

## Quantitative Homotopy Theory in Topological Data Analysis

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**Abstract** This paper lays the foundations of an approach to applying Gromov’s ideas on quantitative topology to topological data analysis. We introduce the “contiguity complex”, a simplicial complex of maps between simplicial complexes defined in terms of the combinatorial notion of contiguity. We generalize the Simplicial Approximation Theorem to show that the contiguity complex approximates the homotopy type of the mapping space as we subdivide the domain. We describe algorithms for approximating the rate of growth of the components of the contiguity complex under subdivision of the domain; this procedure allows us to computationally distinguish spaces with isomorphic homology but different homotopy types.

**Keywords** Simplicial complex · Contiguity · Subdivision · Mesh size · Mapping space · Simplicial approximation theorem · Persistent homology

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## 1 Introduction

The topological approach to data analysis is to organize a data set (which we think of as a finite metric space) into an appropriate simplicial complex, either the Čech complex, or more usually (for efficiency reasons) the Rips complex. Homotopy theoretic invariants of the simplicial complex then give abstract information about the data set. Most work to date has focused on homology, which although easy to compute is an extremely weak invariant of a complex  $Y$ . A complete set of invariants is given by looking at homotopy classes of maps from test complexes  $X$  into  $Y$  (this is called the Yoneda Lemma [4, Sect. III.2]). Of course, we cannot hope to have all this information, and computing homotopy classes of maps is an intractable problem. A less intractable compromise is to consider the homology of the *space* of maps from  $X$  to  $Y$  for suitable specific test spaces  $X$ . When  $X$  is contractible, this just recovers the homology of  $Y$ . But when  $X$  is topological nontrivial, the homology of the mapping space can provide information that is invisible to the homology of  $Y$ . For instance, the homology of the mapping space from  $S^1$  to  $Y$  captures information about the fundamental group of  $Y$ ; this invariant can detect the difference between spaces  $Y$  and  $Y'$  with identical homology. (See Sect. 9 for detailed computational exploration of such an example.)

A key benefit of homotopy invariants is their global nature. On the other hand, in data analysis, we often also care about feature scale. The notion of *quantitative homotopy theory* introduced by Gromov [3] suggests a way of organizing global homotopy information of varying scales by looking at maps and homotopies that have a bounded Lipschitz constant and considering the asymptotics as the bound increases. For a data set, instead of looking at Lipschitz bounds on continuous maps into an associated simplicial complex, it makes more sense to look at *subdivisions* and simplicial maps. For abstract simplicial complexes, a simplicial map  $X \rightarrow Y$  in particular has Lipschitz constant bounded by 1 for the standard metric (induced from the embedding of the geometric realization in Euclidean space) on the geometric realizations  $|X| \rightarrow |Y|$ ; a simplicial map from a subdivision of  $X$  has a higher bound on its Lipschitz constant, which depends on the size of the simplices of the subdivision. When  $Y$  is a Čech or Rips complex of a data set (i.e., finite metric space), analogous remarks hold using the “intrinsic” metric on  $|Y|$  induced from the metric on the data set. Considering continuous maps  $|X| \rightarrow |Y|$  with Lipschitz constant bounded above by  $\lambda$  is analogous to considering simplicial maps to  $Y$  from subdivisions of  $X$  with mesh size bounded below by  $1/\lambda$ . Thus, instead of studying the subspaces of continuous maps from  $|X|$  to  $|Y|$  as we vary the Lipschitz bound, we can study the space of simplicial maps from subdivisions  $X'$  of  $X$  to  $Y$  as we refine the subdivision.

This paper sets the foundation for an approach to quantitative homotopy theory in terms of simplicial maps and subdivisions. As such, it accomplishes two things. Firstly, it presents a definition of a space of simplicial maps between two simplicial complexes (Definition 1.1). Secondly, it shows that as the mesh size of the subdivision tends to zero, this space of simplicial maps approximates the space of continuous maps between the geometric realizations (Theorem 1.2).

The first issue of defining the space of simplicial maps makes use of the a natural notion of “closeness” for simplicial maps called *contiguity*. Simplicial maps

$f_0, f_1, \dots, f_n$  from  $X$  to  $Y$  are *mutually contiguous* means that for every simplex  $v_0, \dots, v_m$  of  $X$ , the set  $\{f_i(v_j)\}$  of vertices of  $Y$  is a simplex (cf. [8, 3.5]). The notion of contiguity is particularly well-suited to the situation when the target is the Rips complex of a finite metric space. In that case, contiguity is a metric property: the maps are mutually contiguous if and only if any two points of  $\{f_i(v_j)\}$  are within  $\epsilon$ , the scale parameter of the Rips complex.

**Definition 1.1** For simplicial complexes  $X$  and  $Y$ , define the *contiguity complex*  $\text{Map}_{\text{SC}}(X, Y)$  to be the simplicial complex with vertices the simplicial maps and simplices the collections of mutually contiguous maps.

As we show in Construction 6.1 below, the geometric realization of  $\text{Map}_{\text{SC}}(X, Y)$  has a canonical natural map to the space  $\text{Map}_{\text{Top}}(|X|, |Y|)$  of continuous maps on geometric realizations (with the compact open topology). The second issue this paper addresses is how good an approximation this becomes as we subdivide  $X$ . While  $\text{Map}_{\text{SC}}(X, Y)$  is not expected to be a good approximation of  $\text{Map}_{\text{Top}}(|X|, |Y|)$  (just as the subspace of Lipschitz constant  $\leq 1$  maps from  $X$  to  $Y$  is not), for successive refinements  $X_1, X_2, \dots$ , of  $X$ , we claim that  $\text{Map}_{\text{SC}}(X_n, Y)$  approximates  $\text{Map}_{\text{Top}}(|X|, |Y|)$  up to homotopy. Specifically, we do not expect every homotopy class of map from  $|X|$  to  $|Y|$  to be represented by a simplicial map from  $X$  to  $Y$ . Rather, the Simplicial Approximation Theorem (e.g., [8, 3.4.8] and Theorem 4.5 below) says that subdivision of  $X$  may be required before a given continuous map  $X \rightarrow Y$  is represented up to homotopy by a simplicial map. We prove the following analogue for the contiguity mapping spaces; see Sects. 3 and 4 for details of terminology and Sect. 7 for the construction of the map.

**Theorem 1.2** *Let  $X$  be a finite simplicial complex and  $Y$  any simplicial complex. Let  $X_n$  be a sequence of successive subdivisions of  $X$  and choose compatible simplicial approximations  $X_{n+1} \rightarrow X_n$ . If the limit of the mesh size of  $X_n$  goes to zero, then  $\bigcup \text{Map}_{\text{SC}}(X_n, Y)$  is homotopy equivalent to the space  $\text{Map}_{\text{Top}}(|X|, |Y|)$  of continuous maps from  $|X|$  to  $|Y|$ . (Here the union is taken over maps  $\text{Map}_{\text{SC}}(X_n, Y) \rightarrow \text{Map}_{\text{SC}}(X_{n+1}, Y)$  induced from the simplicial approximations.)*

Definition 1.1 and Theorem 1.2 lay the foundations for quantitative homotopy theory in topological data analysis by producing a framework for approximating the homotopy type of mapping spaces by contiguity complexes. We discuss in Sect. 9 a detailed example of the computational application of our techniques, and in Sect. 10 we sketch how to integrate this work with the approach of persistent homology. In future work, we plan to develop the relationship to persistent homology and homotopy theory of sample spaces.

## 2 Review of Simplicial Complexes

This section reviews the basic definitions of simplicial complex and geometric realization. We review the notation for geometric simplices, their open simplices, and

their star neighborhoods. In this section and the next two, we follow Spanier [8] in spirit (though not in notation or details) and substitute references there for proofs whenever possible.

**Definition 2.1** A *simplicial complex* consists of a set of *vertices*  $V$  together with a subset  $S$  of the set of non-empty finite subsets of  $V$ , satisfying the following properties.

- (i) The singleton subsets of  $V$  are in  $S$ .
- (ii) If  $\sigma \in S$  and  $\tau \subset \sigma$ ,  $\tau \neq \emptyset$ , then  $\tau \in S$ .

The elements of  $S$  are called *simplices*, and  $\sigma \in S$  is called an  $n$ -simplex if  $\#\sigma = n + 1$ . If  $\sigma \in S$  and  $\tau \subsetneq \sigma$ , then  $\tau$  is called a *face* of  $\sigma$ . A simplicial complex  $X = (V, S)$  is *finite* if  $V$  is a finite set or is *locally finite* if each vertex is contained in at most finitely many simplices.

A map of simplicial complexes  $X = (V, S)$  to  $X' = (V', S')$  is a map  $f: V \rightarrow V'$  such that for each  $\sigma \in S$ ,  $f(\sigma) \in S'$ .

A subcomplex of  $X$  is a simplicial complex  $A = (U, R)$  such that  $U \subset V$  and  $R \subset S$ .

*Example 2.2* The *standard  $n$ -simplex* is the simplicial complex  $\Delta[n]$  with vertex set  $\{0, \dots, n\}$  and simplex set the set of all non-empty subsets of  $\{0, \dots, n\}$ .

A simplicial complex is an abstraction of a certain kind of polytope; given a simplicial complex, the associated polytope is called the *geometric realization*.

**Definition 2.3** Let  $X = (V, S)$  be a simplicial complex and write  $\mathbb{R}^V$  for the real vector space with basis the elements of  $V$ , which we topologize with the standard topology if  $V$  is finite, or with the union topology if  $V$  is infinite: a subset  $U$  of  $\mathbb{R}^V$  is open (or closed) if and only if its intersection with every finite dimensional subspace is open (respectively, closed) in the standard topology. The *geometric realization*  $|X|$  is the subspace of  $\mathbb{R}^V$  of elements that can be written as  $t_0v_0 + \dots + t_nv_n$  where  $t_i \geq 0$ ,  $t_0 + \dots + t_n = 1$ , and  $\{v_0, \dots, v_n\} \in S$  is an  $n$ -simplex of  $X$ .

The topology on  $\mathbb{R}^V$  ensures that  $|X|$  has the *union topology*: a subset  $U$  in  $|X|$  is open (or closed) if and only if its intersection with every finite subcomplex is open (respectively, closed). It follows that the geometric realization can alternatively be defined as a quotient space of the disjoint union of geometric simplices (see below), one for each simplex of  $X$ , with faces identified as per the face relations in  $X$ .

A function from  $V$  into a vector space  $E$  induces a *linear map*  $|X| \rightarrow E$  as the restriction of the linear map  $\mathbb{R}^V \rightarrow E$ . For a map of simplicial complexes  $X \rightarrow X'$ , the map  $|X| \rightarrow \mathbb{R}^{V'}$  factors through  $|X'|$  and the map  $|X| \rightarrow |X'|$  is continuous.

Regarding  $V$  as an orthonormal set in  $\mathbb{R}^V$  endows  $|X|$  with the *standard metric* (via restriction). On finite subcomplexes of  $|X|$ , the subspace topology agrees with the metric topology; in general, the metric topology is coarser than the topology on  $|X|$  when  $X$  is not a locally finite complex. In the context of data sets,  $V$  may have a meaningful distance function, inducing a different (but equivalent, since  $V$  is finite)

metric on  $|X|$ , which we refer to as the *intrinsic metric*. For definiteness, in what follows we will always use the standard metric on  $|X|$ , though for any equivalent metric, all assertions hold with minor modifications.

