Found Comput Math (2014) 14:1209–1242
DOI 10.1007/s10208-014-9194-z

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

# The Number of Singular Vector Tuples and Uniqueness of Best Rank-One Approximation of Tensors

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Received: 6 November 2012 / Revised: 9 November 2013 / Accepted: 27 January 2014 /
Published online: 27 March 2014


**Abstract** In this paper we discuss the notion of singular vector tuples of a complex-valued $d$-mode tensor of dimension $m_1 \times \cdots \times m_d$. We show that a generic tensor has a finite number of singular vector tuples, viewed as points in the corresponding Segre product. We give the formula for the number of singular vector tuples. We show similar results for tensors with partial symmetry. We give analogous results for the homogeneous pencil eigenvalue problem for cubic tensors, i.e., $m_1 = \cdots = m_d$. We show the uniqueness of best approximations for almost all real tensors in the following cases: rank-one approximation; rank-one approximation for partially symmetric tensors (this approximation is also partially symmetric); rank-$(r_1, \ldots, r_d)$ approximation for $d$-mode tensors.

**Keywords** Singular vector tuples · Vector bundles · Chern classes · Partially symmetric tensors · Homogeneous pencil eigenvalue problem for cubic tensors · Singular value decomposition · Best rank-one approximation · Best rank-$(r_1, \ldots, r_d)$ approximation

**Mathematics Subject Classification** 14D21 · 15A18 · 15A69 · 65D15 · 65H10 · 65K05

Communicated by Peter Buergisser.

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## 1 Introduction

The object of this paper is to study two closely related topics: counting the number of singular vector tuples of complex tensor and the uniqueness of a best rank-one approximation of real tensors. To state our results, we introduce notation that will be used in the paper. Let $\mathbb{F}$ be either the field of real or complex numbers, denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively, unless stated otherwise. For each $\mathbf{x} \in \mathbb{F}^m \setminus \{\mathbf{0}\}$ we denote by $[\mathbf{x}] := \text{span}(\mathbf{x})$ the line through the origin spanned by $\mathbf{x}$ in $\mathbb{F}^m$. Then $\mathbb{P}(\mathbb{F}^m)$ is the space of all lines through the origin in $\mathbb{F}^m$. We say that $\mathbf{x} \in \mathbb{F}^m$, $[\mathbf{y}] \in \mathbb{P}(\mathbb{F}^m)$ are generic if there exist subvarietes $U \subsetneq \mathbb{F}^m$, $V \subsetneq \mathbb{P}(\mathbb{F}^m)$ such that $\mathbf{x} \in \mathbb{F}^m \setminus U$, $[\mathbf{y}] \in \mathbb{P}(\mathbb{F}^m) \setminus V$. A set $S \subset \mathbb{F}^m$ is called closed if it is a closed set in the Euclidean topology. We say that a property $P$ holds almost everywhere (a.e.) in $\mathbb{R}^n$ if $P$ does not hold on a measurable set $S \subset \mathbb{R}^n$ of a zero Lebesgue measure. Equivalently, we say that almost all (a.a.) $\mathbf{x} \in \mathbb{R}^n$ satisfy $P$.

For $d \in \mathbb{N}$ denote $[d] := \{1, \ldots, d\}$. Let $m_i \geq 2$ be an integer for $i \in [d]$. Denote $\mathbf{m} := (m_1, \ldots, m_d)$. Let $\Pi_{\mathbb{F}}(\mathbf{m}) := \mathbb{P}(\mathbb{F}^{m_1}) \times \ldots \times \mathbb{P}(\mathbb{F}^{m_d})$. We call $\Pi_{\mathbb{F}}(\mathbf{m})$ the Segre product. Set $\Pi(\mathbf{m}) := \Pi_{\mathbb{C}}(\mathbf{m})$. Denote by $\mathbb{F}^{\mathbf{m}} = \mathbb{F}^{m_1 \times \ldots \times m_d} := \otimes_{i=1}^d \mathbb{F}^{m_i}$ the vector space of $d$-mode tensors $\mathcal{T} = [t_{i_1,\ldots,i_d}], i_j = 1, \ldots, m_j, j = 1, \ldots, d$ over $\mathbb{F}$. (We assume that $d \geq 3$ unless stated otherwise.) For an integer $p \in [d]$ and for $\mathbf{x}_{j_r} \in \mathbb{F}^{m_{j_r}}, r \in [p]$, we use the notation $\otimes_{j_r, r\in[p]} \mathbf{x}_{j_r} := \mathbf{x}_{j_1} \otimes \ldots \otimes \mathbf{x}_{j_p}$. For a subset $P = \{j_1, \ldots, j_p\} \subseteq [d]$ of cardinality $p = |P|$, consider a $p$-mode tensor $\mathcal{X} = [x_{i_{j_1},\ldots,i_{j_p}}] \in \otimes_{j_r, r\in[p]} \mathbb{F}^{m_{j_r}}$, where $j_1 < \ldots < j_p$. Define

$$\mathcal{T} \times \mathcal{X} := \sum_{i_{j_r} \in [m_{j_r}], r \in [p]} t_{i_1,\ldots,i_d} x_{i_{j_1},\ldots,i_{j_p}}$$

as a $(d-p)$-mode tensor obtained by contraction on the indices $i_{j_1}, \ldots, i_{j_p}$.

To motivate our results, let us consider the classical case of matrices, i.e., $d = 2$ and $A \in \mathbb{R}^{m_1 \times m_2}$. We call a pair $(\mathbf{x}_1, \mathbf{x}_2) \in (\mathbb{R}^{m_1} \setminus \{\mathbf{0}\}) \times (\mathbb{R}^{m_2} \setminus \{\mathbf{0}\})$ a *singular vector pair* if

$$A\mathbf{x}_2 = \lambda_1 \mathbf{x}_1, \quad A^\top \mathbf{x}_1 = \lambda_2 \mathbf{x}_2 \tag{1.1}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. For $\mathbf{x} \in \mathbb{R}^m$ let $\|\mathbf{x}\| := \sqrt{\mathbf{x}^\top \mathbf{x}}$ be the Euclidean norm on $\mathbb{R}^m$. Choosing $\mathbf{x}_1, \mathbf{x}_2$ to be of Euclidean length one we deduce that $\lambda_1 = \lambda_2$, where $|\lambda_1|$ is equal to some singular value of $A$. It is natural to identify all singular vector pairs of the form $(a_1\mathbf{x}_1, a_2\mathbf{x}_2)$, where $a_1 a_2 \neq 0$, as the class of singular vector pairs. Thus $([\mathbf{x}_1], [\mathbf{x}_2]) \in \mathbb{P}(\mathbb{R}^{m_1}) \times \mathbb{P}(\mathbb{R}^{m_2})$ is called a singular vector pair of $A$.

For a generic $A$, i.e., $A$ of the maximal rank $r = \min(m_1, m_2)$ and $r$ distinct positive singular values, $A$ has exactly $r$ distinct singular vector pairs. Furthermore, under these conditions $A$ has a unique best rank-one approximation in the Frobenius norm given by the singular vector pair corresponding to the maximal singular value [10].

Assume now that $m = m_1 = m_2$ and $A$ is a real symmetric matrix. Then the singular values of $A$ are the absolute values of the eigenvalues of $A$. Furthermore, if all the absolute values of the eigenvalues of $A$ are pairwise distinct, then $A$ has a unique best rank-one approximation, which is symmetric. Hence, for any real symmetric matrix $A$ there exists a best rank-one approximation which is symmetric.

In this paper we derive similar results for tensors. Let $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$. We first define the notion of a singular vector tuple $(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in (\mathbb{F}^{m_1} \setminus \{\mathbf{0}\}) \times \ldots \times (\mathbb{F}^{m_d} \setminus \{\mathbf{0}\})$ [16]:

$$\mathcal{T} \times \otimes_{j\in[d]\setminus\{i\}} \mathbf{x}_j = \lambda_i \mathbf{x}_i, \quad i = 1, \ldots, d. \tag{1.2}$$

As for matrices we identify all singular vector tuples of the form $(a_1\mathbf{x}_1, \ldots, a_d\mathbf{x}_d)$, $a_1 \ldots a_d \neq 0$ as one class of singular vector tuple in $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi_{\mathbb{F}}(\mathbf{m})$. (Note that for $d = 2$ and $\mathbb{F} = \mathbb{C}$ our notion of singular vector pair differs from the classical notion of singular vectors for complex-valued matrices; see § 3.)

Let $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi(\mathbf{m})$ be a singular vector tuple of $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$. This tuple corresponds to a zero (nonzero) singular value if $\prod_{i\in[d]} \lambda_i = 0$ $(\neq 0)$. This tuple is called a simple singular vector tuple (or just simple) if the corresponding global section corresponding to $\mathcal{T}$ has a simple zero at $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$; see Lemma 11 in § 3.

Our first major result is the following theorem.

**Theorem 1** *Let $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ be generic. Then $\mathcal{T}$ has exactly $c(\mathbf{m})$ simple singular vector tuples that correspond to nonzero singular values. Furthermore, $\mathcal{T}$ does not have a zero singular value. In particular, a generic real-valued tensor $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ has at most $c(\mathbf{m})$ real singular vector tuples corresponding to nonzero singular values, and all of them are simple. The integer $c(\mathbf{m})$ is the coefficient of the monomial $\prod_{i=1}^{d} t_i^{m_i-1}$ in the polynomial*

$$\prod_{i\in[d]} \frac{\hat{t}_i^{m_i} - t_i^{m_i}}{\hat{t}_i - t_i}, \quad \hat{t}_i = \sum_{j\in[d]\setminus\{i\}} t_j, \; i \in [d]. \tag{1.3}$$

At the end of §3 we list the first values of $c(\mathbf{m})$ for $d = 3$. We generalize the preceding results to the class of tensors with given partial symmetry.

We now consider the cubic case where $m_1 = \cdots = m_d = m$. For an integer $m \geq 2$ let $m^{\times d} := (\underbrace{m, \ldots, m}_{d})$. Then $\mathcal{T} \in \mathbb{F}^{m^{\times d}}$ is called $d$-cube, or simply a cube tensor. For a vector $\mathbf{x} \in \mathbb{C}^m$ let $\otimes^k \mathbf{x} := \underbrace{\mathbf{x} \otimes \ldots \otimes \mathbf{x}}_{k}$. Assume that $\mathcal{T}, \mathcal{S} \in \mathbb{C}^{m^{\times d}}$. Then the homogeneous pencil eigenvalue problem is to find all vectors $\mathbf{x}$ and scalars $\lambda$ satisfying $\mathcal{T} \times \otimes^{d-1}\mathbf{x} = \lambda\mathcal{S} \times \otimes^{d-1}\mathbf{x}$. The contraction here is with respect to the last $d-1$ indices of $\mathcal{T}, \mathcal{S}$. We assume without loss of generality that $\mathcal{T} = [t_{i_1,\ldots,i_d}]$, $\mathcal{S} = [s_{i_1,\ldots,i_d}]$ are symmetric with respect to the indices $i_2, \ldots, i_d$. $\mathcal{S}$ is called nonsingular if the system $\mathcal{S} \times \otimes^{d-1}\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = 0$. Assume that $\mathcal{S}$ is nonsingular and fixed. Then $\mathcal{T}$ has exactly $m(d-1)^{m-1}$ eigenvalues counted with their multiplicities. $\mathcal{T}$ has $m(d-1)^{m-1}$ distinct eigenvectors in $\mathbb{P}(\mathbb{C}^m)$ for a generic $\mathcal{T}$. See [21] for the case where $\mathcal{S}$ is the identity tensor.

View $\mathbb{R}^{m_1\times\ldots m_d}$ as an inner product space, where for two $d$-mode tensors $\mathcal{T}, \mathcal{S} \in \mathbb{R}^{m_1\times\ldots\times m_d}$ we let $\langle\mathcal{T}, \mathcal{S}\rangle := \mathcal{T} \times \mathcal{S}$. Then the Hilbert–Schmidt norm is defined as $\|\mathcal{T}\| := \sqrt{\langle\mathcal{T}, \mathcal{T}\rangle}$. [Recall that for $d = 2$ (matrices) the Hilbert–Schmidt norm is called the Frobenius norm.] A best rank-one approximation is a solution to the minimal problem

$$\min_{\mathbf{x}_i \in \mathbb{R}^{m_i}, i\in[d]} \|\mathcal{T} - \otimes_{i\in[d]}\mathbf{x}_i\| = \|\mathcal{T} - \otimes_{i\in[d]}\mathbf{u}_i\|. \tag{1.4}$$

$\otimes_{i\in[d]}\mathbf{u}_i$ is called a best rank-one approximation of $\mathcal{T}$. Our second major result is as follows.

**Theorem 2** *1. For a.a.* $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ *a best rank-one approximation is unique.*
*2. Let* $\mathrm{S}^d(\mathbb{R}^m) \subset \mathbb{R}^{m^{\times d}}$ *be the space of* $d$*-mode symmetric tensors. For a.a.* $\mathcal{S} \in \mathrm{S}^d(\mathbb{R}^m)$ *a best rank-one approximation of* $\mathcal{S}$ *is unique and symmetric. In particular, for each* $\mathcal{S} \in \mathrm{S}^d(\mathbb{R}^m)$ *there exists a best rank-one approximation that is symmetric.*

The last statement of part *2* of this theorem was demonstrated by the first named author in [7]. Actually, this result is equivalent to Banach's theorem [1]. See [23] for another proof of Banach's theorem. In Theorem 12 we generalize part *2* of Theorem 2 to the class of tensors with given partial symmetry.

Let $\mathbf{r} = (r_1, \ldots, r_d)$, where $r_i \in [m_i]$ for $i \in [d]$. In the last section of this paper we study a best rank-$\mathbf{r}$ approximation for a real $d$-mode tensor [6]. We show that for a.a. tensors a best rank-$\mathbf{r}$ approximation is unique.

We now describe briefly the contents of our paper. In § 2 we give a layman's introduction to some basic notions of vector bundles over compact complex manifolds and Chern classes of certain bundles over the Segre product needed for this paper. We hope that this introduction will make our paper accessible to a wider audience. § 3 discusses the first main contribution of this paper, namely, the number of singular vector tuples of a generic complex tensor is finite and is equal to $c(\mathbf{m})$. We give a closed formula for $c(\mathbf{m})$, as in (1.3). § 4 generalizes these results to partially symmetric tensors. In particular, we reproduce the result of Cartwright and Sturmfels for symmetric tensors [3]. In § 5 we discuss a homogeneous pencil eigenvalue problem. In § 6 we give certain conditions on a general best approximation problem in $\mathbb{R}^n$, which are probably well known to the experts. In § 7 we give uniqueness results on the best rank-one approximation of partially symmetric tensors. In § 8 we discuss a best rank-$\mathbf{r}$ approximation.

We thank J. Draisma, who pointed out the importance of distinguishing between isotropic and nonisotropic vectors, as we do in § 3.

## 2 Vector Bundles over Compact Complex Manifolds

In this section we recall some basic results on complex manifolds and holomorphic tangent bundles that we use in this paper. Our object is to give the simplest possible intuitive description of basic results in algebraic geometry needed in this paper, sometimes compromising the rigor. An interested reader can consult with [11] for general facts about complex manifolds and complex vector bundles, and for a simple axiomatic exposition on complex vector bundles with [15]. For a Bertini-type theorem we refer the reader to Fulton [8] and Hartshorne [12].

## 2.1 Complex Compact Manifolds

Let $M$ be a compact complex manifold of dimension $n$. Thus, there exists a finite open cover $\{U_i\}, i \in [N]$ with coordinate homeomorphism $\phi_i : U_i \to \mathbb{C}^n$ such that $\phi_i \circ \phi_j^{-1}$ is holomorphic on $\phi_j(U_i \cap U_j)$ for all $i, j$.

As an example, consider the $m-1$-dimensional complex projective space $\mathbb{P}(\mathbb{C}^m)$, which is the set of all complex lines in $\mathbb{C}^m$ through the origin. Any point in $\mathbb{P}(\mathbb{C}^m)$ is represented by a one-dimensional subspace spanned by the vector $\mathbf{x} = (x_1, \ldots, x_m)^\top \in \mathbb{C}^m \setminus \{\mathbf{0}\}$. The standard open cover of $\mathbb{P}(\mathbb{C}^m)$ consists of $m$ open covers $U_1, \ldots, U_m$, where $U_i$ corresponds to the lines spanned by $\mathbf{x}$ with $x_i \neq 0$. The homeomorphism $\phi_i$ is given by $\phi_i(\mathbf{x}) = (\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_m}{x_i})^\top$. Thus, each $U_i$ is homeomorphic to $\mathbb{C}^{m-1}$.

Let $M$ be an $n$-dimensional compact complex manifold, as previously. For $\zeta \in U_i$, the coordinates of the vector $\phi_i(\zeta) = \mathbf{z} = (z_1, \ldots, z_n)^\top$ are called the local coordinates of $\zeta$. Since $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, $M$ is a real manifold of real dimension $2n$. Let $z_j = x_j + \mathbf{i}y_j, \bar{z}_j = x_j - \mathbf{i}y_j, j \in [n]$, where $\mathbf{i} = \sqrt{-1}$. For simplicity of notation we let $\mathbf{u} = (u_1, \ldots, u_{2n}) = (x_1, y_1, \ldots, x_n, y_n)$ be the real local coordinates on $U_i$. Any function $f : U_i \to \mathbb{C}$ in the local coordinates is viewed as $f(\mathbf{u}) = g(\mathbf{u}) + \mathbf{i}h(\mathbf{u})$, where $h, g : U_i \to \mathbb{R}$. Thus, $df = \sum_{j \in [2n]} \frac{\partial f}{\partial u_j} du_j$. For a positive integer $p$, a (differential) $p$-form $\omega$ on $U_i$ is given in the local coordinates as follows:

$$\omega = \sum_{1 \le i_1 < \ldots < i_p \le 2n} f_{i_1, \ldots, i_p}(\mathbf{u}) du_{i_1} \wedge \ldots \wedge du_{i_p}.$$

($f_{i_1, \ldots, i_p}(\mathbf{u})$ are differentiable functions in local coordinates $\mathbf{u}$ for $1 \le i_1 < \ldots < i_p \le 2n$.) Recall that the wedge product of two differential is anticommutative, i.e., $du_k \wedge du_l = -du_l \wedge du_k$. Then

$$d\omega = \sum_{1 \le i_1 < \ldots < i_p \le 2n} (df_{i_1, \ldots, i_p}) \wedge du_{i_1} \wedge \ldots \wedge du_{i_p}.$$

(Recall that a differential 0-form is a function.) Note that for $p > 2n$ any differential $p$-form is a zero form. A straightforward calculation shows that $d(d\omega) = 0$. $\omega$ is a $p$-form on $M$ if its restriction to each $U_i$ is a $p$-form, and the restrictions of these two forms on $U_i \cap U_j$ are obtained from another by the change of coordinates $\phi_i \circ \phi_j^{-1}$. $\omega$ is called closed if $d\omega = 0$, and $d\omega$ is called an exact form. The space of closed $p$-forms modulo exact $p$-forms is a finite-dimensional vector space over $\mathbb{C}$, which is denoted by $\mathrm{H}^p(M)$. Each element of $\mathrm{H}^p(M)$ is represented by a closed $p$-form, and the difference between two representatives is an exact form. Since the product of two forms is also a form, it follows that the space of all closed forms modulo exact forms is a finite-dimensional algebra, where the identity 1 corresponds to the constant function with value 1 on $M$.

