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Computation of the Highest Coefficients of Weighted Ehrhart Quasi-polynomials of Rational Polyhedra

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Abstract This article concerns the computational problem of counting the lattice points inside convex polytopes, when each point must be counted with a weight associated to it. We describe an efficient algorithm for computing the highest degree coefficients of the weighted Ehrhart quasi-polynomial for a rational simple polytope in varying dimension, when the weights of the lattice points are given by a polynomial function h. Our technique is based on a refinement of an algorithm of A. Barvinok in the unweighted case (i.e., $h \equiv 1$). In contrast to Barvinok's method, our method is

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local, obtains an approximation on the level of generating functions, handles the general weighted case, and provides the coefficients in closed form as step polynomials of the dilation. To demonstrate the practicality of our approach, we report on computational experiments which show that even our simple implementation can compete with state-of-the-art software.

Keywords Ehrhart functions · Exponential sums and integrals · Intermediate sums · Polynomial-time algorithms

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1 Introduction

Computations with lattice points in convex polyhedra arise in various areas of computer science, mathematics, and statistics (see, e.g., [14, 23, 32] and the many references therein). Given $\mathfrak{p}$, a rational convex polytope in $\mathbb{R}^d$, and $h(x)$, a polynomial function on $\mathbb{R}^d$ (often called a *weight function*), this article considers the important problem of computing, or estimating, the sum of the values of $h(x)$ over the lattice points belonging to $\mathfrak{p}$, namely

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

The function $S(\mathfrak{p}, h)$ has already been studied extensively in the *unweighted* case, i.e., when $h(x)$ takes only the constant value 1 (in that case $S(\mathfrak{p}, 1)$ is just the number of lattice points of $\mathfrak{p}$). Many papers and books have been written about the structure of that function (see, e.g., [12, 14] and the many references therein). Nevertheless, in many applications $h(x)$ can be a much more complicated function. Important examples of such a situation appear, for instance, in enumerative combinatorics [1], statistics [22, 27], symbolic integration [6], and nonlinear optimization [25]. Still, only a few algorithmic results exist on the case of an arbitrary polynomial h.

It is well known that when the polyhedron $\mathfrak{p}$ is dilated by an integer factor $n \in \mathbb{N}$, we obtain a function of n, called the *weighted Ehrhart quasi-polynomial* of the pair $(\mathfrak{p}, h)$, namely

$$S(n\mathfrak{p}, h) = \sum_{x \in n\mathfrak{p} \cap \mathbb{Z}^d} h(x) = \sum_{m=0}^{d+M} E_m(n \bmod q)\, n^m.$$

This is a quasi-polynomial in the sense that the function is a sum of monomials up to degree $d + M$, where $M = \deg h$, but whose coefficients E_m are periodic functions of n. The coefficient functions E_m are periodic functions with period q, where $q \in \mathbb{N}$ is the smallest positive integer such that $q\mathfrak{p}$ is a *lattice polytope*, i.e., its vertices are lattice points. We will make this more precise later (we recommend [12, 14] for excellent introductions to this topic).

To begin realizing the richness of $S(n\mathfrak{p}, h)$, note that its leading highest degree coefficient E_{d+M} (which actually does not depend on n) is precisely equal to $\int_{\mathfrak{p}} h(x)\,\mathrm{d}x$, i.e., the integral of h over the polytope $\mathfrak{p}$, when h is homogeneous of degree M. These integrals were studied in [7, 8] and more recently in [6]. Still most other coefficients are difficult to understand, even for easy polytopes, such as simplices (see [21] for a survey of results and challenges). The key aim of this article is to achieve a fast computation of the first few top-degree (weighted) coefficients E_m via an approximation of $S(n\mathfrak{p}, h)$ by a quasi-polynomial that shares the highest coefficients with $S(n\mathfrak{p}, h)$.

We now explain the known results achieved so far in the literature. We stress that computing *all* the coefficients E_m for $m = 0, \dots, d + M$ is an NP-hard problem; thus, the best one can achieve for theoretical results is to obtain an *approximation*, as we propose to do here. Until now most results dealt only with the *unweighted case*, i.e., $h(x) = 1$, and we summarize these results here. A. Barvinok first obtained for lattice polytopes $\mathfrak{p}$ a polynomial-time algorithm that for a fixed integer k_0 can compute the highest k_0 coefficients E_m (see [9]). For this he used Morelli's identities [33] and relied on an oracle that computes the volumes of faces.

Later, in [11], Barvinok obtained a formula relating the k highest degree coefficients of the (unweighted) Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by certain affine lattice subspaces of codimension $< k$. As a consequence, he proved that the k highest degree coefficients of the unweighted Ehrhart quasi-polynomial of a *rational simplex* can be computed by a polynomial algorithm, when the dimension d is part of the input, but k is fixed. More precisely, given a *dilation class* n mod q with $n \in \mathbb{N}$, Barvinok's algorithm computes the *numbers* $E_m(n \bmod q)$ by an interpolation technique. However, neither a closed formula for these E_m, depending on n, nor a generating function for the coefficients became available from [11]. In fact, that article [11, Sect. 8.2] raised the question of efficiently computing such a closed-form expression.

A key point of both Barvinok's and our method is the following. The sum $S(\mathfrak{p}, h)$ has natural generalizations, the *intermediate* sums $S^L(\mathfrak{p}, h)$, where $L \subseteq V = \mathbb{R}^d$ is a rational vector subspace. For a polytope $\mathfrak{p} \subset V$ and a polynomial $h(x)$, we define

$$S^L(\mathfrak{p}, h) = \sum_x \int_{\mathfrak{p}\cap(x+L)} h(y)\,\mathrm{d}y,$$

where the summation index x runs over the projected lattice in V/L. In other words, the polytope $\mathfrak{p}$ is sliced along lattice affine subspaces parallel to L, and the integrals of h over the slices are added up. For $L = V$, there is only one term and $S^V(\mathfrak{p}, h)$ is just the integral of $h(x)$ over $\mathfrak{p}$, while, for $L = \{0\}$, we recover $S(\mathfrak{p}, h)$. Barvinok's method in [11] was to introduce particular linear combinations of the intermediate sums,

$$\sum_{L\in\mathcal{L}} \lambda(L) S^L(\mathfrak{p}, h).$$

It is natural to replace the polynomial weight $h(x)$ with an exponential function $x \mapsto \mathrm{e}^{\langle\xi,x\rangle}$ and consider the corresponding holomorphic functions of ξ in the dual

V^*. Moreover, one can allow $\mathfrak{p}$ to be unbounded; then the sums

$$S^L(\mathfrak{p})(\xi) = \sum_x \int_{\mathfrak{p}\cap(x+L)} \mathrm{e}^{\langle \xi, y\rangle}\,\mathrm{d}y$$

still make sense as meromorphic functions on V^*. The map $\mathfrak{p} \mapsto S^L(\mathfrak{p})(\xi)$ is a valuation.

In [2], it was proved that a version of Barvinok's construction *on the level of generating functions*, namely $\sum_{L\in\mathcal{L}} \lambda(L) S^L(\mathfrak{p})(\xi)$, approximates $S(\mathfrak{p})(\xi)$ in a certain ring of meromorphic functions (a precise statement is given below). The proof in [2] relied on the Euler–Maclaurin expansion of these functions. Another proof, using the Poisson summation formula, will appear in [15].

Here we introduce a *simplified* way to approximate $S(\mathfrak{p})(\xi)$ for the case of a simplicial affine cone $\mathfrak{p} = s + \mathfrak{c}$, which levels the way for a practical and efficient implementation. Via Brion's theorem, it is sufficient to sum up these *local* contributions of the tangent cones of the vertices. We present a method for computing the highest degree coefficients of the Ehrhart quasi-polynomial of a rational simple polytope, by applying the approximation theorem to each of the cones at vertices of $\mathfrak{p}$. The complexity depends on the number of vertices of the polytope; thus, if the simple polytope is presented by its vertices (rather than by linear inequalities), we obtain a polynomial-time algorithm. In particular, the algorithm is polynomial-time for the case of a simplex. We obtain the Ehrhart coefficient functions $E_m(n \bmod q)$ in a *closed form* as *step polynomials*, by which we mean sums of functions of the form

$$f\big((\zeta_1 n) \bmod q_1, \dots, (\zeta_k n) \bmod q_k\big),$$

where f is a polynomial and $\zeta_1, \dots, \zeta_k \in \mathbb{Z}$, $q_1, \dots, q_k \in \mathbb{N}$. Having a closed formula available considerably strengthens Barvinok's result in [11], even in the unweighted case $h = 1$.

The structure of this paper is as follows. In Sect. 2, we first present some necessary preliminaries. Section 3 explains the intermediate generating function $S^L(\mathfrak{p})(\xi)$ in more detail. Then we show how to use an expansion of $S(\mathfrak{p})(\xi)$ into homogeneous components to extract the highest degree coefficients of the weighted Ehrhart polynomial in the case of a lattice polytope. This motivates the approximation results for generating functions. In Sect. 4, we give a simple proof of the approximation theorem of [2], in the case of a simplicial cone (see Theorem 26). The theorem uses the notion of a *patching function* (essentially a form of Möbius inversion formulas described in Sect. 4.1). We exhibit an explicit and easily computable patching function. Using these tools, we show in Sect. 5 that the approximation for a cone $s + \mathfrak{c}$ (on the level of generating functions) can be computed efficiently as a closed formula. The formula makes the periodic dependence on the vertex s explicit. Finally, in Sect. 6, we give the polynomial-time algorithm to compute the coefficients $E_m(n \bmod q)$ as step polynomials. Our main result (Theorem 38) says that, for every fixed number k_0, there exists a polynomial-time algorithm that, given a simple polytope $\mathfrak{p}$ of arbitrary dimension, a linear form $\ell \in V^*$, and a nonnegative integer M, computes the highest $k_0 + 1$ coefficients $E_{M+d-k_0}, \dots, E_{M+d}$ of the weighted Ehrhart quasi-polynomial $S(n\mathfrak{p}, h = \ell^M)$ in the form of step polynomials.

Four comments are in order about the applicability and potential practicality of the main results. First, although the weight h used in Theorem 38 is a power of a linear form, as is carefully explained in [6], one can obtain similar complexity of computation for polynomials that depend on a fixed number of variables, or with fixed degree (Corollary 45). Second, note that using perturbations (see, e.g., [28]), triangulations [26], or simplicial cone decompositions of polyhedra (see, e.g., [30]), one can extend computations from simple polytopes to arbitrary polytopes. Third, since our approximation is done at the level of generating functions, it extends the complexity result from [11] to the weighted case. Finally, at the end of the article we report on experiments using a simple implementation of the algorithm in *Maple*, demonstrating it to be competitive with more sophisticated software tools. This indicates a potential to use this algorithm to experimentally verify conjectures on the positivity of the Ehrhart coefficients of certain polytopes, for examples where the computation of the full Ehrhart polynomials is out of reach. The algorithms presented here require a rich mixture of computational geometry and algebraic-symbolic computation.

2 Preliminaries

2.1 Rational Convex Polyhedra

We consider a *rational vector space* V of dimension d, that is, a finite-dimensional real vector space with a lattice denoted by Λ. We will need to consider subspaces and quotient spaces of V; thus we cannot simply let $V=\mathbb{R}^d$ and $\Lambda=\mathbb{Z}^d$. A point $v\in V$ is called *rational* if there exists a nonzero integer q such that $qv\in\Lambda$. The set of rational points in V is denoted by $V_{\mathbb{Q}}$. A subspace L of V is called *rational* if $L\cap\Lambda$ is a lattice in L, or, equivalently, if L is spanned by vectors in Λ. If L is a rational subspace, the image of Λ in V/L is a lattice in V/L, so that V/L is a rational vector space. The image of Λ in V/L is called the *projected lattice*. A rational space V, with lattice Λ, has a canonical Lebesgue measure $\mathrm{d}x=\mathrm{d}m_\Lambda(x)$, for which V/Λ has measure 1.

A *convex rational polyhedron* $\mathfrak{p}$ in V (we will simply say *polyhedron*) is, by definition, the intersection of a finite number of closed half-spaces bounded by rational affine hyperplanes. We say that $\mathfrak{p}$ is *full dimensional* (in V) if the affine span of $\mathfrak{p}$ is V.

In this article, a *cone* is a polyhedral cone (with vertex 0), and an *affine cone* is a translated set $s+\mathfrak{c}$ of a cone $\mathfrak{c}$. A cone $\mathfrak{c}$ is called *simplicial* if it is generated by linearly independent elements of V. A simplicial cone $\mathfrak{c}$ is called *unimodular* if it is generated by linearly independent lattice vectors $v_1,\dots,v_k$ such that $\{v_1,\dots,v_k\}$ can be completed to a basis of Λ. An affine cone $\mathfrak{a}$ is called simplicial (respectively, simplicial unimodular) if the associated cone is. A *polytope* $\mathfrak{p}$ is a compact polyhedron. The set of vertices of $\mathfrak{p}$ is denoted by $\mathcal{V}(\mathfrak{p})$. For each vertex s, the cone of feasible directions at s is denoted by $\mathfrak{c}_s$. For details of all these notions see, e.g., [12].

2.2 Generating Functions: Exponential Sums and Integrals

Definition 1 We denote by $\mathcal{H}(V^*)$ the ring of holomorphic functions defined around $0\in V^*$. We denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around

$0 \in V^*$ and by $\mathcal{M}_\ell(V^*) \subset \mathcal{M}(V^*)$ the subring consisting of meromorphic functions $\phi(\xi)$ which can be written as a quotient of a holomorphic function and a product of linear forms.

