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On the Rank of a Binary Form

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Abstract We describe in the space of binary forms of degree d the strata of forms having a given rank. We also give a simple algorithm for determining the rank of a given form.

Keywords Tensor rank · Waring's problem · Binary forms

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1 Introduction

Let Q be a homogeneous polynomial of degree d, in n variables, with complex coefficients. We define the *rank* of Q (denoted $\operatorname{rk} Q$) to be the smallest integer r such that Q can be written as a sum of r dth powers of linear forms:

$$Q = L_1^d + \cdots + L_r^d. \tag{1}$$

The terminology comes from the case of quadrics. A quadratic form Q can be written as $Q(X) = X^t AX$ for A a symmetric matrix, and the rank of Q is the rank of the

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matrix A. That is, the rank of A is r if and only if Q is the sum of r squares of linear forms (and not less than r).

If we let $d > 3$, we can state the following classical problem associated to the rank of a form.

Waring's Problem for forms: what is the least integer $E(d,n)$ such that a generic form of degree d in n variables has rank less than or equal to $E(d,n)$?

For binary forms, that is $n = 2$, the solution to this problem was known to Sylvester ([19, 20], see also [9, 10, 13–15]). He showed that $E(2k-1,2) = k$ and that a general form of degree $d = 2k - 1$ has a unique decomposition as in (1). In this case we say that a general binary form of degree $2k - 1$ has a *canonical form* $L_1^d + \cdots + L_k^d$. In the case of even degree ($d = 2k$) he introduced an invariant (the determinant of a catalecticant matrix) and showed that among the forms having this determinant equal to zero a general one has rank k. And once again such a general form admits a unique decomposition as in (1), that is, it has a canonical form. A general binary form of degree $d = 2k$ has rank $k + 1$ (and therefore $E(2k,2) = k + 1$), but there is no canonical form: such a form can be written as the sum of $k + 1$ dth powers of linear forms in an infinite number of ways ([13, 15] or formula (9)).

For $n > 2$ this problem remained unsolved until 1995, when Alexander and Hirschowitz ([1]) computed the value of $E(n,d)$ for every pair (n,d). They actually computed, for each $r \leq E(n,d)$, the dimension of the closure of the set of forms having rank r.

One may generalize the definition of rank in the following way: given a form Q we define its *border rank* as the least integer r such that Q may be written either as a sum of r dth powers of linear forms or as a limit of such sums. It is clear that every form has a border rank less than or equal to $E(d,n)$. From this definition one can ask, for each $r \leq E(d,n)$, for algebraic conditions on a form Q so that its border rank is r.

The answer to this question in the case of binary forms also follows from the work of Sylvester. Given a binary form Q one can arrange its coefficients in a certain catalecticant matrix. The border rank of a binary form Q is the rank of this matrix (see [13] or Sect. 3).

Another solution, using covariants, was given by Gundelfinger ([11] or [10, 14]). For each r such that $2r \leq d$ he gave an equation G_r involving partial derivatives of the form Q. The form G_1 is the Hessian and the other forms are generalizations. Proceeding in this way, the border rank of a form Q is the first r such that $G_r(Q) = 0$. Notice that for $r = 1$ we get the well known fact that a binary form is the dth power of a linear form if and only if its Hessian vanishes [15].

The border rank and the rank of a form may be different. For example the binary form $X^{d-1}Y$ has border rank 2, since it may be written as the limit $\frac{(X+tY)^d-(X-tY)^d}{2d}$ as $t \to 0$. But its rank is d (see [12, p. 147], or Theorem 2). In this case we can write $X^{d-1}Y$ as the sum of d dth powers of linear forms in ∞^{d-1} ways, so for a form having a rank greater than its border rank we will not get a canonical form as in (1).

However, if we admit more general expressions we can get a canonical form. In [13] it is shown that any binary form of degree d having border rank r, with $2r - 1 \leq d$, admits a unique decomposition, thus:

$$Q = L_1^{d-(\mu_i-1)} P_1 + \cdots + L_k^{d-(\mu_i-1)} P_k, \tag{2}$$

where L_i are linear forms, P_i are degree $\mu_i - 1$ forms and $r = \mu_1 + \cdots + \mu_k$ (the number r is also called the *length* of a form). In the case where all the μ_i's are equal to one, the form Q has a rank equal to its length. Notice that the form $X^{d-1}Y$ has length two, as was expected. In this article we show that in general for every r, such that $2r - 1 \leq d$, binary forms having length or border rank r but greater rank all have rank equal to $d - r + 2$ (Theorem 2). For forms in more than two variables little is known about the rank of a form whose border rank is not the same as its rank.

There are several other generalizations of Waring's Problem for forms. For example, given vector spaces $V_1, \ldots, V_m$ and a tensor $T \in V_1 \otimes \cdots \otimes V_m$, we define its rank as the least integer r such that T is written as the sum of r indecomposable tensors $v_1 \otimes \cdots \otimes v_m$. The same questions we asked for the rank of binary forms can be asked in this setting, and so far no general answers have been given. For an introduction to the tensor rank, see [4]. The case $V_1 = \cdots = V_r = \mathbb{C}^2$ is of particular interest in quantum computing (see [6]). For the connection between tensor rank and algebraic complexity theory and other problems see [16]. The article [18] relates the rank of forms with the problem of finding polynomial solutions to partial differential equations with constant coefficients. For more generalizations of Waring's Problem for forms and its relation with the problem of polynomial interpolation, see [7].