We use the following notation for certain subsets of  $|X|$ .

**Notation 2.4** For  $\sigma = \{v_0, \dots, v_n\}$  an  $n$ -simplex in  $X$ , write  $|\sigma|$  for the *geometric simplex*, the (closed) subset of  $|X|$  consisting of elements of the form  $t_0v_0 + \dots + t_nv_n$ . Write  $\text{Op}(\sigma)$  for the *open simplex*, the subset of  $|\sigma|$  consisting of those elements  $t_0v_0 + \dots + t_nv_n$  with all  $t_i$  non-zero (and therefore strictly positive);  $\text{Op}(\sigma)$  is an open subset of  $|\sigma|$ , but not necessarily an open subset of  $|X|$ . For  $v$  a vertex in  $X$ , let  $\text{St}(v)$  denote the *star neighborhood* of  $v$ , the subset of  $|X|$  of elements in  $\mathbb{R}^V$  whose  $v$  coordinate is non-zero (and therefore strictly positive);  $\text{St}(v)$  is an open subset of  $|X|$ . Write  $\text{St}(\sigma)$  for the *star neighborhood* of  $\text{Op}(\sigma)$ , the subset of  $|X|$ ,

$$\text{St}(\sigma) = \bigcap_{v_i \in \sigma} \text{St}(v_i) = \bigcup_{\tau \supset \sigma} \text{Op}(\tau)$$

(where  $\tau$  ranges over the simplices of  $X$  that contain  $\sigma$ );  $\text{St}(\sigma)$  is an open subset of  $|X|$  containing  $\text{Op}(\sigma)$ .

For convenience later, we note here the following fact.

**Proposition 2.5** *For a simplicial complex  $X = (V, S)$ , the collection  $\{\text{Op}(\sigma) \mid \sigma \in S\}$  is a partition of  $|X|$ ; i.e., an element  $x \in |X|$  is in  $\text{Op}(\sigma)$  for one and only one simplex  $\sigma$ .*

*Proof* For  $x \in |X|$  (or indeed in  $\mathbb{R}^V$ ), there is one and only one way to write  $x$  as  $t_0v_0 + \dots + t_nv_n$  for distinct elements  $v_0, \dots, v_n$  of  $V$  with  $t_0 \dots t_n \neq 0$ .  $\square$

We use the following easy observation in the next section.

**Lemma 2.6** *Let  $X = (V, S)$  be a simplicial complex and let  $C$  be a compact subset of  $|X|$  that is convex as a subset of  $\mathbb{R}^V$ . Then  $C$  is contained within a geometric simplex of  $|X|$ .*

*Proof* By compactness  $C$  must be contained in  $\mathbb{R}^{V_0}$  for some finite subset  $V_0 = \{v_0, \dots, v_n\}$  of  $V$  (with the  $v_i$ 's distinct). Making  $V_0$  smaller if necessary, we can assume without loss of generality that for each  $i = 0, \dots, n$  some element  $x_i$  of  $C$  has non-zero  $v_i$  coordinate (which then must be positive). By convexity of  $C$ , we must have  $(x_0 + x_1)/2$  in  $C$ , as this is the midpoint of the line from  $x_0$  to  $x_1$ . Likewise, the point

$$x = \frac{x_0 + x_1}{2^n} + \frac{x_2}{2^{n-1}} + \dots + \frac{x_n}{2}$$

is in  $C$ . Since  $x \in C \subset |X|$  has  $v_i$  coordinate positive for each  $i$  (and all other coordinates zero), we must have  $\sigma = \{v_0, \dots, v_n\} = V_0$  is a simplex of  $X$ . Since  $C \subset |X| \cap \mathbb{R}^{V_0} = |\sigma|$ , we see that  $C$  is contained within a geometric simplex of  $|X|$ .  $\square$

### 3 Review of Subdivision

This section reviews the notion of a subdivision of a simplicial complex and some of its basic properties.

**Definition 3.1** For a simplicial complex  $X = (V, S)$ , a *subdivision* is a simplicial complex  $X' = (V', S')$  with  $V' \subset |X|$  such that the linear map  $|X'| \rightarrow \mathbb{R}^V$  factors as a homeomorphism  $|X'| \cong |X|$ .

Lemma 2.6 above implies that every geometric simplex  $|\sigma|$  of  $|X'|$  has its image contained within a geometric simplex  $|\tau|$  of  $|X|$ , for some  $\tau$  depending on  $\sigma$ . (In particular, the definition above agrees with [8, 3.3].) An easy linear algebra argument [8, 3.3.3] in fact shows that  $\text{Op}(\sigma)$  lands in  $\text{Op}(\tau)$  for the minimal such simplex  $\tau$ . The following theorem is useful for identifying and conceptualizing subdivisions; see [8, 3.3.4] for a proof.

**Theorem 3.2** Let  $X = (V, S)$  and  $X' = (V', S')$  be simplicial complexes with  $V' \subset |X|$  and assume that the linear map  $|X'| \rightarrow \mathbb{R}^V$  factors through  $|X|$ . Then  $X'$  is a subdivision of  $X$  if and only if for every simplex  $\sigma$  of  $X$ , the set  $\{\text{Op}(\sigma') \mid \text{Op}(\sigma') \subset \text{Op}(\sigma)\}$  is a finite partition of  $\text{Op}(\sigma)$ .

In particular, since for a zero simplex  $\text{Op}(\sigma) = |\sigma|$ , we have the following immediate consequence (which is also easy to see on its own).

**Proposition 3.3** If  $X' = (V', S')$  is a subdivision of  $X = (V, S)$ , then  $V' \supset V$ .

Important examples of systematic subdivisions include the following.

*Example 3.4*  $X$  is a subdivision of itself.

*Example 3.5* Barycentric Subdivision For an  $n$ -simplex  $\sigma = \{v_0, \dots, v_n\}$  of  $X = (V, S)$ , let  $b_\sigma = (v_0 + \dots + v_n)/(n + 1) \in \mathbb{R}^V$  be its *barycenter*, and let  $V' = \{b_\sigma \mid \sigma \in S\}$  be the set of all barycenters of all simplices. Let

$$S' = \{ \{b_{\sigma_0}, \dots, b_{\sigma_n}\} \subset V' \mid \sigma_0 \subset \dots \subset \sigma_n \subset V \}.$$

Then  $X' = (V', S')$  is a subdivision of  $X$  [8, 3.3.9] called the *barycentric subdivision*.

For  $X' = (V', S')$  a subdivision of  $X = (V, S)$ , using the homeomorphism  $|X'| \cong |X|$ , we can regard any subdivision of  $X'$  as a subdivision of  $X$  whose vertex set contains  $V'$ ; likewise, any subdivision of  $X$  whose vertex set contains  $V'$  can be regarded as a subdivision of  $X'$ . Iterating barycentric subdivision then produces subdivisions with successively finer partitions of the open simplices of  $|X|$ . We use the standard metric on  $|X|$  to compare sizes of simplices of different subdivisions.

**Definition 3.6** Let  $X'$  be a subdivision of  $X$ . The *mesh size* of  $X'$ ,  $\text{Mesh}(X')$ , is the supremum of the diameters in  $|X|$  of images of the geometric simplices of  $|X'|$ .

Since the standard metric on  $|X|$  comes from a norm on  $\mathbb{R}^V$ , we find that the mesh size of  $X'$  is the supremum of the distances in  $|X|$  between all pairs of vertices of  $X'$  that span a 1-simplex of  $X'$ ,

$$\text{Mesh}(X') = \sup\{d_{|X|}(v, w) \mid \{v, w\} \in S'\}.$$

In the case when  $X' = X$ , the mesh size is  $\sqrt{2}$ , and in general the mesh size must be  $\leq \sqrt{2}$ . In the case when  $X'$  is the barycentric subdivision of  $X$ , an easy calculation shows that the mesh size is bounded above by 1. For the  $n$ th iterated barycentric subdivision, the mesh size is bounded above by  $1/\sqrt{2^{n-1}}$ , which goes to zero.

### 4 Review of Simplicial Approximation

This section reviews the definition of a simplicial approximation of a continuous map between the geometric realizations of simplicial complexes. We then state and prove a formulation of the Simplicial Approximation Theorem.

**Definition 4.1** Let  $X$  and  $Y$  be simplicial complexes and let  $\phi: |X| \rightarrow |Y|$  be a continuous map. A map of simplicial complexes  $f: X \rightarrow Y$  is called a *simplicial approximation* of  $\phi$  when for any  $x \in |X|$  and simplex  $\sigma$  in  $Y$ ,  $\phi(x) \in \text{Op}(\sigma)$  implies  $|f|(x) \in |\sigma|$ .

If  $\phi$  happens to send a vertex  $v$  in  $|X|$  to a vertex in  $|Y|$ , then  $f$  must send  $v$  to the same vertex of  $Y$ . As a consequence we get the following proposition.

**Proposition 4.2** Let  $f: X \rightarrow Y$  be a simplicial approximation of a continuous map  $\phi: |X| \rightarrow |Y|$ . Let  $A$  be a subcomplex of  $X$  such that  $\phi$  restricted to  $|A| \subset |X|$  is the geometric realization of a map of simplicial complexes  $g: A \rightarrow Y$ . Then  $f$  restricted to  $A$  is  $g$ .

Given  $x \in |X|$  and  $\sigma$  the unique simplex in  $Y$  such that  $\phi(x) \in \text{Op}(\sigma)$ , by convexity of  $|\sigma|$  in  $\mathbb{R}^W$ , the entire line segment between  $\phi(x)$  and  $|f|(x)$  is contained in  $|\sigma|$ . It follows that the continuous map  $H: |X| \times [0, 1] \rightarrow \mathbb{R}^W$  defined by

$$H(x, t) = (1 - t) \cdot \phi(x) + t \cdot |f|(x)$$

factors through  $|Y|$ . This then defines a homotopy from  $\phi$  to  $|f|$ , proving the following proposition.

**Proposition 4.3** Let  $f: X \rightarrow Y$  be a simplicial approximation of a continuous map  $\phi: |X| \rightarrow |Y|$ . Then  $\phi$  and  $|f|$  are homotopic. Moreover, if  $\phi$  and  $|f|$  agree on a subset  $A$  of  $|X|$ , then  $\phi$  and  $|f|$  are homotopic rel  $A$ .

It is not always possible to find a simplicial approximation for a given continuous map. The following alternative characterization of a simplicial approximation is useful in this regard. The proof is straightforward; see [8, 3.4.3].