## 2.2 Holomorphic Vector Bundles

A holomorphic vector bundle $E$ on $M$ of rank $k$, where $k$ is a nonnegative integer, is a complex manifold of dimension $n+k$, which can be simply described as follows. There exists a finite open cover $\{U_i\}$, $i \in [N]$ of $M$ with the aforementioned properties satisfying the following additional conditions. At each $\zeta \in U_i$ we are given $k$-dimensional vector space $E_\zeta$, called a fiber of $E$ over $\zeta$, all of which can be identified with a fixed vector space $\mathbf{V}_i$, having a basis $[\mathbf{e}_{1,i}, \ldots, \mathbf{e}_{k,i}]$. For $\zeta \in U_i \cap U_j$, $i \neq j$ the transition matrix from $[\mathbf{e}_{1,i}, \ldots, \mathbf{e}_{k,i}]$ to $[\mathbf{e}_{1,j}, \ldots, \mathbf{e}_{k,j}]$ is given by a $k \times k$ invertible matrix $g_{U_jU_i}(\zeta)$. Thus, $[\mathbf{e}_{1,i}, \ldots, \mathbf{e}_{k,i}] = [\mathbf{e}_{1,j}, \ldots, \mathbf{e}_{k,j}]g_{U_jU_i}(\zeta)$. Each entry of $g_{U_jU_i}(\zeta)$ is a holomorphic function in the local coordinates of $U_j$. We have the following relations:

$$\begin{aligned} g_{U_iU_j}(\zeta)g_{U_jU_i}(\zeta) &= g_{U_iU_j}(\eta), g_{U_jU_p}(\eta)g_{U_pU_i}(\eta) \\ &= I_k \text{ for } \zeta \in U_i \cap U_j, \eta \in U_i \cap U_j \cap U_p. \end{aligned}$$

($I_k$ is an identity matrix of order $k$.)

For $k = 0$, $E$ is called a zero bundle. $E$ is called a line bundle if $k = 1$. $E$ is called a trivial bundle if there exists a finite open cover such that each $g_{U_iU_j}(\zeta)$ is an identity matrix. A vector bundle $F$ on $M$ is called a subbundle of $E$ if $F$ is a submanifold of $E$ such that $F_\zeta$ is a subspace of $E_\zeta$ for each $\zeta \in M$. Assume that $F$ is a subbundle of $E$. Then $G := E/F$ is the quotient bundle of $E$ and $F$, where $G_\zeta$ is the quotient vector space $E_\zeta/F_\zeta$. Let $E_1, E_2$ be two vector bundles on $M$. We can create the following new bundles on $M$: $E := E_1 \oplus E_2$, $F := E_1 \otimes E_2$, $H := \mathrm{Hom}(E_1, E_2)$. Here, $E_\zeta = E_{1,\zeta} \oplus E_{2,\zeta}$, $F_\zeta = E_{1,\zeta} \otimes E_{2,\zeta}$, and $H_\zeta$ consists of all linear transformations from $E_{1,\zeta}$ to $E_{2,\zeta}$. In particular, the vector bundle $\mathrm{Hom}(E_1, E_2)$, where $E_2$ is the one-dimensional trivial bundle, is called a dual bundle of $E_1$ and is denoted by $E_1^\vee$. Recall that $\mathrm{Hom}(E_1, E_2)$ is isomorphic to $E_2 \otimes E_1^\vee$. For a given vector bundle $E$ on $M$ we can define the bundle $F := \otimes^d E$. Here $F_\zeta = \otimes^d E_\zeta$ is a fiber of $d$-mode tensors.

Let $M, M'$ be compact complex manifolds, and assume that $f : M' \to M$ is holomorphic. Assume that $\pi : E \to M$ is a holomorphic vector bundle. Then one can pull back $E$ to obtain a bundle $\pi' : E' \to M'$, where $E' = f^*E$.

Given a manifold $M_i$ with a vector bundle $E_i$ for $i = 1, 2$, we can define the bundle $F := E_1 \oplus E_2$, $G := E_1 \otimes E_2$ on $M := M_1 \times M_2$ by the equality

$$F_{(\zeta_1,\zeta_2)} = E_{1,\zeta_1} \oplus E_{2,\zeta_2}, G_{(\zeta_1,\zeta_2)} = E_{1,\zeta_1} \otimes E_{2,\zeta_2}.$$

A special case for $F$ occurs when one of the factors $E_i$ is a zero bundle, say $E_2 = 0$. Then $E_1 \oplus 0$ is the pullback of the bundle $E_1$ on $M_1$ obtained by using the projection $\pi_1 : M_1 \times M_2$ and is denoted as the bundle $\pi_1^* E_1$ on $M_1 \times M_2$. Thus, $E_1 \oplus E_2$ is the bundle $\pi_1^*E_1 \oplus \pi_2^*E_2$ on $M_1 \times M_2$. Similarly, $E_1 \otimes E_2$ is the bundle $\pi_1^*E_1 \otimes \pi_2^*E_2$.

We now discuss a basic example used in this paper. Consider the trivial bundle $F(m)$ on $\mathbb{P}(\mathbb{C}^m)$ of rank $m$. Thus, $F(m)_\zeta = \mathbb{C}^m$. The tautological line bundle $T(m)$ on $\mathbb{P}(\mathbb{C}^m)$, customarily denoted by $\mathcal{O}(-1)$, is given by $T(m)_{[\mathbf{x}]} = \mathrm{span}(\mathbf{x}) \subset \mathbb{C}^m$. Thus, $T(m)$ is a subbundle of $F(m)$. Denote by $Q(m)$ the quotient bundle $F(m)/T(m)$. Hence, rank $Q(m) = m - 1$. We have an exact sequence of the following bundles on $\mathbb{P}(\mathbb{C}^m)$:

$$0 \to T(m) \to F(m) \to Q(m) \to 0. \tag{2.1}$$

The dual of the bundle of $T(m)$, also called the hyperplane line bundle, is denoted here by $H(m)$. [$H(m)$ is customarily denoted by $\mathcal{O}(1)$ in the algebraic geometry literature.]

### 2.3 Chern Polynomials

We now return to a holomorphic vector bundle $E$ on a compact complex manifold $M$. The seminal work of Chern [5] associates with each $\pi : E \to M$ the Chern class $c_j(E)$ for each $j \in [\dim M]$. One can view $c_j(E)$ as an element in $\mathrm{H}^{2j}(M)$. The Chern classes needed in this paper can be determined by the following well-known rules [15].

One associate with $E$ the Chern polynomial $C(t, E) = 1 + \sum_{j=1}^{\mathrm{rank}\ E} c_j(E)t^j$. Note that $c_j(E) = 0$ for $j > \dim M$. The total Chern class $c(E)$ is $C(1, E) = \sum_{j=0}^{\infty} c_j(E)$. Consider the formal factorization $C(t, E) = \prod_{j=1}^{\mathrm{rank}\ E}(1 + \xi_j(E)t)$. Then the Chern character $ch(E)$ of $E$ is defined as $\sum_{j=1}^{\mathrm{rank}\ E} e^{\xi_j(E)}$.

$C(t, E) = 1$ if $E$ is a trivial bundle. The Chern polynomial of the dual bundle is given by $C(t, E^{\vee}) = C(-t, E)$. Given an exact sequence of bundles

$$0 \to E \to F \to G \to 0,$$

we have the identity

$$C(t, F) = C(t, E)C(t, G), \tag{2.2}$$

which is equivalent to $c(F) = c(E)c(G)$.

The product formula is the identity $ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$. Let $f : M' \to M$. Then $c_j(f^*E)$, viewed as a differential form in $\mathrm{H}^{2j}(M')$, is obtained by pullback of the differential form $c_j(E)$. In particular, for the pullback bundle $\pi_1^* E_1$ described previously, we have the equality $c_j(\pi_1^* E_1) = c_j(E_1)$ when we use the local coordinates $\zeta = (\zeta_1, \zeta_2)$ on $M_1 \times M_2$.

Assume that rank $E = \dim M = n$. Then $c_n(E) = \nu(E)\omega$, where $\omega \in \mathrm{H}^{2n}(M)$ is the volume form on $M$ such that $\omega$ is a generator of $\mathrm{H}^{2n}(M, \mathbf{Z})$. Then $\nu(E)$ is an integer, which is called the top Chern number of $E$.

Denote by $s_m$ the first Chern class of $H(m)$, which belongs to $\mathrm{H}^2(\mathbb{P}(\mathbb{C}^m))$. Then $s_m^k$ represents the differential form $\wedge^k s_m \in \mathrm{H}^{2k}(\mathbb{P}(\mathbb{C}^m))$. Observe that $s_m^m = 0$. Moreover the algebra of all closed forms modulo the exact forms on $\mathbb{P}(\mathbb{C}^m)$ is $\mathbb{C}[s_m]/(s_m^m)$, i.e. all polynomials in the variable $s_m$ modulo the relation $s_m^m = 0$. So $C(t, H(m)) = 1 + s_m t$ and $C(t, T(m)) = 1 - s_m t$. The exact sequence (2.1) and the formula (2.2) imply that

$$1 = C(t, F(m)) = C(t, T(m))C(t, Q(m)) = (1 - s_m t)C(t, Q(m)).$$

Therefore

$$C(t, Q(m)) = \frac{1}{1 - s_m t} = 1 + \sum_{j=1}^{m-1} s_m^j t^j. \tag{2.3}$$

## 2.4 Certain Bundles on Segre Product

Let $m_1, \ldots, m_d \geq 2$ be given integers with $d > 1$. Use the notation $\mathbf{m}_i = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_d)$ for $i \in [d]$. Consider the Segre product $\Pi(\mathbf{m}) := \mathbb{P}(\mathbb{C}^{m_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{m_d})$. Let $\pi_i : \Pi(\mathbf{m}) \to \mathbb{P}(\mathbb{C}^{m_i})$ be the projections on the $i$th component. Then $\pi_i^* H(m_i), \pi_i^* Q(m_i), \pi_i^* F(m_i)$ are the pullback of the bundles $H(m_i), Q(m_i), F(m_i)$ on $\mathbb{P}(\mathbb{C}^{m_i})$ to $\Pi(\mathbf{m})$, respectively.

Consider the map $\iota_{\mathbf{m}} : \Pi(\mathbf{m}) \to \mathbb{P}(\mathbb{C}^{\mathbf{m}})$ given by $\iota_{\mathbf{m}}([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) = [\otimes_{i\in[d]}\mathbf{x}_i]$. It is straightforward to show that $\iota$ is $1-1$. Then $\Sigma(\mathbf{m}) := \iota_{\mathbf{m}}(\Pi(\mathbf{m})) \subset \mathbb{P}(\mathbb{C}^{\mathbf{m}})$ is the Segre variety. Let $T(\mathbf{m})$ be a tautological line bundle on $\mathbb{P}(\mathbb{C}^{\mathbf{m}})$. The identity $\mathrm{span}(\otimes_{j\in[d]}\mathbf{x}_j) = \otimes_{j\in[d]}\mathrm{span}(\mathbf{x}_j)$ implies that the line bundle $\iota^* T(\mathbf{m})$ is isomorphic to $\otimes_{j\in[d]}\pi_j^* T(m_j)$. Hence the dual bundles $\iota^* H(\mathbf{m})$ and $\otimes_{j\in[d]}\pi_j^* H(m_j)$ are isomorphic. Consider next the bundle $\hat{T}(\mathbf{m}_i)$ on $\Pi(\mathbf{m})$, which is

$$\hat{T}(\mathbf{m}_i) := \otimes_{j\in[d]\setminus\{i\}}\pi_j^* T(m_j). \tag{2.4}$$

Hence the dual bundle $\hat{T}(\mathbf{m}_i)^\vee$ is isomorphic to $\otimes_{j\in[d]\setminus\{i\}}\pi_j^* H(m_j)$. In particular,

$$c_1(\hat{T}(\mathbf{m}_i)^\vee) = c_1(\otimes_{j\in[d]\setminus\{i\}}\pi_j^* H(m_j)). \tag{2.5}$$

Define the following vector bundles on $\Pi(\mathbf{m})$:

$$\begin{aligned} &R(i,\mathbf{m}) = \mathrm{Hom}(\hat{T}(\mathbf{m}_i), \pi_i^* Q(m_i)), \quad R(i,\mathbf{m})' = \mathrm{Hom}(\hat{T}(\mathbf{m}), \pi_i^* F\ (m_i)),\\ &R(\mathbf{m}) = \oplus_{i\in[d]} R(i,\mathbf{m}), \quad R_i(\mathbf{m})' := (\oplus_{j\in[d]\setminus\{i\}} R(j,\mathbf{m})) \oplus R(i,\mathbf{m})'. \end{aligned} \tag{2.6}$$

Observe that

$$\begin{aligned} \mathrm{rank}\ R(i,\mathbf{m}) &= \mathrm{rank}\ R(i,\mathbf{m})' - 1 = m_i - 1,\\ \mathrm{rank}\ R(\mathbf{m}) &= \mathrm{rank}\ R_i(\mathbf{m})' - 1 = \dim \Pi(\mathbf{m}). \end{aligned} \tag{2.7}$$

Since $\mathrm{Hom}(E_1, E_2) \sim E_2 \otimes E_1^\vee$, we obtain the following relations:

$$C(t, R(i,\mathbf{m})) = C(t, \pi_i^* Q(m_i) \otimes (\hat{T}(\mathbf{m})^\vee) = C(t, \pi_i^* Q(m_i) \otimes (\otimes_{j\in[d]\setminus\{i\}}\pi_j^* H(m_j)). \tag{2.8}$$

The formula (2.2) yields

$$C(t, R(\mathbf{m})) = \prod_{i\in[d]} C(t, R(i,\mathbf{m})). \tag{2.9}$$

Use the notation $t_i = c_1(\pi_i^* H(m_i))$. The cohomology ring $H^*(\Pi(\mathbf{m}))$ is generated by $t_1, \ldots, t_d$, with the relations $t_i^{m_i} = 0$, that is, $H^*(\Pi(\mathbf{m})) \simeq \mathbb{C}[t_1, \ldots, t_d]/(t_1^{m_1}, \ldots, t_d^{m_d})$, and in what follows we interpret $t_i$ just as variables. Correspondingly, the $k$th Chern class $c_k(E)$ is equal to $p_k(t_1, \ldots, t_d)$ for some homogeneous polynomial $p_k$ of degree $k$ for $k = 1, \ldots, \dim \Pi(\mathbf{m})$. [Recall that $c_0(E) = 1$ and $c_k(E) = 0$ for $k > \dim \Pi(\mathbf{m})$.]

In what follows we need to compute the top Chern class of $R(\mathbf{m})$. Since rank $R(\mathbf{m}) = \dim \Pi(\mathbf{m})$ and $\Pi(\mathbf{m})$ is a manifold, it follows that the top Chern class of $R(\mathbf{m})$ is of the form

$$c(\mathbf{m}) \prod_{i\in[d]} t_i^{m_i-1}, \tag{2.10}$$

where $c(\mathbf{m})$ is an integer. Thus, $c(\mathbf{m}) = \nu(R(\mathbf{m}))$ is the top Chern number of $R(\mathbf{m})$.

**Lemma 3** *Let $R(i, \mathbf{m})$ and $R(\mathbf{m})$ be the vector bundles on the Segre product $\Pi(\mathbf{m})$ given by (2.6). Then the total Chern classes of these vector bundles are given as follows:*

$$c(R(i,\mathbf{m})) = \sum_{j=0}^{m_i-1} (1+\hat{t}_i)^{m_i-1-j} t_i^j, \quad \hat{t}_i := \sum_{k\in[d]\setminus\{i\}} t_k, \tag{2.11}$$

$$c(R(\mathbf{m})) = \prod_{i\in[d]} \left( \sum_{j=0}^{m_i-1} (1+\hat{t}_i)^{m_i-1-j} t_i^j \right). \tag{2.12}$$

*The top Chern number of $R(\mathbf{m})$, $c(\mathbf{m})$, is the coefficient of the monomial $\prod_{i\in[d]} t_i^{m_i-1}$ in the polynomial $\prod_{i\in[d]} \frac{\hat{t}_i^{m_i} - t_i^{m_i}}{\hat{t}_i - t_i}$. (In this formula of $c(\mathbf{m})$ we do not assume the identities $t_i^{m_i} = 0$ for $i \in [d]$.)*

*Proof* Let $\zeta_i := e^{\frac{2\pi \mathbf{i}}{m_i}}$ be the primitive $m_i$th root of unity. Then

$$\prod_{k=0}^{m_i-1} (1-\zeta_i^k x) = 1 - x^{m_i}, \quad \sum_{k=0}^{m_i-1} x^k = \frac{1-x^{m_i}}{1-x} = \prod_{k\in[m_i-1]} (1-\zeta_i^k x). \tag{2.13}$$

The second equality of (2.13) and (2.3) yield that

$$C(t, \pi_i^* Q(m_i)) = \prod_{k\in[m_i-1]} (1-\zeta_i^k t_i t).$$

Hence, $ch(\pi_i^* Q(m_i)) = \sum_{k\in[m_i-1]} e^{-\zeta_i^k t_i}$. Clearly, $ch(H(m_j)) = e^{t_j}$. The product formula for Chern characters yields

$$\begin{aligned} &ch(\otimes_{j\in[d]\setminus\{i\}} \pi_j H(m_j)) = e^{\sum_{j\in[d]\setminus\{i\}} t_j} = e^{\hat{t}_i}, \\ &ch(\pi_i^* Q(m_i) \otimes (\otimes_{j\in[d]\setminus\{i\}} \pi_j^* H(m_j))) = ch(\pi_i^* Q(m_i)) ch(\otimes_{j\in[d]\setminus\{i\}} \pi_j^* H(m_j)) \\ &= \sum_{k\in[m_i-1]} e^{\hat{t}_i - \zeta_i^k t_i}. \end{aligned}$$