The main result of this paper is the complete description of the strata of binary forms having constant rank. Let $S_{d,r}$ denote the set of degree d binary forms having rank r, and let $S^\circ_{d,r}$ denote the set of forms having border rank r. We show that for each $r \geq 2$ such that $2r - 1 \leq d$ we have

$$S^\circ_{d,r} = S_{d,r} \cup S_{d,d-r+2}. \tag{3}$$

If $d = 2r - 2$, we get again Sylvester's result:

$$S^\circ_{d,r} = S_{d,r}. \tag{4}$$

It is clear from the definitions that the closure of $S_{d,r}$ contains $S^\circ_{d,k}$ for each $k < r$. If we combine this fact with formula (3), we get, for each $r \geq 2$ such that $2r - 1 \leq d$, the following characterization of $\overline{S}_{d,r}$:

$$\overline{S}_{d,r} = \Bigl(\bigcup_{k\leq r} S_{d,k}\Bigr) \cup \Bigl(\bigcup_{k\geq d-r+2} S_{d,k}\Bigr). \tag{5}$$

On the other hand, if $r \leq d$ and $2r - 1 > d$, from formulas (3) and (4) we know that $S_{d,r} \subset S^\circ_{d,d-r+2}$. Therefore, the closure of $S_{d,r}$ contains $S^\circ_{d,k}$ for each $k \leq d - r + 1$. So we get the following characterization of $\overline{S}_{d,r}$:

$$\overline{S}_{d,r} = \Bigl(\bigcup_{k\leq d-r+1} S_{d,k}\Bigr) \cup \Bigl(\bigcup_{k\geq r} S_{d,k}\Bigr). \tag{6}$$

A close examination of formulas (5) and (6) gives us the desired description (see Theorem 2):

- If $r \geq 2$ and $2r-1 \leq d$,

$$S_{d,r} = \overline{S}_{d,r} \setminus \overline{S}_{d,d-r+2},$$
$$S_{d,d-r+2} = \overline{S}_{d,d-r+2} \setminus \overline{S}_{d,r-1}.$$

- If $2r-2 = d$,

$$S_{d,r} = \overline{S}_{d,r} \setminus \overline{S}_{d,r-1}.$$

We show how to determine the rank of a given form Q by computing the rank of an explicit matrix, solving a linear system of equations, and deciding if a polynomial constructed from this solution has multiple roots. The algorithm also lets us know the number of ways in which a rank r form can be written as in (1) using r different linear forms.

2 Rank of a Binary Form

Let $V = \mathbb{C}^2$ and let $S_1 = V^*$ be the vector space of linear polynomials in two variables x, y. Let S_d be the space of homogeneous polynomials of degree d in variables x, y. We define the rank of a binary form $Q \in S_d$ as follows.

Definition 1 The rank of Q is the least integer r such that Q can be written as

$$Q = L_1^d + \cdots + L_r^d$$

where $L_i \in S_1, i = 1, \dots, r$.

Let $C \subset \mathbb{P}(S_d)$ be the Veronese curve, that is, the image of the map

$$\mathbb{P}(S_1) \to \mathbb{P}(S_d),$$
$$[L] \mapsto [L^d].$$

Then the rank of a binary form Q is the least integer r such that $[Q]$ lies on a linear space spanned by r elements lying on C. For instance, the forms of rank one are exactly those lying on C; and the forms lying on a secant line to C have rank less than or equal to 2. However, a form lying on a tangent line to C will have rank d although it is a limit of rank two forms. Also, since the union of the tangent lines to C is a closed surface, the limit of rank d forms has either rank one or rank d.

We will prove the following theorem:

Theorem 2 *Let $\mathbb{P}(S_d)$ be the projective space of binary forms of degree d, $C \subset \mathbb{P}(S_d)$ the Veronese curve of dth powers of linear forms and for each $1 \leq r \leq d$, let $S_{d,r} \subset \mathbb{P}(S_d)$ be the projectivization of the set of degree d forms having rank r.*

(1) *For each $r \geq 2$ such that $d \geq 2r-1$ we have*

$$\overline{S}_{d,r} \setminus \overline{S}_{d,r-1} = S_{d,r} \cup S_{d,d-r+2},$$
$$S_{d,r} = \overline{S}_{d,r} \setminus \overline{S}_{d,d-r+2},$$
$$S_{d,d-r+2} = \overline{S}_{d,d-r+2} \setminus \overline{S}_{d,r-1}.$$