**Theorem 4.4** *Let  $X = (V, S)$  and  $Y = (W, T)$  be simplicial complexes. A map of vertex sets  $f: V \rightarrow W$  is a simplicial approximation of a continuous map  $\phi: |X| \rightarrow |Y|$  if and only if for every vertex  $v$  in  $X$ ,  $\phi(\text{St}(v)) \subset \text{St}(f(v))$ .*

In particular, a continuous map  $\phi: |X| \rightarrow |Y|$  admits a simplicial approximation if and only if for every vertex  $v$  of  $X$ , there exists a vertex  $w$  of  $Y$  such that  $\text{St}(v) \subset \phi^{-1}(\text{St}(w))$ . This leads directly to the following theorem, the Simplicial Approximation Theorem. For this, recall that for a metric space  $M$ , the *Lebesgue number* of an open cover  $\mathcal{U} = \{U_\alpha\}$  of  $M$  is a number  $\epsilon > 0$  (if one exists) such that whenever a subset  $D$  of  $M$  has diameter  $\leq \epsilon$ , there exists  $U_\alpha$  in  $\mathcal{U}$  such that  $D \subset U_\alpha$ . When  $M$  is compact, every open cover has a Lebesgue number [7, 27.5].

**Theorem 4.5** (Simplicial Approximation Theorem) *Let  $X$  and  $Y$  be simplicial complexes, and  $\phi: |X| \rightarrow |Y|$  a continuous map. Assume that the open cover  $\{\phi^{-1}(\text{St}(w))\}$  of  $|X|$  (where  $w$  ranges over the vertices of  $Y$ ) has a Lebesgue number  $\lambda > 0$ . If  $X'$  is a subdivision of  $X$  with  $\text{Mesh}(X') < \lambda/2$ , then the composite map  $|X'| \cong |X| \rightarrow |Y|$  admits a simplicial approximation.*

*Proof* We apply Theorem 4.4: for every vertex  $v$  of  $X'$ , the diameter of  $\text{St}(v)$  in  $|X|$  is  $\leq 2 \text{Mesh}(X')$ . □

The following example illustrates the way in which successive subdivision is required to realize “larger” homotopy classes of maps.

*Example 4.6* The simplicial fundamental group Consider the model of the circle given by the boundary of the standard two-simplex  $\partial\Delta[2]$ , i.e., the simplicial complex determined by vertices  $\{v_0, v_1, v_2\}$  and one simplices  $\{(v_0, v_1), (v_1, v_2), (v_2, v_0)\}$ . The  $k$ th barycentric subdivision  $\text{Sd}^k \partial\Delta[2]$  has  $3(2^k)$  vertices. The map  $S^1 \rightarrow S^1$  of degree  $d$  (i.e., a map which “winds around” the circle  $d$  times) does not admit a simplicial approximation  $\text{Sd}^{d-1} \partial\Delta[2] \rightarrow \partial\Delta[2]$  but does have a simplicial approximation  $\text{Sd}^d \partial\Delta[2] \rightarrow \partial\Delta[2]$ .

We need the following additional existence statement for simplicial approximations implicitly used in the statement of Theorem 1.2.

**Theorem 4.7** *If  $X'$  is a subdivision of  $X$ , then the homeomorphism  $|X'| \cong |X|$  admits a simplicial approximation.*

*Proof* We use the notation from Definition 3.1. For every vertex  $v'$  in  $V'$ , let  $\sigma_{v'}$  be the unique simplex of  $X$  with  $v' \in \text{Op}(\sigma_{v'})$ . Choose  $g: V' \rightarrow V$  to be any function that takes each  $v'$  to some vertex of  $\sigma_{v'}$ . We claim that  $g$  is a simplicial map and a simplicial approximation of the homeomorphism  $\phi: |X'| \cong |X|$ . To see this, let  $\sigma' = \{v'_0, \dots, v'_n\}$  be an  $n$ -simplex in  $X'$  and let  $|\tau|$  be a geometric simplex of  $|X|$  that contains the image of  $|\sigma'|$  under  $\phi$ . Since  $|\tau|$  contains  $v'_i$  for all  $i$ , it must also contain  $|\sigma_{v'_i}|$  for all  $i$ ; hence  $\{g(v'_0), \dots, g(v'_n)\} \subset \tau$  is a simplex in  $X$ , which shows that  $g$  is a simplicial map. We note also that  $|g|$  sends  $|\sigma'|$  into  $|\tau|$ . Next we see that  $g$  is



a simplicial approximation of the homeomorphism  $|X'| \cong |X|$ . For  $x \in |X'|$ , without loss of generality, we can assume  $x \in \text{Op}(\sigma')$ , and we note that by Theorem 3.2, under the homeomorphism  $|X'| \cong |X|$ ,  $\text{Op}(\sigma') \subset \text{Op}(\sigma)$  for some simplex  $\sigma$  of  $X$ . Then  $|\sigma|$  contains  $v'_0, \dots, v'_n$ , and so (as above)  $|\sigma|$  contains the image of  $|\sigma'|$  under  $|g|$ . In particular  $x$  lands in  $\text{Op}(\sigma)$  under the homeomorphism  $|X'| \cong |X|$  and  $|g|(x) \in |\sigma|$ .  $\square$

Simplicial approximations, when they exist, need not be unique, but we do have the following theorem, which gives uniqueness up to contiguity. Recall from the introduction that simplicial maps  $f_0, \dots, f_n: X \rightarrow Y$  are *mutually contiguous* when for any simplex  $\sigma = \{v_0, \dots, v_m\}$  of  $X$ , the set

$$\{f_0(v_0), \dots, f_0(v_m), f_1(v_0), \dots, f_n(v_m)\}$$

forms a simplex in  $Y$ .

**Theorem 4.8** *If  $f_0, \dots, f_n: X \rightarrow Y$  are all simplicial approximations to  $\phi: |X| \rightarrow |Y|$ , then they are mutually contiguous.*

*Proof* For an  $m$ -simplex  $\sigma = \{v_0, \dots, v_m\}$  of  $X$ , choose a point  $x \in \text{Op}(\sigma) \subset |X|$ , and let  $\tau$  be the simplex in  $Y$  such that  $\phi(x) \in \text{Op}(\tau)$ . Then  $|f_i|(x) \in |\tau|$  so  $f_i$  must send  $v_0, \dots, v_m$  to vertices of  $\tau$ . It follows that  $\{f_i(v_j)\} \subset \tau$ .  $\square$

Contiguity is a combinatorial analogue of homotopy; a precise statement of the relationship is the content of the main theorem of this paper. However, in contrast to homotopy of maps, contiguity is evidently *not* an equivalence relation (as it is not transitive). The following simple example illustrates the relationship between homotopy classes and contiguity classes [8, 3.5.5].

*Example 4.9* Consider again the model of the circle given by the boundary of the standard two-simplex  $\partial\Delta[2]$ . Suppose that the simplicial maps  $f, g: \partial\Delta[2] \rightarrow \partial\Delta[2]$  both land in proper subcomplexes of  $\partial\Delta[2]$ . Then it is easy to check that  $f$  and  $g$  must each be contiguous to a constant map (sending all vertices to the same vertex) and that any pair of constant maps are contiguous. Thus, all such simplicial maps form one “contiguity class” that represents the null-homotopic map from  $S^1 \rightarrow S^1$ . On the other hand, if we take

$$\begin{aligned} f(v_0) &= v_0, & f(v_1) &= v_1, & f(v_2) &= v_0, \\ g(v_0) &= v_0, & g(v_1) &= v_2, & g(v_2) &= v_0, \end{aligned}$$

then  $f$  and  $g$  are not contiguous because for the 1-simplex  $\{v_0, v_1\}$  in the domain, the subset  $\{f(v_0), f(v_1), g(v_0), g(v_1)\} = \{v_0, v_1, v_2\}$  of vertices in the codomain is not a simplex. The remaining maps  $\partial\Delta[2] \rightarrow \partial\Delta[2]$  are all given by permutations of the vertex set. The even permutations induce degree 1 self-maps of  $S^1$  and the odd permutations induce degree  $-1$  self-maps of  $S^1$ . All maps of the same degree are homotopic, but an easy check shows that no two of these maps are contiguous, and so each vertex permutation is its own contiguity class. As we subdivide the domain, the image of these maps will merge into a single contiguity class for each degree.

### 5 The Product of Simplicial Complexes

In this section we study the product of simplicial complexes and construct maps relating its geometric realization to the product of the geometric realization and to the geometric realization of the product of subdivisions.

**Definition 5.1** Let  $X = (V, S)$  and  $Y = (W, T)$  be simplicial complexes. Define the product complex  $X \boxtimes Y$  to be the simplicial complex  $(V \times W, U)$  where  $\{(v_0, w_0), \dots, (v_n, w_n)\} \in U$  if and only if  $\{v_0, \dots, v_n\} \in S$  and  $\{w_0, \dots, w_n\} \in T$ . (N.B. We do not assume that  $v_0, \dots, v_n$  are distinct elements of  $V$  or  $w_0, \dots, w_n$  are distinct elements of  $W$ .)

The product  $X \boxtimes Y$  comes with canonical maps of simplicial complexes  $X \boxtimes Y \rightarrow X$  and  $X \boxtimes Y \rightarrow Y$  induced by the projections  $V \times W \rightarrow V$  and  $V \times W \rightarrow W$ . These projection maps have the following universal property.

**Proposition 5.2**  $X \boxtimes Y$  is the product of  $X$  and  $Y$  in the category of simplicial complexes: maps of simplicial complexes  $Z \rightarrow X \boxtimes Y$  are in bijective correspondence with pairs of maps of simplicial complexes  $Z \rightarrow X, Z \rightarrow Y$ .

The Cartesian product of spaces has the corresponding universal property in the category of spaces, and so the geometric realization of the projection maps  $|X \boxtimes Y| \rightarrow |X|, |X \boxtimes Y| \rightarrow |Y|$  induce a natural continuous map

$$\rho: |X \boxtimes Y| \rightarrow |X| \times |Y|.$$

Unless  $X$  or  $Y$  is discrete (has only zero simplices), this map is far from a homeomorphism. For example,

$$\Delta[m] \boxtimes \Delta[n] \cong \Delta[(m + 1)(n + 1) - 1],$$

where  $\Delta[\bullet]$  denotes a standard simplex (Example 2.2). Nonetheless, the projection map  $\rho$  does have a natural section

$$\zeta: |X| \times |Y| \rightarrow |X \boxtimes Y|$$

induced by the universal bilinear map

$$\begin{aligned} \mathbb{R}^V \times \mathbb{R}^W &\rightarrow \mathbb{R}^V \otimes \mathbb{R}^W = \mathbb{R}^{V \times W}, \\ (s_0v_0 + \dots + s_mv_m, t_0w_0 + \dots + t_nw_n) &\mapsto \sum s_it_j(v_i, w_j). \end{aligned}$$

This defines a function  $\zeta$  as above since when  $\{v_0, \dots, v_m\}$  and  $\{w_0, \dots, w_n\}$  are simplices of  $X$  and  $Y$ ,  $\{(v_i, w_j)\}$  is a simplex of  $X \boxtimes Y$ , and when  $\sum s_i = \sum t_j = 1$ , then  $\sum s_it_j = 1$ . In general the product topology of  $|X| \times |Y|$  is coarser than the union topology, and  $\zeta$  is not a continuous map (cf. Theorem 2 of [5]). When  $X$  or  $Y$  is locally finite, the product topology on  $|X| \times |Y|$  coincides with the union topology, and  $\zeta$  is a continuous map.