Hence,

$$C(t,R(i,\mathbf{m}))=\prod_{k\in[m_i-1]}(1+(\hat{t}_i-\zeta_i^k t_i)t)=\frac{1}{1+(\hat{t}_i-t_i)t}\prod_{k=0}^{m_i-1}(1+(\hat{t}_i-\zeta_i^k t_i)t),$$

$$c(R(i,\mathbf{m}))=C(1,R(i,\mathbf{m}))=\frac{1}{1+\hat{t}_i-t_i}\prod_{k=0}^{m_i-1}(1+\hat{t}_i-\zeta_i^k t_i)$$

$$=\frac{1}{1+\hat{t}_i-t_i}(1+\hat{t}_i)^{m_i}\prod_{k=0}^{m_i-1}(1-\zeta_i^k x)=\frac{1}{1+\hat{t}_i-t_i}(1+\hat{t}_i)^{m_i}(1-x^{m_i}),$$

where $x=\frac{t_i}{1+\hat{t}_i}$. Since $t_i^{m_i}=0$, we deduce

$$c(R(i,\mathbf{m}))=\frac{(1+\hat{t}_i)^{m_i}}{1-t_i+\hat{t}_i}=\frac{(1+\hat{t}_i)^{m_i-1}}{1-x}=(1+\hat{t}_i)^{m_i-1}\sum_{p=0}^{\infty}x^p$$

$$=(1+\hat{t}_i)^{m_i-1}\sum_{p=0}^{m_i-1}x^p=\sum_{j=0}^{m_i-1}(1+\hat{t}_i)^{m_i-1-j}t_i^j.$$

This establishes (2.11). Equation (2.12) follows from formula (2.2). Note that the degree of the polynomial in $\mathbf{t}:=(t_1,\dots,t_d)$ appearing on the right-hand side of (2.11) is $m_i-1$. The polynomial $\sum_{j=0}^{m_i-1}\hat{t}_i^{m_i-1-j}t_i^j=\frac{\hat{t}_i^{m_i}-t_i^{m_i}}{\hat{t}_i-t_i}$ is the homogeneous polynomial of degree $m_i-1$ appearing on the right-hand side of (2.11). Hence, the homogeneous polynomial of degree $\dim\Pi(\mathbf{m})$ of the right-hand side of (2.12) is $\prod_{i\in[d]}\frac{\hat{t}_i^{m_i}-t_i^{m_i}}{\hat{t}_i-t_i}$. Assuming the relations $t_i^{m_i}=0$, $i\in[d]$, we obtain that this polynomial is $c(\mathbf{m})\prod_{i\in[d]}t_i^{m_i-1}$. This is equivalent to the statement that $c(\mathbf{m})$ is the coefficient of ) $\prod_{i\in[d]}t_i^{m_i-1}$ in the polynomial $\prod_{i\in[d]}\frac{\hat{t}_i^{m_i}-t_i^{m_i}}{\hat{t}_i-t_i}$, where we do not assume the relations $t_i^{m_i}=0$, $i\in[d]$. □

### 2.5 Bertini-Type Theorems

Let $M$ be a compact complex manifold and $E$ a holomorphic bundle on $M$. A holomorphic section $\sigma$ of $E$ on an open set $U\subset E$ is a holomorphic map $\sigma:U\to E$, where $E$ is viewed as a complex manifold. Specifically, let $U_i$, $i\in[N]$ be the finite cover of $M$ such that the bundle $E$ restricted to $U_i$ is $U_i\times\mathbb{C}^k$ with the standard basis $[\mathbf{e}_{1,i},\dots,\mathbf{e}_{k,i}]$, as in §2.2. Then $\sigma(\zeta)=\sum_{j=1}^k\sigma_{j,i}(\zeta)\mathbf{e}_{j,i}$ for $\zeta\in U\cap U_i$, where $\sigma_{j,i}(\zeta)$, $j\in[k]$ are analytic on $U\cap U_i$. $\sigma$ is called a global section if $U=M$. Denote by $\mathrm{H}^0(E)$ the linear space of global sections on $E$. A subspace $\mathbf{V}\subset\mathrm{H}^0(E)$ is said to generate $E$ if $\mathbf{V}(\zeta)$, the value of all sections in $\mathbf{V}$ at each $\zeta\in M$, is equal to $E_\zeta$.

The following proposition is a generalization of the classical Bertini's theorem in algebraic geometry, and it is a standard consequence of the generic smoothness theorem. For the convenience of the reader we state and give a short proof of this proposition.

**Theorem 4** ("Bertini-type" theorem) *Let $E$ be a vector bundle on $M$. Let $\mathbf{V} \subset \mathrm{H}^0(E)$ be a subspace that generates $E$. Then the following statements hold:*

1. *If* rank $E >$ dim $M$ *for the generic $\sigma \in \mathbf{V}$, then the zero locus of $\sigma$ is empty.*
2. *If* rank $E \leq$ dim $M$ *for the generic $\sigma \in \mathbf{V}$, then the zero locus of $\sigma$ is either smooth of codimension* rank $E$ *or it is empty.*
3. *If* rank $E =$ dim $M$, *then the zero locus of the generic $\sigma \in V$ consists of $\nu(E)$ simple points, where $\nu(E)$ is the top Chern number of $E$.*

*Proof* We identify the vector bundle $E$ with its locally free sheaf of sections; see [8, B.3]. We have the projection $E \xrightarrow{\pi} M$, where the fiber $\pi^{-1}(\zeta)$ is isomorphic to the vector space $E_\zeta$. Let $\Pi \subset E$ be the zero section. By assumption we have a natural projection of maximal rank

$$M \times \mathbf{V} \xrightarrow{p} E.$$

Let $Z = p^{-1}(\Pi)$; then $Z$ is isomorphic to the variety $\{(\zeta, \sigma) \in M \times \mathbf{V} | \sigma(\zeta) = 0\}$ and it has dimension equal to dim $M +$ dim $\mathbf{V} -$ rank $E$. Consider the natural projection $Z \xrightarrow{q} \mathbf{V}$; now $\forall \sigma \in V$ the fiber $q^{-1}(\sigma)$ is naturally isomorphic to the zero locus of $\sigma$. We have two cases. If $q$ is dominant (namely the image of $q$ is dense), then by the generic smoothness theorem [12, Corollary III 10.7] $q^{-1}(\sigma)$ is smooth of dimension dim $X -$ rank $E$ for generic $\sigma$.

If $q$ is not dominant (and this always happens in the case rank $E >$ dim $M$), then $q^{-1}(\sigma)$ is empty for generic $\sigma$. This concludes the proof of the first two parts. The third part follows from [8, Example 3.2.16]. □

For our purposes we need the following refinement of Theorem 4.

**Definition 5** Let $\pi : E \to M$ be a vector bundle on a smooth projective variety $M$ such that rank $E \geq$ dim $M$. Let $\mathbf{V} \subset \mathrm{H}^0(E)$ be a subspace. Then $\mathbf{V}$ almost generates $E$ if the following conditions hold. Either $\mathbf{V}$ generates $E$ (in this case $k = 0$) or there exists $k \geq 1$ smooth strict irreducible subvarieties $Y_1, \dots, Y_k$ of $M$ satisfying the following properties. First, on each $Y_j$ there is a vector bundle $E_j$. Second, after assuming $Y_0 = M$ and $E_0 = E$, the following conditions hold:

1. rank $E_j >$ dim $Y_j$ for each $j \geq 1$.
2. Let $\pi_j : E_j \to Y_j$, and for any $i, j \geq 0$ assume that $Y_i$ is a subvariety of $Y_j$. Then $E_i$ is a subbundle of $E_{j|Y_i}$.
3. $\mathbf{V}(\zeta) \subset (E_j)_\zeta$ for $\zeta \in Y_j$.
4. Denote by $P_j \subset [k]$ the set of all $i \in [k]$ such that $Y_i$ are strict subvarieties of $Y_j$. Then $\mathbf{V}(\zeta) = (E_j)_\zeta$ for $\zeta \in Y_j \setminus \cup_{i \in P_j} Y_i$.

**Theorem 6** *Let $E$ be a vector bundle on a smooth projective variety $M$. Assume that* rank $E \geq$ dim $M$. *Let $\mathbf{V} \subset \mathrm{H}^0(E)$ be a subspace that almost generates $E$. Then the following conditions hold:*

1. *If* rank $E > \dim M$*, then for a generic* $\sigma \in \mathbf{V}$ *the zero locus of* $\sigma$ *is empty.*
2. *If* rank $E = \dim M$*, then the zero locus of a generic* $\sigma \in \mathbf{V}$ *consists of* $\nu(E)$ *simple points lying outside* $\cup_{j\in[k]} Y_j$*, where* $\nu(E)$ *is the top Chern number of* $E$*.*

*Proof* As in the proof of Theorem 4 we consider the variety

$$Z = \{(\zeta, \sigma) \in M \times \mathbf{V} | \sigma(\zeta) = 0\}.$$

We consider the two projections

$$\begin{array}{ccc} & Z & \\ \swarrow p & & \searrow q \\ M & & \mathbf{V} \end{array}$$

The fiber $q^{-1}(v)$ can be identified with the zero locus of $v$. If $\zeta \in Y_k$, then, by *4* of Definition 5, the fibers $p^{-1}(\zeta)$ can be identified with a subspace of $\mathbf{V}$ having codimension rank $E_k$. It follows that the dimension of $p^{-1}(Y_k)$ is equal to $\dim \mathbf{V} -$ rank $E_k + \dim Y_k$, which, by *1* of Definition 5, is strictly smaller than $\dim \mathbf{V}$ if $k \geq 1$. Let $Y = \cup_{k\geq 1} Y_k$. Then $p^{-1}(X \setminus Y) \subset Z$ is a fibration and it is smooth. Call $\overline{q}$ the restriction of $q$ to $p^{-1}(X \setminus Y)$. If rank $E > \dim M$, then we obtain that $\overline{q}$ is not dominant and the generic fiber $\overline{q}^{-1}(v)$ is empty. If rank $E = \dim M$, by the generic smoothness theorem applied to $\overline{q}\colon p^{-1}(X \setminus Y) \to \mathbf{V}$, we obtain that there exists $V_0 \subset \mathbf{V}$, with $V_0$ open, such that the fiber $\overline{q}^{-1}(v)$ is smooth for $v \in V_0$.

Moreover, the dimension count yields that $q(p^{-1}(Y))$ is a closed proper subset of $\mathbf{V}$ (note that $q$ is a proper map). Call $V_1 = \mathbf{V} \setminus q(p^{-1}(Y))$, again open.

It follows that for $v \in V_0 \cap V_1$ the fiber $q^{-1}(v)$ coincides with the fiber $\overline{q}^{-1}(v)$, which is smooth by the previous argument, given by finitely many simple points. The number of points is $\nu(E)$, again by [8, Example 3.2.16]. □

## 3 Number of Singular Vector Tuples of a Generic Tensor

In this section we compute the number of singular vector tuples of a generic tensor $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$. In what follows we need the following two lemmas. The first one is well known, and we leave its proof to the reader. Denote by $Q_m := \{\mathbf{x} \in \mathbb{C}^m, \mathbf{x}^\top \mathbf{x} = 0\}$ the quadric of isotropic vectors.

**Lemma 7** *Let* $\mathbf{x} \in \mathbb{C}^m \setminus \{\mathbf{0}\}$*, and use the notation* $\mathbf{U} := \mathbb{C}^m/[\mathbf{x}]$*. For* $\mathbf{y} \in \mathbb{C}^m$ *denote by* $[[\mathbf{y}]]$ *the element in* $\mathbf{U}$ *induced by* $\mathbf{y}$*. Then the following statements hold:*

1. *Any linear functional* $\mathbf{g} : \mathbf{U} \to \mathbb{C}$ *is uniquely represented by* $\mathbf{w} \in \mathbb{C}^m$ *such that* $\mathbf{w}^\top \mathbf{x} = 0$ *and* $\mathbf{g}([[\mathbf{y}]]) = \mathbf{w}^\top \mathbf{y}$*. In particular, if* $\mathbf{x} \in Q_m$*, then the functional* $\mathbf{g}_\mathbf{x} : \mathbf{U} \to \mathbb{C}$ *given by* $g([[\mathbf{y}]]) = \mathbf{x}^\top \mathbf{y}$ *is a linear functional.*
2. *Suppose that* $\mathbf{x} \notin Q_m$ *and* $a \in \mathbb{C}$ *is given. Then for each* $\mathbf{y} \in \mathbb{C}^m$ *there exists a unique* $\mathbf{z} \in \mathbb{C}^m$ *such that* $[[\mathbf{z}]] = [[\mathbf{y}]]$ *and* $\mathbf{x}^\top \mathbf{z} = a$*.*

**Lemma 8** *Let* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$*. Assume that* $\mathbf{x}_i \in \mathbb{F}^{m_i} \setminus \{\mathbf{0}\}, \mathbf{y}_i \in \mathbb{F}^{m_i}$ *are given for* $i \in [d]$*.*

1. *There exists* $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$ *satisfying*

$$\mathcal{T} \times \otimes_{j\in[d]\setminus\{i\}} \mathbf{x}_j = \mathbf{y}_i, \tag{3.1}$$

*for any* $i \in [d]$ *if and only if the following compatibility conditions hold:*

$$\mathbf{x}_1^\top \mathbf{y}_1 = \ldots = \mathbf{x}_d^\top \mathbf{y}_d. \tag{3.2}$$

2. *Let* $P \subset [d]$ *be the set of all* $p \in [d]$ *such that* $\mathbf{x}_p$ *is isotropic. Consider the following system of equations*

$$[[\mathcal{T} \times \otimes_{j\in[d]\setminus\{l\}} \mathbf{x}_j]] = [[\mathbf{y}_l]] \tag{3.3}$$

*for any* $l \in [d]$. *Then there exists* $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$ *satisfying* (3.3) *if and only if one of the following conditions holds:*

a $|P| \le 1$, *i.e., there exists at most one isotropic vector in* $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.
b $|P| = k \ge 2$. *Assume that* $P = \{i_1, \ldots, i_k\}$. *Then*

$$\mathbf{x}_{i_1}^\top \mathbf{y}_{i_1} = \mathbf{x}_{i_2}^\top \mathbf{y}_{i_2} = \ldots = \mathbf{x}_{i_k}^\top \mathbf{y}_{i_k}. \tag{3.4}$$

3. *Fix* $i \in [d]$. *Let* $P \subset [d]\setminus\{i\}$ *be the set of all* $p \in [d]\setminus\{i\}$ *such that* $\mathbf{x}_p$ *is isotropic. Then there exists* $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$ *satisfying condition* (3.1) *and conditions* (3.3) *for all* $l \in [d] \setminus \{i\}$ *if and only if one of the following conditions holds:*

a $|P| = 0$.
b $|P| = k - 1 \ge 1$. *Assume that* $P = \{i_1, \ldots, i_{k-1}\}$. *Let* $i_k = i$. *Then* (3.4) *hold.*

*Proof 1.* Assume first that (3.1) holds. Then $\mathcal{T} \times \otimes_{j\in[d]} \mathbf{x}_j = \mathbf{x}_i^\top \mathbf{y}_i$ for $i \in [d]$. Hence, (3.2) holds. Suppose now that (3.2) holds. We now show that there exists $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$ satisfying (3.1).

Let $U_j = [u_{pq,j}]_{p=q=1}^{m_j} \in \mathbf{GL}(m_j, \mathbb{F})$ for $j \in [d]$. Let $U := \otimes_{i\in[d]} U_i$. Then $U$ acts on $\mathbb{F}^{\mathbf{m}}$ as a matrix acting on the corresponding vector space. That is, let $\mathcal{T}' = U\mathcal{T}$, and assume that $\mathcal{T} = [t_{i_1,\ldots,i_d}]$, $\mathcal{T}' = [t'_{j_1,\ldots,j_p}]$. Then

$$t'_{j_1,\ldots,j_d} = \sum_{i_1\in[m_1],\ldots,i_d\in[m_d]} u_{j_1 i_1,1} \ldots u_{j_d i_d,d} t_{i_1,\ldots,i_d}, \quad j_1 \in [m_1], \ldots, j_d \in [m_d].$$

The conditions (3.1) for $\mathcal{T}'$ become

$$\mathcal{T}' \times \otimes_{j\in[d]\setminus\{i\}} \mathbf{x}'_j = \mathbf{y}'_i,\ i \in [d], \quad \mathbf{x}'_i = (U_i^\top)^{-1} \mathbf{x}_i,\ \mathbf{y}'_i = U_i \mathbf{y}_i,\ i \in [d]. \tag{3.5}$$

Clearly, $\mathbf{x}_i^\top \mathbf{y}_i = (\mathbf{x}'_i)^\top \mathbf{y}'_i$ for $i \in [d]$. Since $\mathbf{x}_i \neq 0$, there exists $U_i \in \mathbf{GL}(m_i, \mathbb{F})$ such that $(U_i^\top)^{-1} \mathbf{x}_i = \mathbf{e}_{1,i} = (1, 0, \ldots, 0)^\top$ for $i \in [d]$. Hence, it is enough to show that (3.1) is satisfied for some $\mathcal{T}$ if $\mathbf{x}_i = \mathbf{e}_{i,1}$ for $i \in [d]$ if $\mathbf{e}_{1,1}^\top \mathbf{y}_1 = \ldots = \mathbf{e}_{d,1}^\top \mathbf{y}_d$. Let $\mathbf{y}_i = (y_{1,i}, \ldots, y_{m_i,i})^\top$ for $i \in [d]$. Then conditions (3.2) imply that $y_{1,1} = \ldots = y_{1,d}$.

Choose a suitable $T = [t_{i_1,\ldots,i_d}]$ as follows. $t_{i_1,\ldots,i_d} = y_{i_j,j}$ if $i_k = 0$ for $k \neq j$, $t_{i_1,\ldots,i_d} = 0$ otherwise. Then (3.1) holds.

*2*. We now consider system (3.3). This system is solvable if and only if we can find $t_1, \ldots, t_d \in \mathbb{F}$ such that

$$\mathbf{x}_1^\top(\mathbf{y}_1 + t_1\mathbf{x}_1) = \ldots = \mathbf{x}_d^\top(\mathbf{y}_d + t_d\mathbf{x}_d). \tag{3.6}$$

Suppose first that $\mathbf{x}_i \notin Q_{m_i}$ for $i \in [d]$. Fix $a \in \mathbb{F}$. Choose $t_i = \frac{a-\mathbf{x}_i^\top\mathbf{y}_i}{\mathbf{x}_i^\top\mathbf{x}_i}$ for $i \in [d]$. Hence system (3.3) is solvable. Suppose next that $\mathbf{x}_j \in Q_{m_j}$. Then $\mathbf{x}_j^\top(\mathbf{y}_j + t_j\mathbf{x}_j) = \mathbf{x}_j^\top\mathbf{y}_j$. Assume that $P = \{j\}$. Let $a = \mathbf{x}_j^\top\mathbf{y}_j$. Choose $t_i, i \neq j$ as above to deduce that (3.6) holds. Hence, (3.3) is solvable.

