Quantifying Transversality by Measuring the Robustness of Intersections

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Abstract By definition, transverse intersections are stable under infinitesimal per-
turbations. Using persistent homology, we extend this notion to a measure. Given a
space of perturbations, we assign to each homology class of the intersection its ro-
bustness, the magnitude of a perturbation in this space necessary to kill it, and then
we prove that the robustness is stable. Among the applications of this result is a sta-
ble notion of robustness for fixed points of continuous mappings and a statement of
stability for contours of smooth mappings.

Keywords Smooth mappings · Transversality · Fixed points · Contours ·
Homology · Filtrations · Zigzag modules · Persistence · Perturbations · Stability

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1 Introduction

The main new concept in this paper is a quantification of the classically differential notion of transversality. This is achieved by extending persistence from filtrations of homology groups to zigzag modules of well groups.

Motivation In hindsight, we place the starting point for the work described in this paper at the difference between qualitative and quantitative statements and their relevance in the sciences; see, e.g., the discussion in Thom's book [12, Chaps. 1.3 and 13.8]. It appears the conscious mind thinks in qualitative terms, delegating the quantitative details to the unconscious, if possible. In the sciences, quantitative statements are a requirement for testing a hypothesis. Without such a test, the hypothesis is not falsifiable and, by popular philosophical interpretation, not scientific [10]. The particular field discussed in [12] is the mathematical study of singularities of smooth mappings, which is dominated by qualitative statements. We refer to the seminal papers by Whitney [14, 15] and the book by Arnold [1] for introductions. A unifying concept in this field is the transversality of an intersection between two spaces. Its origins go far back in history and appear among others in the work of Poincaré about a century ago. It took a good development toward its present form under Pontryagin, Whitney, and Thom; see, e.g., [11]. In his review of Zeeman's book [16], Smale criticizes the unscientific aspects of the work promoted in the then popular area of catastrophe theory, thus significantly contributing to the discussion of qualitative versus quantitative statements and to the fate of that field. At the same time, Smale points to positive aspects and stresses the importance of the concept of transversality in the study of singularities. In a nutshell, an intersection is transverse if it forms a non-zero angle and is therefore stable under infinitesimal perturbations; see Sect. 2 for a formal definition.

Results We view our work as a measure-theoretic extension of the essentially differential concept of transversality. We extend by relaxing the requirements on the perturbations from smooth mappings between manifolds to continuous mappings between topological spaces. At the same time, we are more tolerant to changes in the intersection. To rationalize this tolerance, we measure intersections using real numbers as opposed to 0 and 1 to indicate existence. The measurements are made using the concept of persistent homology; see [5] for the original paper. However, we have need for modifications and use the extension of persistence from filtrations to zigzag modules as proposed in [2]. An important property of persistence, as originally defined for filtrations, is the stability of its diagrams; see [3] for the original proof. There is no comparably general result known for zigzag modules. Our main result is a step in this direction. Specifically, we view the following as the main contributions of this paper:

1. The introduction of well groups that capture the tolerance of intersections to perturbations in a given space of allowable perturbations.
2. The proof that the diagram defined by the well groups is stable.
3. The application of these results to fixed points and periodic orbits of continuous mappings.

In addition, our results have ramifications in the study of the set of critical values, the apparent contour of a smooth mapping. Specifically, the stability of the diagrams mentioned above results in a stability result for the apparent contour of a smooth mapping from an orientable 2-manifold to the plane [6]. The need for these stable diagrams was indeed what triggered the development described in this paper.

Outline   Section 2 provides the relevant background. Section 3 explains how we measure robustness using well groups and zigzag modules. Section 4 proves our main result, the stability of the diagrams defined by the modules. Section 5 discusses applications. Section 6 concludes the paper.

2 Background

We need the algebraic concept of persistent homology to extend the differential notion of transversality, as explained in the introduction. In this section, we give a formal definition of transversality, referring to [8] for a general background in differential topology. We also introduce homology and persistent homology, referring to [9] for a general background in classic algebraic topology and to [4] for a text in computational topology.

Transversality   Let $\mathbb{X}, \mathbb{Y}$ be manifolds, $f : \mathbb{X} \to \mathbb{Y}$ a smooth mapping, and $\mathbb{A} \subseteq \mathbb{Y}$ a smoothly embedded submanifold of the range. We assume the manifolds have finite dimension and no boundary, writing $m = \dim \mathbb{X}$, $n = \dim \mathbb{Y}$, and $k = \dim \mathbb{A}$. Given a point $x \in \mathbb{X}$ and a smooth curve $\gamma : \mathbb{R} \to \mathbb{X}$ with $\gamma(0) = x$, we call $\dot{\gamma}(0)$ the *tangent vector* of γ at x. Varying the curve while maintaining that it passes through x, we get a set of tangent vectors called the *tangent space* of $\mathbb{X}$ at x, denoted as $T_x \mathbb{X}$. Composing the curves with the mapping, $f \circ \gamma : \mathbb{R} \to \mathbb{Y}$, we get a subset of all smooth curves passing through $y = f(x) = f \circ \gamma(0)$. The *derivative* of f at x is $Df(x) : T_x \mathbb{X} \to T_y \mathbb{Y}$ defined by mapping the tangent vector of γ at x to the tangent vector of $f \circ \gamma$ at y. The derivative is a linear map and its image is a subspace of $T_y \mathbb{Y}$. The dimensions of the tangent spaces are $m = \dim T_x \mathbb{X}$ and $n = \dim T_y \mathbb{Y}$, which implies that the dimension of the image of the derivative is $\dim Df(x)(T_x \mathbb{X}) \leq \min\{m, n\}$.

We are interested in properties of f that are stable under perturbations. We call a property *infinitesimally stable* if for every smooth homotopy, $F : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ with $f_0 = f$, there is a real number $\delta > 0$ such that f_t possesses the same property for all $t < \delta$, where $f_t(x) = F(x, t)$ for all $x \in \mathbb{X}$. An important example of such a property is the following. The mapping f is *transverse* to $\mathbb{A}$, denoted as $f \pitchfork \mathbb{A}$, if for each $x \in \mathbb{X}$ with $f(x) \in \mathbb{A}$, the image of the derivative of f at x together with the tangent space of $\mathbb{A}$ at $a = f(x)$ spans the tangent space of $\mathbb{Y}$ at a. More formally, $f \pitchfork \mathbb{A}$ if $Df(x)(T_x \mathbb{X}) + T_a \mathbb{A} = T_a \mathbb{Y}$. It is plausible but also true that transversality is an infinitesimally stable property.