**Theorem 5.3** *The composite function  $\rho \circ \zeta$  is the identity on  $|X| \times |Y|$ . The identity map on  $X \boxtimes Y$  is a simplicial approximation to the composite  $\zeta \circ \rho$ , which is a continuous map.*

*Proof* An arbitrary element of  $|X| \times |Y|$  is of the form

$$(x, y) = (s_0v_0 + \dots + s_mv_m, t_0w_0 + \dots + t_nw_n),$$

where  $\{v_i\}$  is an  $m$ -simplex of  $X$ ,  $\{w_j\}$  is an  $n$ -simplex of  $Y$ , and  $\sum s_i = \sum t_j = 1$ ,  $s_i > 0, t_j > 0$ . This element maps to  $\sum s_it_j(v_i, w_j)$  in  $|X \boxtimes Y|$ . The map induced by the projection  $X \boxtimes Y \rightarrow X$  takes this to the element

$$s_0(t_0 + \dots + t_n)v_0 + \dots + s_m(t_0 + \dots + t_n)v_m = s_0v_0 + \dots + s_mv_m = x$$

in  $|X|$  and likewise the map induced by the projection  $X \boxtimes Y \rightarrow Y$  takes this element to  $y$  in  $|Y|$ . It follows that  $\rho(\zeta(x, y)) = (x, y)$  and therefore that  $\rho \circ \zeta$  is the identity on  $|X| \times |Y|$ .

On the other hand, an arbitrary element of  $|X \boxtimes Y|$  is of the form

$$z = t_0(v_0, w_0) + \dots + t_n(v_n, w_n)$$

with  $\sum t_i = 1$  and  $t_i > 0$  (with the vertices  $(v_i, w_i)$  distinct, though neither the vertices  $v_i$  nor the vertices  $w_i$  need be distinct). The composite  $\zeta \circ \rho$  takes  $z$  to the element  $\sum t_it_j(v_i, w_j)$  of  $|X \boxtimes Y|$ ; in particular, since  $|X \boxtimes Y|$  has the union topology, the composite  $\zeta \circ \rho$  is continuous even when  $\zeta$  is not. Let  $\sigma = \{(v_i, w_j)\}$ , a  $k$ -simplex of  $X \boxtimes Y$  for some  $k \leq (n + 1)^2 - 1$ . Since for all  $i, j, t_it_j > 0$ , we have  $\zeta(\rho(z)) \in \text{Op}(\sigma)$ . Since  $(v_i, w_i) \in \sigma$  for all  $i, j$ , it follows that  $z \in |\sigma|$ . This shows that the identity on  $X \boxtimes Y$  is a simplicial approximation of the composite  $\zeta \circ \rho$ .  $\square$

We also use the following extension of  $\zeta$ .

**Theorem 5.4** *Let  $X'$  be a subdivision of  $X$  and  $Y'$  a subdivision of  $Y$ . The map  $\zeta$  extends to a continuous map*

$$\zeta : |X' \boxtimes Y'| \rightarrow |X \boxtimes Y|,$$

*which sends every geometric simplex to a set of diameter  $\leq \text{Mesh}(X') + \text{Mesh}(Y')$ . Moreover, given  $a : X' \rightarrow X$  and  $b : Y' \rightarrow Y$  simplicial approximations to the homeomorphisms  $|X'| \cong |X|$  and  $|Y'| \cong |Y|$ , then  $a \boxtimes b$  is a simplicial approximation to  $\zeta$ .*

*Proof* Write  $X' = (V', S'), Y' = (W', T')$ , and recall that by definition  $V' \subset \mathbb{R}^V$  and  $W' \subset \mathbb{R}^W$ . The composite map

$$\mathbb{R}^{V'} \times \mathbb{R}^{W'} \rightarrow \mathbb{R}^V \times \mathbb{R}^W \rightarrow \mathbb{R}^V \otimes \mathbb{R}^W = \mathbb{R}^{V \times W}$$

induces a unique bilinear map  $\mathbb{R}^{V' \times W'} \rightarrow \mathbb{R}^{V \times W}$ ; specifically, if

$$v' = s_0v_0 + \dots + s_mv_m \in \mathbb{R}^{V'},$$

$$w' = t_0 w_0 + \dots + t_n w_n \in \mathbb{R}^W,$$

then  $(v', w') \mapsto \sum s_i t_j (v_i, w_j) \in \mathbb{R}^{V \times W}$ . To see that this restricts to a map  $\zeta$  as in the statement, consider an arbitrary element

$$z = r_0 (v'_0, w'_0) + \dots + r_\ell (v'_\ell, w'_\ell) \in |X' \boxtimes Y'|,$$

where  $r_i > 0$  and  $\sum r_i = 1$  (with the vertices  $(v'_i, w'_i)$  distinct). Write

$$v'_i = s_0^i v_0^i + \dots + s_{m_i}^i v_{m_i}^i \in \mathbb{R}^V,$$

$$w'_i = t_0^i w_0^i + \dots + t_{n_i}^i w_{n_i}^i \in \mathbb{R}^W$$

with the usual conventions. Then  $z$  goes to

$$\sum_{i,j,k} r_i s_j^i t_k^i (v_j^i, w_k^i).$$

Since  $\{v'_i \mid i = 0, \dots, \ell\}$  is a simplex in  $X'$ ,  $\{v_j^i \mid i = 0, \dots, \ell, j = 0, \dots, m_i\}$  must be a simplex in  $X$  and likewise  $\{w_k^i\}$  is a simplex in  $Y$ . It follows that  $\{(v_j^i, w_k^i)\}$  is a simplex in  $X \boxtimes Y$ , and since  $\sum r_i s_j^i t_k^i = 1$ , we see that  $z$  lands in  $|X \boxtimes Y|$ . This defines  $\zeta : |X' \boxtimes Y'| \rightarrow |X \boxtimes Y|$ ; note that  $\zeta$  is a linear map and in particular continuous. Since  $\zeta$  is a linear map, the diameter of the image of a simplex  $\{(v'_i, w'_i)\}$  is the supremum (over  $i, j$ ) of the distances

$$d(\zeta(v'_i, w'_i), \zeta(v'_j, w'_j)),$$

which are clearly bounded above by  $\text{Mesh}(X') + \text{Mesh}(Y')$ . For the statement about simplicial approximations, let

$$\sigma = \{(v_j^i, w_k^i) \mid i = 0, \dots, \ell, j = 0, \dots, m_i, k = 0, \dots, n_i\}.$$

We note that in the formula for  $\zeta(z)$  above, we have  $r_i s_j^i t_k^i > 0$  for all  $i, j, k$ , which puts  $\zeta(z)$  in  $\text{Op}(\sigma)$ . Since  $a(v'_i) \in \{v_0^i, \dots, v_{m_i}^i\}$  and  $b(w'_i) \in \{w_0^i, \dots, w_{n_i}^i\}$ , we have  $a \boxtimes b(\{(v'_i, w'_i)\}) \subset \sigma$ , and hence that  $|a \boxtimes b|(z) \in |\sigma|$ . This shows that  $a \boxtimes b$  is a simplicial approximation of  $\zeta$ . □

### 6 Mapping Complexes and Products

We now turn to the contiguity mapping complex  $\text{Map}_{\text{SC}}(X, Y)$  defined in the introduction and the mapping space  $\text{Map}_{\text{Top}}(|X|, |Y|)$ , the space of continuous maps from  $|X|$  to  $|Y|$  with the compact open topology. (Recall that the compact open topology is the smallest topology generated by the sets  $B_{U,K}$  of maps  $f : X \rightarrow Y$  such that  $f(K) \subset U$ , for  $K$  compact and  $U$  open. When  $Y$  is a metric space this topology is equivalent to the topology of uniform convergence on compact sets.) For the statement of Theorem 1.2, we need a comparison map  $\Gamma : |\text{Map}_{\text{SC}}(X, Y)| \rightarrow$

$\text{Map}_{\text{Top}}(|X|, |Y|)$ , which we construct in this section. We study the behavior of this comparison map under the adjunction with the product of topological spaces (when  $X$  is locally finite) and an analogous adjunction in simplicial complexes. We begin with the construction of the comparison map.

**Construction 6.1** Let  $X$  and  $Y$  be simplicial complexes and let  $F$  denote the set of vertices of  $\text{Map}_{\text{SC}}(X, Y)$ , i.e., the set of maps of simplicial complexes  $X \rightarrow Y$ . A typical element  $\phi$  of  $|\text{Map}_{\text{SC}}(X, Y)|$  then is of the form  $t_0 f_0 + \dots + t_n f_n \in \mathbb{R}^F$  where  $f_0, \dots, f_n$  are mutually contiguous and  $\sum t_i = 1, t_i > 0$ . Define  $\Gamma\phi$  to be the function  $|X| \rightarrow |Y|$  that takes an element  $x = s_0 v_0 + \dots + s_m v_m$  of  $|X|$  to the element  $\sum t_i s_j f_i(v_j)$  of  $|Y|$ . We note that  $\Gamma\phi$  is a linear map  $|X| \rightarrow |Y|$  and so in particular  $\Gamma\phi$  is a continuous map. This then constructs a function  $\Gamma : |\text{Map}_{\text{SC}}(X, Y)| \rightarrow \text{Map}_{\text{Top}}(|X|, |Y|)$ . To see that  $\Gamma$  is continuous, it suffices to check it on each simplex, where it is obvious from the formula.

The product of topological spaces and the mapping space fit together into an adjunction as follows. For spaces  $X, Y, Z$ , a continuous map  $\phi : X \times Z \rightarrow Y$  induces a continuous map  $\tilde{\phi} : Z \rightarrow \text{Map}_{\text{Top}}(X, Y)$  defined by

$$(\tilde{\phi}(z))(x) = \phi(x, z).$$

A standard fact from topology [7, 46.11] is that when  $X$  is a locally compact Hausdorff space, this defines a bijection between the set of continuous maps  $X \times Z \rightarrow Y$  and the set of continuous maps  $Z \rightarrow \text{Map}_{\text{Top}}(X, Y)$ . In the case of concern to us, this specializes to the following proposition.

**Proposition 6.2** *Let  $X, Y$ , and  $Z$  be simplicial complexes and assume that  $X$  is locally finite. Continuous maps  $|Z| \rightarrow \text{Map}_{\text{Top}}(|X|, |Y|)$  are in one-to-one correspondence with continuous maps  $|X| \times |Z| \rightarrow |Y|$ .*

We have an analogous relationship between the product of simplicial complexes of the previous section and the contiguity mapping complex, but without the locally finite hypothesis.