This paper relies on the study of important examples of functions in $\mathcal{M}_\ell(V^*)$: the continuous and discrete generating functions $I(\mathfrak{p}, \Lambda)$ and $S(\mathfrak{p}, \Lambda)$ associated to a convex polyhedron $\mathfrak{p}$. Both have an important additivity property which makes them valuations (see [12, Chap. 8] or the survey [13] for a detailed presentation; here we summarize the essentials).

Definition 2 Let M be a vector space. A *valuation* F is a map from the set of polyhedra $\mathfrak{p} \subset V$ to the vector space M such that whenever the indicator functions $[\mathfrak{p}_i]$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i [\mathfrak{p}_i] = 0$, then the elements $F(\mathfrak{p}_i)$ satisfy the same relation $\sum_i r_i F(\mathfrak{p}_i) = 0$.

Proposition 3 *There exists a unique valuation $I(\cdot, \Lambda)$ which associates to every polyhedron $\mathfrak{p} \subset V$ a meromorphic function $I(\mathfrak{p}, \Lambda) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold*:

(i) *If the polyhedron $\mathfrak{p}$ is not full dimensional or if $\mathfrak{p}$ contains a straight line*, *then* $I(\mathfrak{p}, \Lambda) = 0$.
(ii) *If $\xi \in V^*$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is integrable over $\mathfrak{p}$*, *then*

$$I(\mathfrak{p}, \Lambda)(\xi) = \int_{\mathfrak{p}} \mathrm{e}^{\langle \xi, x\rangle}\, \mathrm{d}m_\Lambda(x).$$

(iii) *For every point $s \in V_{\mathbb{Q}}$, one has*

$$I(s + \mathfrak{p}, \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} I(\mathfrak{p}, \Lambda)(\xi).$$

We will call $I(\mathfrak{p}, \Lambda)(\xi)$ the *continuous generating function* of $\mathfrak{p}$.

Proposition 4 *There exists a unique valuation $S(\cdot, \Lambda)$ which associates to every polyhedron $\mathfrak{p} \subset V$ a meromorphic function $S(\mathfrak{p}, \Lambda) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold*:

(i) *If $\mathfrak{p}$ contains a straight line*, *then* $S(\mathfrak{p}, \Lambda) = 0$.
(ii) *If $\xi \in V^*$ is such that $\mathrm{e}^{\langle \xi, x\rangle}$ is summable over the set of lattice points of $\mathfrak{p}$*, *then*

$$S(\mathfrak{p}, \Lambda)(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} \mathrm{e}^{\langle \xi, x\rangle}.$$

(iii) *For every point $s \in \Lambda$, one has*

$$S(s + \mathfrak{p}, \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} S(\mathfrak{p}, \Lambda)(\xi).$$

$S(\mathfrak{p}, \Lambda)(\xi)$ is called the (*discrete*) *generating function* of $\mathfrak{p}$.

2.3 Brion's Theorem

A consequence of the valuation property is the following fundamental theorem. It follows from the Brion–Lawrence–Varchenko decomposition of a polyhedron into the supporting cones at its vertices [12, 18]; see also [19], Proposition 3.1, for a more general Brianchon–Gram type identity.

Theorem 5 *Let $\mathfrak{p}$ be a polyhedron with a set of vertices $\mathcal{V}(\mathfrak{p})$. For each vertex s, let $\mathfrak{c}_s$ be the cone of feasible directions at s. Then*

$$S(\mathfrak{p}, \Lambda) = \sum_{s\in\mathcal{V}(\mathfrak{p})} S(s + \mathfrak{c}_s, \Lambda).$$

2.4 Notation and Basic Facts in the Case of a Simplicial Cone

For all of the notions below, see [12]. Let $v_i \in \Lambda$, $i = 1, \dots, d$ be linearly independent integral vectors and let $\mathfrak{c} = \sum_{i=1}^{d} \mathbb{R}_+ v_i$ be the cone that they span.

Definition 6 The *fundamental parallelepiped* $\mathfrak{b}$ of the cone (with respect to the generators v_i, $i = 1, \dots, d$) is the set

$$\mathfrak{b} = \sum_{i=1}^{d} [0, 1[\, v_i.$$

Note that the set has a half-open boundary. We immediately have the following.

Lemma 7 *Let $s \in V$. Then*

$$I(s + \mathfrak{c}, \Lambda)(\xi) = \mathrm{e}^{\langle\xi,s\rangle} \frac{(-1)^d \operatorname{vol}_\Lambda(\mathfrak{b})}{\prod_{i=1}^{d} \langle\xi, v_i\rangle}, \tag{1}$$

where $\operatorname{vol}_\Lambda(\mathfrak{b})$ is the volume of the fundamental parallelepiped with respect to the Lebesgue measure $\mathrm{d}m_\Lambda$ defined by the lattice.

If $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$, then $\operatorname{vol}_\Lambda(\mathfrak{b}) = |\det(v_1, \dots, v_d)|$, and so

$$I\big(s + \mathfrak{c}, \mathbb{Z}^d\big)(\xi) = \mathrm{e}^{\langle\xi,s\rangle} \frac{(-1)^d |\det(v_1, \dots, v_d)|}{\prod_{i=1}^{d} \langle\xi, v_i\rangle}. \tag{2}$$

We also recall the following elementary but crucial lemma.

Lemma 8

(i) *The affine cone $(s + \mathfrak{c}) \cap \Lambda$ is the disjoint union of the translated parallelepipeds $s + \mathfrak{b} + v$, for $v \in \sum_{j=1}^{d} \mathbb{N} v_j$.*
(ii) *The set of lattice points in the affine cone $s + \mathfrak{c}$ is the disjoint union of the sets $x + \sum_{i=1}^{d} \mathbb{N} v_i$ when x runs over the set $(s + \mathfrak{b}) \cap \Lambda$.*

(iii) *The number of lattice points in the parallelepiped $s+\mathfrak{b}$ is equal to the volume of the parallelepiped with respect to the Lebesgue measure $\mathrm{d}m_\Lambda$ defined by the lattice, that is,*

$$\operatorname{Card}\big((s+\mathfrak{b})\cap\Lambda\big)=\operatorname{vol}_\Lambda(\mathfrak{b}).$$

In particular, when $V=\mathbb{R}^d$ and $\Lambda=\mathbb{Z}^d$, then

$$\operatorname{Card}\big((s+\mathfrak{b})\cap\mathbb{Z}^d\big)=\big|\det(v_1,\dots,v_d)\big|.$$

The study of the generating function $S(s+\mathfrak{c},\Lambda)(\xi)$ of the affine cone $s+\mathfrak{c}$ will be a crucial tool. It relies on expressing $S(s+\mathfrak{c},\Lambda)(\xi)$ in terms of the generating function $S(s+\mathfrak{b},\Lambda)(\xi)=\sum_{x\in(s+\mathfrak{b})\cap\Lambda}\mathrm{e}^{\langle\xi,x\rangle}$ of the fundamental parallelepiped. Lemma 8(ii) immediately gives the following.

Lemma 9

$$S(s+\mathfrak{c},\Lambda)(\xi)=S(s+\mathfrak{b},\Lambda)(\xi)\frac{1}{\prod_{j=1}^d(1-\mathrm{e}^{\langle\xi,v_j\rangle})}. \tag{3}$$

We next rewrite this using the following analytic function.

Definition 10 Let

$$T(\tau,x)=\mathrm{e}^{\tau x}\frac{x}{1-\mathrm{e}^x}=-\sum_{n=0}^{\infty}B_n(\tau)\frac{x^n}{n!}, \tag{4}$$

where $B_n(\tau)$ are the Bernoulli polynomials.

Lemma 11

$$S(s+\mathfrak{c},\Lambda)(\xi)=S(s+\mathfrak{b},\Lambda)(\xi)\prod_{j=1}^{d}T\big(0,\langle\xi,v_j\rangle\big)\cdot\frac{1}{\prod_{j=1}^d\langle\xi,v_j\rangle}. \tag{5}$$

Example 12 Consider the case where $V=\mathbb{R}$ and $\Lambda=\mathbb{Z}$. Let $\mathfrak{c}=\mathbb{R}_+$. Let $s\in\mathbb{R}$; then the "fractional part" $\{s\}\in[0,1[$ is defined as the unique real number such that $s-\{s\}\in\mathbb{Z}$. Then the unique integer $\bar{s}$ in $s+\mathfrak{b}$ is $s+\{-s\}$, and so (1) and (3) give

$$I(s+\mathfrak{c},\mathbb{Z})(\xi)=\mathrm{e}^{\xi s}\frac{-1}{\xi}\quad\text{and}\quad S(s+\mathfrak{c},\mathbb{Z})(\xi)=\mathrm{e}^{\xi(s+\{-s\})}\frac{1}{1-\mathrm{e}^{\xi}}.$$

The latter can be rewritten using (5) as

$$S(s+\mathfrak{c},\mathbb{Z})(\xi)=\mathrm{e}^{\xi(s+\{-s\})}T(0,\xi)\cdot\frac{1}{\xi}=\mathrm{e}^{\xi s}T\big(\{-s\},\xi\big)\cdot\frac{1}{\xi}.$$

3 Key Ideas of the Approximation Theory

3.1 Weighted Ehrhart Quasi-polynomials

Let $\mathfrak{p} \subset V$ be a rational polytope and let $h(x)$ be a polynomial function of degree M on V. We consider the following weighted sum over the set of lattice points of $\mathfrak{p}$:

$$\sum_{x \in \mathfrak{p} \cap \Lambda} h(x).$$

When $\mathfrak{p}$ is dilated by a nonnegative integer $n \in \mathbb{N}$, we obtain a function of n. As mentioned in the introduction, it is a well-known theorem (see, e.g., [14]) that this function is a quasi-polynomial whose coefficient functions are periodic functions with period q, where $q \in \mathbb{N}$ is the smallest positive integer such that $q\mathfrak{p}$ is a lattice polytope. This allows us to make the following definition.

Definition 13 Let q be the smallest positive integer such that $q\mathfrak{p}$ is a lattice polytope. Then we define the Ehrhart quasi-polynomial $E(\mathfrak{p}, h; n)$ and its coefficients $E_m(\mathfrak{p}, h; n \bmod q)$ by

$$E(\mathfrak{p}, h; n) = \sum_{x \in n\mathfrak{p} \cap \Lambda} h(x) = \sum_{m=0}^{d+M} E_m(\mathfrak{p}, h; n \bmod q)\, n^m.$$

We note that the coefficients E_m depend on n, but they actually depend only on $n \bmod q$. If $h(x)$ is homogeneous of degree M, the highest degree coefficient E_{d+M} is equal to the integral $\int_{\mathfrak{p}} h(x)\,\mathrm{d}x$ (see [6] and references therein). We also remark that the quasi-polynomial behavior of $E(\mathfrak{p}, h; n)$ will also follow directly from our explicit calculations in Sect. 6; see Remark 39.

We concentrate on the special case where the polynomial $h(x)$ is a power of a linear form,

$$h(x) = \frac{\langle \xi, x \rangle^M}{M!}.$$

This is not a restriction, because any polynomial can be written as a linear combination of powers of linear forms. In fact, as discussed in [6], whenever the polynomial $h(x)$ is either of fixed degree or only depends on a fixed number of variables (possibly after a linear change of variables), then only a polynomial number of powers of linear forms are needed, and such a decomposition can be computed in polynomial time. We introduce the following notation for this special case.

Definition 14 Let q be as above. We define the Ehrhart quasi-polynomial $E(\mathfrak{p}, \xi, M; n)$ and the coefficients $E_m(\mathfrak{p}, \xi, M; n \bmod q)$ for $m = 0, \dots, M + d$ by

$$E(\mathfrak{p}, \xi, M; n) = \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(\mathfrak{p}, \xi, M; n \bmod q)\, n^m.$$

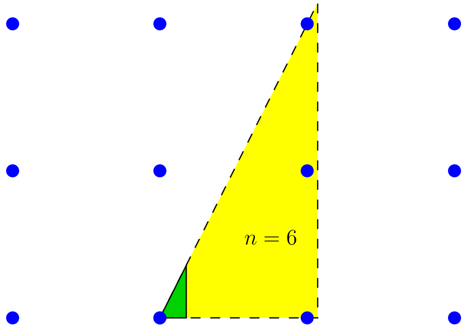


Fig. 1 The example triangle $\mathfrak{t}$ and its dilation $6\mathfrak{t}$

It will be convenient in this paper to introduce the following notation. For a positive integer $q \in \mathbb{N}$ and a real number $n \in \mathbb{R}$, we write

$$\lfloor n \rfloor_q := q \left\lfloor \frac{1}{q} n \right\rfloor \in q\mathbb{Z}, \qquad \{n\}_q := (n \bmod q) \in [0, q),$$

which give the unique decomposition

$$n = \lfloor n \rfloor_q + \{n\}_q.$$

By $\lfloor n \rfloor := \lfloor n \rfloor_1$ and $\{n\} := \{n\}_1$ we obtain the ordinary "floor" and "fractional part" notation. Finally, $\lceil n \rceil := -\lfloor -n \rfloor$ is the "ceiling" notation.