Product Spaces   It is convenient to recast transversality in terms of intersections of subspaces of $\mathbb{X} \times \mathbb{Y}$, and a manifold of dimension $m + n$. Consider the graphs of f
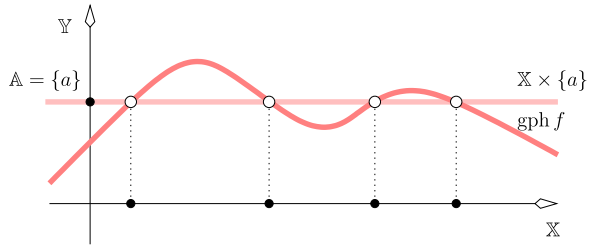
and of its restriction to the preimage of $\mathbb{A}$,

$$\mathrm{gph}\, f = \big\{ (x, y) \in \mathbb{X} \times \mathbb{Y} \mid y = f(x) \big\};$$

$$\mathrm{gph}\, f|_{\mathbb{A}} = \big\{ (x, a) \in \mathbb{X} \times \mathbb{A} \mid a = f(x) \big\}.$$

The intersection of interest is between $\mathrm{gph}\, f$ and $\mathbb{X} \times \mathbb{A}$, two manifolds of dimensions m and $m + k$ embedded in $\mathbb{X} \times \mathbb{Y}$. This intersection is the graph of $f|_{\mathbb{A}}$, which is homeomorphic to the preimage of $\mathbb{A}$. See Fig. 1 for an example in which $m = n = 1$ and $k = 0$. Here, $\mathrm{T}_a \mathbb{A} = 0$ and transversality requires that whenever the curve, $\mathrm{gph}\, f$, intersects the line, $\mathbb{X} \times \mathbb{A}$, it crosses at a non-zero angle. This is the case in Fig. 1, which implies that having a cardinality four preimage of a is an infinitesimally stable property of f. Nevertheless, the left two intersection points are clearly more stable than the right two intersection points, but we will need some algebra to give precise meaning to this statement.

Homology   The algebraic language of homology is a means to define and count holes in a topological space. It is a functor that maps a space to an Abelian group and a continuous map between spaces to a homomorphism between the corresponding groups. There is such a functor for each dimension, p. It is convenient to combine the homology groups of all dimensions into a single algebraic structure. Writing $\mathsf{H}_p(\mathbb{X})$ for the p-dimensional homology group of the topological space $\mathbb{X}$, we form a graded group by taking direct sums,

$$\mathsf{H}(\mathbb{X}) = \bigoplus_{p \geq 0} \mathsf{H}_p(\mathbb{X}).$$

To simplify language and notation, we will suppress dimensions and refer to $\mathsf{H}(\mathbb{X})$ as the *homology group* of $\mathbb{X}$. Its elements are formally written as polynomials, $\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots$, where α_p is a p-dimensional homology class and only finitely many of the classes are non-zero. As usual, adding two polynomials is done componentwise. The groups $\mathsf{H}_p(\mathbb{X})$ depend on a choice of a coefficient group. The theory of persistence introduced below requires that we use field coefficients. An example is modulo 2 arithmetic in which the field is $\mathbb{Z}_2 = \{0, 1\}$. The p-dimensional group is then a vector space, $\mathsf{H}_p(\mathbb{X}) \simeq \mathbb{Z}_2^{\beta_p}$, and its rank, the dimension of the vector space, is the *p-th Betti number*, $\beta_p = \beta_p(\mathbb{X})$. Similarly, $\mathsf{H}(\mathbb{X})$ is a vector space of dimension $\sum_{p \geq 0} \beta_p$. We say that $\mathbb{X}$ and $\mathbb{Y}$ *have the same homology* if there is an isomorphism between $\mathsf{H}(\mathbb{X})$ and $\mathsf{H}(\mathbb{Y})$ whose restrictions to the components are isomorphisms. Equivalently, $\beta_p(\mathbb{X}) = \beta_p(\mathbb{Y})$ for all non-negative integers p.

Fig. 2 The three off-diagonal points represent the births and deaths of three generators. The number of points in the *shaded upper left quadrant* equals the rank of the corresponding homology group
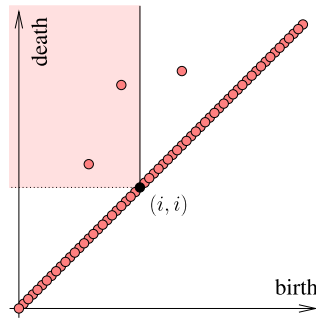
Persistent homology   Now suppose we have a finite sequence of nested spaces, $\mathbb{X}_1 \subseteq \mathbb{X}_2 \subseteq \cdots \subseteq \mathbb{X}_\ell$. Writing $\Phi_i = \mathsf{H}(\mathbb{X}_i)$ for the homology group of the i-th space, we get a sequence of vector spaces connected from left to right by homomorphisms induced by inclusion:

$$\Phi : \Phi_1 \to \Phi_2 \to \cdots \to \Phi_\ell.$$

We call this sequence a *filtration*. To study the evolution of the homology classes as we progress from left to right in the filtration, we let $\varphi_{i,j}$ be the composition of the maps between Φ_i and Φ_j, for $i \le j$. We say that a class $\alpha \in \Phi_i$ is *born* at Φ_i if it does not belong to the image of $\varphi_{i-1,i}$. Furthermore, this class α *dies entering* Φ_j if $\varphi_{i,j-1}(\alpha)$ does not belong to the image of $\varphi_{i-1,j-1}$ but $\varphi_{i,j}(\alpha)$ does belong to the image of $\varphi_{i-1,j}$. We call the images of the maps $\varphi_{i,j}$ the *persistent homology groups* of the filtration and record the evolution of the homology classes in the *persistence diagram* of the filtration, denoted as $\mathrm{Dgm}(\Phi)$. This is a multiset of points in the extended plane, $\bar{\mathbb{R}}^2 = (\mathbb{R} \cup \{-\infty, \infty\})^2$. Marking an increase in rank on the horizontal, birth axis and a drop in rank on the vertical, death axis, each point represents the birth and the death of a generator and records where these events happen; see Fig. 2. For technical reasons that will become clear shortly, we add infinitely many copies of each point on the diagonal to the diagram. Given an index, i, we can read off the rank of $\mathsf{H}(\mathbb{X}_i)$ by counting the points in the half-open upper left quadrant, $[-\infty, i] \times (i, \infty]$, anchored at the point (i, i) on the diagonal. More generally, the rank of the image of $\varphi_{i,j}$ equals the number of points in the upper left quadrant anchored at (i, j).