**Theorem 6.3** *Let  $X, Y$ , and  $Z$  be simplicial complexes. Maps of simplicial complexes  $Z \rightarrow \text{Map}_{\text{SC}}(X, Y)$  are in one-to-one correspondence with maps of simplicial complexes  $X \boxtimes Z \rightarrow Y$ .*

*Proof* Write  $X = (V_X, S_X)$  and similarly for  $Y$  and  $Z$ . Since a map of simplicial complexes is in particular a map of vertex sets, using the exponential law for products and functions of sets, we can identify both the set  $A$  of maps of simplicial complexes  $X \rightarrow \text{Map}_{\text{SC}}(Y, Z)$  and the set  $B$  of maps of simplicial complexes  $X \boxtimes Y \rightarrow Z$  as subsets of the set  $C$  of maps  $V_X \times V_Z \rightarrow V_Y$ . For any  $f \in C$ , for any  $w \in V_Z$ , write  $f_w : V_X \rightarrow V_Y$  for the map  $f(-, w)$ , and let  $\sigma = \{v_0, \dots, v_m\}$  and  $\tau = \{w_0, \dots, w_n\}$  denote arbitrary simplices of  $X$  and  $Z$ , respectively. Starting with  $f \in A, \{f(v_i, w_j) \mid i, j\}$  must be a simplex of  $Y$  since  $f_{w_0}, \dots, f_{w_m}$  are mutually contiguous maps of

simplicial complexes. Since  $\sigma$  and  $\tau$  were arbitrary, we see that  $f$  is a simplicial map  $X \boxtimes Z \rightarrow Y$  and hence  $f \in B$ . On the other hand, if we start with  $f \in B$ , then we know that  $\{f(v_i, w_j) \mid i, j\}$  is a simplex of  $Y$  and so in particular for each  $j$ ,  $\{f(v_i, w_j) \mid i\}$  is a simplex of  $Y$ . Since  $\sigma$  was arbitrary this shows that each  $f_{w_i}$  is a map of simplicial complexes  $X \rightarrow Y$  and that the maps  $f_{w_0}, \dots, f_{w_m}$  are mutually contiguous. Since  $\tau$  was arbitrary, this shows that  $f$  is a map of simplicial complexes  $Z \rightarrow \text{Map}_{\text{SC}}(X, Y)$  and hence  $f \in A$ .  $\square$

We can relate the correspondences in Proposition 6.2 and Theorem 6.3. The following proposition is clear from the formula for  $\zeta$  in the previous section and the formula for the comparison map  $\Gamma$  in Construction 6.1.

**Proposition 6.4** *Let  $X, Y$ , and  $Z$  be simplicial complexes with  $X$  locally finite. If  $f: X \boxtimes Z \rightarrow Y$  and  $g: Z \rightarrow \text{Map}_{\text{SC}}(X, Y)$  correspond under the adjunction of Theorem 6.3, then the composite maps*

$$\begin{aligned} |X| \times |Z| &\xrightarrow{\zeta} |X \boxtimes Z| \xrightarrow{|f|} |Y|, \\ |Z| &\xrightarrow{|g|} |\text{Map}_{\text{SC}}(X, Y)| \xrightarrow{\Gamma} \text{Map}_{\text{Top}}(|X|, |Y|) \end{aligned}$$

correspond under the adjunction of Proposition 6.2.

Proposition 6.2 extends to an exponential correspondence [8, 1.2.8] (tensor and cotensor adjunctions),

$$\text{Map}_{\text{Top}}(|X| \times |Y|, |Z|) \cong \text{Map}_{\text{Top}}(|Z|, \text{Map}_{\text{Top}}(|X|, |Y|))$$

when  $X$  is locally finite. We also have the corresponding extension of Theorem 6.3 (for arbitrary  $X$ ).

**Proposition 6.5** *The correspondence of Theorem 6.3 is the map on vertex sets of an isomorphism of simplicial complexes  $\text{Map}_{\text{SC}}(X \times Y, Z) \cong \text{Map}_{\text{SC}}(Z, \text{Map}_{\text{SC}}(X, Y))$ .*

*Proof* Let  $\alpha = (f_0, \dots, f_\ell)$  be a finite set of maps  $V_X \times V_Z \rightarrow V_Y$ . Then  $\alpha$  is a simplex of  $\text{Map}_{\text{SC}}(X \times Y, Z)$  or  $\text{Map}_{\text{SC}}(Z, \text{Map}_{\text{SC}}(X, Y))$  precisely when for every simplex  $\sigma = \{v_0, \dots, v_m\}$  of  $X$  and every simplex  $\tau = \{w_0, \dots, w_n\}$  of  $Z$ , the subset  $\{f_i(v_j, w_k)\}$  of  $V_Y$  is a simplex of  $Y$ .  $\square$

### 7 A Reduction of Theorem 1.2

In this section, we begin the proof of Theorem 1.2 and reduce it to a statement about homotopy groups which we prove in the next section. For this section and the next, let  $X$  and  $Y$  be simplicial complexes with  $X$  finite. Let  $X_0 = X$  and inductively choose and fix a subdivision  $X_{n+1}$  of  $X_n$  and a simplicial approximation  $f_n: X_{n+1} \rightarrow X_n$  of the given homeomorphism  $|X_{n+1}| \cong |X_n|$ . We assume that viewed as subdivisions of  $X$ ,  $\text{Mesh } X_n$  tends to zero. This is the setup in the hypothesis of Theorem 1.2.

First note that the maps  $f_n^* : \text{Map}_{\text{SC}}(X_n, Y) \rightarrow \text{Map}_{\text{SC}}(X_{n+1}, Y)$  are (up to isomorphism) inclusions of subcomplexes: since the map  $X_{n+1} \rightarrow X_n$  is a surjection on vertex sets, a map of simplicial complexes  $X_n \rightarrow Y$  is completely determined by its composite  $X_{n+1} \rightarrow X_n \rightarrow Y$ . As a consequence, we see in particular that  $\bigcup |\text{Map}_{\text{SC}}(X_n, Y)|$  is a CW complex. Another closely related CW complex is the telescope, constructed as follows.

**Construction 7.1** For a continuous map  $\phi : A \rightarrow B$  between topological spaces  $A$  and  $B$ , the *mapping cylinder*  $M\phi$  is the space  $(A \times [0, 1]) \cup_A B$  obtained by gluing  $A \times [0, 1]$  and  $B$  along the inclusion of  $A$  in  $A \times [0, 1]$  as  $A \times \{1\}$  and the map  $\phi : A \rightarrow B$ . (This gluing is characterized by the universal property given by a description as the pushout

$$\begin{array}{ccc}
 A & \xrightarrow{\text{id} \times \{1\}} & A \times [0, 1] \\
 \phi \downarrow & & \downarrow \\
 B & \longrightarrow & M\phi
 \end{array}$$

in the category of spaces.)

For a sequence of continuous maps  $\phi_0 : A_0 \rightarrow A_1, \phi_1 : A_1 \rightarrow A_2, \dots$ , the *telescope*  $\text{Tel } \phi_n$  is the space

$$M\phi_0 \bigcup_{A_1} M\phi_1 \bigcup_{A_2} M\phi_2 \bigcup \dots,$$

where  $M\phi_n$  and  $M\phi_{n+1}$  are glued together along the inclusions of  $A_{n+1}$  in  $M\phi_n$  and in  $M\phi_{n+1}$  (as  $A_{n+1} \times \{0\}$ ).

Let  $A_n = |\text{Map}_{\text{SC}}(X_n, Y)|$ , let  $\phi_n = |f_n^*|$  (the geometric realization of the map  $f_n^* : \text{Map}_{\text{SC}}(X_n, Y) \rightarrow \text{Map}_{\text{SC}}(X_{n+1}, Y)$  induced by  $f_n$ ) and let  $F = \text{Tel } \phi_n$ . Collapsing the intervals, we obtain a map  $F \rightarrow \bigcup A_n$ .

**Proposition 7.2** *The map  $F = \text{Tel } |f_n^*| \rightarrow \bigcup |\text{Map}_{\text{SC}}(X_n, Y)|$  is a homotopy equivalence.*

*Proof* This is a standard argument from homotopy theory that just uses the fact that each  $f_n^*$  is (up to isomorphism) the inclusion of a subcomplex. Write  $F_0 \subset F_1 \subset \dots \subset F$  for the subspaces

$$\begin{aligned}
 F_0 &= A_0, \\
 F_1 &= M\phi_0, \\
 F_2 &= M\phi_0 \bigcup_{A_1} M\phi_1, \\
 &\vdots
 \end{aligned}$$

Note that there is a canonical inclusion  $a_n$  of  $A_n$  in  $F_n$  via its inclusion as the back face of  $M\phi_n$ . We inductively construct maps  $b_n : A_n \rightarrow F_n$  that are homotopic to the maps  $a_n$  but with the property that  $b_n$  restricts on  $A_{n-1}$  to  $b_{n-1} : A_{n-1} \rightarrow F_{n-1} \subset F_n$ . We start with  $b_0 = a_0 = \text{id} : A_0 \rightarrow F_0$ . For  $n > 0$ , the fact that  $f_{n-1}^*$  is (up to isomorphism) the inclusion of a subcomplex implies that  $M\phi_{n-1} = M|f_{n-1}^*|$  is a deformation retract of  $A_n \times [0, 1]$ , q.v. [8, 3.2.4]. It is slightly more convenient to regard

$$(A_{n-1} \times [-1, 1]) \bigcup_{A_{n-1}} A_n \subset A_n \times [-1, 1]$$

as a deformation retract. Choosing a retraction  $R$ , let

$$r : A_n \rightarrow (A_{n-1} \times [-1, 1]) \bigcup_{A_{n-1}} A_n$$

be its restriction to  $A_n \times \{-1\} \subset A_n \times [-1, 1]$ . Let  $b_n$  be the map obtained as the composite of  $r$  with the map

$$(A_{n-1} \times [-1, 1]) \bigcup_{A_{n-1}} A_n = (A_{n-1} \times [-1, 0]) \bigcup_{A_{n-1}} M\phi_{n-1} \rightarrow F_n$$

that on the  $A_{n-1} \times [-1, 0]$  runs the homotopy from  $b_{n-1}$  to  $a_{n-1}$  and on the  $M\phi_{n-1}$  piece is the inclusion  $M\phi_{n-1} \subset F_n$ . By construction, the restriction of  $b_n$  to  $A_{n-1}$  is  $b_{n-1}$ , and the map  $R$  induces a homotopy  $H_n$  from  $b_n$  to  $a_n$ . Putting the maps  $b_n$  together, we get a map  $b : \bigcup A_n \rightarrow F$ . Note that the composite of the collapse map  $c : F \rightarrow \bigcup A_n$  and the homotopy  $H_n$  restricted to  $A_{n-1}$  is  $c \circ H_{n-1}$  for the first half and the constant homotopy on the inclusion for the second half. Therefore, reparametrizing the  $H_n$ 's, we can fit these homotopies together to be a homotopy from the composite map  $c \circ b$  on  $\bigcup A_n$  to the identity. Similarly, we can use the deformation retractions  $R$  to construct the homotopy between  $b \circ c$  and the identity on  $F$ . □

To define a map out of  $\bigcup A_n$ , we need maps out of each  $A_n$  that are compatible as  $n$  varies. The purpose of the telescope is that to define a map out of  $F$ , we only need maps out of each  $A_n$  and homotopies that make them compatible as  $n$  varies.