Example 15 Consider the rational triangle $\mathfrak{t}$ with vertices $(0, 0)$, $(\frac{5}{28}, 0)$, and $(\frac{5}{28}, \frac{5}{14})$ as shown in Fig. 1. Let us compute $E(\mathfrak{t}, \xi, M; n)$ for this small example. Note that the integer q such that $q\mathfrak{t}$ is a lattice polytope is $q = 28$.

In what follows consider powers of the linear form $\xi = x + y$ as weights for the lattice points. When the power $M = 0$, then we obtain a constant weight, and the quasi-polynomial $E(\mathfrak{t}, \xi, 0; n)$ counts the lattice points inside the various dilations of $\mathfrak{t}$: it is given by the formula

$$\frac{25}{784} n^2 + \left(-\frac{5}{392} \{5n\}_{28} + \frac{5}{14} \right) n + \left(1 + \frac{1}{784} \left(\{5n\}_{28}\right)^2 - \frac{1}{14} \{5n\}_{28} \right).$$

Indeed, when $n = 1$ (no dilation) there is only one lattice point, and the formula above reduces to

$$\frac{1089}{784} - \frac{33}{392} \{5\}_{28} + \frac{1}{784} \left(\{5\}_{28}\right)^2 = \frac{1089}{784} - \frac{33}{392} 5 + \frac{1}{784} 25 = 1.$$

When we dilate the same triangle six times, i.e., $n = 6$, we obtain four lattice points,

$$\frac{841}{196} - \frac{29}{196} \{30\}_{28} + \frac{1}{784} \left(\{30\}_{28}\right)^2 = \frac{841}{196} - \frac{29}{196} \cdot 2 + \frac{1}{784} \cdot 4 = 4.$$

Next let us take $M = 1$; in that case the lattice point (a, b) is counted with weight $a + b$. In this case the top coefficient is equal to the integral of the linear form $\xi =$

$x + y$ over $\mathfrak{t}$. The quasi-polynomial $E(\mathfrak{t}, \xi, 1; n)$ is given by the following formula:

$$\frac{125}{16464} n^3 + \left(-\frac{25}{5488}\{5n\}_{28} + \frac{75}{784}\right) n^2 + \left(\frac{5}{5488}\left(\{5n\}_{28}\right)^2 - \frac{15}{392}\{5n\}_{28} + \frac{25}{84}\right) n + \left(-\frac{1}{16464}\left(\{5n\}_{28}\right)^3 + \frac{3}{784}\left(\{5n\}_{28}\right)^2 - \frac{5}{84}\{5n\}_{28}\right).$$

Again substitute $n = 1$ in the expression, to obtain

$$\frac{275}{686} - \frac{1685}{16464}\{5\}_{28} + \frac{13}{2744}\left(\{5\}_{28}\right)^2 - \frac{1}{16464}\left(\{5\}_{28}\right)^3 = 0.$$

Note that since only the lattice point $(0, 0)$ lies within the triangle at $n = 1$, the quasi-polynomial must evaluate to zero.

In practice, it is impossible to compute $E(\mathfrak{p}, \xi, M; n)$ except when $\mathfrak{p}$ is of small dimension (and M relatively small). Thus we restrict our ambitions.

Let us fix a number k_0. Our goal will be to compute the $k_0 + 1$ highest degree coefficients $E_m(\mathfrak{p}, \xi, M; n \bmod q)$, for $m = M + d, \dots, M + d - k_0$. We will be able to give a polynomial-time algorithm to do so.

3.2 Expansion of the Generating Functions into Homogeneous Components

We will make use of the following key property of $\mathcal{M}_\ell$: A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions as follows. Consider $\phi(t\xi)$ as a meromorphic function of one variable $t \in \mathbb{C}$, which we write as

$$\phi(t\xi) = \sum_{m \geq m_0} t^m \phi_{[m]}(\xi),$$

where m_0 is the lowest degree. We call the function $\phi_{[m]}$ the *homogeneous component* of ϕ of degree m. For instance, $\frac{\xi_1}{\xi_2}$ is homogeneous of degree 0. This example shows that a function in $\mathcal{M}_\ell(V^*)$ which only has nonnegative degree terms need not be analytic.

In particular, consider the generating function $S(s + \mathfrak{c}, \Lambda)(\xi)$ for a simplicial cone $\mathfrak{c}$. By Lemma 11,

$$S(s + \mathfrak{c}, \Lambda)(\xi) = S(s + \mathfrak{b}, \Lambda)(\xi) \prod_{j=1}^{d} T\big(0, \langle \xi, v_j \rangle\big) \cdot \frac{1}{\prod_{j=1}^{d} \langle \xi, v_j \rangle}.$$

This expression, where the first two factors are analytic, shows that indeed $S(s + \mathfrak{c}, \Lambda) \in \mathcal{M}_\ell(V^*)$ and thus has a decomposition into homogeneous components, where the lowest degree is $-d$.

Lemma 16

$$S(s+\mathfrak{c},\Lambda)(\xi)=S(s+\mathfrak{c},\Lambda)_{[-d]}(\xi)+S(s+\mathfrak{c},\Lambda)_{[-d+1]}(\xi)+\cdots,\tag{6}$$

and the lowest degree term $S(s+\mathfrak{c},\Lambda)_{[-d]}(\xi)$ *is equal to* $I(\mathfrak{c},\Lambda)(\xi)$, *i.e., the integral over the unshifted cone* $\mathfrak{c}$.

Proof From the above discussion, we have $S(s+\mathfrak{c},\Lambda)\in\mathcal{M}_\ell(V^*)$. The value at $\xi=0$ of the sum over the parallelepiped is the number of lattice points of the parallelepiped, that is, $\mathrm{vol}_\Lambda(\mathfrak{b})$. This proves the last assertion. □

3.3 Sketch of the Method for Lattice Polytopes

We will now explain the key point of our method, with the simplifying assumption that the vertices of the polytope are lattice points. We will show that the highest degree coefficients of the weighted Ehrhart polynomial can be read out from an approximation of the generating functions of the cones at vertices. In Sect. 4 we will study this approximation, and in Sect. 5 we will show how to efficiently compute it. Then, in Sect. 6, we will return to the computation of Ehrhart coefficients for the general case of rational polytopes.

Proposition 17 *Let* $\mathfrak{p}$ *be a lattice polytope. Then, for* $k\geq 0$, *we have*

$$E_{M+d-k}(\mathfrak{p},\xi,M)=\sum_{s\in\mathcal{V}(\mathfrak{p})}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}S(\mathfrak{c}_s)_{[-d+k]}(\xi).\tag{7}$$

The highest degree coefficient is just the integral

$$E_{M+d}(\mathfrak{p},\xi,M)=\int_{\mathfrak{p}}\frac{\langle\xi,x\rangle^M}{M!}\,\mathrm{d}x.$$

Remark 18 As functions of ξ, the coefficients $E_m(\mathfrak{p},\xi,M)$ are polynomial, homogeneous of degree M. However, in (7), they are expressed as linear combinations of rational functions of ξ, whose poles cancel out.

Proof of Proposition 17 The starting point is Brion's formula. As the vertices are lattice points, we have

$$\sum_{x\in\mathfrak{p}\cap\Lambda}\mathrm{e}^{\langle\xi,x\rangle}=\sum_{s\in\mathcal{V}(\mathfrak{p})}S(s+\mathfrak{c}_s)(\xi)=\sum_{s\in\mathcal{V}(\mathfrak{p})}\mathrm{e}^{\langle\xi,s\rangle}S(\mathfrak{c}_s)(\xi).\tag{8}$$

When $\mathfrak{p}$ is replaced with $n\mathfrak{p}$, the vertex s is replaced with ns, but the cone $\mathfrak{c}_s$ does not change. We obtain

$$\sum_{x\in n\mathfrak{p}\cap\Lambda}\mathrm{e}^{\langle\xi,x\rangle}=\sum_{s\in\mathcal{V}(\mathfrak{p})}\mathrm{e}^{n\langle\xi,s\rangle}S(\mathfrak{c}_s)(\xi).$$

We replace ξ with $t\xi$,

$$\sum_{x\in n\mathfrak{p}\cap\Lambda} \mathrm{e}^{t\langle\xi,x\rangle} = \sum_{s\in\mathcal{V}(\mathfrak{p})} \mathrm{e}^{nt\langle\xi,s\rangle} S(\mathfrak{c}_s)(t\xi). \tag{9}$$

The decomposition into homogeneous components gives

$$S(\mathfrak{c}_s)(t\xi) = t^{-d} I(\mathfrak{c}_s)(\xi) + t^{-d+1} S(\mathfrak{c}_s)_{[-d+1]}(\xi) + \cdots + t^k S(\mathfrak{c}_s)_{[k]}(\xi) + \cdots. \tag{10}$$

Hence, the t^M-term on the right-hand side of (10) is equal to

$$\sum_{k=0}^{M+d} (nt)^{M+d-k} t^{-d+k} \frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!} S(\mathfrak{c}_s)_{[-d+k]}(\xi).$$

Using this equation in (9), we have

$$\begin{aligned}\sum_{x\in n\mathfrak{p}\cap\Lambda} \frac{\langle\xi,x\rangle^M}{M!} = \sum_{s\in\mathcal{V}(\mathfrak{p})} & n^{M+d} \frac{\langle\xi,s\rangle^{M+d}}{(M+d)!} I(\mathfrak{c}_s)(\xi) \\ & + n^{M+d-1} \frac{\langle\xi,s\rangle^{M+d-1}}{(M+d-1)!} S(\mathfrak{c}_s)_{[-d+1]}(\xi) + \cdots + S(\mathfrak{c}_s)_{[M]}(\xi).\end{aligned} \tag{11}$$

From this relation, we read immediately that $\sum_{x\in n\mathfrak{p}\cap\Lambda} \frac{\langle\xi,x\rangle^M}{M!}$ is a polynomial function of n of degree $M+d$, and that the coefficient of n^{M+d-k} is given by (7). The highest degree coefficient is given by

$$E_{M+d}(\mathfrak{p},\xi,M) = \sum_{s\in\mathcal{V}(\mathfrak{p})} \frac{\langle\xi,s\rangle^{M+d}}{(M+d)!} I(\mathfrak{c}_s)(\xi).$$

Applying Brion's formula for the integral, this is equal to the term of ξ-degree M in $I(\mathfrak{p})(\xi)$, which is indeed the integral $\int_{\mathfrak{p}} \frac{\langle\xi,x\rangle^M}{M!}\,\mathrm{d}x$. □

From Proposition 17, we draw an important consequence: In order to compute the k_0+1 highest degree terms of the weighted Ehrhart polynomial for the weight $h(x) = \frac{\langle\xi,x\rangle^M}{M!}$, we *only need* the k_0+1 lowest degree homogeneous terms of the meromorphic function $S(\mathfrak{c}_s)(\xi)$, for every vertex s of $\mathfrak{p}$. We compute such an approximation in Sect. 4; it also turns out to be sufficient in the general case of a rational polytope.

3.4 Intermediate Generating Functions

To obtain the approximation, we study generating functions which interpolate between the integral $I(\mathfrak{p},\Lambda)$ and the discrete sum $S(\mathfrak{p},\Lambda)$. This trend of ideas was first discussed by Barvinok in [11]. Let L be a rational subspace of V. To any polyhedron $\mathfrak{p}$ we associate a meromorphic function $S^L(\mathfrak{p},\Lambda)(\xi) \in \mathcal{M}(V^*)$, which is, roughly speaking, obtained by slicing $\mathfrak{p}$ along affine subspaces parallel to L through lattice points, and adding the integrals of $\mathrm{e}^{\langle \xi, x\rangle}$ along the slices. Recall that the quotient space V/L is endowed with the projected lattice $\Lambda_{V/L}$.

Proposition 19 *Let $L \subseteq V$ be a rational subspace. There exists a unique valuation $S^L(\cdot, \Lambda)$ which to every rational polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $S^L(\mathfrak{p}, \Lambda) \in \mathcal{M}(V^*)$ so that the following properties hold*:

(i) *If $\mathfrak{p}$ contains a line, then $S^L(\mathfrak{p}, \Lambda) = 0$.*
(ii)

$$S^L(\mathfrak{p}, \Lambda)(\xi) = \sum_{x \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (x+L)} \mathrm{e}^{\langle \xi, y\rangle}\, \mathrm{d}y, \tag{12}$$

for every $\xi \in V^*$ such that the above sum converges.
(iii) *For every point $s \in \Lambda$, we have*

$$S^L(s + \mathfrak{p}, \Lambda)(\xi) = \mathrm{e}^{\langle \xi, s\rangle} S^L(\mathfrak{p}, \Lambda)(\xi).$$

We call the function $S^L(\mathfrak{p}, \Lambda)$ an *intermediate generating function*. The proof is entirely analogous to the case $L = \{0\}$, see Theorem 3.1 in [13], and we omit it.

For $L = \{0\}$, we recover the valuation S. For $L = V$, we have $S^V(\mathfrak{p}, \Lambda) = I(\mathfrak{p}, \Lambda)$. In particular, if $\mathfrak{p}$ is not full dimensional, then $S^V(\mathfrak{p}, \Lambda) = 0$.

If $\mathfrak{p}$ is compact, the meromorphic function $S^L(\mathfrak{p}, \Lambda)(\xi)$ is actually regular at $\xi = 0$, and its value for $\xi = 0$ is the $\mathbb{Q}$-valued valuation $E_{L^\perp}(\mathfrak{p})$ considered by Barvinok [11].

