}$$

By the above definitions and by Theorem 2.6, we immediately get:

#### **Proposition 2.9**

$$N\mathcal{B}_{\infty}(f) \subset N\mathcal{S}_{\infty}(f) \subset \mathcal{K}_{\infty}(f)$$
.

Remark 2.10 If f has a compact singular set Sing f or, more generally, if  $J(f_{|Singf}) = \emptyset$ , then  $NS_{\infty}(f) = S_{\infty}(f)$ , and  $NB_{\infty}(f) = B_{\infty}(f)$ . However, these equalities may fail whenever  $J(f_{|Singf}) \neq \emptyset$ .

In this matter, let us point out here that the proofs of [7, Proposition 3.1, Theorem 3.4] run actually for the set  $NS_{\infty}(f)$ ; therefore, in the statements of those results, one has to read  $NS_{\infty}(f)$  instead of  $S_{\infty}(f)$ .

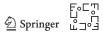
#### 3 Detection of Bifurcation Values at Infinity by Parametrized Curves

#### 3.1 Effective Curve Selection Lemma at Infinity Via the Milnor Set

If  $t_0 \in NS_{\infty}(f)$ , then  $t_0 \in NS_a(f)$  for any  $a \in \mathbb{R}^n$  and in particular for  $a \in \Omega_f$ , where  $\Omega_f$  is as in Lemma 2.4.

**Theorem 3.1** Let  $f = (f_1, ..., f_p) : \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial mapping such that  $\deg f_i \leq d, \forall i = 1, ..., p$ , and n > p. Let  $t_0 \in NS_a(f)$  for some  $a \in \Omega_f$ . Then, there exists an analytic path:

$$x(t) = \sum_{-\infty \le i \le s} a_i t^i, \tag{10}$$



with

$$s \le [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}$$

and such that:

- (a)  $x(t) \in \mathcal{M}_a(f) \setminus \text{Sing } f$ , for any  $t \geq R$ , for some large enough  $R \in \mathbb{R}_+$ ;
- (b)  $||x(t)|| \to \infty$ , as  $t \to \infty$ ;
- (c)  $f(x(t)) \to t_0$ , as  $t \to \infty$ .

*Proof* The case p = 1 is [7, Theorem 3.4]. We assume in the following that p > 1.

From Lemma 2.4, we have that  $\mathcal{M}_a(f) \setminus \operatorname{Sing} f$  is a smooth semi-algebraic set of dimension p since non-empty by our hypothesis on  $t_0$ . Moreover, by the proof of the same lemma,  $\mathcal{M}_a(f) \setminus \operatorname{Sing} f$  is locally a complete intersection defined by (n-p) equations, each of which is of degree at most p(d-1)+1. So let us denote by  $g_1, \ldots, g_{n-p}$  these functions.

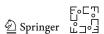
Let  $\mathbb{X} = \overline{\text{graph }f}$  be the closure of the graph of f in  $\mathbb{P}^n \times \mathbb{R}^p$  and let  $\mathbb{X}^\infty$  the intersection of  $\mathbb{X}$  with the hyperplane at infinity  $\{x_0 = 0\}$ . Let  $i : \mathbb{R}^n \to \mathbb{X} \subset \mathbb{P}^n \times \mathbb{R}^p$ ,  $x \mapsto (x, f(x))$  be the graph embedding. We need to work in the following with the closure in  $\mathbb{X}$  of the image  $i(\mathcal{M}_a(f)\backslash \mathrm{Sing} f)$ , which we denote (abusively) by  $\overline{\mathcal{M}_a(f)}\backslash \mathrm{Sing} f$ , in particular since we have to keep track of condition (c), namely  $f(x(t)) \to t_0$ .