Assume finally that $k \geq 2$ and $P = \{i_1, \ldots, i_k\}$. Equation (3.6) yields that if (3.3) is solvable, then (3.4) holds. Suppose that (3.4) holds. Let $a = \mathbf{x}_{i_1}^\top\mathbf{y}_{i_1} = \ldots = \mathbf{x}_{i_k}^\top\mathbf{y}_{i_k}$. For $i \notin P$ let $t_i = \frac{a-\mathbf{x}_i^\top\mathbf{y}_i}{\mathbf{x}_i^\top\mathbf{x}_i}$ to deduce that condition (3.6) holds. Hence, (3.3) is solvable.

*3*. Consider Eq. (3.1) and Eqs. (3.3) for $l \in [d] \setminus \{i\}$. Then this system is solvable if and only if system (3.6) is solvable for $t_i = 0$ and some $t_l \in \mathbb{F}$ for $l \in [d] \setminus \{i\}$. Let $a = \mathbf{x}_i^\top\mathbf{y}_i$. Assume that $|P| = 0$. Choose $t_l = \frac{a-\mathbf{x}_l^\top\mathbf{y}_l}{\mathbf{x}_l^\top\mathbf{x}_l}$ for $l \in [d] \setminus \{i\}$ as above to deduce that this system is solvable. Assume that $P = \{i_1, \ldots, i_{k-1}\}$ for $k \geq 2$. Suppose this system is solvable for some $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$. Then $a = \mathbf{x}_j^\top\mathbf{y}_j$ for each $j \in P$. Let $i_k := i$. Hence (3.6) holds. Conversely, assume that (3.6) holds. Choose $t_l = \frac{a-\mathbf{x}_l^\top\mathbf{y}_l}{\mathbf{x}_l^\top\mathbf{x}_l}$ for $l \notin P \cup \{i\}$. Then (3.6) holds. Hence, our system is solvable. □

**Lemma 9** *Let $R(i, \mathbf{m})$ and $R(\mathbf{m})$ be the vector bundles over the Segre product $\Pi(\mathbf{m})$ defined in (2.6). Denote by $\mathrm{H}^0(R(i, \mathbf{m}))$ and $\mathrm{H}^0(R(\mathbf{m}))$ the linear space of global sections of $R(i, \mathbf{m})$ and $R(\mathbf{m})$, respectively. Then the following conditions hold:*

1. *For each $i \in [d]$ there exists a monomorphism $L_i : \mathbb{C}^{\mathbf{m}} \to \mathrm{H}^0(R(i, \mathbf{m}))$ such that $L_i(\mathbb{C}^{\mathbf{m}})$ generates $R(i, \mathbf{m})$ (see §2.5).*
2. *$L = (L_1, \ldots, L_d)$ is a monomorphism of the direct sum of $d$ copies of $\mathbb{C}^{\mathbf{m}}$ (denoted by $\oplus^d\mathbb{C}^{\mathbf{m}}$) to $\mathrm{H}^0(R(\mathbf{m}))$, which generates $R(\mathbf{m})$.*
3. *Let $\delta : \mathbb{C}^{\mathbf{m}} \to \oplus^d\mathbb{C}^{\mathbf{m}}$ be the diagonal map $\delta(\mathcal{T}) = (\mathcal{T}, \ldots, \mathcal{T})$. Consider $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi(\mathbf{m})$.*

*(a) If at most one of $\mathbf{x}_1, \ldots, \mathbf{x}_d$ is isotropic, then $L \circ \delta(\mathbb{C}^{\mathbf{m}})$ [as a space of sections of $R(\mathbf{m})$] generates $R(\mathbf{m})$ at $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$.*

*(b) Let $P \subset [d]$ be the set of all $i \in [d]$ such that $\mathbf{x}_i$ is isotropic. Assume that $P = \{i_1, \ldots, i_k\}$, where $k \geq 2$. Let $\mathbf{g}_{\mathbf{x}_{i_p}}$ be the linear functional on the fiber of $\pi_{i_p}^* Q(m_{i_p})$ at $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ as defined in Lemma 7 for $p = 1, \ldots, k$. Let $\mathbf{U}(P)$ be the subspace of all linear transformations $\tau = (\tau_1, \ldots, \tau_d) \in R(\mathbf{m})_{([\mathbf{x}_1],\ldots,[\mathbf{x}_d])}$, $\tau_i \in R(i, \mathbf{m})_{([\mathbf{x}_1],\ldots,[\mathbf{x}_d])}$, $i \in [d]$ satisfying*

$$\mathbf{g}_{\mathbf{x}_{i_1}}(\tau_{i_1}(\otimes_{j\in[d]\setminus\{i_1\}}\mathbf{x}_j)) = \ldots = \mathbf{g}_{\mathbf{x}_{i_k}}(\tau_{i_k}(\otimes_{j\in[d]\setminus\{i_k\}}\mathbf{x}_j)). \tag{3.7}$$

*Then $L \circ \delta(\mathcal{T})([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \mathbf{U}(P)$ for each $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$. Furthermore, $L \circ \delta(\mathbb{C}^{\mathbf{m}})([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) = \mathbf{U}(P)$.*

*Proof* For $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ we define the section $L_i(\mathcal{T})(([\mathbf{x}_1], \ldots, [\mathbf{x}_d])) \in R(i, \mathbf{m})_{([\mathbf{x}_1],\ldots,[\mathbf{x}_d])}$ as follows:

$$L_i(\mathcal{T})(([\mathbf{x}_1], \ldots, [\mathbf{x}_d]))(\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j) := [[\mathcal{T} \times \otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j]]. \tag{3.8}$$

It is straightforward to check that $L_i(\mathcal{T})$ is a global section of $R(i, \mathbf{m})$.

Assume $\mathcal{T} \neq 0$. Then there exist $\mathbf{v}_j \in \mathbb{C}^{m_j}$, $j \in [d]$ such that $\mathcal{T} \times \otimes_{j\in[d]}\mathbf{v}_j \neq 0$. Hence, $\mathbf{u}_i := \mathcal{T} \times \otimes_{j\in[d]\setminus\{i\}}\mathbf{v}_j \in \mathbb{C}^{m_i} \setminus \{\mathbf{0}\}$. Let $\mathbf{x}_j = \mathbf{v}_j$ for $j \neq i$. Choose $\mathbf{x}_i \in \mathbb{C}^{m_i} \setminus \{[\mathbf{u}_i]\}$. Then $L_i(\mathcal{T})(([\mathbf{x}_1], \ldots, [\mathbf{x}_d])) \neq 0$. Hence, $L_i$ is injective.

We now show that $L_i(\mathbb{C}^{\mathbf{m}})$ generates $R(i, \mathbf{m})$. Let $\mathbf{y}_i \in \mathbb{C}^{m_i}$. Choose $\mathbf{g}_j \in \mathbb{C}^{m_j}$ such that $\mathbf{g}_j^\top \mathbf{x}_j = 1$ for $j \in [d]$. Set $\mathcal{T} = (\otimes_{j\in[i-1]}\mathbf{g}_j) \otimes \mathbf{y}_i \otimes (\otimes_{j\in[d]\setminus[i]}\mathbf{g}_j)$. Then $L_i(\mathcal{T})(([\mathbf{x}_1], \ldots, [\mathbf{x}_d])) = [[\mathbf{y}_i]]$. This proves *1*.

Define $L((\mathcal{T}_1, \ldots, \mathcal{T}_d))(([\mathbf{x}_1], \ldots, [\mathbf{x}_d])) = \oplus_{i\in[d]}L_i(\mathcal{T}_i)(([\mathbf{x}_1], \ldots, [\mathbf{x}_d]))$. Then $L((\mathcal{T}_1, \ldots, \mathcal{T}_d)) \in \mathrm{H}^0(R(\mathbf{m}))$. Clearly $L$ is a monomorphism. Furthermore, $L(\oplus^d \mathbb{C}^{\mathbf{m}})$ generates $\mathbb{R}(\mathbf{m})$. This shows *2*.

Cases *3a* and *3b* of our lemma follow from parts *2a* and *2b* of Lemma 8, respectively. □

Künneth's formula [14] yields the equalities

$$L_i(\mathbb{C}^{\mathbf{m}}) = \mathrm{H}^0(R(i, \mathbf{m})), \ i \in [d], \quad L(\oplus^d \mathbb{C}^{\mathbf{m}}) = H^0(R(\mathbf{m})). \tag{3.9}$$

The following result is a corollary to Lemma 8.

**Corollary 10** *Assume that* $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi(\mathbf{m})$ *is a singular vector tuple of a tensor* $\mathcal{T}$ *corresponding to a nonzero singular value. Then one of the following holds:*

1. *All* $\mathbf{x}_i$ *are isotropic.*
2. *All* $\mathbf{x}_i$ *are nonisotropic.*

For $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ with a real singular vector tuple $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi_{\mathbb{R}}(\mathbf{m})$ the condition $\prod_{i\in[d]} \lambda_i = 0$ implies that $\lambda_i = 0$ for each $i$. Indeed, since $\mathbf{x}_i \in \mathbb{R}^{m_i} \setminus \{\mathbf{0}\}$, it follows from (1.2) that $\lambda_i = \frac{\mathcal{T}\times\otimes_{j\in[d]}\mathbf{x}_j}{\mathbf{x}_i^\top\mathbf{x}_i}$ for each $i \in [d]$. Thus $\lambda_k = 0$ for some $k \in [d]$ yields that $\mathcal{T} \times \otimes_{j\in[d]}\mathbf{x}_j = 0$. Hence, each $\lambda_i = 0$.

However, this observation is not valid for complex tensors, already in the case of complex-valued matrices ($d = 2$); see subsequent example. It is straightforward to see that a singular vector pair $([\mathbf{x}_1], [\mathbf{x}_2])$ of $A \in \mathbb{C}^{m_1\times m_2}$ is given by the following conditions:

$$A\mathbf{x}_2 = \lambda_1\mathbf{x}_1, \quad A^\top\mathbf{x}_1 = \lambda_2\mathbf{x}_2, \quad \mathbf{x}_i \in \mathbb{C}^{m_i} \setminus \{\mathbf{0}\}, \lambda_i \in \mathbb{C}, \quad i = 1, 2. \tag{3.10}$$

Consider the following simple example:

$$A = \begin{bmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ \mathbf{i} \end{bmatrix}.$$

Then $A^\top\mathbf{x}_1 = \mathbf{x}_2$, $A\mathbf{x}_2 = \mathbf{0}$, i.e., $\lambda_1 = 1, \lambda_2 = 0$.

**Lemma 11** *Let $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$, and consider the section $\hat{\mathcal{T}} := L \circ \delta(\mathcal{T}) \in \mathrm{H}^0(R(\mathbf{m}))$. We have that $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi(\mathbf{m})$ is a zero of $\hat{\mathcal{T}}$ if and only if $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ is a singular vector tuple corresponding to $\mathcal{T}$.*

*Proof* Suppose first that $\hat{\mathcal{T}}(([\mathbf{x}_1], \ldots, [\mathbf{x}_d])) = 0$. Then $L_i(\mathcal{T})(([\mathbf{x}_1], \ldots, [\mathbf{x}_d]))$ is a zero vector in the fiber $R(i, \mathbf{m})$ at $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$. Suppose first that $\mathcal{T} \times (\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j) \neq \mathbf{0}$. Then $\mathcal{T} \times (\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j) = \lambda_i\mathbf{x}_i$ for some $\lambda_i \neq 0$. Otherwise, the previous equality holds with $\lambda_i = 0$. Hence, $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ is a singular vector tuple corresponding to $\mathcal{T}$. Conversely, it is straightforward to see that if $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ is a singular vector tuple corresponding to $\mathcal{T}$, then the section $\hat{\mathcal{T}}$ vanishes at $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi(\mathbf{m})$. □

We now present the proof of Theorem 1, which was stated in § 1.

**Proof of Theorem 1.** Let $V = L\circ\delta(\mathbb{C}^{\mathbf{m}})$ be the subspace of sections of $R(\mathbf{m})$ given by tensors (embedded diagonally). We now show that $\mathbf{V}$ almost generates $R(\mathbf{m})$ as defined in Definition 5. First, rank $R(\mathbf{m}) = \dim \Pi(\mathbf{m})$. Second, let $2^{[d]_k}$ be the set of all subsets of $[d]$ of cardinality $k$ for each $k \in [d]$. Let $\alpha \in 2^{[d]_k}$. Define $Y_\alpha = X_1 \times \ldots X_d$, where $X_i = \mathbb{P}(Q_{m_i})$ if $i \in \alpha$ and $X_i = \mathbb{P}(\mathbb{C}^{m_i})$ otherwise. Clearly, $Y_\alpha$ is a strict smooth subvariety of $\Pi(\mathbf{m})$ of codimension $k$. Note that $Y_\beta \subsetneq Y_\alpha$ if and only if $\alpha \subsetneq \beta$. We now define the subbundle $E_\alpha$ of $\pi^{-1}(Y_\alpha)$. If $\alpha \in 2^{[d]_1}$, then $E_\alpha = \pi^{-1}(Y_\alpha)$. Assume now that $k > 1$. Let $\alpha = \{i_1, \ldots, i_k\}$. Let $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in Y_\alpha$. Thus, $\mathbf{x}_{i_l} \in Q_{m_{i_l}}$ for $l = 1, \ldots, k$. Then the fiber $E_\alpha$ at $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ is the set of all vectors satisfying (3.7). Note that rank $E_\alpha = \dim Y_\alpha + 1$. Assume that $\alpha \subsetneq \beta$. Clearly, $E_\beta$ is a strict subbundle of $\pi_\alpha^{-1}(Y_\beta)$. Hence conditions *1* and *2* of Definition 5 hold. Lemma 9 implies that conditions *3* and *4* of Definition 5 hold. Theorem 6 implies that for a generic $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ the section $L \circ \delta(\mathcal{T})$ has a finite number of simple zeros. Moreover, this number is equal to the top Chern number of $R(\mathbf{m})$. Lemma 3 yields that the top Chern number of $R(\mathbf{m})$ is $c(\mathbf{m})$.

It remains to show that a generic $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ does not have a zero singular value. Fix $i \in [d]$, and consider the set of all $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ that have a singular vector tuple $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in \Pi(\mathbf{m})$ with $\lambda_i = 0$.

Let $R(i, \mathbf{m})'$ and $R_i(\mathbf{m})'$ be defined in (2.6). Similar to definition (3.8), we can define a monomorphism $L_i' : \mathbb{C}^{\mathbf{m}} \to \mathrm{H}^0(R(i, m)')$ by the equality

$$L_i'(\mathcal{T})(([\mathbf{x}_1], \ldots, [\mathbf{x}_d]))(\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j) := \mathcal{T} \times \otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j.$$

Let $\tilde{L}_i = (L_1, \ldots, L_{i-1}, L_i', L_{i+1}, \ldots, L_d) : \oplus_{j\in[d]}\mathbb{C}^{\mathbf{m}} \to \mathrm{H}^0(R_i(\mathbf{m})')$.

We claim that $\tilde{L}_i \circ \delta(\mathbb{C}^{\mathbf{m}})$ almost generates $R_i(\mathbf{m})'$. Clearly, rank $R_i(\mathbf{m})' = \dim \Pi(\mathbf{m})+1$. Recall that a vector $(\tau_1, \ldots, \tau_d) \in R_i(\mathbf{m})'_{([\mathbf{x}_1],\ldots,[\mathbf{x}_d])}$ is of the form

$$\tau_j : \hat{T}(\mathbf{m}_j) \to \pi_j^* Q(m_j) \text{ for } j \in [d] \setminus \{i\}, \quad \tau_i : \hat{T}(\mathbf{m}_i) \to \pi_j^* F(m_i). \tag{3.11}$$

Let $\alpha \subset [d] \setminus \{i\}$ be a nonempty set. Then $Y_\alpha = X_1 \times \ldots \times X_d$, where $X_j = \mathbb{P}(Q_{m_j})$ if $j \in \alpha$ and $X_j = \mathbb{P}(\mathbb{C}^{m_j})$ if $j \notin \alpha$. [Note that $X_i = \mathbb{P}(\mathbb{C}^{m_i})$.] We now define the vector bundles $\pi_\alpha : E_\alpha \to Y_\alpha$. Let $\pi : R_i(\mathbf{m})' \to \Pi(\mathbf{m})$. Assume that $\alpha = \{i_1, \ldots, i_{k-1}\} \subset [d] \setminus \{i\}$, where $k - 1 \geq 1$. Then $E_\alpha$ is the subbundle $\pi^{-1}(Y_\alpha)$

defined as follows. For $([\mathbf{x}_1], \ldots, [\mathbf{x}_d]) \in Y_\alpha$ it consists of all sections of the form (3.11) satisfying a variation of the condition (3.7):

$$\mathbf{g}_{\mathbf{x}_{i_1}}(\tau_{i_1}(\otimes_{j\in[d]\setminus\{i_1\}}\mathbf{x}_j)) = \ldots = \mathbf{g}_{\mathbf{x}_{i_{k-1}}}(\tau_{i_{k-1}}(\otimes_{j\in[d]\setminus\{i_{k-1}\}}\mathbf{x}_j)) = \mathbf{x}_i^\top \tau_i(\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j).$$

Note that rank $E_\alpha = \dim Y_\alpha + 1$. Clearly, the conditions of *1–2* of Definition 5 hold. Part *3* of Lemma 8 implies conditions *3* and *4* of Definition 5. Theorem 6 yields that a generic section of $\tilde{L}_i \circ \delta(\mathcal{T})$ does not have zero. Thus, $\mathcal{T}$ does not have a singular vector tuple satisfying (1.2) with $\lambda_i = 0$. Hence, a generic tensor $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ does not have a zero singular value.

Clearly, a generic $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ has exactly $c(\mathbf{m})$ simple complex-valued singular vector tuples. Only some of those can be realized as points in $\Pi_{\mathbb{R}}(\mathbf{m})$. □

We first observe that Theorem 1 agrees with the standard theory of singular values for $m \times n$ real matrices. That is, a generic $A \in \mathbb{R}^{m\times n}$ has exactly $\min(m, n)$ nonzero singular values all of which are positive and pairwise distinct. The corresponding singular vector pairs are simple.

We now point out a matrix proof of Theorem 1 for $d = 2$. Let $\mathrm{O}(m) \subset \mathbb{C}^{m\times m}$ be a variety of $m \times m$ orthogonal matrices and $\mathrm{D}_{m,n} \subset \mathbb{C}^{m\times n}$ the linear subspace of all diagonal matrices. Consider the trilinear polynomial map $F : \mathrm{O}(m_1) \times \mathrm{D}_{m_1,m_2} \times \mathrm{O}(m_2) \to \mathbb{C}^{m_1\times m_2}$ given by $(U_1, D, U_2) \mapsto U_1 D U_2^\top$. Singular value decomposition yields that any $A \in \mathbb{R}^{m_1\times m_2}$ is of the form $U_1 D U_2^\top$, where $U_1, U_2$ are real orthogonal and $D$ is a nonnegative diagonal matrix. Hence, $F(\mathrm{O}(m_1) \times \mathrm{D}_{m_1,m_2} \times \mathrm{O}(m_2)) = \mathbb{R}^{m_1\times m_2}$. Therefore, the image of $F$ is dense in $\mathbb{C}^{m_1\times m_2}$. Hence, a generic $A \in \mathbb{C}^{m_1\times m_2}$ is of the form $U_1^\top D U_2$. Furthermore, we can assume that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_l)$, $l = \min(m_1, m_2)$, where the diagonal entries are nonzero and pairwise distinct. Assume that $\mathbf{x}_i, \mathbf{y}_i$ are the $i$th columns of $U_1, U_2$ respectively for $i = 1, \ldots, l$. Then $([\mathbf{x}_i], [\mathbf{y}_i])$ is a simple singular vector tuple corresponding to $\lambda_i$ for $i = 1, \ldots, l$.