Stability   Consider now the case in which the spaces in the sequence are sublevel sets of a real-valued function $\varphi : \mathbb{X} \to \mathbb{R}$, that is, there are values r_i such that $\mathbb{X}_i = \varphi^{-1}(\infty, r_i]$ for each i. A *homological critical value* of φ is a value r such that for every sufficiently small $\delta > 0$, the homomorphism from $\mathsf{H}(\varphi^{-1}(-\infty, r - \delta])$ to $\mathsf{H}(\varphi^{-1}(-\infty, r + \delta])$ induced by inclusion is not an isomorphism. We assume that φ is *tame,* by which we mean that each sublevel set has finite rank homology and there are only finitely many homological critical values, denoted as $r_1 < r_2 < \cdots < r_\ell$. We can represent the evolution of the homology classes by the finite filtration consisting of the groups $\Phi_i = \mathsf{H}(\varphi^{-1}(-\infty, r_i])$, for $1 \le i \le \ell$, and by the persistence diagram of that filtration, $D = \mathrm{Dgm}(\Phi)$. Letting $\psi : \mathbb{X} \to \mathbb{R}$ be another tame function, we get another filtration, Ψ, and another persistence diagram, $E = \mathrm{Dgm}(\Psi)$. The *bottleneck distance* between the two is the infimum, over all bijections, $\mu : D \to E$, of the

longest length edge in the matching,

$$W_\infty(D, E) = \inf_\mu \sup_{a \in D} \left\| a - \mu(a) \right\|_\infty.$$

An important result is the stability of the persistence diagram under perturbations of the function.

Stability Theorem for Tame Functions [3] *Let φ and ψ be tame, real-valued functions on $\mathbb{X}$. Then the bottleneck distance between their persistence diagrams is bounded from above by $\|\varphi - \psi\|_\infty$.*

Here, $\|\varphi - \psi\|_\infty = \sup_{x \in \mathbb{X}} |\varphi(x) - \psi(x)|$, as usual. The original form of this result is slightly stronger, as it restricts itself to dimension-preserving bijections. The theorem implies that the bottleneck distance between the diagrams defined by φ and ψ goes to zero as the difference between the two functions approaches zero.

3 Measuring Robustness

The main new concept in this section is the well group defined by a mapping $f : \mathbb{X} \to \mathbb{Y}$, a subspace $\mathbb{A} \subseteq \mathbb{Y}$, and a parameter r. It encodes the persistent homology of the preimage of the subspace.

Admissible Mappings   In this paper, we limit the class of mappings to those with manageable properties. While our goal is a statement of our results in a context that is sufficiently broad to support interesting applications, we are aware of the technical burden that comes with generality. We hope that the following class of mappings provides a "happy medium" between the conflicting goals of generality and transparency.

Definition Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces and $\mathbb{A}$ a subspace of $\mathbb{Y}$. A continuous mapping $f : \mathbb{X} \to \mathbb{Y}$ is *admissible* if $f^{-1}(\mathbb{A})$ has a finite rank homology group.

Requiring that the preimage of $\mathbb{A}$ has finite rank homology is strictly weaker than demanding tameness of the well function defined next.

Perturbations and Well Groups   We are interested in how we explore the neighborhood of $f^{-1}(\mathbb{A})$ as we perturb f. For this purpose, we introduce a subspace $\mathcal{P} = \mathcal{P}(f)$ of $C(\mathbb{X}, \mathbb{Y})$, the space of continuous mappings from $\mathbb{X}$ to $\mathbb{Y}$. For example, $\mathcal{P}$ may be the space of continuous mappings h homotopic to f; that is, there exists a continuous mapping $H : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ with $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in \mathbb{X}$. We assume a metric on $\mathcal{P}$, writing $\|f - h\|_\mathcal{P}$ for the distance between two mappings $f, h \in \mathcal{P}$. For example, we could construct one by assuming a metric on $\mathbb{Y}$, lifting it to define the distance between mappings in $C(\mathbb{X}, \mathbb{Y})$, and taking $\|f - h\|_\mathcal{P}$ to be the infimum length of all curves of mappings that connect f and h within $\mathcal{P}$. We call h an *r-perturbation* of f if $\|f - h\|_\mathcal{P} \le r$.

Table 1 The ranks of the homology and well groups defined for the mapping f and the submanifold $\mathbb{A} = \{a\}$ in Fig. 1

	$(0, r_3)$	(r_3, r_2)	(r_2, r_1)	(r_1, ∞)
$\mathsf{F}(r)$	4	3	2	1
$\mathsf{U}(r)$	4	2	2	0

We use these definitions to introduce the *well function* $f_{\mathbb{A}} : \mathbb{X} \to \mathbb{R}$ of f and $\mathbb{A}$ by setting $f_{\mathbb{A}}(x)$ to the infimum value of r for which there is an r-perturbation $h \in \mathcal{P}$ such that $h(x) \in \mathbb{A}$. The *level set* of $f_{\mathbb{A}}$ at a value r is the preimage of that value: $f_{\mathbb{A}}^{-1}(r)$. The *sublevel set* for the same value r is the union of the level sets at values at most r: $\mathbb{X}_r = f_{\mathbb{A}}^{-1}[0, r]$. Note that $h_{\mathbb{A}}^{-1}(0) = h^{-1}(\mathbb{A})$ is contained in $\mathbb{X}_r$ for every r-perturbation $h \in \mathcal{P}$. This inclusion induces a homomorphism between the corresponding homology groups:

$$j_h : \mathsf{H}\big(h_{\mathbb{A}}^{-1}(0)\big) \to \mathsf{F}(r),$$

where we simplify notation by writing $\mathsf{F}(r)$ for $\mathsf{H}(f_{\mathbb{A}}^{-1}[0, r])$. The image of this map, denoted by $\mathrm{im}\, j_h$, is a subgroup of $\mathsf{F}(r)$. The intersection of subgroups is again a subgroup.

Definition Given a metric space of perturbations, $\mathcal{P} = \mathcal{P}(f)$, the *well group* of $\mathbb{X}_r$ is the subgroup $\mathsf{U}(r) \subseteq \mathsf{F}(r)$ obtained by intersecting the images, $\mathrm{im}\, j_h$, over all r-perturbations h of f in $\mathcal{P}$.

A different space of perturbations gives different well groups and therefore a different interpretation of their meaning.

Example To illustrate the definitions, let us consider again the example in Fig. 1 of the mapping from the real line to itself. The preimage of $\mathbb{A} = \{a\}$ is a set of four points separated by three critical points of f. From left to right, the values of f at these critical points are $a + r_1$, $a - r_2$, $a + r_3$. Correspondingly, the well function, $f_a : \mathbb{R} \to \mathbb{R}$, has three homological critical values, namely $r_1 > r_2 > r_3$. Setting $\mathcal{P} = \mathcal{P}(f)$ to the set of all continuous mappings $h : \mathbb{R} \to \mathbb{R}$ and measuring distance by $\|f - h\|_{\mathcal{P}} = \sup_x |f(x) - h(x)|$, we have a well group $\mathsf{U}(r)$ for each radius $r \geq 0$. Table 1 shows the ranks of $\mathsf{F}(r)$ and $\mathsf{U}(r)$ for values of r in the interior of the four intervals delimited by the homological critical values. Starting with $r = 0$, we have four points, each forming a component represented by a class in the homology group and in the well group of the sublevel set of f_a. Therefore, both groups are the same and have rank 4; see the first column in Table 1. Growing r turns the points into intervals, but leaves the groups the same until r passes r_3, the smallest of the three homological critical values. The two right intervals merge into one, so the rank of the homology group drops to 3. We can find an r-perturbation, $r > r_3$, whose level set at a consists of the left two points of $f^{-1}(a)$, but the right two points have disappeared. Indeed, the level set of every r-perturbation, for $r > r_3$, has a non-empty intersection with the first two, but can have empty intersection with the merged interval on the right. Hence, the left two intervals have a representation in the well group, the merged

interval does not, and the rank of the well group is 2; see the second column in
Table 1. The next change happens when r passes r_2. The middle interval merges with
the merged interval on the right. The rank of the homology group drops to 2, while
the rank of the well group remains unchanged at 2; see the third column in Table 1.
Finally, when r passes r_1, the remaining two intervals merge into one, so the rank of
the homology group drops to 1. We can find an r-perturbation, $r > r_1$, whose level set
at a is empty, so the rank of the well group drops to 0; see the last column in Table 1.