Let therefore  $w:=(\underline{x},t_0)\in\overline{\mathcal{M}_a(f)}\backslash \overline{\mathrm{Sing}f}\cap\mathbb{X}^\infty$ . Let  $U\times\mathbb{R}^p$  be a chart at  $(\underline{x},t_0)\in\mathbb{P}^n\times\mathbb{R}^p$ , where  $U\simeq\mathbb{R}^n$  is an affine chart at infinity of  $\mathbb{P}^n$  and assume (without loss of generality) that the point  $\underline{x}$  is the origin. We may then use an "effective curve selection lemma" to show that there is a curve  $\Gamma\subset\mathcal{M}_a(f)\backslash \overline{\mathrm{Sing}f}$  such that  $w\in\overline{\Gamma}\subset\overline{\mathcal{M}_a(f)}\backslash \overline{\mathrm{Sing}f}$  and that this curve has a one-sided bounded parametrization. To do so, we combine Milnor's basic construction in [17] with the idea of Jelonek and Kurdyka given in [14, Lemma 6.4].

Namely, we consider small enough spheres centered at  $\underline{x} \in U$  of equation  $\rho_w = \beta$  and a function  $h_l := x_0 l$ , for some linear function l in the coordinates of U. One can then prove like in [14, Lemma 6.4] (where an apparently more particular situation was considered, but the proof works as well) that, for a general such linear function l, the set of critical points of the map  $(\rho_w, h_l) : U \times \mathbb{R}^p \cap \overline{\mathcal{M}_a(f)} \setminus \overline{\operatorname{Sing} f} \to \mathbb{R}_+ \times \mathbb{R}$  is an analytic curve and its branches at  $w = (\underline{x}, t_0)$  are the families of singular points of the restrictions of the quadratic function  $h_l$  to the levels  $\{\rho_w = \beta\} \cap \mathcal{M}_a(f) \setminus \overline{\operatorname{Sing} f}$ . It is shown in [14, Lemmas 6.5 and 6.6] that these singular points are all Morse for a generic choice of l, and that there is at least one Morse point on each level, for small enough  $\beta > 0$ .

Let us then consider a branch of this analytic curve as our x(t). By its definition, this curve verifies conditions (a), (b) and (c) and is a solution of the following system of equations:  $g_1 = 0, \ldots, g_{n-p} = 0$  and  $dg_1 \wedge \cdots \wedge dg_{n-p} \wedge d\rho_w \wedge dh_l = 0$ , the first of which are of degree at most p(d-1)+1 and the last one means the annulation of p-1 minors of degree at most p(d-1)(n-p)+2. Thus, our algebraic set of solutions has degree  $\delta$  verifying the inequality:

$$\delta \le [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}.$$



Finally, by using the effective Curve Selection Lemma of Jelonek and Kurdyka [14, Lemma 3.1 and Lemma 3.2] which says that there exists a parametrization of our curve x(t) bounded by the degree  $\delta$  of the curve, we get exactly an expansion like (10). This finishes the proof of our theorem.

## 3.2 Finite Length Expansion for Curves Detecting Asymptotically Critical Values

We need a preliminary result which follows by applying [14, Lemma 3.3] to each function  $h_i$  in the following statement:

**Lemma 3.2** Let  $h = (h_1, \ldots, h_m) \colon \mathbb{R}^k \to \mathbb{R}^m$  be a polynomial map and  $\deg h_i \le \tilde{d}$ ,  $\forall i. Let \ x(t) = \sum_{-\infty \le i \le s} a_i t^i$ , where  $t \in \mathbb{R}$ ,  $a_i \in \mathbb{R}^k$ , s > 0 and that  $\|x(t)\| \to \infty$  and  $h(x(t)) \to b$ . Then, for any  $D \le -\tilde{d}s + s$ , the truncated curve

$$\tilde{x}(t) = \sum_{D \le i \le s} a_i t^i,$$

*verifies*  $\|\tilde{x}(t)\| \to \infty$  *and*  $h(\tilde{x}(t)) \to b$ .