(2) *If* $2r-2=d$, *that is*, $r=d-r-2$, *we have*

$$S_{d,r} = \overline{S}_{d,r} \setminus \overline{S}_{d,r-1}.$$

We will use the following alternative way of describing the Veronese curve. Consider the map

$$\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}(S_d^*),$$
$$[\alpha] \mapsto \bigl[\mathrm{ev}(\alpha)\bigr],$$

where $\mathrm{ev}(\alpha)$ is the linear functional given by evaluation of polynomials at $\alpha \in \mathbb{C}^2$. The Veronese curve is the image of this map. More precisely, let $\{x, y\}$ be a basis for S_1, and $\{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d\}$ a basis for S_d. We consider in S_d^* the basis dual to that of S_d. Then a point in $\mathbb{P}(S_1)$ with homogeneous coordinates $[t, u]$ is mapped by the first map to a point in $\mathbb{P}(S_d)$ with homogeneous coordinates $\bigl[t^d, \binom{d}{1}t^{d-1}u, \dots, \binom{d}{d-1}tu^{d-1}, u^d\bigr]$. On the other hand, a point in $\mathbb{P}^1(\mathbb{C})$ with homogeneous coordinates $[t, u]$ is mapped by the second map to a point in $\mathbb{P}(S_d^*)$ with homogeneous coordinates $[t^d, t^{d-1}u, \dots, tu^{d-1}, u^d]$.

Therefore, the isomorphism given by

$$\mathbb{P}(S_d^*) \to \mathbb{P}(S_d),$$
$$[Z_0, Z_1, \dots, Z_{d-1}, Z_d] \mapsto \left[Z_0, \binom{d}{1}Z_1, \dots, \binom{d}{d-1}Z_{d-1}, Z_d\right]$$

restricts to an isomorphism between the image of the second Veronese map and the first one.

Using this isomorphism, we define the rank of a functional $\varphi \in S_d^*$ as follows:

Definition 3 Let $\varphi \in S_d^*$ be a linear functional. The rank of φ is the least integer r such that φ can be written as

$$\varphi = \mathrm{ev}(\alpha_1) + \cdots + \mathrm{ev}(\alpha_r),$$

where $\alpha_i \in \mathbb{C}^2$, $i = 1, \dots, r$.

Notice that if $\varphi = \lambda_1 \mathrm{ev}(\alpha_1) + \cdots + \lambda_r \mathrm{ev}(\alpha_r)$ with $\lambda_i \neq 0\ \forall i$, then φ is the sum of $\mathrm{ev}(\alpha_i')$, where $\alpha_i' = \omega_i \alpha_i$ and $\omega_i^d = \lambda_i$.

Let $\mathrm{Sec}^r(C)$ denote the r-secant variety of C, that is, the variety defined as the closure of the union of $(r-1)$-planes spanned by r points in C. It is an irreducible variety of dimension $\min\{2r-1, d\}$ ([12, Proposition 11.32, p. 147]). Therefore when we refer to a secant variety we will assume that $2r-1 \le d$. Notice that a point in $\mathrm{Sec}^r(C)$ will lie either on a plane spanned by r different points in C, or in a limit of such planes. Therefore, $\mathrm{Sec}^r(C)$ is the closure of all points having rank less than or equal to r.

We can arrange the homogeneous coordinates $[Z_0, Z_1, \dots, Z_d]$ of a point $[\varphi] \in \mathbb{P}(S_d^*)$ in a catalecticant matrix

$$M_\varphi = \begin{bmatrix} Z_0 & Z_1 & \dots & Z_{d-s} \\ Z_1 & Z_2 & \dots & Z_{d-s+1} \\ \vdots & \vdots & & \vdots \\ Z_s & Z_{s+1} & \dots & Z_d \end{bmatrix}. \tag{7}$$

It is a known fact that if $r \le \min\{s, d-s\}$, $[\varphi] \in \mathrm{Sec}^r(C)$ if and only if $\mathrm{rk}\, M_\varphi \le r$ ([12, Proposition 9.7, p. 103], or [13]).

We can then give the following characterization of the secant variety $\mathrm{Sec}^r(C)$, which will allow us to recognize which points in $\mathrm{Sec}^r(C)$ lie on secant planes to C spanned by r different points.

Lemma 4 *Let $[\varphi] \in \mathbb{P}(S_d^*)$ and let $r \ge 2$ such that $d > 2r-1$. Then $[\varphi] \in \mathrm{Sec}^r(C)$ if and only if $\varphi(S_{d-r} \cdot G) = 0$ for some $G \in S_r$.*

Proof Consider the bilinear form $B : S_{d-r} \times S_r \to \mathbb{C}$ given by $B(F, G) = \varphi(F \cdot G)$. The matrix of B in a basis

$$\{x^{d-r}, x^{d-r-1}y, \dots, y^{d-r}\}\ \{x^r, x^{r-1}y, \dots, y^r\}$$

is

$$M = \begin{bmatrix} Z_0 & Z_1 & \dots & Z_r \\ Z_1 & Z_2 & \dots & Z_{r+1} \\ \vdots & \vdots & & \vdots \\ Z_{d-r} & Z_{d-r+1} & \dots & Z_d \end{bmatrix},$$

where the $Z_i = \varphi(x^{d-i}y^i)$ denote the homogeneous coordinates of $[\varphi]$. Since $d > 2r-1$, we have $r \le d-r$. We know that $[\varphi] \in \mathrm{Sec}^r(C)$ if and only if the rank M is less than or equal to r. But the rank of M is less than or equal to r if and only if there is a polynomial $G \in S_r$ such that $\varphi(S_{d-r} \cdot G) = 0$. □

In the following lemma we give a necessary and sufficient condition for the rank of φ to be less than or equal to a given integer r. We shall denote by Δ_r the discriminant hypersurface in the affine space S_r, defined as the locus of degree r polynomials with multiple roots in $\mathbb{P}^1(\mathbb{C})$. It is a well known fact that Δ_r is an irreducible hypersurface defined by a homogeneous polynomial of degree $2r-2$. Therefore, Δ_r is not a hyperplane.