Terminal Critical Values   Recall that we assume that the mapping $f : \mathbb{X} \to \mathbb{Y}$ is ad-
missible. The initial homology group, $\mathsf{F}(0) = \mathsf{H}(f_{\mathbb{A}}^{-1}(0))$, therefore has finite rank,
and because $\mathsf{U}(0) \subseteq \mathsf{F}(0)$, the initial well group has finite rank. For convenience, we
permit negative parameter values by stipulating $\mathsf{F}(r) = \mathsf{F}(0)$ and $\mathsf{U}(r) = \mathsf{U}(0)$ when-
ever $r \leq 0$. Imagine that we grow the sublevel set by gradually increasing r to infinity.
Since the admissibility of f does not imply the tameness of the well function, this
leaves open the possibility that $f_{\mathbb{A}}$ has infinitely many homological critical values.
We call a radius, r, a *terminal critical value* of $f_{\mathbb{A}}$ if for every sufficiently small
$\delta > 0$ the homomorphism from $\mathsf{F}(r - \delta)$ to $\mathsf{F}(r + \delta)$ applied to $\mathsf{U}(r - \delta)$ does not
give $\mathsf{U}(r + \delta)$. In contrast to the homological critical values, there can only be a finite
number of terminal critical values. To see this, we note that the set of perturbations
that define the well groups grows with increasing r. It follows that the well groups
cannot increase in rank. To state this relationship between well groups more formally,
we write $\mathsf{f}(r, s) : \mathsf{F}(r) \to \mathsf{F}(s)$ for the homomorphism induced by inclusion.

Shrinking Wellness Lemma *For each choice of radii $r \leq s$, the image of the well
group at r contains the well group at s, that is, $\mathsf{U}(s) \subseteq \mathsf{f}(r, s)(\mathsf{U}(r))$.*

The only way the well group can change is by lowering its rank. Since we start with a
finite rank, there can only be finitely many terminal critical values, which we denote
as $u_1 < u_2 < \cdots < u_l$. To this sequence, we add $u_0 = -\infty$ on the left and $u_{l+1} = \infty$
on the right. We choose an interleaved sequence

$$u_0 < r_0 < u_1 < \cdots < u_l < r_l < u_{l+1}$$

and index the homology and the well groups accordingly, writing $\mathsf{F}_i = \mathsf{F}(r_i)$ and
$\mathsf{U}_i = \mathsf{U}(r_i)$, for all i. To these sequences, we add $\mathsf{F}_{-1} = \mathsf{U}_{-1} = 0$ on the left and
$\mathsf{F}_{l+1} = \mathsf{U}_{l+1} = 0$ on the right. Furthermore, we write $\mathsf{f}_{i,j} : \mathsf{F}_i \to \mathsf{F}_j$ instead of $\mathsf{f}(r_i, r_j)$
for all feasible choices of $i \leq j$.

Well Module   In contrast to the homology groups, the well groups of the sublevel
sets do not form a filtration. Instead, they form a special kind of zigzag module. By the
definition of terminal critical values, the rank of U_i exceeds the rank of U_{i+1}. The rank
of the image, $\mathsf{f}_{i,i+1}(\mathsf{U}_i)$, is somewhere between these two ranks. We call a difference
between U_i and its image a *conventional death*, in which a class maps to zero, and a
difference between the image and U_{i+1} an *unconventional death*, in which the image
of a class lies outside the next well group. We capture both cases by inserting a new
group between the contiguous well groups; see Fig. 3. To this end, we consider the
restriction of $f_{i,i+1}$ to U_i and in particular its kernel, $\mathsf{K}_i = \mathsf{U}_i \cap \ker f_{i,i+1}$, which we

Fig. 3 Connecting two consecutive well groups to the quotient group introduced between them. The class α dies a conventional death, and the class β dies an unconventional death
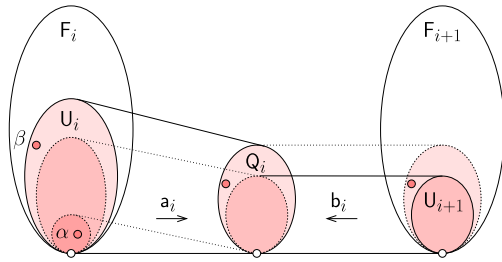
refer to as the *vanishing subgroup* of U_i. Using this subgroup, we construct $Q_i = U_i/K_i$. The forward map, $a_i : U_i \to Q_i$, is defined by mapping a class ξ to $\xi + K_i$. It is clearly surjective. The backward map, $b_i : U_{i+1} \to Q_i$, is defined by mapping a class η to $\xi + K_i$, where ξ belongs to $f_{i,i+1}^{-1}(\eta)$. This map is clearly injective. Instead of a filtration, in which all maps go from left to right, we get a sequence, in which the maps alternate between going forward and backward. As indicated below, every other group in the sequence is a subgroup of the corresponding homology group:

$$
\begin{array}{ccccccccc}
Q_{i-1} & \xleftarrow{b_{i-1}} & U_i & \xrightarrow{a_i} & Q_i & \xleftarrow{b_i} & U_{i+1} & \xrightarrow{a_{i+1}} & Q_{i+1} \\
 & & \downarrow & & & & \downarrow & & \\
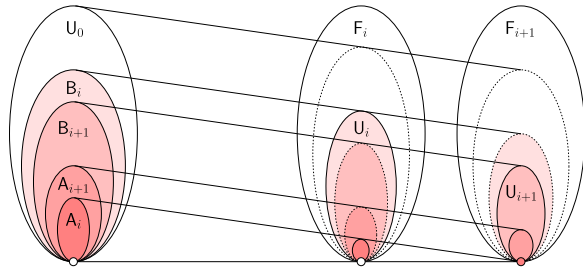\to & & F_i & & \to & & F_{i+1} & & \to .
\end{array}
$$

We call this sequence the *well module* of $f_{\mathbb{A}}$, denoted as U. We remark that U is a special case of a zigzag module as introduced in [2]. It is special because all forward maps are surjective and all backward maps are injective. Equivalently, there are no births other than at the start, when we go from U_{-1} to U_0.