Remark 20 The function $S^L(\mathfrak{p}, \Lambda)$ is actually an element of $\mathcal{M}_\ell(V^*)$, just like the functions $S(\mathfrak{p}, \Lambda)$ and $I(\mathfrak{p}, \Lambda)$. This follows from an interesting decomposition that allows us to write S^L as a combination of terms using S and I for certain cones. This and other properties of the valuation $S^L(\cdot, \Lambda)$ will be discussed in a forthcoming article [4].

4 Approximation of the Generating Function of a Simplicial Affine Cone

Let $\mathfrak{c} \subset V$ be a simplicial cone with integral generators v_j, $j = 1, \dots, d$, and let $s \in V_{\mathbb{Q}}$. Let $k_0 \le d$. In this section we will obtain an expression for the $k_0 + 1$ lowest degree homogeneous terms of the meromorphic function $S(s + \mathfrak{c})(\xi)$. Recall that if $\mathfrak{c}$ is unimodular, the function $S(s + \mathfrak{c})(\xi)$ has a "short" expression,

$$S(s + \mathfrak{c})(\xi) = \mathrm{e}^{\langle \xi, \bar{s}\rangle} \prod_{j=1}^{d} \frac{1}{1 - \mathrm{e}^{\langle \xi, v_j\rangle}},$$

where v_i, $i = 1, \dots, d$ are the primitive integral generators of the edges and $\bar{s}$ is the unique lattice point in the corresponding parallelepiped $s + \mathfrak{b}$. This is a particular case of Lemma 8.

When $\mathfrak{c}$ is not unimodular, it is hard to compute the first k_0 terms of the Laurent expansion of the function $S(s+\mathfrak{c})(\xi)$, if k_0 is part of the input as well as the dimension d. In contrast, *if* k_0 *is fixed*, we will obtain an expression for the terms of degree $\le -d+k_0$ which only involves a discrete summation over cones in dimension $\le k_0$ and determinants. For example, the lowest degree term is $|\det(v_j)|\prod_j \frac{-1}{\langle \xi, v_j\rangle}$.

4.1 Patching Functions

For constructing the approximation, we will use a *patching function*. For $I\subseteq \{1,\dots,d\}$, we denote by L_I the linear span of the vectors v_i, $i\in I$ and by $L_I^\perp\subseteq V^*$ the orthogonal subspace. We denote by I^{c} the complement of I in $\{1,\dots,d\}$.

Definition 21 We denote by $\mathcal{J}^d_{\ge d_0}$ the set of subsets $I\subseteq\{1,\dots,d\}$ of cardinality $|I|\ge d_0$. A function $I\mapsto \lambda(I)$ on $\mathcal{J}^d_{\ge d_0}$ is called a *patching function* if it satisfies the following condition:

$$\Biggl[\bigcup_{I\in\mathcal{J}^d_{\ge d_0}} L_I^\perp = \sum_{I\in\mathcal{J}^d_{\ge d_0}} \lambda(I)\Biggr] L_I^\perp, \tag{13}$$

where $[\cdot]$ denotes the indicator function of a set.

Remark 22 The family of subspaces L_I, $|I|\ge d_0$ is closed under sum, and the family of orthogonals $L_I^\perp$ is closed under intersection. The value $\lambda(I)$ plays the same role as the Möbius function $\mu(L)$ for $L_I^\perp$ that Barvinok [11, Sect. 7] computes algorithmically for a certain family of subspaces L by walking the poset. From this relation to Möbius functions, it follows that patching functions do exist. The precise relation between Barvinok's construction and the construction of the present paper will be studied in the forthcoming paper [5].

We will compute a canonical patching function below, in Proposition 29.
Let us first state some interesting properties.

Lemma 23 *Let* $I\mapsto\lambda(I)$ *be a function on* $\mathcal{J}^d_{\ge d_0}$. *The following conditions are equivalent*:

(i) λ *is a patching function.*
(ii) $\sum_{I\in\mathcal{J}^d_{\ge d_0}, I\subseteq I_0}\lambda(I)=1$ *for every* $I_0\in\mathcal{J}^d_{\ge d_0}$.
(iii) *For* $1\le i\le d$, *let* $F_i(z)\in\mathbb{C}[[z]]$ *be a formal power series* (*in one variable*) *with constant term equal to* 1. *Then*

$$\prod_{1\le i\le d} F_i(z_i)\equiv \sum_{I\in\mathcal{J}^d_{\ge d_0}}\lambda(I)\prod_{i\in I^{\mathrm{c}}} F_i(z_i)$$
$$\text{mod terms of } z\text{-degree } \ge d-d_0+1. \tag{14}$$

(iv) *Let* $z_{I^c} = \sum_{i \in I^c} z_i$. *Then*

$$e^{z_1+\cdots+z_d} \equiv \sum_{I \in \mathcal{J}^d_{\geq d_0}} \lambda(I) e^{z_{I^c}} \text{ mod terms of } z\text{-degree} \geq d - d_0 + 1. \tag{15}$$

Proof Let $I_0 \in \mathcal{J}^d_{\geq d_0}$. Then there exists $\xi \in L^\perp_{I_0}$ such that $\xi \in L^\perp_I$ if and only if $L^\perp_{I_0} \subseteq L^\perp_I$, i.e., if and only if $I \subseteq I_0$. Thus (i) ⇔ (ii).

Let us prove that (ii) ⇒ (iii). We write $F_i(z_i) = 1 + z_i g_i(z_i)$. We have

$$\prod_{1 \leq i \leq d} \bigl(1 + z_i g_i(z_i)\bigr) = \sum_{K \subseteq \{1,\ldots,d\}} \prod_{i \in K} z_i g_i(z_i). \tag{16}$$

Consider a monomial $z_1^{k_1} \cdots z_d^{k_d}$ of total degree $k_1 + \cdots + k_d \leq d - d_0$. Let us denote its coefficient in the product $\prod_{i \in K} z_i g_i(z_i)$ by α_K. Let K_0 be the set of indices such that $k_i \neq 0$. Then $|K_0| \leq d - d_0$. Moreover, on the right-hand side of (16), our monomial appears only in the terms where $K \subseteq K_0$. Therefore the coefficient of $z_1^{k_1} \cdots z_d^{k_d}$ in $\prod_{1 \leq i \leq d} F_i(z_i)$ is equal to $\sum_{K \subseteq K_0} \alpha_K$. Furthermore, the coefficient of $z_1^{k_1} \cdots z_d^{k_d}$ on the right-hand side of (14) is equal to

$$\sum_{I \in \mathcal{J}^d_{\geq d_0}} \lambda(I) \sum_{K \subseteq K_0 \cap I^c} \alpha_K = \sum_{K \subseteq K_0} \alpha_K \sum_{\substack{I \in \mathcal{J}^d_{\geq d_0} \\ K \subseteq I^c}} \lambda(I).$$

By condition (ii) we have

$$\sum_{\substack{I \in \mathcal{J}^d_{\geq d_0} \\ K \subseteq I^c}} \lambda(I) = 1 \quad \text{for every } K \subseteq K_0.$$

Thus we have proved that (ii) ⇒ (iii). Next, (iv) is a particular case of (iii), so it remains only to prove that (iv) implies (ii).

By expanding the exponentials in condition (iv), we obtain

$$(z_1 + \cdots + z_d)^{d-d_0} = \sum_{I \in \mathcal{J}^d_{\geq d_0}} \lambda(I) \Bigl(\sum_{i \in I^c} z_i\Bigr)^{d-d_0}.$$

Condition (ii) follows easily from this relation. □

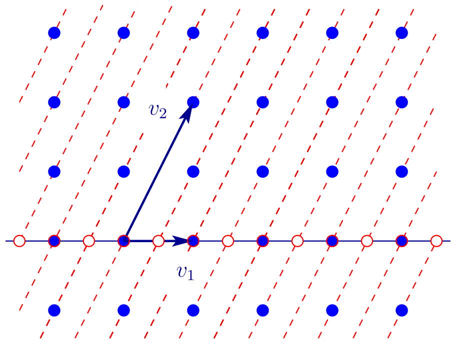
4.2 Formula for Intermediate Sums

In preparation for the approximation theorem, we need some notation and an expression for intermediate sums $S^L(s + \mathfrak{c}, \Lambda)(\xi)$.

We have $V = L_I \oplus L_{I^c}$. For $x \in V$ we denote the components by

$$x = x_I + x_{I^c}.$$

Fig. 2 The projected lattice $\Lambda_{\{1\}}$



Thus we identify the quotient V/L_I with L_{I^c}, and we denote the projected lattice by $\Lambda_{I^c} \subset L_{I^c}$. Note that $L_{I^c} \cap \Lambda \subseteq \Lambda_{I^c}$, but the inclusion is strict in general.

Example 24 Let $v_1 = (1,0)$, $v_2 = (1,2)$, $I = \{2\}$. The projected lattice Λ_{I^c} on $L_{I^c} = \mathbb{R}v_1$ is $\mathbb{Z}\frac{v_1}{2}$. See Fig. 2.

We denote by $\mathfrak{c}_I$ the cone generated by the vectors v_j, for $j \in I$ and by $\mathfrak{b}_I$ the parallelepiped $\mathfrak{b}_I = \sum_{i\in I}[0,1[\, v_i$. Similarly, we denote by $\mathfrak{c}_{I^c}$ the cone generated by the vectors v_j, for $j \in I^c$ and $\mathfrak{b}_{I^c} = \sum_{i\in I^c}[0,1[\, v_i$. The projection of the cone $\mathfrak{c}$ on $V/L_I = L_{I^c}$ identifies with $\mathfrak{c}_{I^c}$. Note that the generators v_i, $i \in I^c$, may be non-primitive for the projected lattice Λ_{I^c}, even if they are primitive for Λ, as we see in the previous example. We write $s = s_I + s_{I^c}$.

We first show that the intermediate generating function $S^{L_I}(s+\mathfrak{c},\Lambda)$ decomposes as a product.

The function $S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)$ is a meromorphic function on the space $(L_{I^c})^*$. The integral $I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi)$ is a meromorphic function on the space $(L_I)^*$. We consider both as functions on V^* through the decomposition $V = L_I \oplus L_{I^c}$.

Proposition 25 *The intermediate sum for the full cone $s+\mathfrak{c}$ breaks up into the product*

$$S^{L_I}(s+\mathfrak{c},\Lambda)(\xi) = S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)\, I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi). \tag{17}$$

Proof The projection of the cone $s+\mathfrak{c}$ into L_{I^c} is the cone $s_{I^c} + \mathfrak{c}_{I^c}$. For each $x_{I^c} \in (s_{I^c} + \mathfrak{c}_{I^c}) \cap \Lambda_{I^c}$, the slice $(s+\mathfrak{c}) \cap (x_{I^c} + L_I)$ is the cone $x_{I^c} + s_I + \mathfrak{c}_I$. Let us compute the integral on the slice,

$$\int_{(s+\mathfrak{c})\cap(x_{I^c}+L_I)} \mathrm{e}^{\langle \xi, y\rangle}\, \mathrm{d}m_{L_I\cap\Lambda}(y). \tag{18}$$

We write $y = x_{I^c} + s_I + \sum_{j\in I} y_j v_j$. Then

$$\mathrm{d}m_{L_I\cap\Lambda}(y) = \mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I) \prod_{j\in I} \mathrm{d}y_j.$$

Hence (18) is equal to

$$e^{\langle \xi, x_{I^c}\rangle} e^{\langle \xi, s_I\rangle} \operatorname{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)(-1)^{|I|} \prod_{j\in I} \frac{1}{\langle \xi, v_j\rangle}.$$

We observe that only the first factor, $e^{\langle \xi, x_{I^c}\rangle}$, depends on x_{I^c}. The sum of these factors over all $x_{I^c} \in (s_{I^c} + \mathfrak{c}_{I^c}) \cap \Lambda_{I^c}$ gives $S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)$, and using formula (1) for the integral, we obtain (17). □

4.3 Approximation Theorem

We can now state and prove the approximation theorem.

Theorem 26 (Approximation by a patched generating function) *Let $\mathfrak{c} \subset V$ be a rational simplicial cone with edge generators $v_1, \dots, v_d$. Let $s \in V_{\mathbb{Q}}$. Let $I \mapsto \lambda(I)$ be a patching function on $\mathcal{J}^d_{\geq d_0}$. For $I \in \mathcal{J}^d_{\geq d_0}$ let L_I be the linear span of $\{v_i\}_{i\in I}$. Then we have*

$$S(s+\mathfrak{c}, \Lambda)(\xi) \equiv A^{\lambda}(s+\mathfrak{c}, \Lambda)(\xi) := \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I)\, S^{L_I}(s+\mathfrak{c}, \Lambda)(\xi)$$
$$\textit{mod terms of } \xi\textit{-degree } \geq -d_0+1. \tag{19}$$

We call the function $A^{\lambda}(s+\mathfrak{c}, \Lambda)(\xi)$ on the right-hand side of (19) the *patched generating function* of $s+\mathfrak{c}$ (with respect to λ).