**Construction 7.3** We construct the continuous map  $\Phi : F \rightarrow \text{Map}_{\text{Top}}(|X|, |Y|)$  by specifying compatible maps on each piece  $M\phi_n$ . On the  $A_n \times [0, 1]$  part of  $M\phi_n$ ,  $\Phi$  does the homotopy (induced by the homotopy of Proposition 4.3) from the map

$$A_n = |\text{Map}_{\text{SC}}(X_n, Y)| \xrightarrow{f} \text{Map}_{\text{Top}}(|X_n|, |Y|) \cong |\text{Map}_{\text{Top}}(|X|, |Y|)|,$$

to the map

$$A_n = |\text{Map}_{\text{SC}}(X_n, Y)| \xrightarrow{f} \text{Map}_{\text{Top}}(|X_n|, |Y|) \xrightarrow{|f_n|^*} |\text{Map}_{\text{Top}}(|X_{n+1}|, |Y|)| \cong |\text{Map}_{\text{Top}}(|X|, |Y|)|,$$



where the last homeomorphisms are induced by the subdivision homeomorphisms  $|X_n| \cong |X|$  and  $|X_{n+1}| \cong |X|$ , respectively. On the  $A_{n+1}$  part of  $M\phi_n$ , we do the map

$$A_{n+1} = |\text{Map}_{\text{SC}}(X_{n+1}, Y)| \xrightarrow{F} \text{Map}_{\text{Top}}(|X_{n+1}|, |Y|) \cong |\text{Map}_{\text{Top}}(|X|, |Y|)|.$$

These pieces then fit together to define a continuous map  $\Phi$  on  $F$ .

Theorem 1.2 now reduces to showing that the map  $\Phi$  is a homotopy equivalence. Since  $X$  is finite, Milnor [6] shows that  $\text{Map}_{\text{Top}}(|X|, |Y|)$  has the homotopy type of a CW complex, and the Whitehead Theorem [8, 7.6.24] shows that a weak equivalence (see below) between spaces of the homotopy type of a CW complex must be a homotopy equivalence. Thus, Theorem 1.2 reduces to the following theorem.

**Theorem 7.4** *The map  $\Phi$  is a weak equivalence.*

As we explain, the proof of this theorem consists of Lemmas 7.5 and 7.6 below.

For the map  $\Phi$  to be a *weak equivalence* means that it induces a bijection on path components and for each element  $x \in F$  and each  $q \geq 1$ , the induced map on homotopy groups

$$\pi_q(F, x) \rightarrow \pi_q(\text{Map}_{\text{Top}}(|X|, |Y|), \Phi(x))$$

is an isomorphism. (See [8, 7.2] for a definition of the homotopy groups  $\pi_q$ .) We have the following first step in this direction.

**Lemma 7.5**  *$\Phi$  is a bijection on path components.*

*Proof* The Simplicial Approximation Theorem (Theorem 4.5) and Proposition 4.3 shows that  $\Phi$  is surjective on path components. Theorem 3.5.6 of [8] shows that  $\Phi$  is injective on path components. □

The map on homotopy groups  $\pi_q(F, x) \rightarrow \pi_q(\text{Map}_{\text{Top}}(|X|, |Y|), \Phi(x))$  will be an isomorphism for one  $x$  in a path component if and only if it is an isomorphism for all  $x$  in that path component. For  $x \in A_k$ , we can compute

$$\pi_q(F, x) = \text{colim}_{n \geq k} \pi_q(A_n, \phi_k^n(x))$$

where  $\phi_k^n$  is the composite map  $\phi_{n-1} \circ \dots \circ \phi_k$ . If we write  $f_k^n = f_k \circ \dots \circ f_{n-1}$  and assume that  $x$  is a vertex  $g : X_k \rightarrow Y$ , then  $\phi_k^n(g) = g \circ f_k^n$ . The proof of Theorem 7.4 therefore reduces to the following lemma.

**Lemma 7.6** *Let  $g : X_k \rightarrow Y$  be a map of simplicial complexes and let  $q > 0$ . The map*

$$\text{colim}_{n \geq k} \pi_q(|\text{Map}_{\text{SC}}(X_n, Y)|, g \circ f_k^n) \rightarrow \pi_q(\text{Map}_{\text{Top}}(|X_k|, |Y|), |g|)$$

*is a bijection.*

The proof of this lemma occupies the next section.

### 8 The Proof of Lemma 7.6

This section contains the proof of Lemma 7.6. Notation and assumptions are as in the previous section. For fixed  $q \geq 1, k \geq 0$ , and  $g : X_k \rightarrow Y$ , we need to prove injectivity and surjectivity of the map

$$\text{colim}_{n \geq k} \pi_q(|\text{Map}_{\text{SC}}(X_n, Y)|, g \circ f_k^n) \rightarrow \pi_q(\text{Map}_{\text{Top}}(|X_k|, |Y|), |g|).$$

Since the simplicial complexes  $X_n$  for  $n < k$  play no role, by renumbering, we assume without loss of generality that  $k = 0$ .

We begin by proving surjectivity. Let  $\psi : S^q \rightarrow \text{Map}_{\text{Top}}(|X|, |Y|)$  be a continuous map which sends the chosen base point of  $S^q$  to  $|g|$ . Choose and fix a simplicial complex  $Z$  with a homeomorphism from its geometric realization to  $S^q$  having the base point at a vertex  $z$ , e.g.,  $Z = \partial \Delta[q + 1]$ . We then get a map  $\alpha : |X| \times |Z| \rightarrow |Y|$  which restricted to  $|X| \times \{z\}$  is  $|g|$ . Since  $|X| \times |Z|$  is compact, giving it the metric given by the max of the standard metrics on  $|X|$  and  $|Z|$ , we can choose a Lebesgue number  $\lambda > 0$  of the open cover  $\{\alpha^{-1}(\text{St}(w))\}$  (where  $w$  ranges over the vertices of  $Y$ ). Choose  $n$  large enough so that  $\text{Mesh}(X_n) < \lambda/4$  and choose a subdivision  $Z'$  of  $Z$  so that  $\text{Mesh}(Z') < \lambda/4$ . Then the map

$$\zeta : |X_n \boxtimes Z'| \rightarrow |X \boxtimes Z|$$

of Theorem 5.4 sends each simplex to a set of diameter  $< \lambda/2$ , and so the composite map

$$|X_n \boxtimes Z'| \xrightarrow{\zeta} |X \boxtimes Z| \xrightarrow{\rho} |X| \times |Y|$$

sends each simplex to a set of diameter  $< \lambda/2$ . The latter map  $\rho \circ \zeta$  therefore sends the star of each vertex to a set of diameter  $< \lambda$ . By Theorem 4.4, the composite map

$$\alpha \circ \rho \circ \zeta : |X_n \boxtimes Z'| \rightarrow |Y|$$

admits a simplicial approximation  $a : X_n \boxtimes Z' \rightarrow Y$ . Let  $b : Z' \rightarrow \text{Map}_{\text{SC}}(X_n, Y)$  be the adjoint map under the correspondence of Theorem 6.3. On geometric realization, the composite map

$$\phi : S^q \cong |Z'| \rightarrow |\text{Map}_{\text{SC}}(X_n, Y)| \rightarrow \text{Map}_{\text{Top}}(|X_n|, |Y|) \cong \text{Map}_{\text{Top}}(|X|, |Y|)$$

corresponds to the map

$$|X_n| \times |Z'| \xrightarrow{\zeta} |X_n \boxtimes Z'| \xrightarrow{|a|} |Y|.$$

Since the composite

$$|X| \times S^q \cong |X_n| \times |Z'| \xrightarrow{\zeta} |X_n \boxtimes Z'| \xrightarrow{|a|} |Y|$$

is homotopic to  $\alpha$  via the homotopy of Proposition 4.3, it follows that  $\phi$  is homotopic to the original map  $\psi$ . We are not quite finished because  $|b|$  is not a based map;

however,  $b(z)$  and  $g \circ f_0^n$  are both simplicial approximations of the map

$$|X_n| \cong |X| \xrightarrow{g} |Y|$$

and therefore span a 1-simplex in  $\text{Map}_{\text{SC}}(X_n, Y)$  by Theorem 4.8. The change of base point of  $|b|$  along this path produces an element of  $\pi_q(|\text{Map}_{\text{SC}}(X_n, Y)|, g \circ f_0^n)$  sent by the homomorphism to the original element of  $\pi_q(\text{Map}_{\text{Top}}(|X|, |Y|), |g|)$  represented by  $\psi$ .

We now prove injectivity. Given elements

$$[\psi], [\psi'] \in \text{colim } \pi_q(|\text{Map}_{\text{SC}}(X_n, Y)|, g \circ f_0^n),$$

we can choose  $k$  so that both are represented by based maps  $S^q \rightarrow |\text{Map}_{\text{SC}}(X_k, Y)|$ , and then renumbering, we can assume without loss of generality that  $k = 0$ . Using the Simplicial Approximation Theorem, we can choose a simplicial complex  $Z$  homeomorphic to  $S^q$  with base point at a vertex  $z$ , and based maps  $p, p': Z \rightarrow \text{Map}_{\text{SC}}(X, Y)$  of simplicial complexes representing  $[\psi]$  and  $[\psi']$ , respectively. We assume that the composite maps

$$\Gamma \circ |p|, \Gamma \circ |p'|: S^q = |Z| \rightarrow |\text{Map}_{\text{SC}}(X, Y)| \rightarrow \text{Map}_{\text{Top}}(|X|, |Y|)$$

both represent the same element of  $\pi_q(\text{Map}_{\text{Top}}(|X|, |Y|), |g|)$ , and we need to see that  $|p|$  and  $|p'|$  eventually represent the same element in the colimit. Choose a homotopy in  $\text{Map}_{\text{Top}}(|X|, |Y|)$ , and let  $H$  denote the adjoint map

$$|X| \times S^q \times [0, 1] \rightarrow |Y|.$$

Arguing as above, for some  $n$  and some refinement  $Z'$  of  $Z$  and  $I'$  of  $[0, 1]$ , we can construct a map of simplicial complexes

$$h: X_n \boxtimes Z' \boxtimes I' \rightarrow Y$$

giving a simplicial approximation of the map

$$|X_n \boxtimes Z' \boxtimes I'| \rightarrow |X_n| \times |Z'| \times |I'| \cong |X| \times S^q \times [0, 1] \xrightarrow{H} |Y|.$$

In particular, the composite map

$$|X| \times S^q \times [0, 1] \cong |X_n| \times |Z'| \times |I'| \xrightarrow{\zeta} |X_n \boxtimes Z' \boxtimes I'| \xrightarrow{|h|} |Y|$$

is homotopy equivalent to  $H$ . We obtain an adjoint map

$$Z' \boxtimes I' \rightarrow \text{Map}_{\text{SC}}(X_n, Y)$$

and on geometric realization a map

$$\eta: S^q \times [0, 1] \cong |Z'| \times |I'| \rightarrow |Z' \boxtimes I'| \rightarrow |\text{Map}_{\text{SC}}(X_n, Y)|.$$

The restriction  $\eta_0$  of  $\eta$  to  $S^q \times \{0\}$  and  $(f_0^n)^*(p)$  are both adjoint to simplicial approximations of the same map

$$|X_n \boxtimes Z'| \xrightarrow{\rho} |X_n| \times S^q \cong |X| \times S^q \rightarrow |Y|.$$