Left Filtration   Perhaps surprisingly, the evolution of the homology classes can still be fully described by pairing births with deaths, just as for a filtration. To shed light on this construction, we follow [2] and turn a zigzag module into a filtration. In our case, all births happen at U_0, so this transformation is easier than for general zigzag modules. Write $u_{0,i} : U_0 \to F_i$ for the restriction of $f_{0,i}$ to the initial well group. By the Shrinking Wellness Lemma, the image of this map contains the i-th well group, that is, $U_i \subseteq u_{0,i}(U_0)$. We consider the preimages of the well groups in U_0 together with the preimages of their vanishing subgroups, $A_i = u_{0,i}^{-1}(K_i)$ and $B_i = u_{0,i}^{-1}(U_i)$; see Fig. 4. We note that $A_i/A_{i-1} \simeq \ker a_i$ and $B_i/B_{i+1} \simeq \operatorname{cok} b_i$. In words, the first quotient represents the homology classes that die a conventional death, going from U_i to U_{i+1}, and the second quotient represents the homology classes that die an unconventional death in the same transition. As illustrated in Fig. 4, the preimages form a nested sequence of subgroups of U_0. Together with the inclusion maps, this gives the *left filtration* of the zigzag module,

$$
0 \to A_0 \to \cdots \to A_l = B_l \to \cdots \to B_0 = U_0.
$$

We can recover the well groups with $U_i \simeq B_i/A_{i-1}$. Recall that $U_{l+1} = 0$, which implies $K_l = U_l$. It follows that the middle two groups in the left filtration, A_l and B_l, are indeed equal.

Compatible Bases A useful property of the left filtration is the existence of compatible bases of all its groups. By this we mean a basis of U_0 that contains a basis for each A_i and each B_i. Specifically, we rewrite U_0 as a direct sum of kernels of forward maps and cokernels of backward maps:

$$U_0 \simeq \ker a_0 \oplus \cdots \oplus \ker a_l \oplus \operatorname{cok} b_{l-1} \oplus \cdots \oplus \operatorname{cok} b_0.$$

Reading this decomposition from left to right, we encounter the A_i and the B_i in the sequence in which they occur in the left filtration. Choosing a basis for each kernel and each cokernel, we thus get compatible bases for all groups in the left filtration. We call this the *left filtration basis* of U_0. It is unique up to choosing bases for the kernels and cokernels.

Consider now a homology class α in U_0 and its representation as a sum of basis vectors. We write $\alpha(a_i)$ for the projection of α to the preimage of the kernel of the i-th forward map, which is obtained by removing all vectors that do not belong to the basis of $\ker a_i$. Similarly, we write $\alpha(b_i)$ for the projection of α to the preimage of $\operatorname{cok} b_i$. Letting j be the minimum index such that $\alpha(a_i) = \alpha(b_i) = 0$ for all $i \geq j$, we say that α *falls ill* at U_j.

Well Diagrams Constructing the birth-death pairs that describe the well module is now easy. All classes are born at U_0; however, to distinguish the changes in the well group from those in the homology group, we say that all the classes *get well* at U_0. They fall ill later, and once they fall ill, they do not get well any more. The drop in rank from U_{i-1} to U_i is $\mu_i = \operatorname{rank}(\ker a_{i-1}) + \operatorname{rank}(\operatorname{cok} b_{i-1})$. We thus have μ_i copies of the point $(0, u_i)$ in the diagram. There is no information in the first coordinates, which are all zero. We thus define the *well diagram* as the multiset of points u_i with multiplicities μ_i, denoting it as $\operatorname{Dgm}(U)$. For technical reasons that will become obvious in the next section, we add infinitely many copies of 0 to this diagram. Hence, each point in $\operatorname{Dgm}(U)$ is either 0, a positive real number, or ∞, and the diagram itself is a multiset of points on the extended line, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. It has infinitely many points at 0 and a finite number of non-zero points.

As suggested by the heading of this section, we think of each point in the diagram as a measure of how resilient a homology class of $f^{-1}(\mathbb{A})$ is against perturbations in $\mathcal{P}$. At each well group U_i, an entire set of homology classes falls ill, and we call u_i the *robustness* of each class α in this set, denoting it as $\varrho(\alpha) = u_i$.

4 Proving Stability

We are interested in relating the distance between mappings to the distance between their well diagrams, both defined using a common perturbation space $\mathcal{P}$. After quantifying these distances, we connect parallel well modules to form new modules, and we finally prove that the well diagram is stable.

Distance Between Functions   Let $f, g : \mathbb{X} \to \mathbb{Y}$ be two admissible mappings between topological spaces, $\mathbb{A} \subseteq \mathbb{Y}$ a subspace, and $\mathcal{P}$ a metric space of perturbations that contains both f and g. Using $\mathbb{A}$, we get the two well functions: $f_\mathbb{A}, g_\mathbb{A} : \mathbb{X} \to \mathbb{R}$. The distance between them is the supremum difference between corresponding values:

$$\|f_\mathbb{A} - g_\mathbb{A}\|_\infty = \sup_{x \in \mathbb{X}} \left| f_\mathbb{A}(x) - g_\mathbb{A}(x) \right|.$$

The distance between the two well functions is related to the distance, $\|f - g\|_\mathcal{P}$, between the two mappings f and g in $\mathcal{P}$. Specifically, the distance between the well functions cannot exceed the distance between the mappings.

Distance Lemma *Assuming the preceding notation, we have* $\|f_\mathbb{A} - g_\mathbb{A}\|_\infty \leq \|f - g\|_\mathcal{P}$.

Proof Fix a point $x \in \mathbb{X}$ and let $r = f_\mathbb{A}(x)$. By construction of $f_\mathbb{A}$, there exists an r-perturbation h of f in $\mathcal{P}$ with $x \in h^{-1}(\mathbb{A})$. By the triangle inequality, $\|g - h\|_\mathcal{P} \leq \|f - g\|_\mathcal{P} + r$. Hence, $g_\mathbb{A}(x) \leq r + \|f - g\|_\mathcal{P}$. Thus, $| f_\mathbb{A}(x) - g_\mathbb{A}(x) | \leq \|f - g\|_\mathcal{P}$. $\qquad\square$

Distance Between Diagrams   Let $\mathsf{G}(r)$ be the homology group and $\mathsf{V}(r) \subseteq \mathsf{G}(r)$ the well group of $g_\mathbb{A}^{-1}[0, r]$. As for f, we insert quotients between contiguous well groups and connect them with forward and backward maps to form a well module, denoted as V. The corresponding well diagram, $\mathrm{Dgm}(\mathsf{V})$, is again a multiset of points in $\bar{\mathbb{R}}$, consisting of infinitely many copies of 0 and finitely many non-zero points. Recall that the bottleneck distance between the diagrams of f and g is the length of the longest edge in the minimizing matching. Because our diagrams are one dimensional, the bottleneck distance is easy to compute. To describe the algorithm, we order the positive points in both diagrams, obtaining

$$0 \leq u_1 \leq u_2 \leq \cdots \leq u_M;$$
$$0 \leq v_1 \leq v_2 \leq \cdots \leq v_M,$$

where we add zeros to make sure we have two sequences of the same length. The *inversion-free matching* pairs u_i with v_i for all i. We prove that this matching gives the bottleneck distance.