Proof of Theorem 26 We write the vertex as $s = \sum_i s_i v_i$. Let $a = \sum_i a_i v_i \in V$. We apply (14) to the functions

$$F_i(z_i) = e^{(a_i - s_i) z_i} \frac{-z_i}{1-e^{z_i}},$$

and we substitute $z_i = \langle \xi, v_i\rangle$. We obtain

$$e^{\langle \xi, a-s\rangle} \prod_{i=1}^{d} \frac{-\langle \xi, v_i\rangle}{1-e^{\langle \xi, v_i\rangle}} \equiv \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I) e^{\langle \xi, a_{I^c} - s_{I^c}\rangle} \prod_{i\in I^c} \frac{-\langle \xi, v_i\rangle}{1-e^{\langle \xi, v_i\rangle}}$$
$$\text{mod terms of } \xi\text{-degree } \geq d-d_0+1.$$

We multiply both sides first by $e^{\langle \xi, s\rangle}$; because this is analytic in ξ and thus of nonnegative ξ-degree, the identity modulo terms of high ξ-degree still holds true. Then we multiply by $1/\prod_{i=1}^{d}(-\langle \xi, v_i\rangle)$, which is homogeneous of degree $-d$ in ξ. We obtain

$$e^{\langle \xi, a\rangle} \prod_{i=1}^{d} \frac{1}{1-e^{\langle \xi, v_i\rangle}} \equiv \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I) e^{\langle \xi, a_{I^c}\rangle} \prod_{i\in I^c} \frac{1}{1-e^{\langle \xi, v_i\rangle}} e^{\langle \xi, s_I\rangle} \prod_{i\in I} \frac{-1}{\langle \xi, v_i\rangle}$$
$$\text{mod terms of } \xi\text{-degree } \geq -d_0+1. \tag{20}$$

Now, we sum up equalities (20) when a runs over the set $(s+\mathfrak{b})\cap\Lambda$ of integral points in the fundamental parallelepiped $s+\mathfrak{b}$ of the affine cone $s+\mathfrak{c}$. On the left-hand side we obtain

$$\sum_{a\in(s+\mathfrak{b})\cap\Lambda} \mathrm{e}^{\langle\xi,a\rangle}\prod_{i=1}^{d}\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}.$$

By Lemma 9, this is precisely $S(s+\mathfrak{c},\Lambda)(\xi)$. On the right-hand side, for each I, we have a sum over $a\in(s+\mathfrak{b})\cap\Lambda$ of the function

$$\mathrm{e}^{\langle\xi,a_{I^{\mathrm{c}}}\rangle}\prod_{i\in I^{\mathrm{c}}}\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}\mathrm{e}^{\langle\xi,s_I\rangle}\prod_{i\in I}\frac{-1}{\langle\xi,v_i\rangle},$$

which depends only on the projection $a_{I^{\mathrm{c}}}$ of a in the decomposition $a=a_I+a_{I^{\mathrm{c}}}\in L_I\oplus L_{I^{\mathrm{c}}}$. When a runs over $(s+\mathfrak{b})\cap\Lambda$, its projection $a_{I^{\mathrm{c}}}$ runs over $(s_{I^{\mathrm{c}}}+\mathfrak{b}_{I^{\mathrm{c}}})\cap\Lambda_{I^{\mathrm{c}}}$. Let us show that the fibers have the same number of points, equal to $\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)$. For a given $a_{I^{\mathrm{c}}}\in(s_{I^{\mathrm{c}}}+\mathfrak{b}_{I^{\mathrm{c}}})\cap\Lambda_{I^{\mathrm{c}}}$, let us compute the fiber

$$\{y\in(s+\mathfrak{b})\cap\Lambda : y_{I^{\mathrm{c}}}=a_{I^{\mathrm{c}}}\}.$$

Fix a point $a_I+a_{I^{\mathrm{c}}}$ in this fiber. Then $y=a_{I^{\mathrm{c}}}+y_I$ lies in the fiber if and only if $y_I-a_I\in(s_I-a_I+\mathfrak{b}_I)\cap\Lambda$. By Lemma 8(ii), the cardinality of the fiber is equal to $\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)$. Thus we obtain

$$\begin{aligned} &S(s+\mathfrak{c})(\xi)\\ &\quad\equiv\sum_{I\in\mathcal{J}^d_{\geq d_0}}\lambda(I)S(s_{I^{\mathrm{c}}}+\mathfrak{b}_{I^{\mathrm{c}}})(\xi)\prod_{i\in I^{\mathrm{c}}}\frac{1}{1-\mathrm{e}^{\langle\xi,v_i\rangle}}\mathrm{e}^{\langle\xi,s_I\rangle}\,\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)\prod_{i\in I}\frac{-1}{\langle\xi,v_i\rangle}\\ &\qquad\text{mod terms of }\xi\text{-degree }\geq -d_0+1. \end{aligned}\tag{21}$$

By Proposition 25 and Lemmas 7 and 9, the term corresponding to an $I\in\mathcal{J}^d_{\geq d_0}$ on the right-hand side of (21) is precisely

$$S^{L_I}(s+\mathfrak{c},\Lambda)(\xi)=S(s_{I^{\mathrm{c}}}+\mathfrak{c}_{I^{\mathrm{c}}},\Lambda_{I^{\mathrm{c}}})(\xi)\,I(s_I+\mathfrak{c}_I,L_I\cap\Lambda)(\xi),$$

which completes the proof. □

Remark 27 For $d_0=0$, we obtain the poset $\mathcal{J}^d_{\geq 0}$ of all subsets of $\{1,\dots,d\}$. The unique patching function on $\mathcal{J}^d_{\geq 0}$ is given by $\lambda(\emptyset)=1$ and $\lambda(I)=0$ for all $I\neq\emptyset$. Then the approximation is trivial, i.e., $S(s+\mathfrak{c},\Lambda)(\xi)=A^{\lambda}(s+\mathfrak{c},\Lambda)(\xi)$.

Example 28 Let $\mathfrak{c}$ be the first quadrant in $\mathbb{R}^2$, and $d_0=1$. Thus $\mathcal{J}^2_{\geq 1}$ consists of three subsets, $\{1\},\{2\}$, and $\{1,2\}$. A patching function is given by $\lambda(\{i\})=1$ and $\lambda(\{1,2\})=-1$. We consider the affine cone $s+\mathfrak{c}$ with $s=(-\frac{1}{2},-\frac{1}{2})$. Let $\xi=(\xi_1,\xi_2)$. We have

$$I(s_i+\mathfrak{c}_{\{i\}})(\xi)=\frac{-\mathrm{e}^{-\xi_i/2}}{\xi_i},\qquad I(s+\mathfrak{c})(\xi)=\frac{\mathrm{e}^{-\xi_1/2-\xi_2/2}}{\xi_1\xi_2},$$

$$S(s_i+\mathfrak{c}_{\{i\}})(\xi)=\frac{1}{1-\mathrm{e}^{\xi_i}},\qquad S(s+\mathfrak{c})(\xi)=\frac{1}{(1-\mathrm{e}^{\xi_1})(1-\mathrm{e}^{\xi_2})}.$$

The approximation theorem claims that

$$\frac{1}{(1-\mathrm{e}^{\xi_1})(1-\mathrm{e}^{\xi_2})}\equiv\frac{1}{1-\mathrm{e}^{\xi_2}}\cdot\frac{-\mathrm{e}^{-\xi_1/2}}{\xi_1}+\frac{1}{1-\mathrm{e}^{\xi_1}}\cdot\frac{-\mathrm{e}^{-\xi_2/2}}{\xi_2}-\frac{\mathrm{e}^{-\xi_1/2-\xi_2/2}}{\xi_1\xi_2}$$
$$\text{mod terms of } \xi\text{-degree } \geq 0.$$

Indeed, the difference between the two sides is equal to

$$\left(\frac{1}{1-\mathrm{e}^{\xi_1}}+\frac{\mathrm{e}^{-\xi_1/2}}{\xi_1}\right)\left(\frac{1}{1-\mathrm{e}^{\xi_2}}+\frac{\mathrm{e}^{-\xi_2/2}}{\xi_2}\right)$$

which is analytic near 0.

4.4 An Explicit Patching Function

Next we compute an explicit patching function on $\mathcal{J}^d_{\geq d_0}$. It is related to the Möbius function of the poset $\mathcal{J}^d_{\geq d_0}$, so we call it the *Möbius patching function* and denote it by $\lambda_{\text{Möbius}}$. We will denote the corresponding patched generating function $A^{\lambda_{\text{Möbius}}}(s+\mathfrak{c},\Lambda)$ by $A_{\geq d_0}(s+\mathfrak{c},\Lambda)$.

Proposition 29 *For $I\in\mathcal{J}^d_{\geq d_0}$, let*

$$\lambda_{\text{Möbius}}(I)=(-1)^{|I|-d_0}\binom{|I|-1}{d_0-1}.$$

Then $\lambda_{\text{Möbius}}$ is a patching function on $\mathcal{J}^d_{\geq d_0}$.

Proof We prove that $\lambda_{\text{Möbius}}$ satisfies condition (iv) of Lemma 23. The trick is to write $\mathrm{e}^z=1+t(\mathrm{e}^z-1)|_{t=1}$. Thus

$$\mathrm{e}^{z_1+\cdots+z_d}=\prod_{i=1}^d\mathrm{e}^{z_i}=\prod_{i=1}^d\bigl(1+t\bigl(\mathrm{e}^{z_i}-1\bigr)\bigr)\Big|_{t=1}.$$

Let us consider $P(t):=\prod_{i=1}^d(1+t(\mathrm{e}^{z_i}-1))=\sum_{q=0}^d C_q(z)t^q$ as a polynomial in the indeterminate t. As $\mathrm{e}^{z_i}-1$ is a sum of terms of z_i-degree >0, we have

$$\mathrm{e}^{z_1+\cdots+z_d}\equiv\sum_{q=0}^{k_0}C_q(z)\text{ mod terms of } z\text{-degree } \geq k_0+1. \tag{22}$$

Next, we write

$$P(t)=\prod_{i=1}^d\bigl(1+t\bigl(\mathrm{e}^{z_i}-1\bigr)\bigr)=\prod_{i=1}^d\bigl((1-t)+t\mathrm{e}^{z_i}\bigr).$$

By expanding the product, we obtain

$$C_q(z) = \sum_{|K| \le q} (-1)^{q-|K|} \binom{d-|K|}{q-|K|} e^{z_K}.$$

Summing up these coefficients for $0 \le q \le k_0 = d - d_0$, we obtain

$$\sum_{q=0}^{k_0} C_q(z) = \sum_{|K| \le k_0} \left(\sum_{q=|K|}^{k_0} (-1)^{q-|K|} \binom{d-|K|}{q-|K|} \right) e^{z_K}.$$

By substituting $K = I^{\mathrm{c}}$ and $d - q = m$, we obtain

$$\sum_{q=0}^{k_0} C_q(z) = \sum_{|I| \ge d_0} f\big(|I|\big) e^{z_{I^{\mathrm{c}}}},$$

with

$$f(j) = \sum_{m=d_0}^{j} (-1)^{j-m} \binom{j}{j-m} = \sum_{m=d_0}^{j} (-1)^{j-m} \binom{j}{m}.$$

The truncated binomial sum $f(j)$ is easy to compute, using the recursion relation $\binom{j}{m} = \binom{j-1}{m-1} + \binom{j-1}{m}$. We obtain

$$f(j) = (-1)^{j-d_0} \binom{j-1}{d_0-1}.$$

Thus, $\lambda_{\text{Möbius}}(I) = f(|I|)$ satisfies condition (ii) for a patching function. □

Remark 30 Proposition 29 can also be deduced from results in [16].

5 Computation of the Patched Generating Function

In this section, we show that if $k_0 = d - d_0$ is fixed, the patched generating function $A_{\ge d_0}(s + \mathfrak{c}, \Lambda)$ can be efficiently computed for a simplicial cone $s + \mathfrak{c}$. This will be a consequence of Barvinok's polynomial-time decomposition of cones in fixed dimension [10, 12]. We exhibit the dependence of the patched generating function on the vertex s explicitly as a "step function" in two useful ways, using the "ceiling" function $\lceil\cdot\rceil$ and the "fractional part" function $\{\cdot\}$, respectively.

We start with the following result.

Theorem 31 (Short formula for $S^{L_I}(s + \mathfrak{c}, \mathbb{Z}^d)(\xi)$ for varying s) *Fix a nonnegative integer k_0. There exists a polynomial-time algorithm for the following problem. Given the following input:*

(I_1) *a number d in unary encoding,*

(I_2) *a simplicial cone* $\mathfrak{c} = \mathfrak{c}(v_1, \ldots, v_d) \subset \mathbb{R}^d$, *represented by the vectors* $v_1, \ldots, v_d \in \mathbb{Z}^d$ *in binary encoding*,

(I_3) *a subspace* $L_I = \operatorname{lin}(v_i : i \in I) \subseteq \mathbb{R}^d$ *of codimension* k_0, *represented by an index set* $I \subseteq \{1, \ldots, d\}$ *of cardinality* $d_0 = d - k_0$,

compute the following output in binary encoding:

(O_1) *a finite set* Γ,

(O_2) *for every* γ *in* Γ, *integers* $\alpha^{(\gamma)}$, *rational vectors* $\eta_i^{(\gamma)}$ *and* $w_i^{(\gamma)}$ *for* $i = 1, \ldots, d$, *where* $\eta_i^{(\gamma)} \in \mathbb{Z}^d$ *for* $i \in I^{\mathrm{c}}$

such that for every $s \in \mathbb{Q}^d$, *we have the following equality of meromorphic functions of* ξ:

$$\begin{aligned} & S^{L_I}\big(s + \mathfrak{c}, \mathbb{Z}^d\big)(\xi) \\ &\quad = \sum_{\gamma \in \Gamma} \alpha^{(\gamma)} \prod_{i \in I^{\mathrm{c}}} T\big(\big\lceil\big\langle\eta_i^{(\gamma)}, s\big\rangle\big\rceil, \big\langle\xi, w_i^{(\gamma)}\big\rangle\big) \\ &\qquad \cdot \prod_{i \in I} \exp\big(\big\langle\eta_i^{(\gamma)}, s\big\rangle\big\langle\xi, w_i^{(\gamma)}\big\rangle\big) \cdot \frac{1}{\prod_{i=1}^{d} \langle\xi, w_i^{(\gamma)}\rangle} \qquad (23a) \\ &\quad = \mathrm{e}^{\langle\xi, s\rangle} \sum_{\gamma \in \Gamma} \alpha^{(\gamma)} \prod_{i \in I^{\mathrm{c}}} T\big(\big\{-\big\langle\eta_i^{(\gamma)}, s\big\rangle\big\}, \big\langle\xi, w_i^{(\gamma)}\big\rangle\big) \cdot \frac{1}{\prod_{i=1}^{d} \langle\xi, w_i^{(\gamma)}\rangle}. \qquad (23b) \end{aligned}$$

Of course, for $I = \emptyset$ we have $L = \{0\}$, and so we recover formulas for $S(s + \mathfrak{c}, \mathbb{Z}^d)(\xi)$. If we set $I = \{1, \ldots, d\}$, then $L = \mathbb{R}^d$, and we get formulas for $I(s + \mathfrak{c}, \mathbb{Z}^d)(\xi)$.