Lemma 5 *Let $\varphi \in S_d^*$. Then $\mathrm{rk}(\varphi) \le r$ if and only if the set*

$$W = \{G \in S_r : \varphi(S_{d-r} \cdot G) = 0\}$$

is not contained in Δ_r.

In particular, every φ has rank less than or equal to d.

Proof Let $\varphi \in S_d^*$ such that $\mathrm{rk}(\varphi) \leq r$. Then we can write $\varphi = \sum_{i=1}^r \mathrm{ev}(\alpha_i)$, where $[\alpha_1], \dots, [\alpha_r] \in \mathbb{P}^1(\mathbb{C})$ are r different points. The polynomial G with roots $[\alpha_1], \dots, [\alpha_r]$ lies on W and therefore W is not contained in Δ_r.

Now let us consider a polynomial $G \in W \setminus \Delta_r$, and let $[\alpha_1] = [t_1, u_1], \dots, [\alpha_r] = [t_r, u_r] \in \mathbb{P}^1(\mathbb{C})$ be the roots of G. We will show that $\varphi = \sum_{i=1}^r \lambda_i \mathrm{ev}(\alpha_i)$ by proving that $\{\varphi, \mathrm{ev}(\alpha_1), \dots, \mathrm{ev}(\alpha_r)\}$ is a linearly dependent set. Consider the $(r+1) \times (d+1)$ matrix having as its rows the coordinates of these functionals in the basis dual to $\{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d\}$:

$$\begin{bmatrix} \varphi(x^d) & \varphi(x^{d-1}y) & \dots & \varphi(y^d) \\ t_1^d & t_1^{d-1}u_1 & \dots & u_1^d \\ \vdots & \vdots & \ddots & \vdots \\ t_r^d & t_r^{d-1}u_r & \dots & u_r^d \end{bmatrix}.$$

We claim that this matrix does not have maximal rank. Let us consider the maximal minor obtained by choosing $r+1$ columns in this matrix. Using the linearity of φ, this minor can be expressed as $\varphi(F)$, where $F \in S_d$ is a polynomial having $[\alpha_1], \dots, [\alpha_r]$ as roots in $\mathbb{P}^1(\mathbb{C})$. Thus, $\varphi(F) = \varphi(F' \cdot G) = 0$.

When $r = d$, W is the kernel of φ, which is a hyperplane. Therefore $W \not\subset \Delta_d$, and then φ is a linear combination of d elements in C. □

Notice that we proved that if we fix a polynomial $G \in S_r$ with r different roots $[\alpha_1], \dots, [\alpha_r]$, then the plane spanned in $\mathbb{P}(S_d^*)$ by $[\mathrm{ev}(\alpha_1)], \dots, [\mathrm{ev}(\alpha_r)]$ can be characterized as

$$\langle [\mathrm{ev}(\alpha_1)], \dots, [\mathrm{ev}(\alpha_r)] \rangle = \{ [\varphi] \in \mathbb{P}(S_d^*) : \varphi(S_{d-r} \cdot G) = 0 \}.$$

If we let $G \in S_r$ have multiple roots, the expression on the right still makes sense, and it defines a linear variety of dimension $r-1$ if $r \leq d$, that is a limit of $(r-1)$-planes spanned by r different points in C. We will denote by Λ_G the linear variety on the right side of the equation for $G \in S_r$ an arbitrary polynomial. For example, if G has only one root $[\alpha]$ with multiplicity r, the space Λ_G is the $(r-1)$-osculating plane to C at the point $[\mathrm{ev}(\alpha)]$, since we can obtain Λ_G as a limit of $\Lambda_{G(t)}$, for $G(t)$ polynomials with simple roots. Notice that with this notation we can redefine $\mathrm{Sec}^r(C)$ as the reduced union of the planes Λ_G as G ranges over S_r. The following lemma lets us conclude that if G has roots $[\alpha_1], \dots, [\alpha_k]$ with multiplicities $m_1, \dots, m_k$, respectively, then Λ_G is the linear span of the $(m_i - 1)$-osculating planes to C at the points $[\mathrm{ev}(\alpha_i)]$.

Lemma 6 *Let $F \in S_k$, $G \in S_l$, P the least common multiple of F and G, and H their greatest common divisor. Then*

(1) *If* $\deg P \geq d+1$, *then* $\langle \Lambda_F, \Lambda_G \rangle = \mathbb{P}(S_d^*)$.
(2) *If* $\deg P \leq d$, *then* $\langle \Lambda_F, \Lambda_G \rangle = \Lambda_P$.
(3) *If* $\deg P \leq d+1$, *then* $\Lambda_F \cap \Lambda_G = \Lambda_H$.