Matching Lemma *Assuming the preceding notation, the bottleneck distance between* $\mathrm{Dgm}(\mathsf{U})$ *and* $\mathrm{Dgm}(\mathsf{V})$ *is equal to* $\max_{1 \leq i \leq M} |u_i - v_i|$.

Proof For a given matching, we consider the vector of absolute differences, which we sort largest first. Comparing two such vectors lexicographically, we now prove that the inversion-free matching gives the minimum vector. This implies the claimed equality,

$$W_\infty\big(\mathrm{Dgm}(\mathsf{U}), \mathrm{Dgm}(\mathsf{V})\big) = \max_{1 \le i \le M} |u_i - v_i|.$$

To prove minimality, we consider a matching that has at least one inversion, that is, pairs (u_i, v_t) and (u_j, v_s) with $i < j$ and $s < t$. If $u_i = u_j$ or $v_s = v_t$, then switching to the pairs (u_i, v_s) and (u_j, v_t) preserves the sorted vector of absolute differences. Otherwise, the new vector is lexicographically smaller than the old vector. Indeed, the minimum of the four points is u_i or v_s and the maximum is u_j or v_t. If the minimum and the maximum are from opposite diagrams, then they delimit the largest of the four absolute differences, and this largest difference belongs to the old vector. Otherwise, both absolute differences shrink when we switch the pairs. Repeatedly removing inversions as described eventually leads to the inversion-free matching, which shows that it minimizes the vector and its largest entry is the bottleneck distance. □

Bridges The main tool in the proof of stability is the concept of a short bridge between parallel filtrations. The length of these bridges relates to the distance between the functions defining the filtrations. Let $\varepsilon = \|f - g\|_\mathcal{P}$. By the Distance Lemma, we have $\|f_\mathbb{A} - g_\mathbb{A}\|_\infty \le \varepsilon$, which implies that the sublevel set of $g_\mathbb{A}$ for radius r is contained in the sublevel set of $f_\mathbb{A}$ for radius $r + \varepsilon$. Hence, there is a homomorphism $\mathcal{B}_r : \mathsf{G}(r) \to \mathsf{F}(r + \varepsilon)$, which we call the *bridge* from G to F at radius r. We use the bridge to connect the initial segment of G to the terminal segment of F. The endpoints of the bridge satisfy the property expressed in the Shrinking Wellness Lemma.

Bridge Lemma *Let $\mathcal{B}_r : \mathsf{G}(r) \to \mathsf{F}(r + \varepsilon)$ be the bridge at r, where $\varepsilon = \|f - g\|_\mathcal{P}$ and $f, g \in \mathcal{P}$. Then $\mathsf{U}(r + \varepsilon) \subseteq \mathcal{B}_r(\mathsf{V}(r))$.*

Proof Let α be a homology class in $\mathsf{U}(r + \varepsilon)$. By definition of the well group, α belongs to the image of $\mathsf{H}(h^{-1}(\mathbb{A}))$ in $\mathsf{F}(r + \varepsilon)$ for every $(r + \varepsilon)$-perturbation h of f in $\mathcal{P}$. This includes all r-perturbations of g. It follows that the preimage of α in $\mathsf{G}(r)$ belongs to the well group, that is, $\mathcal{B}_r^{-1}(\alpha) \in \mathsf{V}(r)$. □

Everything we said about bridges is of course symmetric in F and G. In other words, $f_\mathbb{A}^{-1}[0, r] \subseteq g_\mathbb{A}^{-1}[0, r + \varepsilon]$ and there is a bridge from $\mathsf{F}(r)$ to $\mathsf{G}(r + \varepsilon)$ for every $r \ge 0$.

New Modules We use the Bridge Lemma to construct new zigzag modules from the well modules of f and g. Specifically, we use $\mathcal{B}_r$ to connect the initial segment of V, from $\mathsf{V}(0)$ to $\mathsf{V}(r)$, to the terminal segment of U, from $\mathsf{U}(r + \varepsilon)$ to $\mathsf{U}(\infty)$. To complete the module, we insert $\mathsf{Q}(r) = \mathsf{V}(r)/(\mathsf{V}(r) \cap \ker \mathcal{B}_r)$ between $\mathsf{V}(r)$ and $\mathsf{U}(r + \varepsilon)$. The forward map, from $\mathsf{V}(r)$ to $\mathsf{Q}(r)$, is surjective, and the backward map, from $\mathsf{U}(r + \varepsilon)$ to $\mathsf{Q}(r)$, is injective; see Fig. 5. The new zigzag module is thus of the same type as the well modules, implying that it has a left filtration basis that gives rise to a family of compatible bases for the groups in the left filtration.

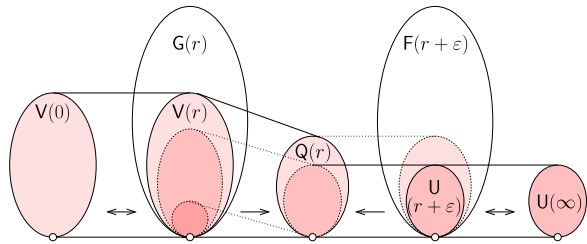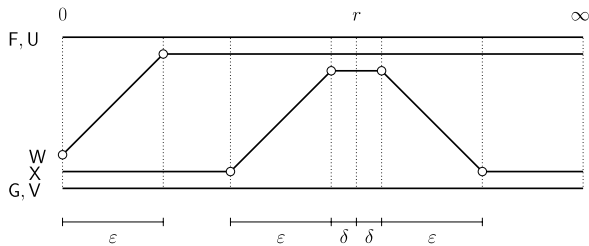**Fig. 5** The zigzag module obtained by connecting an initial segment of V to a terminal segment of U

Fig. 6 The four curves represent four filtrations as well as the four zigzag modules. The middle two are constructed from the outer two by adding bridges connecting the dots