We list for the convenience of the reader a few values $c(\mathbf{m})$. First,

$$c(\underbrace{2, \ldots, 2}_{d}) = d! \tag{3.12}$$

Indeed, $\frac{\hat{t}_i^2 - t_i^2}{\hat{t}_i - t_i} = (\hat{t}_i + t_i) = \sum_{j\in[d]} t_j$. Therefore, $\prod_{j\in[d]} \frac{\hat{t}_i^2 - t_i^2}{\hat{t}_i - t_i} = (\sum_{j\in[d]} t_j)^d$. Clearly, the coefficient of $t_1 \ldots t_d$ in this polynomial is $d!$.

Second, we list in Table 1 the first values in the case where $d = 3$. From this table one sees that $c(m_1, m_2, m_3)$ stabilizes for $m_3 \geq m_1 + m_2 - 1$, and the case where equality holds is called the boundary format case in the theory of hyperdeterminants ([9]). It is the case where a “diagonal” naturally occurs, as in Fig. 1:

In the $d = 2$ case, a boundary format means a square.

## 4 Partially Symmetric Singular Vector Tuples

For an integer $m \geq 2$ let $m^{\times d} := (\underbrace{m, \ldots, m}_{d})$. Then $\mathcal{T} \in \mathbb{F}^{m^{\times d}}$ is called a $d$-cube, or simply a cube tensor. Denote by $\mathrm{S}^d(\mathbb{F}^m) \subset \mathbb{F}^{m^{\times d}}$ the subspace of symmetric

**Table 1** Values of $c(d_1, d_2, d_3)$

| $d_1, d_2, d_3$ | $c(d_1, d_2, d_3)$ | |
|---|---|---|
| 2, 2, 2 | 6 | |
| 2, 2, n | 8 | $n \geq 3$ |
| 2, 3, 3 | 15 | |
| 2, 3, $n$ | 18 | $n \geq 4$ |
| 2, 4, 4 | 28 | |
| 2, 4, $n$ | 32 | $n \geq 5$ |
| 2, 5, 5 | 45 | |
| 2, 5, $n$ | 50 | $n \geq 6$ |
| $2, m, m+1$ | $2m^2$ | |
| 3, 3, 3 | 37 | |
| 3, 3, 4 | 55 | |
| 3, 3, $n$ | 61 | $n \geq 5$ |
| 3, 4, 4 | 104 | |
| 3, 4, 5 | 138 | |
| 3, 4, $n$ | 148 | $n \geq 6$ |
| 3, 5, 5 | 225 | |
| 3,5,6 | 280 | |
| 3,5,$n$ | 295 | $n \geq 7$ |
| $3, m, m+2$ | $\frac{8}{3}m^3 - 2m^2 + \frac{7}{3}m$ | |
| 4, 4, 4 | 240 | |
| 4, 4, 5 | 380 | |
| 4,4,6 | 460 | |
| 4,4,$n$ | 480 | $n \geq 7$ |
| 4, 5, 5 | 725 | |
| 4,5,6 | 1030 | |
| 4,5,7 | 1185 | |
| 4,5,$n$ | 1220 | $n \geq 8$ |
| 5, 5, 5 | 1621 | |
| 5,5,6 | 2671 | |
| 5,5,7 | 3461 | |
| 5,5,8 | 3811 | |
| 5,5,$n$ | 3881 | $n \geq 9$ |

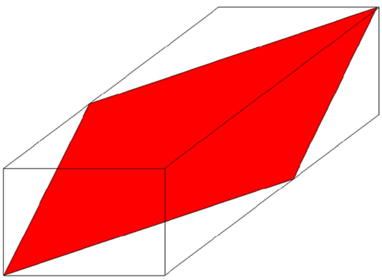

**Fig. 1** A diagonal in three-dimensional case

tensors. For $\mathcal{T} \in \mathrm{S}^d(\mathbb{F}^m)$ it is natural to consider a singular vector tuple (1.2) where $\mathbf{x}_1 = \ldots = \mathbf{x}_d = \mathbf{x}$ [16, Formula (7), with $p = 2$]. This is equivalent to the system

$$\mathcal{T} \times \otimes^{d-1}\mathbf{x} = \lambda\mathbf{x}, \quad \mathbf{x} \neq 0. \tag{4.1}$$

Here $\otimes^{d-1}\mathbf{x} := \underbrace{\mathbf{x} \otimes \ldots \otimes \mathbf{x}}_{d-1}$. Furthermore, the contraction in (4.1) is on the last $d-1$ indices. Equation (4.1) makes sense for any cube tensor $\mathcal{T} \in \mathbb{C}^{m^{\times d}}$ [16,19,22]. For $d = 2$, $\mathbf{x}$ is an eigenvector of the square matrix $\mathcal{T}$. Hence, for a $d$-cube tensor ($d \geq 3$) $\mathbf{x}$ is referred to as a nonlinear eigenvalue of $\mathcal{T}$. Abusing slightly our notation we call $([\mathbf{x}], \ldots, [\mathbf{x}]) \in \Pi(m^{\times d})$ a symmetric singular vector tuple of $\mathcal{T}$. [Note that if $\mathcal{T} \in \mathrm{S}^d(\mathbb{C}^m)$, then $([\mathbf{x}], \ldots, [\mathbf{x}])$ is a proper symmetric singular vector tuple of $\mathcal{T}$.]

Let $s_{d-1}(\mathcal{T}) = [t'_{i_1,\ldots,i_d}]$ be the symmetrization of a $d$-cube $\mathcal{T} = [t_{i_1,\ldots,i_d}]$ with respect to the last $d-1$ indices

$$t'_{i_1,\ldots,i_d} = \frac{1}{p(i_2,\ldots,i_d)} \sum_{\{j_2,\ldots,j_d\}=\{i_2,\ldots,i_d\}} t_{i_1,j_2,\ldots,j_d}. \tag{4.2}$$

Here $p(i_2,\ldots,i_d)$ is the number of multisets $\{j_2,\ldots,j_d\}$ that are equal to $\{i_2,\ldots,i_d\}$. [Note that for $d = 2$, $s_1(\mathcal{T}) = \mathcal{T}$.] It is straightforward to see that

$$\mathcal{T} \times \otimes^{d-1}\mathbf{y} = s_{d-1}(\mathcal{T}) \otimes^{d-1} \mathbf{y} \text{ for all } \mathbf{y}. \tag{4.3}$$

Hence, in (4.1) we can assume that $\mathcal{T}$ is symmetric with respect to the last $d-1$ indices.

As for singular vector tuples we view the eigenvectors of $\mathcal{T}$ as elements of $\mathbb{P}(\mathbb{C}^m)$. It was shown by Cartwright and Sturmfels [3] that a generic $\mathcal{T} \in \mathbb{C}^{m^{\times d}}$ has exactly $\frac{(d-1)^m-1}{d-2}$ distinct eigenvectors. (This formula was conjectured in [19].)

The aim of this section is to consider *partially symmetric singular vectors* and their numbers for a generic tensor. This number will interpolate our formula $c(\mathbf{m})$ for the number of singular vector tuples for a generic $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ and the number of eigenvalues of generic $\mathcal{T} \in \mathbb{C}^{m^{\times d}}$ given in [3].

Let $d = \omega_1 + \ldots + \omega_p$ be a partition of $d$. Thus, each $\omega_i$ is a positive integer. Let $\omega_0 = m'_0 = 0$ and $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_p)$, and denote by $\mathbf{m}(\boldsymbol{\omega})$ the $d$-tuple

$$\mathbf{m}(\boldsymbol{\omega}) = (\underbrace{m'_1,\ldots,m'_1}_{\omega_1},\ldots,\underbrace{m'_p,\ldots,m'_p}_{\omega_p}) = (m_1,\ldots,m_d). \tag{4.4}$$

Denote by $\mathrm{S}^{\boldsymbol{\omega}}(\mathbb{F}) \subset \mathbb{F}^{\mathbf{m}(\boldsymbol{\omega})}$ the subspace of tensors that are partially symmetric with respect to the partition $\boldsymbol{\omega}$. That is, the entries of $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{F})$ are invariant if we permute indices in the $k$th group of indices $[\sum_{j=0}^{k}\omega_j] \setminus [\sum_{j=0}^{k-1}\omega_j]$ for $k \in [p]$. Note that $\mathrm{S}^{\boldsymbol{\omega}}(\mathbb{F}) = \mathrm{S}^d(\mathbb{F}^{\mathbf{m}}))$ for $p = 1$ and $\mathrm{S}^{\boldsymbol{\omega}}(\mathbb{F}) = \mathbb{F}^{\mathbf{m}}$ for $p = d$. We call $\boldsymbol{\omega} = (1,\ldots,1)$, i.e., $p = d$, a *trivial* partition.

For simplicity of notation we let $\mathrm{S}^{\boldsymbol{\omega}} := \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{C})$. Assume that $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$. Consider a singular vector tuple $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ satisfying (1.2) and $\boldsymbol{\omega}$-symmetric conditions

$$\mathbf{x}_j = \mathbf{z}_k \text{ for } j \in \left[\sum_{i=1}^{k} \omega_i m_i'\right] \setminus \left[\sum_{i=0}^{k-1} \omega_i m_i'\right], \ k \in [p]. \tag{4.5}$$

We rewrite (1.2) for an $\boldsymbol{\omega}$-symmetric singular vector tuple $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ as follows. Define

$$\otimes_{l\in[p]} (\otimes^{\omega_l - \delta_{lk}} \mathbf{z}_l) := \otimes_{j\in[d]\setminus\{1+\sum_{i=0}^{k-1} \omega_i m_i'\}} \mathbf{x}_j, \ \text{ for } k \in [p]. \tag{4.6}$$

Hence our equations for an $\boldsymbol{\omega}$-symmetric singular vector tuple for $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ is given by

$$\mathcal{T} \times \otimes_{l\in[p]}(\otimes^{\omega_l - \delta_{lk}} \mathbf{z}_l) = \lambda_k \mathbf{z}_k \quad k \in [p]. \tag{4.7}$$

In view of the definition of $\otimes_{l\in[p]}(\otimes^{\omega_l-\delta_{lk}}\mathbf{z}_l)$, we agree that the contraction on the left-hand side of (4.7) is done on all indices except the index $1 + \sum_{i=0}^{k-1} \omega_i m_i'$. As for the $d$-cube tensor, system (4.7) makes sense for any $\mathcal{T} \in \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})}$.

Let $\mathbf{m}' := (m_1', \ldots, m_p')$. We call $([\mathbf{z}_1], \ldots, [\mathbf{z}_p]) \in \Pi(\mathbf{m}')$ satisfying (4.7), a $\boldsymbol{\omega}$-symmetric singular vector tuple of $\mathcal{T} \in \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})}$. We say that $([\mathbf{z}_1], \ldots, [\mathbf{z}_p])$ corresponds to a zero (nonzero) singular value if $\prod_{i=1}^{p} \lambda_i = 0 \ (\neq 0)$.

The aim of this section is to generalize Theorem 1 to tensors in $\mathrm{S}^{\boldsymbol{\omega}}$.

**Theorem 12** *Let $d \geq 3$ be an integer, and assume that $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)$ is a partition of $d$. Let $\mathbf{m}(\boldsymbol{\omega})$ be defined by (4.6). Denote by $\mathrm{S}^{\boldsymbol{\omega}} \subset \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})}$ the subspace of tensors partially symmetric with respect to $\boldsymbol{\omega}$. Let $c(\mathbf{m}', \boldsymbol{\omega})$ be the coefficient of the monomial $\prod_{i=1}^{p} t_i^{m_i'-1}$ in the polynomial*

$$\prod_{i\in[p]} \frac{\hat{t}_i^{m_i'} - t_i^{m_i'}}{\hat{t}_i - t_i}, \quad \hat{t}_i = (\omega_i - 1)t_i + \sum_{j\in[p]\setminus\{i\}} \omega_j t_j, \ i \in [p]. \tag{4.8}$$

*A generic $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ has exactly $c(\mathbf{m}', \boldsymbol{\omega})$ simple $\boldsymbol{\omega}$-symmetric singular vector tuples that correspond to nonzero singular values. A generic $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ does not have a zero singular value. In particular, a generic real-valued tensor $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}_{\mathbb{R}}$ has at most $c(\mathbf{m}', \boldsymbol{\omega})$ real singular vector tuples, and all of them are simple.*

*Proof* The proof of this theorem is analogous to the proof of Theorem 1, so we point out briefly the needed modifications. Let $H(m_i')$, $Q(m_i')$, and $F(m_i')$ be the vector bundles defined in §2.4. Let $\pi_i$ be the projection of $\Pi(\mathbf{m}')$ on the component $\mathbb{P}(\mathbb{C}^{m_i'})$. Then $\pi_i^* H(m_i'), \pi_i^* Q(m_i'), \pi_i^* F(m_i')$ are the pullbacks of the vector bundles $H(m_i')$, $Q(m_i')$, $F(m_i')$ to $\Pi(\mathbf{m}')$, respectively. Clearly, $c(\pi_i^* H(m_i')) = 1 + t_i$, and moreover $c(\otimes^k \pi_i^* H(m_i')) = 1 + kt_i$, where $t_i^{m_i'} = 0$.

We next observe that we can view $\Pi(\mathbf{m}')$ as a submanifold of $\Pi(\mathbf{m}(\omega))$ using the embedding

$$\eta : \Pi(\mathbf{m}') \to \Pi(\mathbf{m}(\boldsymbol{\omega})), \quad \eta(([\mathbf{z}_1], \ldots, [\mathbf{z}_p])) = ([\mathbf{x}_1], \ldots, [\mathbf{x}_d]), \tag{4.9}$$

where we assume relations (4.5). Let $\tilde{R}(i, \mathbf{m}')$ and $\tilde{R}(i, \mathbf{m}')'$ be the pullbacks of $R(j, \mathbf{m})$ and $R(j, \mathbf{m})'$, respectively, where $j = 1 + \sum_{k=0}^{i-1} \omega_k m_k'$ [see (2.6)]. Then

$$\tilde{R}(i, \mathbf{m}') := \mathrm{Hom}(\eta^* \hat{T}(\mathbf{m}_j), \pi_i^* Q(m_i')), \quad \tilde{R}(i, \mathbf{m}')' := \mathrm{Hom}(\eta^* \hat{T}(\mathbf{m}_j), \pi_i^* F(m_i')),$$
$$\tilde{R}(\mathbf{m}') := \oplus_{i\in[p]} \tilde{R}(i, \mathbf{m}'), \quad \tilde{R}_i(\mathbf{m}')' := (\oplus_{j\in[p]\setminus\{i\}} \tilde{R}(i, \mathbf{m}')) \oplus \tilde{R}(i, \mathbf{m}')'. \tag{4.10}$$

Note that

$$\text{rank } \tilde{R}(i, \mathbf{m}') = \text{rank } \tilde{R}(i, \mathbf{m}')' - 1 = m_i' - 1,$$
$$\text{rank } \tilde{R}(\mathbf{m}') = \text{rank } \tilde{R}_i(\mathbf{m}')' - 1 = \dim \Pi(\mathbf{m}').$$

As in the proof of Lemma 3 we deduce that the top Chern class of $\tilde{R}(i, \mathbf{m}')$ is given by the polynomial

$$\sum_{j=0}^{m_i'-1} (\sum_{k\in[p]} (\omega_k - \delta_{ki}) t_k)^j t_i^{m_i'-1-j}), \quad i \in [p], \tag{4.11}$$

where we assume the relations $t_i^{m_i'} = 0$ for $i \in [p]$. Use (2.2) to deduce that the top Chern number of $\tilde{R}(\mathbf{m}')$ is $c(\mathbf{m}', \boldsymbol{\omega})$.

From the results of §3, in particular Lemma 9, we deduce that there exists a monomorphism $L_i : \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})} \to \mathrm{H}^0(\tilde{R}(i, \mathbf{m}'))$. Furthermore, $L_i(\mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})})$ generates $\tilde{R}(i, \mathbf{m}')$. Let $L = (L_1, \ldots, L_p) : \oplus_{i\in[p]} \mathbb{C}^{\mathbf{m}(\omega)} \to \mathrm{H}^0(\tilde{R}(\mathbf{m}'))$. Then $L(\oplus_{i\in[p]} \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})})$ generates $\mathrm{H}^0(\tilde{R}(\mathbf{m}'))$. Let $\delta : \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})} \oplus^p \mathbb{C}^{\mathbf{m}(\boldsymbol{\omega})}$ be the diagonal map. We claim that $L \circ \delta$ almost generates $\mathrm{H}^0(\tilde{R}(\mathbf{m}'))$.

First, we consider a special case of Lemma 8 for $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$. Here we assume that $\mathbf{x}_1, \ldots, \mathbf{x}_d$ and $\mathbf{y}_1, \ldots, \mathbf{y}_d$ satsify the conditions induced by the equalities (4.5):

$$\mathbf{x}_1 = \ldots = \mathbf{x}_{\omega_1} (= \mathbf{z}_1), \ldots, \mathbf{x}_{d-\omega_p+1} = \ldots = \mathbf{x}_d = (\mathbf{z}_p),$$
$$\mathbf{y}_1 = \ldots = \mathbf{y}_{\omega_1} (= \mathbf{w}_1), \ldots, \mathbf{y}_{d-\omega_p+1} = \ldots = \mathbf{y}_d = (\mathbf{w}_p).$$

Then all parts of the lemma need to be stated in terms of $\mathbf{z}_1, \ldots, \mathbf{z}_p$ and $\mathbf{w}_1, \ldots, \mathbf{w}_p$. Second, we restate Lemma 9 for $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_d$ and $\mathbf{y}_1, \ldots, \mathbf{y}_d$ of the preceding form. Third, let $Y_\alpha \subsetneq \Pi(\mathbf{m}')$, where $\alpha$ are nonempty subsets of $[p]$, be the varieties defined in the proof of Theorem 1. The proof of Theorem 1 yields that $L \circ \delta(\mathrm{S}^{\boldsymbol{\omega}})$ almost generates $\tilde{R}(\mathbf{m}')$ with respect to the varieties $Y_\alpha$. Theorem 6 yields that a generic $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ has exactly $c(\mathbf{m}', \boldsymbol{\omega})$ simple $\boldsymbol{\omega}$-symmetric singular vector tuples. The proof that a generic $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ does not have a zero singular value is analogous to the proof given in Theorem 1. □

*Remark 13* In the special case where $\boldsymbol{\omega} = (1, 1, \ldots, 1)$, we have $c(\mathbf{m}', \boldsymbol{\omega})) = c(\mathbf{m}')$, and Theorem 12 reduces to Theorem 1. In the case where $\boldsymbol{\omega} = (d)$, we have $c(m, \boldsymbol{\omega}) = \frac{(d-1)^m - 1}{d-2}$, and Theorem 12 reduces to the results in [3]. This last reduction was already performed in [20].