If we try to replace x(t) given in (10) by a truncated path, we may go out of the set  $\mathcal{M}_a(f) \setminus \operatorname{Sing} f$ . Bearing in mind the inclusion  $\mathcal{S}_a(f) \subset \mathcal{K}_\infty(f)$  of Theorem 2.6, instead of searching in vain a truncated expansion inside the Milnor set, we may show that there exists a truncation which verifies the Malgrange–Rabier condition (7). The proof of the following result employs the technique of [6, Theorem 3.2] and [5, Theorem 2.4.8], where we have used the t-regularity to find a geometric interpretation for  $\mathcal{K}_\infty(f)$ .

**Proposition 3.3** Let  $f = (f_1, ..., f_p) \colon \mathbb{R}^n \to \mathbb{R}^p$  be a polynomial map such that n > p and that  $\deg f_i \leq d, \forall i$ . Let

$$x(t) = (x_1(t), \dots, x_n(t)) = \sum_{-\infty \le i \le s} a_i t^i,$$

where  $t \in \mathbb{R}$ ,  $a_i \in \mathbb{R}^n$ , s > 0 and such that:

- (a)  $||x(t)|| \to \infty$ , as  $t \to \infty$ ;
- (b)  $f(x(t)) \to b$ , as  $t \to \infty$ ;
- (c)  $||x(t)||v(Df(x(t))) \rightarrow 0$ , as  $t \rightarrow \infty$ .

Then, the truncated expansion

$$\tilde{x}(t) = \sum_{-ds < i < s} a_i t^i,$$

verifies the following conditions:

(i)  $\|\tilde{x}(t)\| \to \infty$ , as  $t \to \infty$ ;



- (ii)  $f(\tilde{x}(t)) \to b$ , as  $t \to \infty$ ;
- (iii)  $\|\tilde{x}(t)\| \nu(D f(\tilde{x}(t))) \to 0$ , as  $t \to \infty$ .

*Proof* We treat here the case p > 1. See Remark 3.4 for the case p = 1.

By the definition of  $\nu$  (Definition 2.5 and (8)), condition (c) means:

$$||x(t)|| \left( \inf_{||y||=1} ||Df(x(t))^*(y)|| \right) \to 0, \text{ as } t \to \infty,$$
 (11)

where D  $f(x(t))^*$  denotes the adjoint of D f(x(t)).

Since  $\nu$  is a semi-algebraic mapping (see, for example, [16, Proposition 2.4]), the Curve Selection Lemma and (11) imply that there exists an analytic path (see also the proofs of [6, Theorem 3.2] and [3, Proposition 2.4] for this argument):

$$y(t) = \sum_{-\infty < i < 0} b_j t^j = (y_1(t), \dots, y_p(t)), b_j \in \mathbb{R}^p,$$

such that  $||y(t)|| = 1, \forall t \gg 0$ , and that:

$$\|x(t)\| \left\| y_1(t) \frac{\partial f_1}{\partial x}(x(t)) + \dots + y_p(t) \frac{\partial f_p}{\partial x}(x(t)) \right\| \to 0, \text{ as } t \to \infty,$$
 (12)

where  $\frac{\partial f_i}{\partial x}(x(t)) := \left(\frac{\partial f_i}{\partial x_1}(x(t)), \dots, \frac{\partial f_i}{\partial x_n}(x(t))\right)$  for  $i = 1, \dots, p$ . For any fixed  $j \in \{1, \dots, n\}$ , we set  $\phi_j : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ ,

$$\phi_j(x,y) := \left( y_1 \frac{\partial f_1}{\partial x_j}(x) + \dots + y_p \frac{\partial f_p}{\partial x_j}(x) \right). \tag{13}$$

It then follows that  $\deg \phi_i \leq d$  and that our path:

$$(x(t), y(t)) := \left(\sum_{-\infty \le i \le s} a_i t^i, \sum_{-\infty \le i \le 0} b_j t^j\right)$$

verifies the conditions:

- (1)  $||x(t)|| \to \infty$  as  $t \to \infty$ , and ||y(t)|| = 1;
- (2)  $x_i(t)\phi_j(x(t), y(t)) \to 0$  as  $t \to \infty$ , for any  $i, j \in \{1, \dots, n\}$ .