The simplicial approximations are contiguous, and so the maps  $\eta_0$  and  $(f_0^n)^*(p)$  are homotopic maps  $S^q \rightarrow |\text{Map}_{\text{SC}}(X_n, Y)|$ . Likewise the restriction  $\eta_1$  of  $\eta$  to  $S^q \times \{1\}$  and  $(f_0^n)^*(p')$  are homotopic maps  $S^q \rightarrow |\text{Map}_{\text{SC}}(X_n, Y)|$ . Extending  $\eta$  with these new homotopies, we get a homotopy

$$\eta: S^q \times [-1, 2] \rightarrow |\text{Map}_{\text{SC}}(X_n, Y)|$$

from  $|(f_0^n)^*(p)|$  to  $|(f_0^n)^*(p')|$ . Again, we are not quite done, because we need to worry about the base point. Restricting to the base point of  $S^q$ , the map

$$\alpha: [-1, 2] \rightarrow |\text{Map}_{\text{SC}}(X_n, Y)|$$

defines a loop based at  $g \circ f_0^n$ . The argument is completed by showing that  $\alpha$  is contractible. By construction, the restriction to  $[0, 1]$  is adjoint to the restriction of  $h$ ,

$$X_n \boxtimes I' \rightarrow Y$$

which is a simplicial approximation to the constant homotopy on the map

$$|X_n| \cong |X| \xrightarrow{|g|} |Y|.$$

The composite

$$X_n \boxtimes I' \rightarrow X_n \xrightarrow{g \circ f_0^n} Y$$

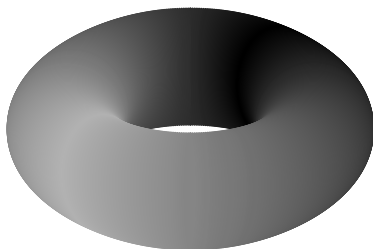
is also a simplicial approximation to the same map; these simplicial maps are contiguous, and so their adjoints are homotopic maps from  $[0, 1]$  to  $|\text{Map}_{\text{SC}}(X_n, Y)|$ . On  $\{0\}$  and  $\{1\}$ , this homotopy is the restriction of  $\alpha$  to  $[-1, 0]$  and  $[1, 2]$ , respectively, and this gives a contraction of the loop  $\alpha$ .

### 9 The Contiguity Complex and Topological Data Analysis

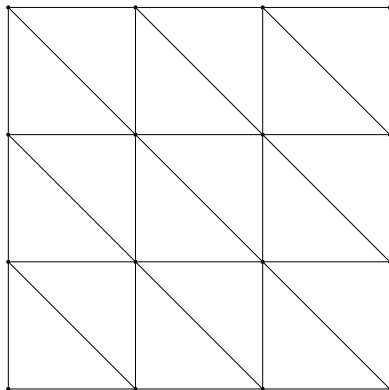
As we indicated in the introduction, much of our interest in the contiguity complex comes from its application to topological data analysis. When  $Y$  is a simplicial complex extracted from a data set  $M$ , the homology of the mapping spaces  $\text{Map}_{\text{Top}}(|X|, |Y|)$  provides refined invariants of  $Y$  for suitable test spaces  $X$ . This homology itself is difficult to compute, but the work of the preceding sections justifies studying approximations given as the homology of the contiguity complexes  $\text{Map}_{\text{SC}}(X_n, Y)$  for simplicial subdivisions  $X_n$  of  $X$ .

In this section, we describe a computational application of the contiguity complex, using the circle as a test space. Specifically, we present numerical experiments that illustrate how studying the growth rate of the number of contiguity classes of

**Fig. 1** Torus; homeomorphic to the geometric realization of the simplicial complex  $T$



**Fig. 2** The simplicial complex  $T$ , where the *top* is identified with the *bottom* and the *left-hand side* identified with the *right-hand side*

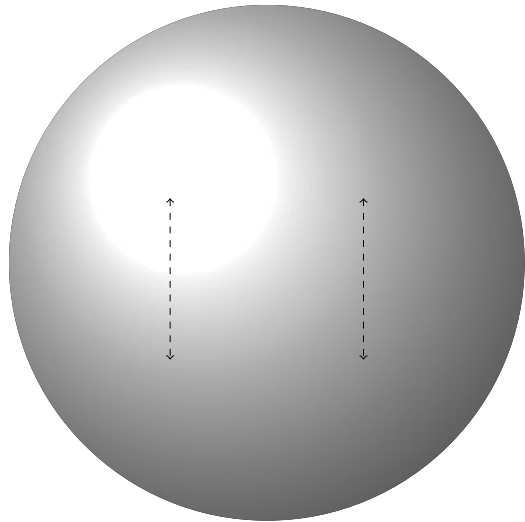


maps from successive subdivisions of  $S^1$  to two simplicial complexes with identical homology can distinguish them. This example demonstrates both the power of our methodology as well as some of the difficulties imposed by the computational complexity of applying it.

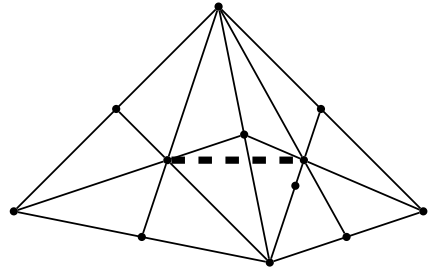
Let  $T$  denote the simplicial complex with geometric realization homeomorphic to the 2-torus (see Fig. 1), represented by the standard simplicial complex obtained by identifying the edges of a rectangle. This complex has 9 vertices, 27 1-simplices, and 18 2-simplices; see Fig. 2 below. Let  $P$  denote the simplicial complex with geometric realization homotopy equivalent to a “doubly pinched sphere” (see Fig. 3), formed by taking the barycentric subdivision of  $\partial\Delta[3]$ , and then adding two 1-simplices to connect the barycenters of two distinct pairs of 2-simplices. This complex has 14 vertices, 38 1-simplices, and 24 2-simplices; see Fig. 4 below. (The geometric realization of this complex is actually homeomorphic to a sphere with two disjoint internal chords.)

A straightforward computation shows that the simplicial complexes  $P$  and  $T$  have isomorphic homology; computing the simplicial homology cannot distinguish these spaces. However,  $P$  and  $T$  are not homotopy equivalent. To distinguish them, we will study the rank of the simplicial complex of based maps out of subdivisions of a circle. (For  $X$  and  $Y$  simplicial complexes, each having a pre-chosen base point vertex, we let  $\text{Map}_{\text{SC}*}(X, Y)$  denote the subcomplex of  $\text{Map}_{\text{SC}}(X, Y)$  consisting of those maps that send the chosen base point of  $X$  to the chosen base point of  $Y$ .) In our case, we consider the following family of subdivisions of the usual simplicial model of the circle  $\partial\Delta[2]$ : let  $S_k^1$  denote the simplicial complex with vertices  $\{v_0, v_1, \dots, v_{k-1}\}$

**Fig. 3** Doubly pinched sphere, with *dashed lines* indicating identified points; homotopy equivalent to the geometric realization of the simplicial complex  $P$



**Fig. 4** The simplicial complex  $P$ , with only one of the extra connecting edges shown



and edges  $(v_i, v_{i+1})$  for  $0 \leq i < k - 1$  along with  $(v_{k-1}, v_0)$ . We compute the ranks of  $H_0 \text{Map}_{\text{SC}^*}(S_k^1, P)$  and  $H_0 \text{Map}_{\text{SC}^*}(S_k^1, T)$ .

Computing these ranks exactly is computationally intractable for even relatively small  $k$ . A simplicial map from  $S_k^1$  to any simplicial complex  $X$  is given by a cycle in the graph specified by the 1-skeleton of  $X$ , and as  $k$  grows the search space of possible cycles to check grows exponentially. Moreover, since contiguity is not an equivalence relation on simplicial maps, we have to compute the transitive closure of the contiguity relation. Instead, we use a randomized algorithm.

First, we note that given two simplicial maps  $f$  and  $g$ , determining if  $f$  and  $g$  are contiguous can be done very efficiently. For one thing, it suffices to check the contiguity property only on certain simplices:

**Lemma 9.1** *Let  $f, g: X \rightarrow Y$  be simplicial maps. Then  $f$  and  $g$  are contiguous if and only if  $f$  and  $g$  satisfy the contiguity property for every maximal simplex of  $X$  (simplex which is not the face of any other simplex).*

When studying a fixed simplicial complex, the maximal simplices can be precomputed. Moreover, when the dimension of  $X$  is small, checking the contiguity property for a given simplex of  $X$  can be reduced to a table lookup (where the table required is

of size  $v^{2(d+1)}$ , for  $v$  the number of vertices in  $Y$  and  $d$  the dimension of a maximal simplex in  $X$ ).

Next, note that  $f$  and  $g$  are in the same contiguity class if and only if there exists a path of simplicial maps  $f_0, f_1, \dots, f_\ell$  such that  $f_0 = f$ ,  $f_\ell = g$ , and  $f_i$  is contiguous to  $f_{i+1}$  for all  $0 \leq i < \ell$ . We have the following randomized “local search” algorithm to determine if  $f$  and  $g$  are in the same contiguity class. The idea is to do a random walk on the space of simplicial maps, starting at  $f$  and mostly taking “greedy” steps that get us closer to  $g$ . For this, define

$$d(f, g) = \sum_{v \in X} d_Y(f(v), g(v)),$$

where  $d_Y$  is the graph distance on the 1-skeleton of  $Y$ . In the case when  $X$  is the circle  $S_k^1$  and we are considering based maps, we can decompose the step of going from  $f_i$  to  $f_{i+1}$  into a sequence of mutually contiguous maps  $f_i = f_{i,0}, f_{i,1}, \dots, f_{i,k-1} = f_{i+1}$  where each  $f_{i,j+1}$  differs from  $f_{i,j}$  only in a single vertex; specifically,  $f_{i,j}$  agrees with  $f_{i+1}$  on vertices  $0, \dots, j$  and with  $f_i$  on vertices  $j + 1, \dots, k - 1$ . Thus, for any  $f$  and  $g$  in the same contiguity class, it is possible to find a path of contiguous simplicial maps from  $f$  to  $g$ , where at each step we change a single vertex. This justifies the following randomized algorithm that checks whether maps  $f$  and  $g$  belong to the same contiguity class.

**Algorithm 9.2** Fix a parameter  $\kappa \in [0, 1]$ —typically values of  $\kappa$  are around 0.1. (The value of  $\kappa$  controls the frequency with which we take completely random steps as opposed to “greedy” steps that improve the distance to the target map.) Fix a maximum number of iterations  $M$ .

1. Set  $f_0 = f$ , and  $i = 0$ .
2. If  $f_i = g$ , return 1.
3. If  $i \geq M$ , return 0.
4. Set  $i = i + 1$ .
5. Construct  $f_i$  as follows:
  - (a) Choose a uniform random vertex  $v$  of  $X$ .
  - (b) Build a candidate map  $\tilde{f}$  from  $f_{i-1}$  by altering the value at  $v$ , uniformly selecting amongst possible values for  $\tilde{f}(v)$  such that  $\tilde{f}(v) \neq f_{i-1}(v)$ . (Possible values are those such that  $\tilde{f}$  is a simplicial map.)
  - (c) Set  $\alpha$  to be a uniform random value in  $[0, 1]$ .
  - (d) If  $\alpha < \kappa$  or  $d(\tilde{f}, g) \leq d(f_{i-1}, g)$ , set  $f_i = \tilde{f}$ . Otherwise, goto step (a) above.
6. Goto step (2) above.