**Lemma 14** *In the case where* $\boldsymbol{\omega} = (d-1, 1)$*, we have*

$$c((m_1, m_2), (d-1, 1)) = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} \binom{i}{j} (d-2)^j (d-1)^{i-j}.$$

*If* $m_1 \le m_2$*, then we have* $c((m_1, m_2), (d-1, 1)) = \frac{(2d-3)^{m_1}-1}{2d-4}$.
*If* $m_1 = m_2 + 1$*, then we have* $c((m_1, m_2), (d-1, 1)) = \frac{(2d-3)^{m_1}-1}{2d-4} - (d-1)^{m_1-1}$.

We now compare our formulas for the $3 \times 3 \times 3$ partially symmetric tensors. Consider first the case where $c((3), (3)) = \frac{2^3-1}{2-1} = 7$, i.e., the Cartwright–Sturmfels formula. That is, a generic symmetric $3 \times 3 \times 3$ tensor has 7 singular vector triples of the form $([\mathbf{x}], [\mathbf{x}], [\mathbf{x}])$. Second, consider a generic $(2, 1)$ partially symmetric tensor. The previous lemma gives $c((3, 3), (2, 1)) = 13$, i.e., a generic partially symmetric tensor has 13 singular vector triples of the form $([\mathbf{x}], [\mathbf{x}], [\mathbf{y}])$. Third, consider a generic $3 \times 3 \times 3$ tensor. In this case, our formula gives $c(3, 3, 3) = c((3, 3, 3), (1, 1, 1)) = 37$ singular vector triples of the form $[\mathbf{x}], [\mathbf{y}], [\mathbf{z}]$.

Let us assume that we have a generic symmetric $3 \times 3 \times 3$ tensor. Let us estimate the total number of singular vector triples it may have, assuming that it behaves as a generic partially symmetric tensor and a nonsymmetric one. First it has 7 singular vector triples of the form $[\mathbf{x}], [\mathbf{x}], [\mathbf{x}]$. Second, it has $3 \cdot 6 = 18$ singular vector triples of the form $[\mathbf{x}], [\mathbf{y}], [\mathbf{z}]$, where exactly two out of these three classes are the same. Third, it has 12 singular vector triples of the form $[\mathbf{x}], [\mathbf{y}], [\mathbf{z}]$, where all three classes are distinct. Note also that the number 37 was computed, in a similar setting, in [18].

The previously discussed situation indeed occurs for the diagonal tensor $\mathcal{T} = [\delta_{i_1, i_2} \delta_{i_2 i_3}] \in \mathbb{C}^{3 \times 3 \times 3}$.

We list in Table 2 the singular vector triples of this tensor $\mathcal{T}$. The first 7 singular vector triples have equal entries, and they are the ones counted by the formula in [3]. The first $7 + 6 = 13$ singular vector have the form $([\mathbf{x}], [\mathbf{x}], [\mathbf{y}])$. Any singular vector of this form gives 3 singular vector triples $([\mathbf{x}], [\mathbf{x}], [\mathbf{y}])$, $([\mathbf{x}], [\mathbf{y}], [\mathbf{x}])$, $([\mathbf{y}], [\mathbf{x}], [\mathbf{x}])$. Note that six singular vector triples have zero singular value, but this does not correspond to the generic case; indeed, for a generic tensor all 37 singular vector triples correspond to a nonzero singular value.

In the case of $4 \times 4 \times 4$ tensors, the diagonal tensor has 156 singular vector triples corresponding to a nonzero singular value and infinitely many singular vector triples corresponding to zero singular values. These infinitely many singular vector triples fill exactly 36 projective lines in the Segre product $\mathbb{P}(\mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^4)$, which "count" in this case for the remaining $240 - 156 = 84$ singular vector triples.

## 5 A Homogeneous Pencil Eigenvalue Problem

By $\mathbf{x} = (x_1, \ldots, x_m)^\top \in \mathbb{C}^m$ denote $\mathbf{x}^{\circ(d-1)} := (x_1^{d-1}, \ldots, x_m^{d-1})^\top$. Let $\mathcal{T} \in \mathbb{C}^{m^{\times d}}$. The eigenvalues of $\mathcal{T}$ satisfying (4.1) are called the $E$-eigenvalues in [22]. The homogeneous eigenvalue problem introduced in [16,17,21], sometimes referred to as $N$-eigenvalues, is

$$\mathcal{T} \times \otimes^{d-1} \mathbf{x} = \lambda \mathbf{x}^{\circ(d-1)}, \quad \mathbf{x} \neq \mathbf{0}. \tag{5.1}$$

**Table 2** List of 37 singular vector triples of a $3 \times 3 \times 3$ diagonal tensor

| (x0,x1,x2)(y0,y1,y2)(z0,z1,z2) | Singular value | |
|---|---|---|
| (1,0,0)(1,0,0)(1,0,0) | 1 | |
| (0,1,0)(0,1,0)(0,1,0) | 1 | |
| (0,0,1)(0,0,1)(0,0,1) | 1 | |
| (1,1,0)(1,1,0)(1,1,0) | 1 | |
| (1,0,1)(1,0,1)(1,0,1) | 1 | |
| (0,1,1)(0,1,1)(0,1,1) | 1 | |
| (1,1,1)(1,1,1)(1,1,1) | 1 | |
| (1,1,0)(1,−1,0)(1,−1,0) | 1 | 3 permutations |
| (1,0,1)(1,0,−1)(1,0,−1) | 1 | 3 permutations |
| (0,1,1)(0,1,−1)(0,1,−1) | 1 | 3 permutations |
| (1,1,1)(1,1,−1)(1,1,−1) | 1 | 3 permutations |
| (1,1,1)(1,−1,1)(1,−1,1) | 1 | 3 permutations |
| (1,1,1)(−1,1,1)(−1,1,1) | 1 | 3 permutations |
| (1,0,0)(0,1,0)(0,0,1) | 0 | 6 permutations |
| (1,1,−1)(1,−1,1)(−1,1,1) | −1 | 6 permutations |

Let $\mathcal{S} \in \mathbb{C}^{m^{\times d}}$. Then a generalized $d-1$ *pencil* eigenvalue problem is

$$\mathcal{T} \times \otimes^{d-1}\mathbf{x} = \lambda\mathcal{S} \times \otimes^{d-1}\mathbf{x}. \tag{5.2}$$

For $d = 2$ the preceding homogeneous system is the standard eigenvalue problem for a pencil of matrices $\mathcal{T} - \lambda\mathcal{S}$.

A tensor $\mathcal{S}$ is called *singular* if the system

$$\mathcal{S} \times \otimes^{d-1}\mathbf{x} = \mathbf{0} \tag{5.3}$$

has a nontrivial solution. Otherwise, $\mathcal{S}$ is called nonsingular. It is very easy to give an example of a symmetric nonsingular $\mathcal{S}$ [7]. Let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be linearly independent in $\mathbb{C}^m$. Then $\mathcal{S} = \sum_{i=1}^m \otimes^d \mathbf{w}_i$ is nonsingular. The set of singular tensors in $\mathbb{C}^{m^{\times d}}$ is given by the zero set of some multidimensional resultant [9, Chapter 13]. It can be obtained by elimination of variables. Let us denote by $\mathrm{res}_{m,d} \in \mathbb{C}[\mathbb{C}^{m^{\times d}}]$ the multidimensional resultant corresponding to system (5.3), which is a homogeneous polynomial in the entries of $\mathcal{S}$ of degree $\mu(m,d) = m(d-1)^{m-1}$; see formula (2.12) of [9, Chapter 9]. Denote by $Z(\mathrm{res}_{m,d})$ the zero set of the polynomial $\mathrm{res}_{m,d}$. Then $\mathrm{res}_{m,d}$ is an irreducible polynomial such that system (5.3) has a nonzero solution if and only if $\mathrm{res}_{m,d}(\mathcal{S}) = 0$. Furthermore, for a generic point $\mathcal{S} \in Z(\mathrm{res}_{m,d})$ system (5.3) has exactly one simple solution in $\mathbb{P}(\mathbb{C}^m)$. The eigenvalue problem (5.2) consists of two steps. First, find all $\lambda$ satisfying $\mathrm{res}_{m,d}(\lambda\mathcal{S} - \mathcal{T}) = 0$. Clearly, $\mathrm{res}_{m,d}(\lambda\mathcal{S} - \mathcal{T})$ is a polynomial in $\lambda$ of degree at most $\mu(m,d)$. (It is possible that this polynomial in $\lambda$ is a zero polynomial. This is the case where there exists a nontrivial solution to the system $\mathcal{S} \otimes^{d-1} \mathbf{x} = \mathcal{T} \otimes^{d-1} \mathbf{x} = \mathbf{0}$.) Then one needs to find the nonzero solutions of the system $(\lambda\mathcal{S}-\mathcal{T})\otimes^{d-1}\mathbf{x}=0$, which are viewed as eigenvectors in $\mathbb{P}(\mathbb{C}^m)$. Assume that $\mathcal{S}$ is nonsingular. Then $\mathrm{res}_{m,d}(\lambda\mathcal{S}-\mathcal{T})=\mathrm{res}_{m,d}(\mathcal{S})\lambda^{\mu(m,d)}+$ polynomial in $\lambda$ of degree at most $\mu(m,d)-1$. We show below a result, known to experts, that for generic $\mathcal{S},\mathcal{T}$ each eigenvalue $\lambda$ of the system $(\lambda\mathcal{S}-\mathcal{T})\otimes^{d-1}\mathbf{x}=0$ has exactly one corresponding eigenvector in $\mathbb{P}(\mathbb{C}^m)$. We outline a short proof of the following known theorem, which basically uses only the existence of the resultant for system (5.3). For an identity tensor $\mathcal{S}$, i.e., (5.1), see [21].

**Theorem 15** *Let $\mathcal{S},\mathcal{T}\in\mathbb{C}^{m^{\times d}}$, and assume that $\mathcal{S}$ is nonsingular. Then* $\mathrm{res}_{m,d}(\lambda\mathcal{S}-\mathcal{T})$ *is a polynomial in $\lambda$ of degree $m(d-1)^{m-1}$. For a generic $\mathcal{S}$ and $\mathcal{T}$ to each eigenvalue $\lambda$ of the pencil* (5.1) *corresponds one eigenvector in $\mathbb{P}(\mathbb{C}^m)$.*

*Proof* Consider the space $\mathbb{P}(\mathbb{C}^2)\times\mathbb{P}(\mathbb{C}^{m^{\times d}}\times\mathbb{C}^{m^{\times d}})\times\mathbb{P}(\mathbb{C}^m)$ with the local coordinates $((u,v),(\mathcal{S},\mathcal{T}),\mathbf{x})$. Consider the system of $m$ equations that are homogeneous in $(u,v),(\mathcal{S},\mathcal{T}),\mathbf{x}$ given by

$$(u\mathcal{S}-v\mathcal{T})\times\otimes^{d-1}\mathbf{x}=\mathbf{0}. \tag{5.4}$$

The existence of the multidimensional resultant is equivalent to the assumption that the preceding variety $V(m,d)$ is an irreducible variety of dimension $2m^d-1$ in $\mathbb{P}(\mathbb{C}^2)\times\mathbb{P}(\mathbb{C}^{m^{\times d}}\times\mathbb{C}^{m^{\times d}})\times\mathbb{P}(\mathbb{C}^m)$. Thus, it is enough to find a good point $(\mathcal{S}_0,\mathcal{T}_0)$ such that it has exactly $\mu(m,d)=m(d-1)^{m-1}$ smooth points $((u_i,v_i),(\mathcal{S}_0,\mathcal{T}_0),\mathbf{x}_i)$ in $V(m,d)$.

We call $\mathcal{T}=[t_{i_1,\dots,i_d}]\in\mathbb{C}^{m^{\times d}}$ an almost diagonal tensor if $t_{i_1,\dots,i_d}=0$ whenever $i_p\ne i_q$ for some $1<p<q\le d$. An almost diagonal tensor $\mathcal{T}$ is represented by a matrix $B=[b_{ij}]\in\mathbb{C}^{m\times m}$, where $t_{i,j,\dots,j}=b_{ij}$. Assume now that $\mathcal{S}_0,\mathcal{T}_0$ are almost diagonal tensors represented by the matrices $A,B$, respectively. Then

$$\mathcal{S}_0\times\otimes^{d-1}\mathbf{x}=A\mathbf{x}^{\circ(d-1)},\ \ \mathcal{T}_0\times\otimes^{d-1}\mathbf{x}=B\mathbf{x}^{\circ(d-1)}. \tag{5.5}$$

Assume furthermore that $A=I$, and $B$ is a cyclic permutation matrix, i.e., $B(x_1,\dots,x_m)^\top=(x_2,\dots,x_m,x_1)^\top$. Then $B$ has $m$ distinct eigenvalues, the $m$th roots of unity. $\mathbf{x}$ is an eigenvector of (5.5) if and only if $\mathbf{x}^{\circ(d-1)}$ is an eigenvector of $B$. Fix an eigenvalue of $B$. One can set $x_1=1$. Then we have exactly $(d-1)^{m-1}$ eigenvectors in $\mathbb{P}(\mathbb{C}^m)$ corresponding to each eigenvalue $\lambda$ of $B$. Thus, all together we have $m(d-1)^{m-1}$ distinct eigenvectors. It remains to show that each point $((u_i,v_i),(\mathcal{S}_0,\mathcal{T}_0),\mathbf{x}_i)$ is a simple point of $V(m,d)$. For that we need to show that the Jacobian of system (5.4) at each point has rank $m$, the maximal possible rank, at $((u_i,v_i),(\mathcal{S}_0,\mathcal{T}_0),\mathbf{x}_i)$. For that we assume that $u_i=\lambda_i, v_i=1, x_1=1$. This easily follows from the fact that each eigenvalue of $B$ is a simple eigenvalue. Hence, the projection of $V(m,d)$ on $\mathbb{P}(\mathbb{C}^{m^{\times d}}\times\mathbb{C}^{m^{\times d}})$ is $m(d-1)^{m-1}$ valued.

Note that in this example each eigenvalue $\lambda$ of (5.5) is of multiplicity $(d-1)^{m-1}$. It remains to show that when we consider the pair $\mathcal{S}_0,\mathcal{T}$, where $\mathcal{T}$ varies in the neighborhood of $\mathcal{T}_0$, we obtain $m(d-1)^{m-1}$ different eigenvalues. Since the Jacobian of system (5.5) has rank $m$ at each eigenvalue $\lambda_i=\frac{u_i}{v_i}$ and the corresponding eigenvector $\mathbf{x}_i$, one has a simple variation formula for each $\delta\lambda_i$ using the implicit function theorem.

Set $x_1 = 1$ and denote $\mathbf{F}(\mathbf{x}, \lambda, \mathcal{T}) = (F_1, \ldots, F_m) := (\lambda\mathcal{S}_0 - \mathcal{T}) \times \otimes^{d-1}\mathbf{x}$. Thus, we have the system of $m$ equations $\mathbf{F}(\mathbf{x}, \lambda) = 0$ in $m$ variables $x_2, \ldots, x_m, \lambda$. We let $\mathcal{T} = \mathcal{T}_0 + t\mathcal{T}_1$, and we want to find the first term of $\lambda_i(t) = \lambda_i + \alpha_i t + O(t^2)$. We also assume that $\mathbf{x}_i(t) = \mathbf{x}_i + t\mathbf{y}_i + O(t^2)$, where $\mathbf{y}_i = (0, y_{2,i}, \ldots, y_{m,i})^\top$. Let

$$\mathbf{z}_i = \sum_{j\in[d-1]} \otimes^{j-1}\mathbf{x}_i \otimes \mathbf{y}_i \otimes^{d-1-j} \mathbf{x}_i.$$

The first-order computation yields the equation

$$\mathcal{T}_1 \times \otimes^{d-1}\mathbf{x}_i + \mathcal{T}_0 \times \mathbf{z}_i = \alpha_i\mathcal{S}_0 \times \otimes^{d-1}\mathbf{x}_i + \lambda_i\mathcal{S}_0 \times \mathbf{z}_i. \tag{5.6}$$

Let $\mathbf{w} = (w_1, \ldots, w_m)^\top$ be the left eigenvector of $B$ corresponding to $\lambda_i$, i.e., $\mathbf{w}^\top B = \lambda_i\mathbf{w}^\top$ normalized by the condition $\mathbf{w}^\top(\mathcal{S}_0\times\otimes^{d-1}\mathbf{x}_i) = (\mathcal{S}_0\times\otimes^{d-1}\mathbf{x}_i)\times\mathbf{w} = 1$. Contracting both sides of (5.6) using the vector $\mathbf{w}$ we obtain

$$\alpha_i = \mathcal{T}_1 \times (\mathbf{w} \otimes (\otimes^{d-1}\mathbf{x}_i)). \tag{5.7}$$

It is straightforward to show that $\alpha_1, \ldots, \alpha_{m(d-1)^{m-1}}$ are pairwise distinct for a generic $\mathcal{T}_1$. ◻

The proof of Theorem 15 yields the following corollary.