Applying Lemma 3.2 to the mapping  $(x_i \phi_j)_{i,j=1}^n$ , we get that, for any  $D \le -(d+1)s + s = -ds$ , the truncated path:

$$(\tilde{x}(t), \tilde{y}(t)) := \left(\sum_{D \le i \le s} a_i t^i, \sum_{D \le i \le 0} b_j t^j\right)$$

verifies the conditions:

- (1')  $\|\tilde{x}(t)\| \to \infty$  and  $\|\tilde{y}(t)\| \to 1$  as  $t \to \infty$ ;
- (2')  $\tilde{x}_i(t)\phi_i(\tilde{x}(t), \tilde{y}(t)) \to 0$ , as  $t \to \infty$ , for any  $i, j \in \{1, 2, \dots, n\}$ .

These imply:

$$\|\tilde{x}(t)\| \left\| \tilde{y}_1(t) \frac{\partial f_1}{\partial x} (\tilde{x}(t)) + \dots + \tilde{y}_p(t) \frac{\partial f_p}{\partial x} (\tilde{x}(t)) \right\| \to 0 \text{ as } t \to \infty, \tag{14}$$

and since  $\|\tilde{y}(t)\| \to 1$ , we obtain:

$$\|\tilde{x}(t)\| \frac{1}{\|\tilde{y}(t)\|} \left\| \tilde{y}_1(t) \frac{\partial f_1}{\partial x} (\tilde{x}(t)) + \dots + \tilde{y}_p(t) \frac{\partial f_p}{\partial x} (\tilde{x}(t)) \right\| \to 0, \text{ as } t \to \infty.$$
 (15)

The later implies that  $\|\tilde{x}(t)\|\nu(Df(\tilde{x}(t))) \to 0$ , as  $t \to \infty$ , which shows (iii).

Next, (i) follows by (1'), and (ii) follows from Lemma 3.2 for 
$$h := f$$
, since  $-ds < -ds + s$ .

Remark 3.4 In case p=1, in the proof of Proposition 3.3 we may consider  $\phi_j: \mathbb{R}^n \to \mathbb{R}$ ,  $\phi_j(x,y) = \frac{\partial f}{\partial x_j}(x)$  since in this case y=1. Then,  $\deg \phi_j \leq d-1$  and by applying Lemma 3.2 as above to the mapping  $(x_i\phi_j)_{i,j=1}^n$  we get that, for any  $D \leq -ds+s$ , the truncation  $\tilde{\tilde{x}}(t) = \sum_{D \leq i \leq s} a_i t^i$  satisfies (i), (ii) and (iii).

In the definition of  $\overline{\operatorname{Arc}(f)}$ , the lower bound is -ds + s instead of -ds. Since the value of the degree s from Theorem 3.1 is  $d^{n-1}$  in case p = 1, we recover the result in [7].

#### 3.3 Arc Space and the Main Result

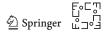
We may now apply to a polynomial map  $f = (f_1, ..., f_p) : \mathbb{R}^n \to \mathbb{R}^p$ , deg  $f_i \le d$ , a similar procedure as the one described by Jelonek and Kurdyka [14] in case p = 1. Thus, in case p > 1, we consider the following space of arcs associated with f:

$$\operatorname{Arc}(f) := \left\{ (x(t), y(t)) = \left( \sum_{-ds \le i \le s} a_i t^i, \sum_{-ds \le j \le 0} b_j t^j \right), (a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}^p \right\},$$
(16)

where  $s := [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}$ , as in Theorem 3.1. Then, Arc(f) is a vector space of finite dimension.

Referring to the notations in (16), we define, in a similar manner as [14, Definition 6.10], the *asymptotic variety of arcs*  $Arc_{\infty}(f) \subset Arc(f)$ , as the algebraic subset of the rational arcs  $(x(t), y(t)) \in Arc(f)$  verifying the following conditions:

- (a')  $\exists k > 0$  such that  $a_k \neq 0 \in \mathbb{R}^n$ , and  $||b_0|| = 1$ .
- (b') ord<sub>t</sub>  $f(x(t)) \leq 0$ .
- (c') ord<sub>t</sub>  $(x_i(t)\phi_j(x(t), y(t))) < 0$ , for any  $i, j \in \{1, ..., n\}$ , where  $\phi_j$  is defined at (13) in the proof of Proposition 3.3.