Proof The first two statements are evident if F and G have simple roots, since any $d+1$ points in the Veronese curve are in general position. The general case is a limit of this particular case.

For the third statement we use the first two to deduce that $\dim\langle \Lambda_F, \Lambda_G\rangle = \deg P - 1$. We also know that

$$\dim\langle \Lambda_F, \Lambda_G\rangle = \dim \Lambda_F + \dim \Lambda_G - \dim(\Lambda_F \cap \Lambda_G),$$

and therefore $\dim(\Lambda_F \cap \Lambda_G) = \deg F + \deg G - \deg P - 1 = \deg H - 1$. On the other hand

$$\Lambda_F \cap \Lambda_G \supset \Lambda_H,$$

and $\dim \Lambda_H = \deg H - 1$. Therefore, $\Lambda_H = \Lambda_F \cap \Lambda_G$. □

In the next proposition we show a lower bound for the increase of rank when passing to a limit position.

Proposition 7 *Let $r \geq 2$ be an integer such that $d \geq 2r - 1$. If $[\varphi] \in \operatorname{Sec}^r(C)$ and its rank is greater than r, then* $\operatorname{rk}\varphi \geq d - r + 2$.

Proof We know by Lemma 4 that φ satisfies the condition

$$\varphi(S_{d-r} \cdot G) = 0$$

for some $G \in S_r$. And since $\operatorname{rk}\varphi > r$, G has multiple roots (Lemma 5).

Assume that $\operatorname{rk}\varphi \leq d - r + 1$, that is, we can write

$$\varphi = \sum_{j=1}^{d-r+1} \lambda_j \operatorname{ev}(\alpha_j).$$

Let $F \in S_{d-r+1}$ such that its roots are $[\alpha_1], \dots, [\alpha_{d-r+1}]$.

We have $[\varphi] \in \Lambda_F \cap \Lambda_G$, and since $\deg F + \deg G = d + 1$ by Lemma 6 we have $[\varphi] \in \Lambda_H$, where H is the greatest common divisor of F and G. Since F has simple roots and since G has at most $r - 1$ different roots, we have $\deg H \leq r - 1$. Therefore, $[\varphi] \in \operatorname{Sec}^{r-1}(C)$, which is a contradiction.

Then we must have $\operatorname{rk}(\varphi) \geq d - r + 2$. □

In order to prove the converse we will need the following lemma.

Lemma 8 *Let $W \subsetneq S_k$ be a subspace of codimension l without common zeroes. Then if $m \geq l$, $S_m \cdot W = S_{k+m}$.*

Proof We will use the fact that if $V \subsetneq S_r$ has no common zeroes, then

$$\dim S_1 \cdot V \geq \dim(V) + 2, \tag{8}$$

which is proved in [12, Lemma 9.8, p. 103].

Assume that $S_m \cdot W \subsetneq S_{k+m}$. Then $S_r \cdot W$ is a proper subspace of S_{k+r} for each $1 \le r \le m$, and therefore, by applying m times (8) we have $\dim W + 2m \le k + m$, that is, $\dim W \le k - m$. But we know that $\dim W = k + 1 - l$, and therefore $m \le l - 1$, which is a contradiction. □

Now we prove the converse of Proposition 7.

Proposition 9 *Let* $r \ge 2$, *such that* $d \ge 2r - 1$, *and let* $[\varphi] \in \mathbb{P}(S_d^*)$, *such that* $\operatorname{rk}(\varphi) \ge d - r + 2$. *Then* $[\varphi] \in \operatorname{Sec}^r(C)$.

Proof Consider the bilinear form $B : S_{r-1} \times S_{d-r+1} \to \mathbb{C}$ given by $B(F, G) = \varphi(F \cdot G)$. We know that if the rank of the matrix of B is less than or equal to $r - 1$, then $[\varphi] \in \operatorname{Sec}^{r-1}(C)$. Therefore, we can assume that the rank of the matrix of B is r and then the subspace $W = \{G \in S_{d-r+1} : \varphi(S_{r-1} \cdot G) = 0\}$ has dimension $d - 2r + 2$. Since $\operatorname{rk}\varphi \ge d - r + 2$, all polynomials in W will have multiple roots. Therefore, by Bertini's theorem, W has common zeroes, and one of them is a common zero for all polynomials in W. Let H be the polynomial having as roots the common zeroes of W, counted with multiplicity. Let n be the degree of H, and let $W' \subset S_{d-r+1-n}$ be the subspace without common zeroes, such that $W = H \cdot W'$. Since $\dim W = \dim W'$, we have

$$d - 2r + 2 = \dim W' \le \dim S_{d-r+1-n} = d - r + 2 - n.$$

Therefore, $n \le r$.