**Corollary 16** *Let $\mathcal{T} \in \mathbb{C}^{m^{\times d}}$ be a generic tensor. Then the homogeneous eigenvalue problem (5.1) has exactly $m(d-1)^{m-1}$ distinct eigenvectors in $\mathbb{P}(\mathbb{C}^m)$, which correspond to distinct eigenvalues.*

We close this section with a heuristic argument that shows that a generic pencil $(\mathcal{S}, \mathcal{T}) \in \mathbb{P}(\mathbb{C}^{m^{\times d}} \times \mathbb{C}^{m^{\times d}})$ has $\mu(m, d) = m(d-1)^{m-1}$ distinct eigenvalues in $\mathbb{P}(\mathbb{C}^m)$. Let $\mathcal{S} \in \mathbb{C}^{m^{\times d}}$ be nonsingular. Then $\mathcal{S}$ induces a linear map $\hat{\mathcal{S}}$ from the line bundle $\otimes^{d-1}T(m)$ to the trivial bundle $\mathbb{C}^m$ over $\mathbb{P}(\mathbb{C}^m)$ by $\otimes^{d-1}\mathbf{x} \mapsto \mathcal{S} \times \otimes^{d-1}\mathbf{x}$. Then we have an exact sequence of line bundles

$$0 \to \otimes^{d-1}T(m) \to \mathbb{C}^m \to Q_{m,d} \to 0,$$

where $Q_{m,d} = \mathbb{C}^m/(\hat{\mathcal{S}}(\otimes^{d-1}T(m)))$. The Chern polynomial of $Q_{m,d}$ is $1+\sum_{i=1}^{m-1}(d-1)^i t^i \alpha^i$. A similar computation for finding the number of eigenvectors of (4.1) shows that the number of eigenvalues of (5.2) is the coefficient of $t_1^{m-1}$ in the polynomial $\frac{\hat{t}_1^m - \tilde{t}_1^m}{\hat{t}_1 - \tilde{t}_1}$. Here, $\hat{t}_1 = \tilde{t}_1 = (d-1)t_1$. Hence, the coefficient of $t_1^{m-1}$ is $\frac{(d-1)^m-(d-1)^m}{(d-1)-(d-1)}$. The calculus interpretation of this formula is the derivative of $t^m$ at $t = d-1$, which gives the value of the coefficient $m(d-1)^{m-1}$.

## 6 Uniqueness of a Best Approximation

Let $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ be the standard inner product and the corresponding Euclidean norm on $\mathbb{R}^n$. For a subspace $\mathbf{U} \subset \mathbb{R}^n$ we denote by $\mathbf{U}^\perp$ the subspace of all vectors orthogonal to $\mathbf{U}$ in $\mathbb{R}^n$. Let $C \subsetneq \mathbb{R}^n$ be a given nonempty closed set (in Euclidean topology, see §1). For each $\mathbf{x} \in \mathbb{R}^n$ we consider the function

$$\mathrm{dist}(\mathbf{x}, C) := \inf\{\|\mathbf{x}-\mathbf{y}\|,\ \mathbf{y} \in C\}\ (\geq 0). \tag{6.1}$$

We first recall that this infimum is achieved for at least one point $\mathbf{y}^\star \in C$, which is called a best approximation of $\mathbf{x}$. Observe that $\|\mathbf{x}-\mathbf{y}\| \geq \|\mathbf{y}\| - \|\mathbf{x}\|$. Hence, in the infimum (6.1) it is enough to restrict the values of $\mathbf{y}$ to the compact set $C(\mathbf{x}) := \{\mathbf{y} \in C,\ \ \|\mathbf{y}\| \leq \|\mathbf{x}\| + \mathrm{dist}(\mathbf{x}, C)\}$. Since $\|\mathbf{x}-\mathbf{y}\|$ is a continuous function on $C(\mathbf{x})$, it achieves its minimum at some point $\mathbf{y}^\star$, which will sometimes be denoted by $\mathbf{y}(\mathbf{x})$.

The following result is probably well known, and we present its short proof for completeness.

**Lemma 17** *Let $C \subsetneq \mathbb{R}^n$ be a given closed set. Let $\mathbf{U} \subset \mathbb{R}^n$ be a subspace with* $\dim \mathbf{U} \in [n]$ *and such that $\mathbf{U}$ is not contained in $C$. Let $d(\mathbf{x})$, $\mathbf{x} \in \mathbf{U}$ be the restriction of* $\mathrm{dist}(\cdot, C)$ *to* $\mathbf{U}$.

1. *The function* $\mathrm{dist}(\cdot, C)$ *is Lipschitz with constant* 1*:*

$$|\mathrm{dist}(\mathbf{x}, C) - \mathrm{dist}(\mathbf{z}, C)| \leq \|\mathbf{x}-\mathbf{z}\|\, for\ all\, \mathbf{x}, \mathbf{z} \in \mathbb{R}^n. \tag{6.2}$$

2. *The function $d(\cdot)$ is differentiable a.e. in* $\mathbf{U}$.
3. *Let $\mathbf{x} \in \mathbf{U} \setminus C$, and assume that $d(\cdot)$ is differentiable at* $\mathbf{x}$. *Denote the differential by $\partial d(\mathbf{x})$, which is viewed as a linear functional on* $\mathbf{U}$. *Let $\mathbf{y}^\star \in C$ be a best approximation to* $\mathbf{x}$. *Then*

$$\partial d(\mathbf{x})(\mathbf{u}) = \langle \mathbf{u}, \frac{1}{\mathrm{dist}(\mathbf{x}, C)}(\mathbf{x}-\mathbf{y}^\star)\rangle\ for\ each\ \mathbf{u} \in \mathbf{U}. \tag{6.3}$$

*If $\mathbf{z}^\star$ is another best approximation to $\mathbf{x}$, then $\mathbf{z}^\star - \mathbf{y}^\star \in \mathbf{U}^\perp$.*

*Proof* Assuming that $\mathrm{dist}(\mathbf{x}, C) = \|\mathbf{x}-\mathbf{y}^\star\|$ we deduce the following inequality:

$$\mathrm{dist}(\mathbf{z}, C) \leq \|\mathbf{z}-\mathbf{y}^\star\| \text{ for each } \mathbf{z} \in \mathbb{R}^n. \tag{6.4}$$

Suppose next that $\mathrm{dist}(\mathbf{z}, C) = \|\mathbf{z}-\mathbf{y}\|, \mathbf{y} \in C$. Hence,

$$\begin{aligned} -\|\mathbf{x}-\mathbf{z}\| &\leq \|\mathbf{x}-\mathbf{y}^\star\| - \|\mathbf{z}-\mathbf{y}^\star\| \leq \mathrm{dist}(\mathbf{x}, C) - \mathrm{dist}(\mathbf{z}, C) \\ &\leq \|\mathbf{x}-\mathbf{y}\| - \|\mathbf{z}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{z}\|. \end{aligned}$$

This proves (6.2) and part *1*. Clearly, $d(\cdot)$ is also Lipschitz on $\mathbf{U}$. Rademacher's theorem yields that $d(\cdot)$ is differentiable a.e., which proves part *2*. To prove part *3* we fix $\mathbf{u} \in \mathbf{U}$. Then

$$\mathrm{dist}(\mathbf{x}+t\mathbf{u}, C) = \mathrm{dist}(\mathbf{x}, C) + t\partial d(\mathbf{x})(\mathbf{u}) + to(t)$$

(6.4) yields the inequality

$$\mathrm{dist}(\mathbf{x}+t\mathbf{u}, C) \le \|\mathbf{x}+t\mathbf{u}-\mathbf{y}^\star\| = \|\mathbf{x}-\mathbf{y}^*\| + t\langle \mathbf{u}, \frac{1}{\mathrm{dist}(\mathbf{x}, C)}(\mathbf{x}-\mathbf{y}^\star)\rangle + O(t^2).$$

Compare this inequality with the previous equality to deduce that

$$t\partial d(\mathbf{x})(\mathbf{u}) \le t\langle \mathbf{u}, \frac{1}{\mathrm{dist}(\mathbf{x}, C)}(\mathbf{x}-\mathbf{y}^\star)\rangle$$

for all $t \in \mathbb{R}$. This implies (6.3). If $\mathbf{z}^\star$ is another best approximation to $\mathbf{x}$, then (6.3) yields that $\mathbf{z}^\star - \mathbf{y}^\star \in \mathbf{U}^\perp$. □

**Corollary 18** *Let $C \subsetneq \mathbb{R}^n$ be a given closed set.*

1. *The function* $\mathrm{dist}(\mathbf{x}, C)$ *is differentiable a.e. in* $\mathbb{R}^n$.
2. *Let* $\mathbf{x} \in \mathbb{R}^n \setminus C$, *and assume that* $\mathrm{dist}(\cdot, C)$ *is differentiable at* $\mathbf{x}$. *Then* $\mathbf{x}$ *has a unique best approximation* $\mathbf{y}(\mathbf{x}) \in C$. *Furthermore,*

$$\partial\mathrm{dist}(\mathbf{x}, C)(\mathbf{u}) = \langle \mathbf{u}, \frac{1}{\mathrm{dist}(\mathbf{x}, C)}(\mathbf{x}-\mathbf{y}(\mathbf{x}))\rangle \text{ for each } \mathbf{u} \in \mathbb{R}^n. \tag{6.5}$$

*In particular, a.a.* $\mathbf{x} \in \mathbb{R}^n$ *have a unique best approximation* $\mathbf{y}(\mathbf{x}) \in C$.

*Proof* Choose $\mathbf{U} = \mathbb{R}^n$, so $d(\cdot) = \mathrm{dist}(\cdot, C)$ is differentiable a.e. by part *2* of Lemma 17. This establishes part *1* of our lemma. Assume that $\mathbf{y}^\star$ and $\mathbf{z}^\star$ are best approximations of $\mathbf{x}$. Then $\mathbf{z}^\star - \mathbf{y}^\star \in (\mathbb{R}^n)^\perp$ by part *3* of Lemma 17. As $(\mathbb{R}^n)^\perp = \{\mathbf{0}\}$, we obtain that $\mathbf{z}^\star = \mathbf{y}^\star$. Furthermore, (6.5) holds. □

## 7 Best Rank-One Approximations of *d*-Mode Tensors

On $\mathbb{C}^{\mathbf{m}}$ define an inner product and its corresponding Hilbert–Schmidt norm $\langle \mathcal{T}, \mathcal{S}\rangle := \mathcal{T} \times \bar{\mathcal{S}}$, $\|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$. We first present some known results of the best rank-one approximations of real tensors. In this section we assume that $\mathbb{F} = \mathbb{R}$ and $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$. Let $\mathrm{S}^{m-1} \subset \mathbb{R}^m$ be the $m-1$-dimensional sphere $\|\mathbf{x}\| = 1$. Denote by $\mathrm{S}(\mathbf{m})$ the $d$-product of the spheres $\mathrm{S}^{m_1-1} \times \ldots \times \mathrm{S}^{m_d-1}$. Let $(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathrm{S}(\mathbf{m})$, and associate with $(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ the $d$ one-dimensional subspaces $\mathbf{U}_i = \mathrm{span}(\mathbf{x}_i)$, $i \in [d]$. Note that

$$\| \otimes_{i\in[d]} \mathbf{x}_i \| = \prod_{i\in[d]} \|\mathbf{x}_i\| = 1.$$

The projection $P_{\otimes_{i\in[d]}\mathbf{U}_i}(\mathcal{T})$ of $\mathcal{T}$ onto the one-dimensional subspace $\mathbf{U} := \otimes_{i\in[d]}\mathbf{U}_i \subset \otimes_{i\in[d]}\mathbb{R}^{m_i}$ is given by

$$f_{\mathcal{T}}(\mathbf{x}_1, \ldots, \mathbf{x}_d) \otimes_{i\in[d]} \mathbf{x}_i,\ f_{\mathcal{T}}(\mathbf{x}_1, \ldots, \mathbf{x}_d) := \langle \mathcal{T}, \otimes_{i\in[d]}\mathbf{x}_i\rangle,\ (\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathrm{S}(\mathbf{m}). \tag{7.1}$$

Let $P_{(\otimes_{i\in[d]}\mathbf{U}_i)^\perp}(\mathcal{T})$ be the orthogonal projection of $\mathcal{T}$ onto the orthogonal complement of $\otimes_{i\in[d]}\mathbf{U}_i$. The Pythagorean identity yields

$$\|\mathcal{T}\|^2 = \|P_{\otimes_{i\in[d]}\mathbf{U}_i}(\mathcal{T})\|^2 + \|P_{(\otimes_{i\in[d]}\mathbf{U}_i)^\perp}(\mathcal{T})\|^2. \tag{7.2}$$

With this notation, a best rank-one approximation of $\mathcal{T}$ from $\mathrm{S}(\mathbf{m})$ is given by

$$\min_{(\mathbf{x}_1,\ldots,\mathbf{x}_d)\in\mathrm{S}(\mathbf{m})} \min_{a\in\mathbb{R}} \|\mathcal{T} - a \otimes_{i\in[d]} \mathbf{x}_i\|.$$

Observing that

$$\min_{a\in\mathbb{R}} \|\mathcal{T} - a \otimes_{i\in[d]} \mathbf{x}_i\| = \|\mathcal{T} - P_{\otimes_{i\in[d]}\mathbf{U}_i}(\mathcal{T})\| = \|P_{(\otimes_{i\in[d]}\mathbf{U}_i)^\perp}(\mathcal{T})\|,$$

it follows that a best rank-one approximation is obtained by the minimization of $\|P_{(\otimes_{i\in[d]}\mathbf{U}_i)^\perp}(\mathcal{T})\|$. In view of (7.2), we deduce that a best rank-one approximation is obtained by the maximization of $\|P_{\otimes_{i\in[d]}\mathbf{U}_i}(\mathcal{T})\|$, and, finally, using (7.1), it follows that a best rank-one approximation is given by

$$\sigma_1(\mathcal{T}) := \max_{(\mathbf{x}_1,\ldots,\mathbf{x}_d)\in\mathrm{S}(\mathbf{m})} f_{\mathcal{T}}(\mathbf{x}_1,\ldots,\mathbf{x}_d). \tag{7.3}$$

As in the matrix case, $\sigma_1(\mathcal{T})$ is called in [13] the *spectral norm*. Furthermore, it is shown in [13] that the computation of $\sigma_1(\mathcal{T})$ in general is NP-hard for $d > 2$.

We will make use of the following result of [16], where we present the proof for completeness.

**Lemma 19** *For $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$, the critical points of $f|_{\mathrm{S}(\mathbf{m})}$, defined in* (7.1)*, are singular vector tuples satisfying*

$$\mathcal{T} \times (\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j) = \lambda\mathbf{x}_i \ \ \textit{for all } i \in [d],\ (\mathbf{x}_1,\ldots,\mathbf{x}_d) \in \mathrm{S}(\mathbf{m}). \tag{7.4}$$

*Proof* We need to find the critical points of $\langle\mathcal{T}, \otimes_{j\in[d]}\mathbf{x}_j\rangle$, where $(\mathbf{x}_1,\ldots,\mathbf{x}_d) \in \mathrm{S}(\mathbf{m})$. Using Lagrange multipliers we consider the auxiliary function

$$g(\mathbf{x}_1,\ldots,\mathbf{x}_d) := \langle\mathcal{T}, \otimes_{j\in[d]}\mathbf{x}_j\rangle - \sum_{j\in[d]} \lambda_j \mathbf{x}_j^\top \mathbf{x}_j.$$

The critical points of $g$ then satisfy

$$\mathcal{T} \times (\otimes_{j\in[d]\setminus\{i\}}\mathbf{x}_j) = \lambda_i\mathbf{x}_i, \quad i \in [d],$$

and hence $\langle\mathcal{T}, \otimes_{j\in[d]}\mathbf{x}_j\rangle = \lambda_i\mathbf{x}_i^\top\mathbf{x}_i = \lambda_i$ for all $i \in [d]$, which implies (7.4). □

Observe next that $(\mathbf{x}_1,\ldots,\mathbf{x}_d)$ satisfies (7.4) if and only if the vectors $(\pm\mathbf{x}_1,\ldots,\pm\mathbf{x}_d)$ satisfy (7.4). In particular, we could choose the signs in $(\pm\mathbf{x}_1,\ldots,\pm\mathbf{x}_d)$ such that each corresponding $\lambda$ is nonnegative and then these $\lambda$ could be interpreted as the singular values of $\mathcal{T}$. The maximal singular value of $\mathcal{T}$ is denoted by $\sigma_1(\mathcal{T})$ and is given by (7.3). Note that to each nonnegative singular value are associated at least $2^{d-1}$ singular vector tuples of the form $(\pm\mathbf{x}_1, \ldots, \pm\mathbf{x}_d)$. Thus, it is more natural to view the singular vector tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ as points $([\mathbf{x}_1], \ldots, [\mathbf{x}_d])$ in the real projective Segre product $\Pi_{\mathbb{R}}(\mathbf{m})$. Furthermore, the projection of $\mathcal{T}$ on the one-dimensional subspace spanned by $\otimes_{i\in[d]}(\pm\mathbf{x}_i)$, where $(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathrm{S}(\mathbf{m})$, is equal to one vector $(\mathcal{T} \times \otimes_{i\in[d]}\mathbf{x}_i) \otimes_{i\in[d]} \mathbf{x}_i$.

**Theorem 20** *For a.a. $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ a best rank-one approximation is unique.*

*Proof* Let

$$C(\mathbf{m}) := \{\mathcal{T} \in \mathbb{R}^{\mathbf{m}},\ \mathcal{T} = \otimes_{j\in[d]}\mathbf{x}_j,\ \mathbf{x}_j \in \mathbb{R}^{m_j},\ j \in [d]\}. \tag{7.5}$$

$C(\mathbf{m})$ is a compact set consisting of rank-one tensors and the zero tensor. Corollary 18 yields that for a.a. $\mathcal{T}$ a best rank-one approximation is unique. □

Note that Theorem 20 implies part *1* of Theorem 2. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)$ be a partition of $d$. For $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ it is natural to consider a best rank-one approximation to $\mathcal{T}$ of the form $\pm\prod_{i\in[p]} \otimes^{\omega_i}\mathbf{x}_i$, where $\mathbf{x}_i \in \mathbb{R}^{m'_i}$, $i \in [p]$. We call such an approximation a best $\boldsymbol{\omega}$-symmetric rank-one approximation. (The factor $\pm$ is needed only if each $\omega_i$ is even.) As in the case $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$, a best $\boldsymbol{\omega}$-symmetric rank-one approximation of $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ is a solution to the following maximum problem:

$$\max_{(\mathbf{x}_1,\ldots,\mathbf{x}_p)\in \mathrm{S}(\mathbf{m}')} |\mathcal{T} \times \otimes_{i\in[p]} \otimes^{\omega_i} \mathbf{x}_i|. \tag{7.6}$$

As before, the critical points of the functions $\pm\mathcal{T} \times \otimes_{i\in[p]} \otimes^{\omega_i} \mathbf{x}_i$ on $\mathrm{S}(\mathbf{m}')$ satisfy

$$\mathcal{T} \times \otimes_{j\in[p]} \otimes^{\omega_j-\delta_{ji}} \mathbf{x}_j = \lambda\mathbf{x}_i, \quad i \in [p], \quad (\mathbf{x}_1, \ldots, \mathbf{x}_p) \in \mathrm{S}(\mathbf{m}'). \tag{7.7}$$

A best $\boldsymbol{\omega}$-symmetric rank-one approximation corresponds to all $\lambda$ for which $|\lambda|$ has a maximal possible value. The arguments of the proof of Theorem 20 imply the following result.