A particular construction starts with the filtrations $\mathsf{F}(0) \to \cdots \to \mathsf{F}(\infty)$ and $\mathsf{G}(0) \to \cdots \to \mathsf{G}(\infty)$ and adds $\mathcal{B}_0 : \mathsf{G}(0) \to \mathsf{F}(\varepsilon)$. Following the bridge from G to F at 0, we get a new filtration and a new zigzag module, denoting the latter as W; see Fig. 6. The decomposition of $\mathsf{W}(0) = \mathsf{V}(0)$ by the left filtration of W is similar to the decomposition of $\mathsf{U}(0)$ by the left filtration of U; see Fig. 4. Letting i be the index such that $u_i \leq \varepsilon < u_{i+1}$, we have $\mathsf{U}(\varepsilon) = \mathsf{U}_i$. The classes in A_{i-1} and in $\mathsf{U}_0/\mathsf{B}_i$ die before we reach $\mathsf{F}(\varepsilon)$. The remaining classes form $\mathsf{U}(\varepsilon) \simeq \mathsf{B}_i/\mathsf{A}_{i-1}$. Correspondingly, there are homology classes in $\mathsf{W}(0)$ that die before we reach $\mathsf{F}(\varepsilon)$, namely the ones in the kernel of the forward map, from $\mathsf{W}(0)$ to $\mathsf{Q}(0)$, and in the preimage of the cokernel of the backward map, from $\mathsf{U}(\varepsilon)$ to $\mathsf{Q}(0)$. The remaining classes form $\mathsf{W}(\varepsilon) \simeq \mathcal{B}_0^{-1}(\mathsf{U}(\varepsilon))/(\mathsf{W}(0) \cap \ker \mathcal{B}_0)$. The two quotient groups, $\mathsf{U}(\varepsilon)$ and $\mathsf{W}(\varepsilon)$, are decomposed in parallel so that choosing a basis for $\mathsf{U}(\varepsilon)$ gives one for $\mathsf{W}(\varepsilon)$. This will be useful shortly.

Main Result   We are now ready to state and prove the stability of the well diagram.

Stability Theorem for Well Diagrams *Let f and g be admissible mappings between topological spaces $\mathbb{X}$ and $\mathbb{Y}$, let $\mathbb{A}$ be a subspace of $\mathbb{Y}$, and let $\mathcal{P}$ be a metric space of perturbations containing both f and g. Then $W_\infty(\mathrm{Dgm}(\mathsf{U}), \mathrm{Dgm}(\mathsf{V})) \leq \|f - g\|_{\mathcal{P}}$ for the well modules U and V of f and g.*

Proof We construct a bijection from $\mathrm{Dgm}(\mathsf{U})$ to $\mathrm{Dgm}(\mathsf{V})$ such that the difference between matched points is at most $\varepsilon = \|f - g\|_{\mathcal{P}}$. Specifically, we match each point $u \leq \varepsilon$ in $\mathrm{Dgm}(\mathsf{U})$ with a copy of 0 in $\mathrm{Dgm}(\mathsf{V})$, and we use the parallel bases of $\mathsf{U}(\varepsilon)$ and $\mathsf{W}(\varepsilon)$ for the rest, where W is the zigzag module obtained by adding the bridge from G to F at radius 0, as described above.

Let α belong to the left filtration basis of $\mathsf{U}(0)$ such that its image belongs to the basis of $\mathsf{U}(\varepsilon)$. Let r be the value at which α falls ill, and note that $r > \varepsilon$. By the Bridge

Lemma, there is a β in the left filtration basis of $V(0) = W(0)$ such that the images of α and β in $W(\varepsilon) = U(\varepsilon)$ coincide. We now construct yet another zigzag module, by adding a first bridge from $G(r - \varepsilon - \delta)$ to $F(r - \delta)$ and a second bridge from $F(r + \delta)$ back to $G(r + \varepsilon + \delta)$, where $\delta > 0$ is sufficiently small that there is no death in the interval $[r - \delta, r + \delta]$, except possibly at r. We denote the resulting module by X; see Fig. 6. We note that all maps between groups are induced by inclusions so that the diagram formed by the filtrations and the bridges between them commutes.

By construction, the image of β in $F(r - \delta)$ is non-zero and belongs to $U(r - \delta)$. By contrast, the image of β in $F(r + \delta)$ is either zero or lies outside $U(r + \delta)$. Applying the Bridge Lemma going backward along the first bridge, we note that the image of $\beta \in W(0) = X(0)$ in $G(r - \varepsilon - \delta)$ is non-zero and belongs to $V(r - \varepsilon - \delta)$. Applying the Bridge Lemma going forward along the second bridge, we note that the image of β in $G(r + \varepsilon + \delta)$ is either zero or lies outside $V(r + \varepsilon + \delta)$. Since we can choose $\delta > 0$ as small as we like, this implies that β falls ill somewhere in the interval $[r - \varepsilon, r + \varepsilon]$. In the matching, this radius is paired with r, the radius at which α falls ill in U. The absolute difference between the two radii is at most ε, as required.   □

5 Applications

In this section, we use the stability of the transversality measure to derive stability results for fixed points, periodic orbits, and apparent contours. All three problems can be recast in terms of intersections between topological spaces and are therefore amenable to the tools developed in this paper.

Fixed Points   A *fixed point* of a continuous mapping from a topological space to itself is a point that is its own image. Assuming that this space is the m-dimensional Euclidean space and b is the mapping, we introduce a mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ defined by $f(x) = x - b(x)$. A fixed point of b is a root of f, that is, $f(x) = 0$. Writing $\mathbb{X} = \mathbb{Y} = \mathbb{R}^m$ and $\mathbb{A} = \{0\}$, the origin of $\mathbb{R}^m$, we get the setting studied in this paper. Each fixed point x of b corresponds to a class in the 0-dimensional homology group of $f^{-1}(0)$. Letting $\varrho(x)$ be the maximum robustness of all the fixed points of f, every perturbation of magnitude less than $\varrho(x)$ has at least one root, and correspondingly, the perturbation of b has at least one fixed point. This implication suffices to give a new proof of a classic topological result on fixed points; see [9]. Let $\mathbb{B}^m$ be the closed unit ball in $\mathbb{R}^m$.

Brouwer's Fixed Point Theorem *Every continuous mapping* $b : \mathbb{B}^m \to \mathbb{B}^m$ *has a fixed point.*

Proof Extend b to a mapping from $\mathbb{R}^m$ to $\mathbb{R}^m$ by defining $b(x)$ equal to its value at $x/\|x\|_2$ whenever $x \notin \mathbb{B}^m$. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $f(x) = x - b(x)$ and let $g : \mathbb{R}^m \to \mathbb{R}^m$ be the identity, defined by $g(x) = x$. We may assume that f is admissible, else the homology group of $f^{-1}(0)$ has infinite rank and f has infinitely many roots. The other mapping, g, is clearly admissible, with a single root at $x = 0$. Letting $\mathcal{P}$ be the space of all continuous mappings from $\mathbb{R}^m$ to $\mathbb{R}^m$, and measuring the

distance between f and g by taking the supremum of the Euclidean distance between corresponding image points, we get

$$\|f - g\|_{\mathcal{P}} = \sup_{x \in \mathbb{R}^m} \|f(x) - g(x)\|_2 = \sup_{x \in \mathbb{R}^m} \|b(x)\|_2,$$