Let us then set  $\alpha_0$ : Arc $_{\infty}(f) \to \mathbb{R}^p$ ,  $\alpha_0(\xi(t)) := \lim_{t \to \infty} f(x(t))$ , where  $\xi(t) = (x(t), y(t))$ .

In view of the above results, we may now give an estimation of the non-trivial  $\rho$ -bifurcation set at infinity  $NS_{\infty}(f)$ , thus of the non-trivial bifurcation locus  $NB_{\infty}(f)$ , cf Proposition 2.9:

**Theorem 3.5**  $NS_{\infty}(f) \subset \alpha_0(Arc_{\infty}(f)) \subset \mathcal{K}_{\infty}(f)$ .

*Proof* If  $\alpha \in NS_{\infty}(f)$ , then  $\alpha \in NS_a(f)$  for any fixed  $a \in \Omega_f$ . By Theorem 3.1, there exists a path

$$x(t) = \sum_{-\infty \le i \le s} a_i t^i \in \mathcal{M}_a(f) \backslash \operatorname{Sing} f,$$

such that  $\lim_{t\to\infty} f(x(t)) = \alpha$ . It follows from Theorem 2.6 that x(t) verifies the conditions (a)–(c) of Proposition 3.3. Moreover, the truncation  $\tilde{x}$  defined in the same Proposition 3.3 verifies the properties (i)–(iii). Since conditions (i)–(iii) are equivalent to conditions (a')–(c'), we conclude that the first inclusion holds.

The second inclusion  $\alpha_0(\operatorname{Arc}_{\infty}(f)) \subset \mathcal{K}_{\infty}(f)$  is a direct consequence of the definitions of  $\operatorname{Arc}_{\infty}(f)$  and  $\mathcal{K}_{\infty}(f)$  since properties (a'), (b') and (c') characterize the values  $\alpha_0 \in \mathcal{K}_{\infty}(f)$  as shown in the proof of Proposition 3.3. This completes our proof.

Let us remark that the first inclusion can be strict, as shown by the next example:

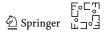
Example 3.6 ([7, Example 2.10]) Let 
$$f: \mathbb{R}^2 \to \mathbb{R}$$
,  $f(x, y) = y(x^2y^2 + 3xy + 3)$ . We have  $N\mathcal{S}_{\infty}(f) = \emptyset$ ,  $0 \in \alpha_0(\operatorname{Arc}_{\infty}(f))$  and  $0 \in \mathcal{K}_{\infty}(f)$ .

In trying to prove the equality in place of the second inclusion in Theorem 3.5, one notices that the inverse inclusion depends on the possibility of truncating paths which detect some value  $\alpha_0 \in \mathcal{K}_{\infty}(f)$  at the order provided by Theorem 3.1. But our Theorem 3.1 is based on paths in the Milnor set  $\mathcal{M}_a(f) \setminus \operatorname{Sing} f$ , which provide in principle lower degrees than working with the Malgrange–Rabier condition (7), and we know that the later is not equivalent to  $\rho$ -regularity (cf Sect. 2). Else, for the same reason, it would be difficult to obtain examples to disprove the inverse inclusion.

**Acknowledgements** LRGD acknowledges support from the Fapemig-Proc APQ-00431-14 Grant. ST acknowledges support from MPIM, Bonn. LRGD and MT acknowledge support from the USP-COFECUB Uc Ma 133/12 Grant. ST and MT acknowledge support from the CNRS-Tubitak No. 25784 Grant, from Université de Lille 1 and from Labex CEMPI (ANR-11-LABX-0007-01). The authors thank the anonymous referees for their valuable suggestions.