The codimension of W' in $S_{d-r+1-n}$ is $r - n$. Since $n \ge 2$, we have $r - 1 \ge \operatorname{codim} W'$; therefore, we can use Lemma 8 to conclude that $S_{r-1} \cdot W' = S_{d-n}$. Now

$$0 = \varphi(S_{r-1} \cdot W) = \varphi(S_{r-1} \cdot W' \cdot H) = \varphi(S_{d-n} \cdot H),$$

that is, $[\varphi] \in \operatorname{Sec}^n(C)$. Since $n \le r$, $[\varphi] \in \operatorname{Sec}^r(C)$, as we wanted. □

We define now a subvariety of $\operatorname{Sec}^r(C)$ that will contain the points in $\operatorname{Sec}^r(C)$ with rank greater than r. In Proposition 5 we proved that $\operatorname{rk}\varphi \le r$ if and only if $[\varphi] \in \Lambda_G$ for G with no multiple roots. Let $\operatorname{Sec}^{r,2}(C)$ be the reduced union of linear subspaces of the form Λ_G with $G \in S_r$ a polynomial with multiple roots. Alternatively, we can define $\operatorname{Sec}^{r,2}(C)$ as the closure of linear subspaces spanned by $r - 2$ points in C and a tangent line to C, since every linear subspace of the form Λ_G for some G with multiple roots is a limit of those. Notice that if $[\varphi] \in \operatorname{Sec}^r(C)$ is such that its rank is greater than r, then it must belong to a plane of the form Λ_G for some $G \in S_r$, and G must have multiple roots, since otherwise $\operatorname{rk}\varphi \le r$. Finally, notice that if $r = 2$, then $\operatorname{Sec}^{r,2}(C)$ is the tangential surface of C.

In the following proposition we calculate the dimension of $\operatorname{Sec}^{r,2}(C)$.

Proposition 10 *If* $2r - 2 \le d$, *then* $\operatorname{Sec}^{r,2}(C)$ *is an irreducible variety of dimension* $2r - 2$.

Proof We consider the following correspondence:

$$\Sigma = \left\{ \left([\varphi], [G]\right) \in \mathbb{P}(S_d^*) \times \mathbb{P}(\Delta_r) : \varphi(S_{d-r} \cdot G) = 0 \right\} \subset \mathbb{P}(S_d^*) \times \mathbb{P}(\Delta_r),$$

where $\mathbb{P}(\Delta_r)$ denotes the discriminant hypersurface in $\mathbb{P}(S_r)$. The fibers of the second projection are linear spaces of dimension $r-1$, and therefore Σ is an irreducible variety of dimension $2r-2$. The image of the first projection is $\operatorname{Sec}^{r,2}(C)$, and therefore is irreducible.

If $2r \le d+1$ and $[\varphi] \in \Lambda_G \cap \Lambda_F$, then $[\varphi] \in \Lambda_H$, where H is the greatest common divisor of F and G. Therefore, $[\varphi] \in \operatorname{Sec}^{r-1}(C)$. Since $\operatorname{Sec}^{r-1}(C)$ is a proper closed subset in $\operatorname{Sec}^{r,2}(C)$, we can consider the open subset $U = \operatorname{Sec}^{r,2}(C) \setminus \operatorname{Sec}^{r-1}(C)$ and therefore the first projection is one to one over U. This shows that $\dim \operatorname{Sec}^{r,2}(C) = 2r-2$ if $2r \le d+1$.

If $2r-2 = d$, then $\operatorname{Sec}^{r-1}(C)$ is an hypersurface. Since $\operatorname{Sec}^{r-1}(C) \subsetneq \operatorname{Sec}^{r,2}(C)$, we must have $\operatorname{Sec}^{r,2}(C) = \mathbb{P}(S_d^*)$, that is, $\dim \operatorname{Sec}^{r,2}(C) = 2r-2$. □

Theorem 2 will be a consequence of the following theorem:

Theorem 11 *Let $C \subset \mathbb{P}(S_d^*)$ be the Veronese curve.*

(1) *For each $r \ge 2$ such that $d \ge 2r-1$ we have*

$$\operatorname{Sec}^r(C) \setminus \operatorname{Sec}^{r-1}(C) = \{[\varphi] : \operatorname{rk}\varphi = r\} \cup \{[\varphi] : \operatorname{rk}\varphi = d-r+2\},$$

$$\{[\varphi] : \operatorname{rk}\varphi = r\} = \operatorname{Sec}^r(C) \setminus \operatorname{Sec}^{r,2}(C),$$

$$\{[\varphi] : \operatorname{rk}\varphi = d-r+2\} = \operatorname{Sec}^{r,2}(C) \setminus \operatorname{Sec}^{r-1}(C).$$

(2) *If $d = 2r-2$ we have*

$$\{[\varphi] : \operatorname{rk}\varphi = r\} = \mathbb{P}(S_d^*) \setminus \operatorname{Sec}^r(C).$$

Proof of Theorem 2 First we assume that we are working with the Veronese curve $C \subset \mathbb{P}(S_d)$ that consists of dth powers of linear forms. Then we can identify the subset $\{[\varphi] : \operatorname{rk}\varphi = k\}$ with $S_{d,k}$.

Therefore, we only have to show that for r such that $2r-1 \le d$ we have $\overline{S}_{d,r} = \operatorname{Sec}^r(C)$ and that $\overline{S}_{d,d-r+2} = \operatorname{Sec}^{r,2}(C)$. Theorem 11 shows that $S_{d,r}$ and $S_{d,d-r+2}$ are open dense subsets of $\operatorname{Sec}^r(C)$ and $\operatorname{Sec}^{r,2}(C)$, respectively.