which is at most 1. Letting $g_0(x)$ be the Euclidean distance of the point $g(x)$ from the origin in $\mathbb{R}^m$, we write $G_r = g_0^{-1}[0, r]$ and note that the top-dimensional relative homology group, $\mathsf{H}_m(\mathbb{R}^m, \mathbb{R}^m - G_r)$, is isomorphic to $\mathbb{Z}_2$. Letting $h : \mathbb{R} \to \mathbb{R}$ be an r-perturbation of the identity mapping, g, we write $H_0 = h^{-1}(0)$. With this, we get the following maps between relative homology groups:

$$\mathsf{H}_m(\mathbb{R}^m, \mathbb{R}^m - G_r) \xrightarrow{} \mathsf{H}_m(\mathbb{R}^m, \mathbb{R}^m - H_0)$$

$$\begin{array}{c} g \\ \cong \qquad\qquad h \\ \mathsf{H}_m(\mathbb{R}^m, \mathbb{R}^m - \{0\}), \end{array}$$

where the unlabeled homomorphism is induced by inclusion of spaces, and the other two are induced by the mappings g and h. The straight line homotopy taking g to h keeps each point in $\mathbb{R}^m - G_r$ in $\mathbb{R}^m - H_0$, for the entire length of the homotopy. This means that the triangle commutes and therefore H_0 cannot be empty. Thus the well diagram of the identity consists of a single, non-zero point at plus infinity. The Stability Theorem for Well Diagrams implies that the well diagram of f also has a point at plus infinity. But this implies that f has a root and, equivalently, that b has a fixed point. $\square$

The above reduction of fixed points to an intersection setting uses the difference between two points, an operation not available if the mapping $b : \mathbb{M} \to \mathbb{M}$ is defined on a general metric space. In this case, we can use the correspondence between the fixed points of b and the intersection points between the graph of b and the diagonal in $\mathbb{M} \times \mathbb{M}$. To apply the results of this paper, we set $\mathbb{X} = \mathbb{M}$, $\mathbb{Y} = \mathbb{M} \times \mathbb{M}$, and $\mathbb{A} = \{(x', x') \mid x' \in \mathbb{M}\}$. Furthermore, we define the distance between two points $x = (x', x'')$ and $y = (y', y'')$ in $\mathbb{M} \times \mathbb{M}$ equal to

$$\|x - y\|_{\mathbb{Y}} = \begin{cases} \infty & \text{if } x' \neq y'; \\ \|x'' - y''\|_{\mathbb{M}} & \text{if } x' = y'. \end{cases}$$

We restrict $\mathcal{P}$ to those mappings $h : \mathbb{M} \to \mathbb{M} \times \mathbb{M}$ that arise as graphs of a continuous mapping from $\mathbb{M}$ to itself. It is not difficult to see that this setting gives the same robustness values for the case $\mathbb{M} = \mathbb{R}^m$ discussed above.

Periodic Orbits   We generalize the above setting by allowing for fixed points of iterations of the mapping. Letting $\mathbb{M}$ be a metric space and $f : \mathbb{M} \to \mathbb{M}$ a mapping, we write $f^j : \mathbb{M} \to \mathbb{M}$ for the j-fold composition of f with itself. A sequence

$$F_j(x) = \big(x, f(x), f^2(x), \ldots, f^{j-1}(x) \big)$$

is an *order-j periodic orbit* of f if $f^j(x) = f \circ f^{j-1}(x) = x$. It is straightforward to see the following relationship between f and its j-fold composite.

Orbit Lemma *A point $x \in \mathbb{M}$ is a fixed point of f^j iff $F_j(x)$ is an order-j periodic orbit of f.*

We can therefore use the methods of this paper to measure the robustness of x, that is, to determine how much f^j needs to be perturbed to remove the fixed point. However, it is more interesting to measure how much f needs to be perturbed to remove the periodic orbit. This is different because not every mapping can be written as the j-fold composite of another mapping. This motivates us to introduce $\mathcal{P}$ as the space of perturbations of f^j that are j-fold composites of perturbations h of f. Using this space $\mathcal{P}$, we intersect the images of the homomorphisms induced by the h^j. With this setup, we construct the well diagram of f^j and interpret the resulting values as the robustness of the order-j periodic orbits of f.

Apparent Contours   As mentioned in the introduction, [6] reduces the stability of the contour of a mapping to the stability of well diagrams, the main result of this paper. We briefly review the reduction. Let $\mathbb{M}$ be a compact, orientable 2-manifold and $f : \mathbb{M} \to \mathbb{R}^2$ a smooth mapping. The derivative of f at a point x is a linear map from the tangent space to $\mathbb{R}^2$. The point x is *critical* if the derivative at x is not surjective, and the *apparent contour* of f is the set of images of critical points. Beyond the smoothness of f, we assume that the well functions it defines are admissible. Specifically, for each $a \in \mathbb{R}^2$, the function $f_a : \mathbb{M} \to \mathbb{R}$ is defined by mapping every point x to $f_a(x) = \|f(x) - a\|_2$ and we assume that $f_a^{-1}(0)$ consists of a finite number of points.

To study the apparent contour, we consider the entire 2-parameter family of well functions. Fixing a value $a \in \mathbb{R}^2$, the sublevel sets of f_a form a filtration of homology groups and a zigzag module of well groups. Each point in the preimage of a falls ill at a particular radius interpreted as the robustness of that point. The main result of this paper implies that this measure is stable, that is, $W_\infty(\mathrm{Dgm}(\mathsf{U}), \mathrm{Dgm}(\mathsf{V})) \leq \|f_a - g_a\|_\infty$, where U and V are the well modules defined by the mappings $f, g : \mathbb{M} \to \mathbb{R}$ and the value $a \in \mathbb{R}^2$. As shown in [6], this implies that the apparent contours of f and of g are close. The sense in which they are close is interesting in its own right, and we refer to that paper for details.

6 Discussion

The main contribution of this paper is the definition of a robustness measure for the homology of the intersection between topological spaces, and a proof that this measure is stable. While robustness and persistence are related, there are also differences between these notions. For example, robustness adapts to a given metric space of perturbations, and this extra degree of freedom is sometimes essential, such as for a meaningful analysis of periodic orbits. The results in this paper raise a number of questions and invite extensions in several directions.

- Fixed points of mappings play an important role in game theory [13]. Can the results of this paper be used to gain insight into the nature of fixed points as they arise in different games? What are contexts in which the robustness of a fixed point is relevant to the understanding of the dynamics of a game?
- The three applications sketched in Sect. 5 barely scratch the surface of the possible. An interesting direction for further research is mappings from lower to higher dimensions. For example, the boundary of a computer-aided design model is the image of a mapping from a 2-manifold to $\mathbb{R}^3$. Can our results be used to detect and remove accidental self-intersections, a problem of significant economic importance [7]?
- Except for a few special settings, we have no algorithms for computing well diagrams. The main obstacle is the possibly infinite set of perturbations that appears in the definition of well groups. However, since the groups that arise for admissible mappings are finite, only a finite number of perturbations are relevant. Can we approach the algorithmic question from this direction?

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