Remark 32 Consider the term corresponding to $\gamma \in \Gamma$ in (23a) or (23b). As will follow from the proof, the vector $w_i^{(\gamma)}$ for $i \in I$ is just the original vector v_i, and the collection $w_i^{(\gamma)}$, $i = 1, \ldots, d$, forms a basis of $\mathbb{R}^d$. Furthermore, the vectors $w_i^{(\gamma)}$, with $i \in I^{\mathrm{c}}$, are in $L_{I^{\mathrm{c}}}$ and form a basis of the projected lattice. The vectors $\eta_i^{(\gamma)}$, $i = 1, \ldots, d$, are the dual (biorthogonal) vectors to the elements $w_j^{(\gamma)}$, $j = 1, \ldots, d$, i.e., $\langle\eta_i^{(\gamma)}, w_j^{(\gamma)}\rangle = \delta_{i,j}$. Thus we only need to compute the integers $\alpha^{(\gamma)}$ and the elements $w_i^{(\gamma)}$, where $i \in I^{\mathrm{c}}$.

Remark 33 Consider the term corresponding to $\gamma \in \Gamma$ in (23b). As the vectors $w_i^{(\gamma)}$, $i \in I^{\mathrm{c}}$, form a basis of the projected lattice, we may identify $V/(L_I + \Lambda)$ to $\bigoplus_{i \in I^{\mathrm{c}}} [0, 1[\, w_i^{(\gamma)}$. Define

$$s^{(\gamma)} = \sum_{i \in I^{\mathrm{c}}} \big\{-\big\langle\eta_i^{(\gamma)}, s\big\rangle\big\}\, w_i^{(\gamma)}.$$

As the $\eta_i^{(\gamma)}$ for $i \in I^{\mathrm{c}}$ are integer vectors, and $\langle\eta_i^{(\gamma)}, v_j\rangle = 0$ if $j \in I$, we can think of $s \mapsto s^{(\gamma)}$ as a linear map on the torus $V/(\Lambda + L_I)$ with integer coefficients. The point $s + s^{(\gamma)}$ is in $\bigoplus_{i \in I^{\mathrm{c}}} \mathbb{Z} w_i^{(\gamma)} \oplus \bigoplus_{i \in I} \mathbb{R} v_i$, and formula (23b) also reads

$$S^{L_I}\big(s+\mathfrak{c},\mathbb{Z}^d\big)(\xi)=\sum_{\gamma\in\Gamma}\alpha^{(\gamma)}\mathrm{e}^{\langle\xi,s+s^{(\gamma)}\rangle}\frac{1}{\prod_{i\in I^{\mathrm{c}}}(1-\mathrm{e}^{\langle\xi,w_i^{(\gamma)}\rangle})}\frac{1}{\prod_{i\in I}\langle\xi,v_i\rangle}. \tag{24}$$

Now we prove the theorem.

Proof of Theorem 31 Let us describe the algorithm along with the proof. Let $\Lambda=\mathbb{Z}^d$. By Proposition 25,

$$S^{L_I}(s+\mathfrak{c},\Lambda)(\xi)=S(s_{I^{\mathrm{c}}}+\mathfrak{c}_{I^{\mathrm{c}}},\Lambda_{I^{\mathrm{c}}})(\xi)\,I(s_I+\mathfrak{c}_I,L_I\cap\Lambda)(\xi). \tag{25}$$

We first discuss $I(s_I+\mathfrak{c}_I,L_I\cap\Lambda)$. We have

$$I(s_I+\mathfrak{c}_I,L_I\cap\Lambda)(\xi)=\mathrm{e}^{\langle\xi,s_I\rangle}\operatorname{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)\prod_{j\in I}\frac{-1}{\langle\xi,v_j\rangle}. \tag{26}$$

Using linear functionals $\eta_i\in\mathbb{Q}^d$, $i\in I$ (the coordinate functions with respect to the basis v_i), write $s_I=\sum_{i\in I}\langle\eta_i,s\rangle v_i$. The η_i can be read off in polynomial time from the inverse of the matrix whose columns are $v_1,\dots,v_d$. Then $\mathrm{e}^{\langle\xi,s_I\rangle}$ takes the form

$$\mathrm{e}^{\langle\xi,s_I\rangle}=\prod_{i\in I}\exp\big(\langle\eta_i,s\rangle\langle\xi,v_i\rangle\big). \tag{27}$$

Now, to handle the factor $S(s_{I^{\mathrm{c}}}+\mathfrak{c}_{I^{\mathrm{c}}},\Lambda_{I^{\mathrm{c}}})$, note that $\mathfrak{c}_{I^{\mathrm{c}}}\subset L_{I^{\mathrm{c}}}$ is a k_0-dimensional cone. By using a Hermite normal form computation, which is polynomial time [29], we can compute a linear change of variables which replaces the projected lattice $\Lambda_{I^{\mathrm{c}}}$ on $L_{I^{\mathrm{c}}}$ by $\mathbb{Z}^{k_0}$. Then, using Barvinok's decomposition [10], we decompose it into a family of cones which are unimodular,

$$[\mathfrak{c}_{I^{\mathrm{c}}}]\equiv\sum_{m\in M}\epsilon_m\big[\mathfrak{c}_{I^{\mathrm{c}}}^{(m)}\big]\quad\text{(modulo cones containing lines)}, \tag{28}$$

where $\epsilon_m\in\{\pm1\}$. As k_0 is fixed, this decomposition can be done with a polynomial-time algorithm. Of course, this step is crucial with respect to the efficiency of the whole algorithm.

Again changing notation, we now denote by $\mathfrak{c}=\mathfrak{c}(\{w_i\}_{i\in I^{\mathrm{c}}})$ one of these unimodular cones $\mathfrak{c}_{I^{\mathrm{c}}}^{(m)}\subset L_{I^{\mathrm{c}}}$, with primitive generators w_i, and also write $\epsilon=\epsilon_m$. We remark that the vectors w_i, $i\in I^{\mathrm{c}}$, generate the projected lattice on $L_{I^{\mathrm{c}}}$. Using linear functionals $\eta_i\in\mathbb{Q}^d$, $i\in I^{\mathrm{c}}$, write $s_{I^{\mathrm{c}}}=\sum_{i\in I^{\mathrm{c}}}\langle\eta_i,s\rangle w_i$. Actually, we have $\eta_i\in\mathbb{Z}^d$. By letting $w_i=v_i$ for the other indices $i\in I$, we can write

$$s=\sum_{i\in I}\langle\eta_i,s\rangle v_i+\sum_{i\in I^{\mathrm{c}}}\langle\eta_i,s\rangle w_i=\sum_{i=1}^{d}\langle\eta_i,s\rangle w_i. \tag{29}$$

Let $s'_{I^{\mathrm{c}}}$ be the unique lattice point in the fundamental parallelepiped of the cone $s_{I^{\mathrm{c}}}+\mathfrak{c}$. We have

$$s'_{I^{\mathrm{c}}}=\sum_{i\in I^{\mathrm{c}}}\big\lceil\langle\eta_i,s\rangle\big\rceil w_i. \tag{30}$$

Using this, we obtain the generating function from Lemma 9 as

$$S(s_{I^c}+\mathfrak{c})(\xi)=\frac{e^{\langle\xi,s'_{I^c}\rangle}}{\prod_{i\in I^c}(1-e^{\langle\xi,w_i\rangle})}.$$

Thus finally, using (26) we have the meromorphic function

$$\epsilon\,\mathrm{vol}_{L_I\cap\Lambda}(\mathfrak{b}_I)(-1)^{|I|}\cdot\frac{e^{\langle\xi,s'_{I^c}\rangle}}{\prod_{i\in I^c}(1-e^{\langle\xi,w_i\rangle})}\cdot\frac{e^{\langle\xi,s_I\rangle}}{\prod_{j\in I}\langle\xi,v_j\rangle}. \tag{31}$$

Then (31) is now written as

$$\alpha\prod_{i\in I^c}T\big(\lceil\langle\eta_i,s\rangle\rceil,\langle\xi,w_i\rangle\big)\cdot\prod_{i\in I}\exp\big(\langle\eta_i,s\rangle\langle\xi,v_i\rangle\big)\cdot\frac{1}{\prod_{i=1}^d\langle\xi,w_i\rangle}, \tag{32}$$

where α collects the multiplicative constants in (31). Collecting these terms gives the desired short formula (23a).

To derive the second form, we note that $\lceil\langle\eta_i,s\rangle\rceil=\langle\eta_i,s\rangle+\{-\langle\eta_i,s\rangle\}$, so we can write

$$T\big(\lceil\langle\eta_i,s\rangle\rceil,\langle\xi,w_i\rangle\big)=T\big(\{-\langle\eta_i,s\rangle\},\langle\xi,w_i\rangle\big)\exp\big(\langle\eta_i,s\rangle\langle\xi,w_i\rangle\big). \tag{33}$$

Thus the term (32) can be written as

$$\alpha\prod_{i\in I^c}T\big(\{-\langle\eta_i,s\rangle\},\langle\xi,w_i\rangle\big)\cdot e^{\langle\xi,s\rangle}\cdot\frac{1}{\prod_{i=1}^d\langle\xi,w_i\rangle}, \tag{34}$$

using (29). Collecting these terms gives the short formula (23b). □

Example 34 Let us give a short example of the output of our algorithm. Consider the three-dimensional cone with rays given by the vectors $(1,1,1)$, $(1,-1,0)$, $(1,1,0)$. This cone is not unimodular. We consider the affine cone $s+\mathfrak{c}$. Our algorithm described in Theorem 31 computes any intermediate generating function $S^L(s+\mathfrak{c},\mathbb{Z}^3)$ when L is a linear span of a face of $\mathfrak{c}$. For $L=\{0\}$ (indexed by the empty set I), we obtain the meromorphic function $S(s+\mathfrak{c},\mathbb{Z}^3)(\xi)$ (the discrete generating function of the cone $s+\mathfrak{c}$). Here $S(s+\mathfrak{c},\mathbb{Z}^3)(\xi)$ depends on $s=(s_1,s_2,s_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$ and is given by

$$\begin{aligned}&\exp(s_1\xi_1+s_2\xi_2+s_3\xi_3)\\&\quad\cdot\bigg(-\frac{T(\{-s_3+s_2\},-\xi_1-\xi_2)T(\{-s_1+s_2\},\xi_1)T(\{-s_3\},\xi_1+\xi_2+\xi_3)}{(-\xi_1-\xi_2)\xi_1(\xi_1+\xi_2+\xi_3)}\\&\quad+\frac{T(\{-s_3+s_2\},\xi_1-\xi_2)T(\{2s_3-s_2-s_1\},\xi_1)T(\{-s_3\},\xi_1+\xi_2+\xi_3)}{(\xi_1-\xi_2)\xi_1(\xi_1+\xi_2+\xi_3)}\bigg).\end{aligned}$$

If $L=\mathbb{R}v_1$ is the subspace of dimension 1 generated by the edge $v_1=(1,1,1)$ of the cone $\mathfrak{c}$ (so that L is indexed by the subset $I=\{1\}$ of $\{1,2,3\}$), the intermediate generating function $S^L(s+\mathfrak{c},\mathbb{Z}^3)$ is given by

$$\exp(s_1\xi_1+s_2\xi_2+s_3\xi_3)\biggl(-\frac{T(\{-s_3+s_2\},-\xi_1-\xi_2)T(\{-s_1+s_2\},\xi_1)}{(-\xi_1-\xi_2)\xi_1(-\xi_1-\xi_2-\xi_3)} + \frac{T(\{-s_3+s_2\},\xi_1-\xi_2)T(\{2s_3-s_2-s_1\},\xi_1)}{(\xi_1-\xi_2)\xi_1(-\xi_1-\xi_2-\xi_3)}\biggr).$$

Remark 35 When $k_0=d-d_0$ is fixed, the set $\mathcal{J}^d_{\geq d_0}$ has a polynomially bounded cardinality, and it can be enumerated by using a straightforward algorithm along with the evaluation of the patching function $\lambda_{\text{Möbius}}$. Thus we can also compute $A_{\geq d_0}(s+\mathfrak{c},\Lambda)(\xi)$ in the same form (23a) or (23b) in polynomial time.