On the other hand if $d = 2r-2$ we have to show that $\overline{S}_{d,r} = \mathbb{P}(S_d^*)$. Using Theorem 11 again, we see that $S_{d,r}$ is an open subset of $\mathbb{P}(S_d^*)$. □

Now we will prove Theorem 11.

Proof of Theorem 11 First assume $2r-1 \le d$. Before we prove the three statements in Theorem 11 let us prove that, if $[\varphi] \in \operatorname{Sec}^r(C) \setminus \operatorname{Sec}^{r-1}(C)$, then there is a unique plane Λ_G such that $[\varphi] \in \Lambda_G$. If $[\varphi] \in \Lambda_G \cap \Lambda_F$, then $[\varphi] \in \Lambda_H$, where H is the greatest common divisor of F and G. Therefore $[\varphi] \in \operatorname{Sec}^{r-1}(C)$.

- The first statement follows from the following:

$$\operatorname{Sec}^r(C) = \{[\varphi] : \operatorname{rk}\varphi \le r\} \cup \{[\varphi] : \operatorname{rk}\varphi \ge d-r+2\},$$

so this is what we will prove.

Let $\varphi \in S_d^*$ such that $[\varphi] \in \mathrm{Sec}^r(C)$. Assume that $\mathrm{rk}\,\varphi > r$. Then Proposition 7 shows that $\mathrm{rk}\,\varphi \geq d - r + 2$, as we wanted.

Now if $\mathrm{rk}\,\varphi \leq r$, then clearly $[\varphi] \in \mathrm{Sec}^r(C)$. If on the other hand $\mathrm{rk}\,\varphi \geq d - r + 2$, then $[\varphi] \in \mathrm{Sec}^r(C)$ by Proposition 9.

- Now we prove the second statement. Let $[\varphi] \in \mathbb{P}(S_d^*)$, such that $\mathrm{rk}\,\varphi = r$. We know that $[\varphi] \in \Lambda_G$ for some $G \in S_r$ with no multiple roots and therefore $[\varphi] \in \mathrm{Sec}^r(C)$. If $[\varphi] \in \mathrm{Sec}^{r,2}(C)$, then $[\varphi] \in \Lambda_F$ for some $F \in S_r$ with multiple roots. But then $[\varphi] \in \mathrm{Sec}^{r-1}(C)$, which is a contradiction, since there are no r rank points in $\mathrm{Sec}^{r-1}(C)$. Therefore, $[\varphi] \in \mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r,2}(C)$.

  Conversely, if $[\varphi] \in \mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r,2}(C)$, then $[\varphi] \in \Lambda_G$ for some $G \in S_r$ with no multiple roots since $[\varphi] \notin \mathrm{Sec}^{r,2}(C)$. Therefore $\mathrm{rk}\,\varphi \leq r$. On the other hand, since $\mathrm{Sec}^{r-1}(C) \subset \mathrm{Sec}^{r,2}(C)$, $\mathrm{rk}\,\varphi \geq r$, and therefore $\mathrm{rk}\,\varphi = r$.
- Finally, we prove the third statement. Let $[\varphi] \in \mathbb{P}(S_d^*)$ such that $\mathrm{rk}\,\varphi = d - r + 2$. By Proposition 9, we have $[\varphi] \in \mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r-1}(C)$. Therefore there is a polynomial $G \in S_r$ such that $[\varphi] \in \Lambda_G$. If G has no multiple roots, then $\mathrm{rk}\,\varphi \leq r$. Since $r < d - r + 2$, g must have multiple roots, and therefore $[\varphi] \in \mathrm{Sec}^{r,2}(C)$.

  Conversely, if $[\varphi] \in \mathrm{Sec}^{r,2}(C) \setminus \mathrm{Sec}^{r-1}(C)$, we know that $\mathrm{rk}\,\varphi = r$ or $\mathrm{rk}\,\varphi = d - r + 2$. On the other hand we know that $[\varphi] \in \Lambda_G$ for some $G \in S_r$ with multiple roots. If $\mathrm{rk}\,\varphi = r$, then $[\varphi] \in \Lambda_F$ for some $F \in S_r$ with no multiple roots. But this means that $[\varphi] \in \mathrm{Sec}^{r-1}(C)$, which is a contradiction. Therefore, $\mathrm{rk}\,\varphi = d - r + 2$.