To distinguish  $P$  and  $T$ , we observe that as  $k$  grows, we expect the number of contiguity classes of based maps from  $S_k^1$  to  $T$  to grow polynomially whereas the number of contiguity classes of maps from  $S_k^1$  to  $P$  will grow exponentially. The point is that new contiguity classes of  $T$  are generated by horizontal or vertical loops, whereas new contiguity classes of  $P$  can be described by words in the “pinch” edges. The expected numerical differences are sufficiently stark that even estimates of the ranks should expose the differences.

We use Algorithm 9.2 to estimate the number of contiguity classes as follows.

**Table 1** Estimates of counts of contiguity classes of maps from  $S_1^k$

$k$	$T$	$P$	$\frac{T}{k^2}$	$\frac{P}{k^2}$
9	41	40	0.506	0.494
12	48	58	0.333	0.403
15	66	124	0.293	0.551
18	103	242	0.318	0.747
21	165	506	0.374	1.143

**Algorithm 9.3** We run the following procedure for a predetermined amount of time.

1. Initialize  $L$  to be the empty list.
2. Let  $\gamma$  be a uniformly randomly chosen  $k$ -cycle in the 1-skeleton of the target complex  $Y$  that starts and ends at an arbitrarily chosen base point, representing based simplicial maps from  $S_k^1 \rightarrow Y$ .
3. If  $\gamma$  is not in the same contiguity class as any of the cycles in  $L$ , append  $\gamma$  to  $L$ .
4. Goto step (2) above.

Although it is necessary to choose the stopping time arbitrarily, in practice we find it most effective to successively increase the stopping times until the size of  $L$  stops changing before the stopping time is reached.

Selected results of running Algorithm 9.3 on based maps from  $S_k^1$  to  $T$  and  $P$  are summarized in Table 1. We used parameters  $M = 500000$  and  $\kappa = 0.1$  in Algorithm 9.2. The implementation was in C (with a Python wrapper) and required about 400 lines of code.

Scaling by the square of the cycle length, we see that the values for  $T$  are growing roughly quadratically and the values for  $P$  are growing faster (in fact, exponentially). These results are consistent with computation by hand of exact values, and they clearly indicate a difference between  $P$  and  $T$ .

Finally, note that for simplicial complexes generated as Rips complexes of point clouds, determining the contiguity of maps  $f$  and  $g$  can be done particularly efficiently even when use of a lookup table is infeasible.

**Lemma 9.4** *Let  $X$  be a simplicial complex and  $(M, \partial_M)$  a finite metric space. Simplicial maps  $f_0, f_1, \dots, f_n: X \rightarrow R_\epsilon(M)$  are mutually contiguous if and only if for every simplex  $\{x_0, \dots, x_m\}$  in  $X$ ,  $\partial_M(f_i(x_k), f_j(x_\ell)) \leq \epsilon$ .*

For expository purposes we chose an example which started with simplicial complexes, but it is straightforward to generate synthetic point clouds that illustrate the same phenomenon.

### 10 The Contiguity Complex and Persistent Homology

The example in the preceding section involved aggregating information from a range of mesh sizes on the test space. Moreover, in practice it is often necessary to assemble



and summarize information from a range of feature scales on the target space. Thinking about how to systematically handle such considerations lead quickly to ideas of persistent homology [2]. In this section, we briefly sketch the connection of our work to persistence; a detailed investigation is the subject of future work.

Recall that given a finite metric space  $(M, \partial_M)$ , for each  $\epsilon > 0$  we can associate the Rips complex  $R_\epsilon(M)$ . This is a simplicial complex with vertices the points of  $M$ , and higher simplices specified by the rule that a subset  $\{v_0, v_1, \dots, v_n\} \subset M$  spans an  $n$ -simplex when

$$\partial_M(v_i, v_j) \leq \epsilon, \quad 0 \leq i < j \leq n.$$

The construction of the Rips complex is functorial in both the metric space and in  $\epsilon$ , with respect to metric maps and increasing order, respectively. Specifically, there are natural maps

$$R_\epsilon(M) \rightarrow R_{\epsilon'}(M)$$

for  $\epsilon \leq \epsilon'$ . The idea of persistence is that since a priori knowledge about the correct value of  $\epsilon$  may be hard to come by, one should study invariants which assemble information as  $\epsilon$  varies. For any finite metric space  $(M, \partial_M)$ , there is a finite collection of values  $\{\epsilon_1, \epsilon_2, \dots, \epsilon_m\}$  at which the Rips complex changes. The persistent homology of  $(M, \partial_M)$  is a collection of homological invariants of the direct system of Rips complexes

$$R_0(M) \rightarrow R_{\epsilon_1}(M) \rightarrow R_{\epsilon_2}(M) \rightarrow \dots \rightarrow R_{\epsilon_m}(M).$$

In fact, persistent homology can be defined for any direct system of simplicial complexes [9]. In the following, we will suppress the choice of coefficients (typically a field).

The contiguity complex is functorial in the target variable: for a simplicial complex  $X$ ,  $\text{Map}_{\text{SC}}(X, -)$  is a covariant functor to simplicial complexes. As a consequence, for  $\epsilon \leq \epsilon'$  there is a natural map

$$\text{Map}_{\text{SC}}(X, R_\epsilon(M)) \rightarrow \text{Map}_{\text{SC}}(X, R_{\epsilon'}(M)).$$

**Definition 10.1** Let  $(M, \partial_M)$  be a finite metric space (the data) and  $Z$  a simplicial complex (the test space). The  $k$ th persistent contiguity homology of maps from  $Z$  to  $M$  is the  $k$ th persistent homology associated to the direct system of mapping complexes

$$\text{Map}_{\text{SC}}(Z, R_{\epsilon_1}(M)) \rightarrow \text{Map}_{\text{SC}}(Z, R_{\epsilon_2}(M)) \rightarrow \dots \rightarrow \text{Map}_{\text{SC}}(Z, R_{\epsilon_m}(M)).$$

One of the appealing features of the contiguity complex (and the persistent contiguity homology) is that computation of persistence becomes particularly computationally tractable when the target is a Rips complex, as indicated by Lemma 9.4.

*Example 10.2* Suppose that  $M$  is a point cloud with the property that for  $\epsilon_1 < \epsilon_2$  the Rips complex  $R_{\epsilon_1}(M)$  is homotopy equivalent to a circle and the Rips complex  $R_{\epsilon_2}(M)$  is homotopy equivalent to a point (and before  $\epsilon_1$  is discrete). Concretely,

suppose that  $M$  consists of samples from the unit disk in  $\mathbb{R}^2$  such that the density of the points is significantly higher on the boundary than on the interior.

Let  $S_k^1$  be the simplicial complex modelling the circle that has  $k$  vertices, as in Sect. 9. Suppose that  $k > d|M|$  for some number  $d$ . Roughly speaking, the barcode associated to the zeroth persistent contiguity homology will have the following behavior. For  $\epsilon < \epsilon_1$ , there are bars (contiguity classes) for each vertex in  $M$ . When  $\epsilon \in (\epsilon_1, \epsilon_2)$ , the bars associated to interior points will remain, but the bars on the boundary will all collapse to a single point. However, new bars will arise for the contiguity classes corresponding to maps of non-zero degree (up to at least  $d$  in absolute value). Note that there will be more of these than the number of homotopy classes, as in Example 4.9. When  $\epsilon > \epsilon_2$ , all of the bars from the discrete phase will merge into the single bar from the boundary points and most of the bars corresponding to the maps on the boundary will merge into a single bar (but not necessarily all, depending on the precise value of  $k$ ).

The work of this paper tells us that to study maps out of a test homotopy type  $|X|$  (presented as the geometric realization of a simplicial complex  $X$ ), the construction of Definition 10.1 should be applied to a subdivision  $X'$  of  $X$  with a sufficiently small mesh size. However, just as it may not be clear how to understand the correct feature scale  $\epsilon$  for the target, a priori information about suitable mesh sizes for  $X'$  may be unavailable. For instance, it is clear from Example 10.2 that choosing the mesh size (i.e., the value of  $k$ ) properly depends on intimate knowledge of the behavior of the data set. Moreover, the example of the previous section indicates that the rate of change of the size of the contiguity complex as the mesh size for  $X'$  is itself a very useful invariant.

To this end, we observe that the contiguity complex is also functorial in the source: for a simplicial complex  $Y$ ,  $\text{Map}_{\text{SC}}(-, Y)$  is a contravariant functor to simplicial complexes. Thus, we can consider the direct system induced by successive subdivision of the target. Let  $X_n$  be a sequence of successive subdivisions of  $X$  and choose compatible simplicial approximations  $X_{n+1} \rightarrow X_n$  of the homeomorphisms  $|X_{n+1}| \cong |X_n|$ .

**Definition 10.3** Let  $X$  and  $Y$  be simplicial complexes. The  $k$ th persistent subdivision homology from  $X$  to  $Y$  relative to subdivisions  $\{X_1, \dots, X_\ell\}$  is given by the  $k$ th persistent homology of the direct system

$$\text{Map}_{\text{SC}}(X, Y) \rightarrow \text{Map}_{\text{SC}}(X_1, Y) \rightarrow \dots \rightarrow \text{Map}_{\text{SC}}(X_\ell, Y).$$

Different choices of subdivisions  $X_n$  and  $X'_n$  will lead to different persistent subdivision invariants. However, Theorem 1.2 and its proof suggest that these different choices lead to essentially the same information when the range of mesh sizes is comparable.

We do not give a detailed example here, but we note that we can begin to understand the behavior of the zeroth persistent subdivision homology from the computations behind the example in Sect. 9. In that case, the number of bars increases as  $k$  increases, but at each stage some existing bars merge, as maps in the same homotopy class but different contiguity classes can enter the same contiguity class. This merging can be explicitly tracked using the algorithms we described above.

Finally, observe that when studying a finite metric space  $M$ , for a test space  $X$  and subdivisions  $X_n$  of  $X$ , we can assemble the collection  $\{\text{Map}_{\text{SC}}(X_n, R_\epsilon(M))\}$  as  $n$  and  $\epsilon$  vary into a diagram suitable for application of multidimensional persistence [1]. We suspect that this persistent homological invariant is the appropriate mapping invariant to study in this context. A detailed study of these persistent mapping invariants is the subject of future work.

## 11 Conclusions

The classical notions of maps of simplicial complexes and contiguity of maps of simplicial complexes generalize to define a simplicial complex of maps between simplicial complexes. The classical Simplicial Approximation Theorem says that maps of simplicial complexes approximate continuous maps between geometric realizations once subdivisions of small enough mesh are considered. This theorem generalizes to mapping spaces to show that the geometric realization of the simplicial complex of maps provides a good approximation of the space of maps when subdivisions of small enough mesh are considered. The number of contiguity classes of maps from small target spaces can be effectively computed and allows us to distinguish between data sets that cannot be distinguished by homology alone. The construction of the simplicial complex of maps is functorial, and so extends by naturality to the setting of persistence.

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