6 Computation of Ehrhart Quasi-polynomials

We now apply the approximation of the generating functions of the cones at vertices to the computation of the highest coefficients for a weighted Ehrhart quasi-polynomial. We first discuss the case when the weight is a power of a linear form.

Theorem 36 *Let $\mathfrak{p}$ be a simple rational polytope, and let $\mathcal{V}(\mathfrak{p})$ denote the set of its vertices. For each vertex $s\in\mathcal{V}(\mathfrak{p})$, let $\mathfrak{c}_s$ be the tangent cone of s, and let $q_s\in\mathbb{N}$ be a positive integer such that $q_s s\in\Lambda$. Fix a linear form $\ell\in V^*$ and M a nonnegative integer. Fix $0\leq k_0\leq d$ and let $d_0=\max\{d-k_0,0\}$. Then the Ehrhart quasi-polynomial*

$$E(\mathfrak{p},\ell,M;n)=\sum_{x\in n\mathfrak{p}\cap\Lambda}\frac{\langle\ell,x\rangle^M}{M!}$$

coincides in degree $\geq M+d-k_0$ with the following quasi-polynomial:

$$\sum_{k=0}^{k_0}\sum_{s\in\mathcal{V}(\mathfrak{p})}\bigl(\lfloor n\rfloor_{q_s}\bigr)^{M+d-k}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}A_{\geq d_0}\bigl(\{n\}_{q_s}s+\mathfrak{c}_s,\Lambda\bigr)_{[-d+k]}(\xi), \tag{35a}$$

evaluated at $\xi=\ell$, which can also be written as

$$\sum_{k=0}^{k_0}n^{M+d-k}\sum_{s\in\mathcal{V}(\mathfrak{p})}\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}\bigl(\mathrm{e}^{-\langle\xi,\{n\}_{q_s}s\rangle}A_{\geq d_0}\bigl(\{n\}_{q_s}s+\mathfrak{c}_s,\Lambda\bigr)(\xi)\bigr)_{[-d+k]}, \tag{35b}$$

evaluated at $\xi=\ell$.

In the following, we will use the second form (35b).

Remark 37 The sum (35) depends polynomially on ℓ. However, for an individual vertex s, the functions

$$\xi\mapsto\frac{\langle\xi,s\rangle^{M+d-k}}{(M+d-k)!}A_{\geq d_0}\bigl(\{n\}_{q_s}s+\mathfrak{c}_s,\Lambda\bigr)_{[-d+k]}(\xi)$$

and

$$\xi \mapsto \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!}\big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s\rangle} A_{\geq d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)(\xi)\big)_{[-d+k]}$$

are meromorphic functions, which are not defined if ξ is singular. Thus in the algorithm we use a deformation procedure; i.e., we evaluate the function at $\xi = \ell + \epsilon \ell'$ for a suitable rational perturbation vector ℓ' and then compute the limit for $\epsilon \to 0$.

Proof of Theorem 36 The sum $\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x\rangle^M}{M!}$ is the term of ξ-degree M in

$$S(n\mathfrak{p})(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(ns + \mathfrak{c}_s)(\xi).$$

Fix a vertex s. We write $n = \lfloor n \rfloor_{q_s} + \{n\}_{q_s}$. As $\lfloor n \rfloor_{q_s} s$ is a lattice point, we have

$$S(ns + \mathfrak{c}_s)(\xi) = \mathrm{e}^{\lfloor n \rfloor_{q_s} \langle \xi, s\rangle} S\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)(\xi). \tag{36}$$

Consider $S(ns + \mathfrak{c}_s)_{[M]}(\xi)$ as a quasi-polynomial in n. By (36), it coincides in degree $\geq M + d - k_0$ with

$$\sum_{k=0}^{k_0} \big(\lfloor n \rfloor_{q_s}\big)^{M+d-k} \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!} S\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)_{[-d+k]}(\xi).$$

Now, for $0 \leq k \leq k_0$, we have

$$S\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)_{[-d+k]}(\xi) = A_{\geq d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)_{[-d+k]}(\xi).$$

By specializing on $\xi = \ell$, we obtain the claim in the form of (35a).

To obtain the second claim in the form of (35b), we write

$$S(ns + \mathfrak{c}_s)(\xi) = \mathrm{e}^{n\langle \xi, s\rangle}\big(\mathrm{e}^{-\langle \xi, s\rangle \{n\}_{q_s}} S\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)(\xi)\big). \tag{37}$$

Again, by expanding, we obtain that the quasi-polynomial $S(ns + \mathfrak{c}_s)_{[M]}(\xi)$ coincides in degree $\geq M + d - k_0$ with

$$\sum_{k=0}^{k_0} n^{M+d-k} \frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!}\big(\mathrm{e}^{-\langle \xi, s\rangle \{n\}_{q_s}} S\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)(\xi)\big)_{[-d+k]}.$$

Since $\mathrm{e}^{-\langle \xi, s\rangle \{n\}_{q_s}}$ is analytic in ξ, we have for $0 \leq k \leq k_0$ that

$$\begin{aligned} &\big(\mathrm{e}^{-\langle \xi, s\rangle \{n\}_{q_s}} S\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)(\xi)\big)_{[-d+k]} \\ &\quad = \big(\mathrm{e}^{-\langle \xi, s\rangle \{n\}_{q_s}} A_{\geq d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s\big)(\xi)\big)_{[-d+k]}. \end{aligned} \tag{38}$$

Again, by specializing on $\xi = \ell$, we obtain the claim in the form of (35b). □

We now derive the coefficient functions E_m of the weighted Ehrhart quasi-polynomial as short *closed formulas* that are step polynomials (cf. [34]). To be precise, we define a step polynomial in n as a finite sum of functions of the form

$$f\big((\zeta_1 n) \bmod q_1, \ldots, (\zeta_k n) \bmod q_k\big),$$

where f is a polynomial and $\zeta_1, \ldots, \zeta_k \in \mathbb{Z}$, $q_1, \ldots, q_k \in \mathbb{N}$. These can then be evaluated efficiently, providing a corollary (Theorem 43) in the same form as Barvinok's theorem in [11].

Theorem 38 *For every fixed number* $k_0 \in \mathbb{N}$, *there exists a polynomial-time algorithm for the following problem.*
Input:

(I_1) *a number* $d \in \mathbb{N}$ *in unary encoding, with* $d \geq k_0$,
(I_2) *a finite index set* $\mathcal{V}$,
(I_3) *a simple polytope* $\mathfrak{p}$, *given by its vertices, rational vectors* $s_j \in \mathbb{Q}^d$ *for* $j \in \mathcal{V}$ *in binary encoding*,
(I_4) *a rational vector* $\ell \in \mathbb{Q}^d$ *in binary encoding*,
(I_5) *a number* $M \in \mathbb{N}$ *in unary encoding*.

Output, *in binary encoding*,

(O_1) *an index set* Γ,
(O_2) *polynomials* $f^{\gamma,m} \in \mathbb{Q}[r_1, \ldots, r_{k_0}]$ *and integer numbers* $\zeta_i^{\gamma,m} \in \mathbb{Z}$, $q_i^{\gamma,m} \in \mathbb{N}$ *for* $\gamma \in \Gamma$ *and* $m = M + d - k_0, \ldots, M + d$ *and* $i = 1, \ldots, k_0$,

such that the Ehrhart quasi-polynomial

$$E(\mathfrak{p}, \ell, M; n) = \sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \ell, x\rangle^M}{M!} = \sum_{m=0}^{M+d} E_m\big(\mathfrak{p}, \ell, M; \{n\}_q\big)\, n^m$$

agrees in n*-degree* $\geq M + d - k_0$ *with the quasi-polynomial*

$$\sum_{\gamma \in \Gamma} \sum_{m=M+d-k_0}^{M+d} f^{\gamma,m}\big(\big\{\zeta_1^{\gamma,m} n\big\}_{q_1^{\gamma,m}}, \ldots, \big\{\zeta_{k_0}^{\gamma,m} n\big\}_{q_{k_0}^{\gamma,m}}\big)\, n^m.$$

Remark 39 For $d \leq k_0$, the algorithm actually computes the complete Ehrhart quasi-polynomial, i.e., the coefficient functions $E_m(\mathfrak{p}, \ell, M; \{n\}_q)$ for $m = 0, \ldots, M + d$. However, the key point of our method is to handle the case where $d > k_0$; then the non-trivial efficiently computable approximations come into play.

Remark 40 The specific form of the quasi-polynomial given by the theorem gives a more precise period q_i for the individual terms, rather than a period q_s that is determined by the vertex. The q_i will always be divisors of q_s. Due to the projections into lattices in small dimension $\leq k_0$, these periods can be much smaller than q_s. In particular, the highest degree coefficient E_{M+d}, of course, is a constant.

We will use the following lemma.

Lemma 41 (Lemma 4 of [6]) *For every fixed number $D \in \mathbb{N}$, there exists a polynomial-time algorithm for the following problem.*
Input: *a number M in unary encoding, a sequence of k polynomials $P_j \in \mathbb{Q}[X_1, \dots, X_D]$ of total degree at most M, in dense monomial representation.*
Output: *the product $P_1 \cdots P_k$ truncated at degree M.*

We can now prove the theorem.

Proof of Theorem 38 Because the polytope $\mathfrak{p}$ is simple, we can use the primal–dual algorithm by Bremner, Fukuda, and Marzetta [17, Corollary 1] to compute the inequality description (H-description) from the given V-description in polynomial time. From the double description, we can compute in polynomial time the description of the tangent cones $\mathfrak{c}_{s_j}$ for $j \in \mathcal{V}$ by the primitive vectors $v_{s_j,1}, \dots, v_{s_j,d} \in \mathbb{Z}^d$ such that $\mathfrak{c}_{s_j} = \mathfrak{c}(v_{s_j,1}, \dots, v_{s_j,d})$.

We now use formula (35b) of Theorem 36, which gives (with $d_0 = d - k_0$)

$$E_m\big(p, \xi, M; \{n\}_q\big) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \frac{\langle \xi, s\rangle^m}{m!} \big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s\rangle} A_{\ge d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)(\xi)\big)_{[-d+k]} \tag{39}$$

for $m = M + d - k$, when $m \ge M + d - k_0$. We compute this separately for each $k = 0, \dots, k_0$, that is, $m = M + d - k_0, \dots, M + d$. Let $s + \mathfrak{c}_s$ be one of these cones. By the algorithm of Theorem 31 and Remark 35, we compute the data describing the parametric short formula (23b) for $A_{\ge d_0}(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda)(\xi)$. We then consider one of the summands of

$$\frac{\langle \xi, s\rangle^m}{m!} \big(\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s\rangle} A_{\ge d_0}\big(\{n\}_{q_s} s + \mathfrak{c}_s, \Lambda\big)(\xi)\big)_{[-d+k]}$$

at a time. Here $\mathrm{e}^{-\langle \xi, \{n\}_{q_s} s\rangle}$ and the term $\mathrm{e}^{\langle \xi, \{n\}_{q_s} s\rangle}$ from (23b) cancel, and thus each summand takes the form

$$\left(\frac{\langle \xi, s\rangle^{M+d-k}}{(M+d-k)!}\right)\left(\prod_{i \in I^{\mathrm{c}}} T\big(\tau_i(n), \langle \xi, w_i\rangle\big)\right)_{[k]}\left(\frac{1}{\prod_{i=1}^{d} \langle \xi, w_i\rangle}\right), \tag{40}$$

where

$$\tau_i(n) := \big\{-\langle \eta_i, s\rangle \{n\}_{q_s}\big\} \quad \text{for } i \in I^{\mathrm{c}}. \tag{41}$$

Let $q_i \in \mathbb{N}$ be the smallest positive integer such that $q_i\langle -\eta_i, s\rangle \in \mathbb{Z}$. Then q_i is a divisor of the number q_s associated with the vertex s, because $\eta_i \in \mathbb{Z}^d$. Then

$$\tau_i(n) = \frac{1}{q_i}\big\{\zeta_i \{n\}_{q_s}\big\}_{q_i} \quad \text{with } \zeta_i = q_i\langle -\eta_i, s\rangle \in \mathbb{Z}.$$

Since q_i is a divisor of q_s, this simplifies to

$$\tau_i(n) = \frac{1}{q_i}\{\zeta_i n\}_{q_i}, \tag{42}$$

where ζ_i can be reduced modulo q_i as well because n is assumed to be an integer. We now treat $r_i := \{\zeta_i n\}_{q_i} \in \mathbb{N}$ as symbolic variables.

In order to evaluate (40) at $\xi = \ell$, we note the following. The first factor is holomorphic in ξ and homogeneous of ξ-degree $m = M + d - k$, the second factor is holomorphic in ξ and homogeneous of degree k, and the third factor is homogeneous of ξ-degree $-d$. If $\langle \ell, w_i\rangle = 0$ for some i, we cannot just substitute $\xi = \ell$ in the formula. Instead we use a perturbation. In polynomial time, we can compute a rational vector $\ell' \in \mathbb{Q}^d$ such that $\langle \ell', w_i\rangle \neq 0$ for all vectors w_i with $\langle \ell, w_i\rangle = 0$. It is important that we choose the same vector once and for all computations with all cones and summands.