**Proposition 21** *For almost all $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ a best rank-one $\boldsymbol{\omega}$-symmetric approximation is unique.*

Assume that $\otimes_{j\in[d]}\mathbf{y}_j \in \mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$ is a best rank-one approximation to a tensor $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$. It is not obvious a priori that $\otimes_{j\in[d]}\mathbf{y}_j$ is $\boldsymbol{\omega}$-symmetric. However, the following result is obvious:

$$\begin{aligned} &\otimes_{j\in[d]}\mathbf{y}_{\sigma(j)} \text{ is a best rank-one approximation of } \mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R}) \\ &\text{for each permutation } \sigma : [d] \to [d], \text{ which preserves } \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R}). \end{aligned} \tag{7.8}$$

**Lemma 22** *For a.a. $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ there exists a unique rank-one tensor $\otimes_{j\in[d]}\mathbf{y}_j \in \mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$ such that all best rank-one approximations of $\mathcal{T}$ are of the form* (7.8).

To prove this lemma, we need an auxiliary lemma.

**Lemma 23** *Let* $\otimes_{j\in[d]}\mathbf{x}_j, \otimes_{j\in[d]}\mathbf{y}_j \in \mathbb{R}^{n^{\times d}}$. *Assume that*

$$\langle \otimes_{j\in d}\mathbf{x}_j, \otimes^d \mathbf{u}\rangle = \langle \otimes_{j\in d}\mathbf{y}_j, \otimes^d \mathbf{u}\rangle \ \forall \mathbf{u}\in\mathbb{R}^n. \tag{7.9}$$

*Then there exists a permutation* $\sigma : [d]\to[d]$ *such that* $\otimes_{j\in[d]}\mathbf{y}_j = \otimes_{j\in[d]}\mathbf{x}_{\sigma(j)}$.

*Proof* Note that condition (7.9) is equivalent to the equality

$$\prod_{j\in[d]} \mathbf{u}^\top \mathbf{x}_j = \prod_{j\in[d]} \mathbf{u}^\top \mathbf{y}_j \ \forall \mathbf{u}\in\mathbb{R}^n. \tag{7.10}$$

If $\otimes_{j\in d}\mathbf{x}_j = 0$, then $\prod_{j\in[d]}\mathbf{u}^\top\mathbf{y}_j = 0$ for all $\mathbf{u}$. Hence, $\mathbf{y}_j = \mathbf{0}$ for some $j$, so $\otimes_{j\in[d]}\mathbf{y}_j = \otimes_{j\in[d]}\mathbf{x}_j = 0$. Thus, we assume that $\otimes_{j\in[d]}\mathbf{x}_j$ and $\otimes_{j\in[d]}\mathbf{y}_j$ are both nonzero.

We now prove the lemma by induction. For $d = 1$ the lemma is trivial. Assume that the lemma holds for $d = k$. Let $d = k+1$. Assume that $\mathbf{u}\in\operatorname{span}(\mathbf{x}_{k+1})^\perp$. Then (7.10) yields that $\prod_{j\in[d]}\mathbf{u}^\top\mathbf{y}_j = 0$. Hence, $\operatorname{span}(\mathbf{x}_{k+1})^\perp \subset \cup_{j\in[k+1]}\operatorname{span}(\mathbf{y}_j)^\perp$. Therefore, there exists $j\in[k+1]$ such that $\operatorname{span}(\mathbf{x}_{k+1})^\perp = \operatorname{span}(\mathbf{y}_j)^\perp$. Thus, $\mathbf{y}_j = t\mathbf{x}_{k+1}$ for some $t\in\mathbb{R}\setminus\{0\}$. Hence, there exist $\mathbf{z}_1,\ldots,\mathbf{z}_{d+1}\in\mathbb{R}^n$ and a permutation $\sigma' : [k+1]\to[k+1]$ such that $\otimes_{j\in[k+1]}\mathbf{z}_{\sigma'(j)} = \otimes_{j\in[k+1]}\mathbf{y}_j$, where $\mathbf{z}_{k+1} = \mathbf{x}_{k+1}$. Thus, $\otimes_{j\in[k+1]}\mathbf{x}_j$ and $\otimes_{j\in[k+1]}\mathbf{z}_j$ satisfy (7.10). Therefore, $\otimes_{j\in[k]}\mathbf{x}_j$ and $\otimes_{j\in[k]}\mathbf{z}_j$ satisfy (7.10). Use the induction hypothesis to deduce the lemma. □

**Proof of Lemma 22.** We use part *3* of Lemma 17 as follows. Let $\mathbb{R}^n = \mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$, and assume that $C = C(\mathbf{m}(\boldsymbol{\omega}))$, as defined in (7.5). We let $\mathbf{U} := \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$. Assume that $d(\cdot)$ is differentiable at $\mathcal{T}\in\mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})\setminus C$. Suppose that $\otimes_{j\in[d]}\mathbf{y}_j, \otimes_{j\in[d]}\mathbf{z}_j$ are best rank-one approximations of $\mathcal{T}$. Thus,

$$\sigma_1(\mathcal{T}) = \|\otimes_{j\in[d]}\mathbf{y}_j\| = \prod_{j\in[d]}\|\mathbf{y}_j\| = \|\otimes_{j\in[d]}\mathbf{z}_j\| = \prod_{j\in[d]}\|\mathbf{z}_j\| > 0.$$

Without loss of generality we may assume that

$$\|\mathbf{y}_j\| = \|\mathbf{z}_j\| = \sigma_1(\mathcal{T})^{\frac{1}{d}} \ \forall j\in[d]. \tag{7.11}$$

Lemma 17 yields that

$$\langle \otimes_{i\in[p]} \otimes^{\omega_i} \mathbf{u}_i, \otimes_{j\in[d]}\mathbf{y}_j - \otimes_{j\in[d]}\mathbf{z}_j\rangle = 0 \ \forall \mathbf{u}_i \in\mathbb{R}^{m_i'} \ i\in[p].$$

The preceding equality is equivalent to

$$\prod_{i\in[p]}\prod_{j_i\in[\omega_i]} \mathbf{u}_i^\top \mathbf{y}_{\alpha_i+j_i} = \prod_{i\in[p]}\prod_{j_i\in[\omega_i]} \mathbf{u}_i^\top \mathbf{z}_{\alpha_i+j_i}, \quad \forall \mathbf{u}_i\in\mathbb{R}^{m_i'},\ i\in[p], \tag{7.12}$$

where $\omega_0 = 0$ and $\alpha_i = \sum_{k=0}^{i-1} \omega_k$ for all $i \in [p]$.

Suppose first that $p = 1$, i.e., $\mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ is the set of all symmetric tensors in $\mathbb{R}^{m_1^{\times\omega_1}}$. (Note that $d = \omega_1$.) Then Lemma 23 and (7.12) yield that $\otimes_{j\in[d]}\mathbf{z}_j = \otimes_{j\in[d]}\mathbf{y}_{\sigma(j)}$ for some permutation $\sigma : [d] \to [d]$. This proves our lemma for $p = 1$.

Assume now that $p > 1$. Fix $k \in [p]$ and $\mathbf{u}_i \in i \in [p] \setminus \{k\}$. Let

$$s_k := \prod_{i\in[p]\setminus\{k\}} \prod_{l_j\in[\omega_j]} \mathbf{u}_{l_j}^\top \mathbf{y}_{\alpha_j+l_j}, \quad t_k := \prod_{i\in[p]\setminus\{k\}} \prod_{l_j\in[\omega_j]} \mathbf{u}_{l_j}^\top \mathbf{z}_{\alpha_j+l_j}.$$

Assume that $s_k \neq 0$. Then the two rank-one tensors $s_k \otimes_{l_k\in[\omega_k]} \mathbf{y}_{\alpha_k+l_k}$ and $t_k \otimes_{l_k\in[\omega_k]} \mathbf{z}_{\alpha_k+l_k} \in \mathbb{R}^{(m'_k)^{\times\omega_k}}$ satisfy the assumptions of Lemma 23. Hence, there exists a permutation $\sigma_k : [\omega_k] \to [\omega_k]$ such that $t_k \otimes_{l_k\in[\omega_k]} \mathbf{z}_{\alpha_k+l_k} = s_k \otimes_{l_k\in[\omega_k]} \mathbf{y}_{\alpha_k+\sigma_k(l_k)}$. In view of (7.11), we deduce the equality $\otimes_{l_k\in[\omega_k]}\mathbf{z}_{\alpha_k+l_k} = \pm \otimes_{l_k\in[\omega_k]} \mathbf{y}_{\alpha_k+\sigma_k(l_k)}$. Hence, there exists $\omega : [d] \to [d]$, which leaves invariant each set $[\alpha_{j+1}]$ for $j \in [p-1]$ such that $\otimes_{j\in[d]}\mathbf{z}_j = \pm \times_{j\in[d]} \mathbf{y}_{\sigma(j)}$. Because $\otimes_{j\in[d]}\mathbf{z}_j$ and $\otimes_{j\in[d]}\mathbf{y}_j$ are best rank-one approximation to $\mathcal{T}$, we deduce that $\otimes_{j\in[d]}\mathbf{z}_j = \otimes_{j\in[d]}\mathbf{y}_{\sigma(j)}$. □

A recent result of the first author claims that each $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ has a best rank-one approximation that is $\boldsymbol{\omega}$-symmetric [7, Theorem 1]. For symmetric tensors this theorem is equivalent to the old theorem of Banach [1]. (See [4, Theorem 4.1] for another proof of Banach's theorem.) We now give a refined version of [7, Theorem 1], whose proof uses the results in [7].

**Theorem 24** *Each $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ has a best rank-one approximation that is $\boldsymbol{\omega}$-symmetric. Furthermore, for a.a. $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ a best rank-one approximation is unique and $\boldsymbol{\omega}$-symmetric.*

*Proof* The claim that each $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ has a best rank-one approximation that is $\boldsymbol{\omega}$-symmetric is proved in [7]. It is left to show that for a.a. $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ a best rank-one approximation is unique and $\boldsymbol{\omega}$-symmetric. Lemma 22 claims that for a.a. $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}(\mathbb{R})$ there exists a unique rank-one tensor $\otimes_{j\in[d]}\mathbf{y}_j \in \mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$ such that all best rank-one approximations of $\mathcal{T}$ are of the form (7.8). The first part of the theorem yields that one of these best rank-one approximations $\otimes_{j\in[d]}\mathbf{y}_j \in \mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$ is $\boldsymbol{\omega}$-symmetric . Hence, all the tensors of the form (7.8) are equal to $\otimes_{j\in[d]}\mathbf{y}_j \in \mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$. □

Note that part 2 of Theorem 2 follows from Theorem 24.

## 8 Best Rank-r Approximation

In the first part of this section we assume that $\mathbb{F}$ is any field. Let $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, $M = \prod_{j\in d} m_j$, $M_i = \frac{M}{m_i}$, and $\mathbf{m}_i = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_d) \in \mathbb{N}^{d-1}$ for $i \in [d]$. Assume that $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{\mathbf{m}}$. Denote by $T_i \in \mathbb{F}^{m_i\times M_i}$ the unfolded matrix of the tensor $\mathcal{T}$ in mode $i$. That is, let $\mathcal{T}_{j,k} \in \mathbb{F}^{\mathbf{m}_k}$ be the following $d-1$ mode tensor. Its entries are $[t_{i_1,\ldots,i_{k-1},j,i_{k+1},\ldots,i_d}]$ for $i_p \in [m_p]$, $p \in [d] \setminus \{k\}$. Thus, $j \in [m_k]$. Then row $j$ of $T_i$ is a tensor $\mathcal{T}_{j,i}$ viewed as a vector in $\mathbb{F}^{\mathbf{m}_i}$. Then $\mathrm{rank}_i \mathcal{T}$ is the rank of the matrix $T_i$. $T_i$ can be seen as the matrix of the contraction map $\otimes_{j\in[d]\setminus\{i\}}(\mathbb{F}^\vee)^{m_j} \to \mathbb{F}^{m_i}$ for $i \in [d]$. Clearly,

$$\operatorname{rank}_i \mathcal{T} \leq \min(m_i, M_i) \quad i \in [d]. \tag{8.1}$$

Carlini and Kleppe characterized the possible $r_i$ occurring as in the following theorem.

**Theorem 25** ([2], Theorem 7) *Suppose that $r_i \in [m_i]$ for $i \in [d]$. Then there exists $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$ such that* $\operatorname{rank}_i \mathcal{T} = r_i$ *for $i \in [d]$ if and only if*

$$r_i^2 \leq \prod_{j \in [d]} r_j \quad \text{for each } i \in [d]. \tag{8.2}$$

We show a related argument working over any infinite field. For each $i$ let $f_i$ be one minor of $T_i$ of order $\min(m_i, M_i)$. Let $f = \prod_{i \in [d]} f_i$, which is a nonzero polynomial in the entries of $\mathcal{T} = [t_{i_1,\ldots,i_d}]$. Let $V(\mathbf{m}) \subset \mathbb{F}^{\mathbf{m}}$ be the zero set of $f$.

**Theorem 26** *Let $\mathbf{m} \in \mathbb{N}^d$, and assume that $V(\mathbf{m}) \subset \mathbb{F}^{\mathbf{m}}$ is defined as previously. Then for each $\mathcal{T} \in \mathbb{F}^{\mathbf{m}} \setminus V(\mathbf{m})$ the following equality holds:*

$$\operatorname{rank}_i \mathcal{T} = \min(m_i, M_i) \text{ for } i \in [d]. \tag{8.3}$$

*In particular, when $\mathbb{F}$ is a infinite field, a generic tensor $\mathcal{T} \in \mathbb{F}^{\mathbf{m}}$ satisfies* (8.3).

*Proof* Suppose first that $m_i \leq M_i$. We claim that the $m_i$ tensors $\mathcal{T}_{1,i}, \ldots, \mathcal{T}_{m_i,i}$ are linearly independent. Suppose not. Then any $m_i \times m_i$ minor of $T_i$ is zero. This contradicts the assumption that $\mathcal{T} \in \mathbb{F}^{\mathbf{m}} \setminus V(\mathbf{m})$. Hence, $\operatorname{rank}_i \mathcal{T} = m_i$. Suppose that $m_i > M_i$. Let $\mathcal{T}_{k_1,i}, \ldots, \mathcal{T}_{k_{M_i},i}$ be the $M_i$ tensors that contribute to the minor $f_i$. Since $f_i(\mathcal{T}) \neq 0$, we deduce that $\mathcal{T}_{k_1,i}, \ldots, \mathcal{T}_{k_{M_i},i}$ are linearly independent. Hence, $\operatorname{rank}_i \mathcal{T}_i = M_i$ for each $i \in [d]$. Since $f$ is a nonzero polynomial, for an infinite field $\mathbb{F}$, $V(\mathbf{m})$ is a proper closed subset of $\mathbb{F}^{\mathbf{m}}$ in the Zariski topology. Hence, (8.3) holds for a generic tensor. □

Over infinite fields, Theorem 25 can be proved as a consequence of Theorem 26. Indeed, let $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{N}^d$, and assume that (8.2) holds. Choose a generic $\mathcal{T}' = [t'_{j_1,\ldots,j_d}] \in \mathbb{F}^{\mathbf{r}}$. Thus, $\operatorname{rank}_i \mathcal{T}' = r_i, i \in [d]$. Extend $\mathcal{T}'$ to $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{\mathbf{m}}$ by adding zero entries, i.e., $t_{j_1,\ldots,j_d} = t'_{j_1,\ldots,j_d}$ for $j_i \in [r_i], i \in [d]$, and all other entries of $\mathcal{T}$ are zero. Then $\operatorname{rank}_i \mathcal{T} = r_i, i \in [d]$.

In what follows we assume that $\mathbb{F} = \mathbb{R}$. Observe that the set of tensors having rank $(r_1, \ldots, r_d)$ contains in the closure exactly all tensors of rank $(a_1, \ldots, a_d)$, with $a_i \leq r_i$. This closure is an algebraic variety, defined as the zero set of all the minors of order $r_i + 1$ of $T_i$ for $i \in [d]$. We denote it by $C_{\mathbf{r}}$. Note that having rank $(1, \ldots, 1)$ is equivalent to having rank 1.

Clearly, $C_{\mathbf{r}}$ is a closed set in $\mathbb{R}^{\mathbf{m}}$. The best $\mathbf{r}$-rank approximation of $\mathcal{T}$ is the closest tensor in $C_{\mathbf{r}}$ to $\mathcal{T}$ in the Hilbert–Schmidt norm [6].

Corollary 18 yields the following theorem.

**Theorem 27** *Let $\mathbf{m} = (m_1, \ldots, m_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$, where $r_i \in [m_i]$ for $i \in [d]$, and they satisfy (8.2). Then almost all $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ have a unique best $\mathbf{r}$-rank approximation.*

Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)$ be a partition of $d$, $\mathbf{m}' = (m'_1, \ldots, m'_p)$, and assume that $\mathbf{m}(\boldsymbol{\omega})$ is defined by (4.4). Assume that $\mathbf{r}' = (r'_1, \ldots, r'_p)$, where $r'_i \in [m'_i]$ for $i \in [p]$. Let $\mathbf{r}(\boldsymbol{\omega}) = (\underbrace{r'_1, \ldots, r'_1}_{\omega_1}, \ldots, \underbrace{r'_p, \ldots, r'_p}_{\omega_p})$.

Let $C'_{\mathbf{r}'} = C_{\mathbf{r}(\boldsymbol{\omega})} \cap \mathrm{S}^{\boldsymbol{\omega}}$. Clearly, $C'_{\mathbf{r}'}$ is a closed set, consisting of $\omega$-symmetric tensors in $\mathbb{R}^{\mathbf{m}(\boldsymbol{\omega})}$ having rank $\mathbf{r}(\boldsymbol{\omega})$ .

Let $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$. Then a best $\omega$-symmetric $\mathbf{r}(\boldsymbol{\omega})$-rank approximation of $\mathcal{T}$ is the closest tensor in $C'_{\mathbf{r}'}$ to $\mathcal{T}$. Corollary 18 yields the following theorem.

**Theorem 28** *Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)$ be a partition of $d$. Assume that $\mathbf{m}' = (m'_1, \ldots, m'_p)$, $\mathbf{r}' = (r'_1, \ldots, r'_p)$, $r'_i \in [m'_i]$, $i \in [p]$ and that $\mathbf{m}(\boldsymbol{\omega})$ satisfies (8.2). Then a.a. $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$ have a unique best $\omega$-symmetric $\mathbf{r}(\boldsymbol{\omega})$-rank approximation.*

We close our paper with the following problem. Let $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$. Does $\mathcal{T}$ have a best $\mathbf{r}(\boldsymbol{\omega})$-rank approximation that is $\omega$-symmetric? If so, is a best $\mathbf{r}(\boldsymbol{\omega})$-rank approximation unique for a.a. $\mathcal{T} \in \mathrm{S}^{\boldsymbol{\omega}}$? In the previous section we showed that for $\mathbf{r}(\omega) = (1, \ldots, 1)$ the answers to these problems are yes.

**Acknowledgments** Shmuel Friedland was supported by National Science Foundation Grant DMS-1216393. Giorgio Ottaviani is member of INDAM.

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