If $d = 2r - 2$, then $\mathrm{Sec}^{r,2}(C) = \mathrm{Sec}^r(C) = \mathbb{P}(S_d^*)$. But we know that $\mathrm{Sec}^{r-1}(C)$ contains all points $[\varphi]$ such that $\mathrm{rk}\,\varphi > r$ or $\mathrm{rk}\,\varphi < r$. Therefore, all points in $\mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r-1}(C)$ have rank r. □

3 Computing the Rank

Next we show how to determine the rank of a binary form $Q \in S_d$. Firstly we find the integer r such that $Q \in \overline{S}_{d,r} \setminus \overline{S}_{d,r-1}$, that is, we compute its length or border rank. Indeed, if $Q = Z_0 x^d + \binom{d}{1} Z_1 x^{d-1} y + \cdots + \binom{d}{d-1} Z_{d-1} x y^{d-1} + Z_d y^d$, then r is the rank of one of these two matrices:

$$\begin{bmatrix} Z_0 & Z_1 & \dots & Z_l \\ Z_1 & Z_2 & \dots & Z_{l+1} \\ \vdots & \vdots & & \vdots \\ Z_l & Z_{l+1} & \dots & Z_d \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} Z_0 & Z_1 & \dots & Z_l \\ Z_1 & Z_2 & \dots & Z_{l+1} \\ \vdots & \vdots & & \vdots \\ Z_{l+1} & Z_{l+2} & \dots & Z_d \end{bmatrix},$$

on wether $d = 2l$ or $d = 2l + 1$.

Let us assume first that $2r - 2 \neq d$. Then we have $r < d - r + 2$. We have to decide if $\mathrm{rk}\,Q = r$ or $\mathrm{rk}\,Q = d - r + 2$. The matrix M

$$M = \begin{bmatrix} Z_0 & Z_1 & \dots & Z_r \\ Z_1 & Z_2 & \dots & Z_{r+1} \\ \vdots & \vdots & & \vdots \\ Z_{d-r-2} & Z_{d-r+1} & \dots & Z_d \end{bmatrix}$$

also has rank r. Therefore, the linear system

$$M \cdot \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{r-1} \\ g_r \end{pmatrix} = 0$$

has a unique solution up to nonzero scalars. Let $G \in S_r$ be the polynomial $G = g_0x^r + g_1x^{r-1}y + \cdots + g_r y^r$. Then $\operatorname{rk} Q = r$ if and only if G has no multiple roots. Indeed, if G has r distinct roots $[t_1, u_1], \dots, [t_r, u_r] \in \mathbb{P}^1(\mathbb{C})$, then Q is a linear combination of the dth powers of the linear forms $L_1 = t_1x + u_1y, \dots, L_r = t_r x + u_r y$. Notice that we also proved that if $\operatorname{rk} Q = r$, and $2r - 1 \le d$, there is a unique way to write Q as a sum of r dth powers of linear forms. This way of getting a canonical form is the same as the one Sylvester used.

If G has multiple roots, then $\operatorname{rk} Q = d - r + 2$. In order to compute the number of ways in which we can write Q as a sum of $d - r + 2$ dth powers of linear forms we will consider the matrix

$$N = \begin{bmatrix} Z_0 & Z_1 & \dots & Z_{d-r+2} \\ Z_1 & Z_2 & \dots & Z_{d-r+3} \\ \vdots & \vdots & & \vdots \\ Z_{r-2} & Z_{r-1} & \dots & Z_d \end{bmatrix}.$$

Then Q is a linear combination of the dth power of the forms $L_i = t_ix + u_iy$, $i = 1, \dots, d - r + 2$ if and only if the coefficients of the polynomial H having $[u_i, t_i]$ as roots verifies

$$N \cdot \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{d-r+1} \\ h_{d-r+2} \end{pmatrix} = 0.$$

The rank of N is $r - 1$ and therefore the projective space $\mathbb{P}(S)$ of solutions of the linear system associated to N is $d - 2r + 3$. So H must lie in $\mathbb{P}(S) \setminus \mathbb{P}(\Delta_{d-r+2})$ which is an open set of dimension $d - 2r + 3$. Then we can write Q as a sum of $d - r + 2$ dth powers of linear forms in ∞^{d-2r+3} different ways.

It remains to analyze the case $l + 1 = r$. In this case, we know that $\operatorname{rk} Q = l + 1 = r$. We consider the matrix

$$N = \begin{bmatrix} Z_0 & Z_1 & \dots & Z_r \\ Z_1 & Z_2 & \dots & Z_{r+1} \\ \vdots & \vdots & & \vdots \\ Z_{r-2} & Z_{r-3} & \dots & Z_d \end{bmatrix}.$$

The rank of N is $r - 1$, and therefore the projective subspace of polynomials $[G] \in \mathbb{P}(S_r)$ such that $N \cdot [G] = 0$ is a line in $\mathbb{P}(S_r)$. Since we know that $\operatorname{rk} Q = r$, this line is not contained in $\mathbb{P}(\Delta_r)$, and therefore Q can be written as a sum of r dth powers of linear forms in ∞^1 ways. This shows that a general binary form of even degree has no canonical form.

We can summarize the previous discussion in the following way. Let $\sigma(r)$ be the number of ways in which a rank r binary form of degree d can be written as a sum of r dth powers of linear forms. Then

$$\sigma(r) = \begin{cases} 1 & \text{if } 2r-1 \leq d, \\ \infty^{2r-1-d} & \text{if } 2r-1 > d. \end{cases} \tag{9}$$

This article is a substantial revision of our ArXiv posting arxiv:math/0112311v1 [math.AG] of 2001. Since this article was submitted, various generalizations of the main result appeared. In [2, 5, 8] Waring's Problem of binary forms over the real field is treated. In [3] and [17] there are generalizations of the Waring Problem for forms in three or more variables.

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