We then set $\xi = t(\ell + \epsilon\ell')$, where t and ϵ are treated as symbolic variables. Here the exponent of the variable t keeps track of the ξ-degrees. We then perform computations with truncated series in $\mathbb{Q}[r_i : i \in I^{\mathrm{c}}][t^{\pm 1}, \epsilon^{\pm 1}]$. We note that this is a polynomial ring in a constant number of variables only, because $|I^{\mathrm{c}}|$ is bounded above by the constant k_0. Thus Lemma 41 gives us a polynomial-time algorithm for multiplying the series. Then (40) can be written as

$$\frac{\langle \ell + \epsilon\ell', s\rangle^m}{m!} \cdot \left(\prod_{i\in I^{\mathrm{c}}} T\bigl(\tau_i(n), \bigl\langle t(\ell + \epsilon\ell'), w_i\bigr\rangle\bigr)\right)_{[k]} \cdot \frac{1}{\prod_{i=1}^d \langle \ell + \epsilon\ell', w_i\rangle} \cdot t^{M-k}, \tag{43}$$

where the subscript $[k]$ now means to take the term of t-degree k. In the end we are interested in the coefficient of the term $t^M\epsilon^0$.

Expanding the factors of (43) gives the following contributions, all of which can be written down in polynomial time. First of all, the rational terms $\langle \ell + \epsilon\ell', w_i\rangle^{-1}$ give the following contribution. If $\langle \ell, w_i\rangle = 0$, we simply get

$$\frac{1}{\langle \ell + \epsilon\ell', w_i\rangle} = \frac{1}{\langle \ell', w_i\rangle}\epsilon^{-1}. \tag{44}$$

If $\langle \ell, w_i\rangle \neq 0$, we get the geometric series in ϵ,

$$\frac{1}{\langle \ell + \epsilon\ell', w_i\rangle} = \frac{1}{\langle \ell, w_i\rangle}\sum_{u=0}^{\infty}\left(-\frac{\langle \ell', w_i\rangle}{\langle \ell, w_i\rangle}\right)^u \epsilon^u.$$

The first and second terms in (43) are holomorphic; thus the only negative degrees in ϵ come from the rational terms (44). Let U be the number of vectors w_i that are orthogonal to ℓ; then ϵ^{-U} is the lowest negative degree. Note that $U \leq d$. Since we wish to find the term of ϵ-degree 0, we can truncate all series after ϵ-degree U:

$$\frac{1}{\langle \ell + \epsilon\ell', w_i\rangle} = \frac{1}{\langle \ell, w_i\rangle}\sum_{u=0}^{U}\left(-\frac{\langle \ell', w_i\rangle}{\langle \ell, w_i\rangle}\right)^u \epsilon^u + o_\epsilon\bigl(\epsilon^U\bigr). \tag{45}$$

We expand the first factor of (43) as follows:

$$\frac{\langle \ell + \epsilon\ell', s\rangle^m}{m!} = \sum_{u=0}^{\min\{m,U\}} \binom{m}{u} \langle \ell, s\rangle^{m-u} \langle \ell', s\rangle^u \epsilon^u + o_\epsilon\big(\epsilon^U\big). \tag{46}$$

Now we consider the holomorphic terms

$$\begin{aligned} &T\big(\tau_i(n), \big\langle t(\ell+\epsilon\ell'), w_i\big\rangle\big) \\ &\quad = -\sum_{j=0}^{\infty} \frac{1}{j!} B_j(\tau_i) \big\langle t(\ell+\epsilon\ell'), w_i\big\rangle^j \\ &\quad = -\sum_{j=0}^{k_0} \frac{1}{j!} B_j\bigg(\frac{1}{q_i} r_i\bigg) \Bigg(\sum_{u=0}^{\min\{j,U\}} \binom{j}{u} \langle \ell, w_i\rangle^{j-u} \langle \ell', w_i\rangle^u \epsilon^u \Bigg) t^j \\ &\qquad + o_t\big(t^{k_0}\big) + o_\epsilon\big(\epsilon^U\big). \end{aligned} \tag{47}$$

The Bernoulli polynomials $B_j(\tau_i)$ of degree $j \le k_0$ that appear in this formula can be efficiently expanded in polynomial time using recursion formulas. We remark that the variables r_i appear with a degree that is at most that of t. Using Lemma 41, we multiply the truncated series (47) for $i \in I^{\mathrm{c}}$ in $\mathbb{Q}[r_i : i \in I^{\mathrm{c}}][t][\epsilon]$, truncating in each step after t^{k_0} and ϵ^U. We thus obtain the second factor of (43),

$$\bigg(\prod_{i \in I^{\mathrm{c}}} T\big(\tau_i(n), \big\langle t(\ell+\epsilon\ell'), w_i\big\rangle\big) \bigg)_{[k]} \quad \text{for all } k = 0, \dots, k_0, \tag{48}$$

as a truncated series in $\mathbb{Q}[r_i : i \in I^{\mathrm{c}}][\epsilon]$.

Then we multiply the truncated series (44), (45), (48), and (46) in polynomial time, truncating in each step after ϵ^U, using Lemma 41. In the end, we read out the coefficient of ϵ^0 as a polynomial in $\mathbb{Q}[r_i : i \in I^{\mathrm{c}}]$. Then we substitute for r_i. Collecting these terms gives the formula for the Ehrhart coefficient $E_m(\mathfrak{p}, \xi, M; \{n\}_q)$. □

Example 42 Let us give a short example of the output of our algorithm for $E_m(\mathfrak{p}, \ell, M, \{n\}_q)$, when $\mathfrak{p}$ is the simplex in $\mathbb{R}^5$ with vertices:

$$(0,0,0,0,0),\ (\tfrac{1}{2},0,0,0,0),\ (0,\tfrac{1}{2},0,0,0),\ (0,0,\tfrac{1}{2},0,0),\ (0,0,0,\tfrac{1}{6},0),$$
$$(0,0,0,0,\tfrac{1}{6}).$$

We consider the linear form ℓ on $\mathbb{R}^5$ given by the scalar product with $(1,1,1,1,1)$.

If $M = 0$, the coefficients of $E_m(\mathfrak{p}, \ell, M = 0; \{n\}_q)$ are just the coefficients of the unweighted Ehrhart quasi-polynomial $S(n\mathfrak{p}, 1)$. We obtain

$$\begin{aligned} S(n\mathfrak{p}, 1) = {}& \frac{1}{34560} n^5 + \bigg(\frac{5}{3456} - \frac{1}{6912} \{n\}_2 \bigg) n^4 \\ & + \bigg(\frac{139}{5184} - \frac{5}{864} \{n\}_2 + \frac{1}{3456} \big(\{n\}_2\big)^2 \bigg) n^3 + \cdots. \end{aligned}$$

Now if $M = 1$, all integral points $(x_1, x_2, x_3, x_4, x_5)$ are weighted with the function $h(x) = x_1 + x_2 + x_3 + x_4 + x_5$, and we obtain

$$S(n\mathfrak{p}, h) = \frac{11}{1244160} n^6 + \Big(\frac{19}{41472} - \frac{11}{207360}\{n\}_2\Big) n^5 + \Big(\frac{553}{62208} - \frac{95}{41472}\{n\}_2 + \frac{11}{82944}(\{n\}_2)^2\Big) n^4 + \cdots.$$

We remark that although $q = 6$ is the smallest integer such that $q\mathfrak{p}$ is a lattice polytope, only periodic functions of n mod 2 enter in the top three Ehrhart coefficients. This indeed conforms to the known periodicity properties of the Ehrhart coefficients.

As a corollary, simply by evaluating the step polynomials, we obtain the following result, which directly extends the complexity result from Barvinok's paper to the weighted case.

Theorem 43 (Evaluation of the Ehrhart coefficients for a given dilation class $\{n\}_q$) *For every fixed number* $k_0 \in \mathbb{N}$, *there exists a polynomial-time algorithm for the following problem.*
Input:

(I_1) *a number $d \in \mathbb{N}$ in unary encoding, with $d \geq k_0$,*
(I_2) *a finite index set $\mathcal{V}$,*
(I_3) *a simple polytope $\mathfrak{p}$, given by its vertices, rational vectors $s_j \in \mathbb{Q}^d$ for $j \in \mathcal{V}$ in binary encoding,*
(I_4) *a rational vector $\ell \in \mathbb{Q}^d$ in binary encoding,*
(I_5) *a number $M \in \mathbb{N}$ in unary encoding,*
(I_6) *a number n in binary encoding.*

Output, in binary encoding,

(O_1) *a positive integer $q \in \mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and*
(O_2) *the numbers $E_m(\mathfrak{p}, \ell, M; \{n\}_q)$ for $m = M + d - k_0, \dots, M + d$.*

Remark 44 A direct algorithm for computing $E_m(\mathfrak{p}, \ell, M; \{n\}_q)$ for just one dilation class $\{n\}_q$ could, of course, use the values $r_i = \{\zeta_i n\}_{q_i} \in \mathbb{Z}$ rather than symbolic variables r_i and would therefore only need to do calculations with truncated series in the two-variable ring $\mathbb{Q}[t^{\pm 1}, \epsilon^{\pm 1}]$.

Via the decomposition of polynomials into powers of linear forms, which is, as discussed in [6], polynomial-time under suitable hypotheses, we obtain the following corollary.

Corollary 45 *For every fixed number* $k_0 \in \mathbb{N}$, *there exist polynomial-time algorithms for the following problems.*
Input:

(I_1) *a number $d \in \mathbb{N}$ in unary encoding, with $d \geq k_0$,*
(I_2) *a simple rational polytope $\mathfrak{p} \subset \mathbb{R}^d$, given by its vertices in binary encoding,*

Table 1 Computation times for Ehrhart polynomials of random lattice simplices. "Full" refers to the computation of the full Ehrhart polynomials using *LattE macchiato*. "Top 3" refers to the computation of the highest three Ehrhart coefficients using the new algorithm, setting $k_0 = 2$

Dimension	Average runtime (CPU seconds)			
	Full (*LattE macchiato*)			Top 3 (new code)
	Dual	Primal	Primal_{1000}	
3	0.16	0.10	0.04	1.12
4	28.00	4.68	0.28	4.31
5		317.5	5.8	13.4
6			198.0	37.4
7				103
8				294
9				393
10				1179
11				1681

(I_3) *a number M in unary encoding*,

(I_4) *a polynomial h of degree $\leq M$ which is given either as*

(a) *a power of a linear form*, *or*

(b) *a sparse polynomial where each monomial only depends on a fixed number of variables*, *or*

(c) *a sparse polynomial of fixed total degree*,

(I_5) *a number n in binary encoding.*

Output, *in binary encoding*,

(O_1) *a positive integer $q \in \mathbb{N}$ such that $q\mathfrak{p}$ is a lattice polytope and*

(O_2) *the numbers $E_m(\mathfrak{p}, h; \{n\}_q)$ for $m = M + d - k_0, \ldots, M + d$.*

7 Experiments

We implemented the algorithms in *Maple*, for the unweighted case and assuming the input to be lattice simplices of full dimension (in this case the quasi-polynomial becomes a polynomial). This assumption was made for simplicity of output in the calculation and because available software to verify the results (e.g., *LattE macchiato* [31]) cannot compute with weights. In addition, the problem of computing Ehrhart polynomials for lattice simplices has already received attention from many researchers, and it is non-trivial (see e.g., the references in [21]). After checking simple low-dimensional examples by hand, we set up automatic scripts for generating random tests. The simplices generated had vertex coordinates drawn uniformly at random from $\{-99, \ldots, 99\}$. We timed the speed of the procedure to compute the top three Ehrhart coefficients (that is, $k_0 = 2$) in 50 random simplices per dimension and recorded the average time of computation. We compared this with the computation of the *full* Ehrhart polynomials using the state-of-the-art algorithms implemented in *LattE macchiato* [31]; see Table 1.

In the table, *Dual* refers to an implementation of Barvinok's decomposition of the duals of the tangent cones into unimodular cones, as implemented first in *LattE* [24], and which is still the default method in *LattE macchiato*.[1] *Primal* refers to a primal variant of Barvinok's decomposition described in [30]; it is more efficient for these examples because the determinants of the dual cones are much larger.[2] Our implementation of the new algorithm in Maple also uses a primal variant of Barvinok's decomposition to unimodular cones, which was introduced in [20]. Thus the new code should be compared to the runtimes listed in column *Primal*. Finally, $Primal_{1000}$ refers to a variant in which Barvinok's decomposition is stopped when a cone has a determinant at most 1000; then the points in the fundamental parallelepipeds are enumerated.[3]

All computations were stopped if unfinished after 30 minutes. Thus, the table ends at dimension 11 because all randomly generated examples we tried in dimension 12 took more than 30 minutes of calculation. The computation times are given in CPU seconds on a computer with AMD Opteron 880 processors running at 2.4 GHz.

In conclusion, the experiments indicate that the algorithms presented here can dramatically improve the computation of full Ehrhart polynomials. The fact that, for very low dimensions, the implementation is slower than *LattE macchiato* is explained by the choice of *Maple* as an implementation language. *Maple* is an interpreted system, which is much slower than C++, the implementation language of *LattE macchiato*. We expect that the speedups of $Primal_{1000}$ compared to *Primal*, which were first documented in [30], will also be obtained in a refined implementation of our new algorithms.

The implementation is available at [3].

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[1] The LattE macchiato command is `count --ehrhart-polynomial`.

[2] The command is `count --ehrhart-polynomial --irrational-primal`.

[3] The command used is `count --ehrhart-polynomial --irrational-primal --maxdet=1000`.

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