#### References

- 1. J. Bochnak, M. Coste and M. F. Roy, Real Algebraic Geometry, Springer-Verlag, Berlin, 1998.
- Y. Chen and M. Tibăr, Bifurcation values of mixed polynomials, Math. Res. Lett. 19 (2012), no.1, 59–79.
- Y. Chen, L.R.G. Dias, K. Takeuchi and M. Tibăr, Invertible polynomial maps via Newton nondegeneracy, Ann. Fourier 64 (2014), no. 5, 1807–1822.



- M. Coste, An introduction to semi-algebraic geometry, RAAG Network School, 2002. https://perso. univ-rennes1.fr/michel.coste/
- L.R.G. Dias, Regularity at infinity and global fibrations of real algebraic maps, PhD thesis, Université Lille 1 (France) and Universidade de São Paulo (Brazil), 2013.
- L.R.G. Dias, M.A.S. Ruas and M. Tibăr, Regularity at infinity of real maps and a Morse-Sard theorem, J. Topol. 5 (2012), no 2, 323–340.
- L.R.G. Dias and M. Tibăr, Detecting bifurcation values at infinity of real polynomials, Math. Z. 279
  (2015), 311–319.
- N. Dutertre, On the topology of semi-algebraic functions on closed semi-algebraic sets. Manuscripta Math. 139 (2012), 415–441.
- 9. T. Gaffney, Fibers of polynomial maps at infinity and a generalized Malgrange condition, Compositio Math. 119(2) (1999), 157–167.
- H.V. Hà and T.S. Pham, On the Łojasiewicz exponent at infinity of real polynomials. Ann. Polon. Math. 94 (2008), no. 3, 197–208.
- 11. H.V. Hà and T.S. Pham, Global optimization of polynomials using the truncated tangency variety and sums of squares. SIAM J. Optim. 19 (2008), no. 2, 941–951.
- 12. Z. Jelonek, Testing sets for properness of polynomial maps, Math. Ann. 315 (1999), no.1, 1-35.
- Z. Jelonek, On asymptotic critical values and the Rabier theorem. Geometric singularity theory, 125– 133, Banach Center Publ., 65, Polish Acad. Sci., Warsaw, 2004.
- Z. Jelonek and K. Kurdyka, Reaching generalized critical values of a polynomial, Math. Z. 276 (2014), no. 1-2, 557–570.
- 15. C. Joita and M. Tibăr, Bifurcation values of families of real curves. arXiv:1403.4808
- K. Kurdyka, P. Orro and S. Simon, Semialgebraic Sard theorem for generalized critical values, J. Differential Geometry 56 (2000), 67–92.
- 17. J. W. Milnor, Singular points of complex hypersurfaces, Ann. of Math. Studies 61, Princeton 1968.
- A. Némethi and A. Zaharia, On the bifurcation set of a polynomial function and Newton boundary, Publ. Res. Inst. Math. Sci. 26 (1990), no. 4, 681–689.
- A. Parusiński, On the bifurcation set of a complex polynomial with isolated singularities at infinity, Compositio Math. 97 (1995), 369–384.
- L. Păunescu and A. Zaharia, On the Łojasiewicz exponent at infinity for polynomial functions, Kodai Math. J. 20(3) (1997), 269–274.
- P. J. Rabier, Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds. Ann. of Math. 146 (1997), 647–691.
- 22. M. Safey El Din, Computing the global optimum of a multivariate polynomial over the reals, in: ISSAC 2008 (D. Jeffrey, ed.), ACM, New York, 2008, pp. 71–78.
- D. Siersma and M. Tibăr, Singularities at infinity and their vanishing cycles, Duke Math. Journal 80(3) (1995), 771–783.
- M. Tibăr, Regularity at infinity of real and complex polynomial maps, in Singularity Theory, The C.T.C Wall Anniversary Volume, LMS Lecture Notes Series 263, Cambridge University Press, 1999, pp. 249–264.
- M. Tibăr and A. Zaharia, Asymptotic behaviour of families of real curves, Manuscripta Math. 99 (1999), no.3, 383–393.

