Found Comput Math (2009) 9: 611–649
DOI 10.1007/s10208-009-9040-x

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

Period Doubling in the Rössler System—A Computer Assisted Proof

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Received: 5 December 2007 / Revised: 8 December 2008 / Accepted: 18 December 2008 /
Published online: 9 April 2009

Abstract Using rigorous numerical methods, we validate a part of the bifurcation diagram for a Poincaré map of the Rössler system (Rössler in Phys. Lett. A 57(5):397–398, 1976)—the existence of two period-doubling bifurcations and the existence of a branch of period two points connecting them. Our approach is based on the Lyapunov–Schmidt reduction and uses the C^r-Lohner algorithm (Wilczak and Zgliczyński, available at http://www.ii.uj.edu.pl/~wilczak) to obtain rigorous bounds for the Rössler system.

Keywords Period doubling · Rigorous numerical analysis

Mathematics Subject Classification (2000) 37M20 · 37N30

Communicated by Konstantin Mischaikow.

Research of D. Wilczak was supported by an annual national scholarship for young scientists from the Foundation for Polish Science.
Research of P. Zgliczyński was supported in part by Polish State Ministry of Science and Information Technology grant N201 024 31/2163.

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1 Introduction

The goal of this paper is to show how to produce a piece of the rigorous bifurcation diagram of the periodic orbits for an ODE. We study the Rössler system [23], one of the textbook examples of ODEs generating nontrivial dynamics, for a parameter range containing two period-doubling bifurcations.

According to the discussion in the Kuzniecov textbook [13, Sect. 2.7], there are two extremes in studying bifurcations in dynamical systems. The first one, going back to Poincaré, is to analyze the appearance (branching) of new invariant objects (equilibria or periodic orbits) from known ones as parameters of the system vary. A good reference for this approach is the textbook by Chow and Hale [6]. On the other extreme is the approach going back to Andronov [2] and Thom [25], which is to study rearrangements (bifurcations) of the whole phase portrait under variations of parameters. It is apparent that the first approach is necessarily one of the initial steps in attempting to describe the bifurcations in the Andronov–Thom sense. In fact, in many-dimensional systems (even for planar maps like the Hénon map) achieving a complete description of the phase space portrait and its changes appears to be hopeless in view of the results on the Henon-like maps [4, 18, 27, 28].

While there exists a vast literature on the bifurcation theory—see, for example, [3, 6, 7, 13] and the references therein, and also a lot of numerical bifurcation diagrams for various systems can be found in the literature (see, for example, the references in [13]), there are virtually no rigorous results on bifurcations of periodic orbits for ODEs in dimension three or higher in the situation where the periodic orbit undergoing the bifurcation is not given to us analytically due to some special symmetries of the system. The basic reason for this is: while numerical experiments and/or normal form computations may clearly show what is happening (in terms of the bifurcations) we usually lack any reasonable rigorous estimates about the observed orbits, which prevents us from turning these observations into rigorous statements. To obtain the necessary estimates, one needs to integrate the variational equations describing the partial derivatives with respect to the initial conditions up to order 3 or higher. This is usually a serious problem for rigorous ODE solvers. It turns out that the naive approach of applying an ODE solver to the system of variational equations does not work because the methods dealing with the wrapping effect used in Lohner-type algorithms (the most effective rigorous ODE solvers) [14, 20, 30] break down for such systems. As a solution to this problem, the C^r-Lohner algorithm was proposed in [29] and it is used in the present work.

Concerning the content of the paper regarding the bifurcation theory itself, we were forced to reformulate some well-known theorems to make them amenable to computer-assisted proofs. It is a common feature of all bifurcation theorems that the bifurcation point (or rather a candidate) and all necessary data like the spectrum and maybe some higher order terms are always given as part of the assumptions. But in a nonlinear system, we usually do not have these data explicitly. In fact, the existence of the bifurcation point has to be proved by looking at the behavior of the system in some neighborhood. This forces us to reformulate some bifurcation theorems in a semi-local way and we have to investigate properties of solutions of implicit equations that are degenerate (due to the presence of bifurcations). This is the reason why, from various approaches to bifurcations, we chose the one developed in [6], which is based on the Lyapunov–Schmidt reduction.

In our work, we focus on the period-doubling bifurcation of periodic orbits for the Rössler equations. Indeed, we study the Poincaré map for the Rössler system. The paper is organized as follows: In Sects. 3, 4, and 5, we discuss the main tools used to produce a validated piece of the bifurcation diagram containing the period-doubling bifurcations. In the remaining sections, we give some details concerning our results for the Rössler system.

2 Basic Definitions

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of natural, integer, rational, real, and complex numbers, respectively. $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are the negative and positive integers, respectively. By S^1, we will denote the unit circle on the complex plane.

For $\mathbb{R}^n$, we will denote the norm of x by $\|x\|$ and if the formula for the norm is not specified in some context, then it means that one can use any norm there. For $x_0 \in \mathbb{R}^s$, let $B_s(x_0, r) = \{z \in \mathbb{R}^s \mid \|x_0 - z\| < r\}$ and $B_s = B_s(0, 1)$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. By $\mathrm{Sp}(A)$, we denote the spectrum of A, which is the set of $\lambda \in \mathbb{C}$ such that there exists $x \neq \mathbb{C}^n \setminus \{0\}$, such that $Ax = \lambda x$.

For a map $f : X \to Y$, by $\mathrm{dom}(f)$ we will denote the domain of f. For a map $F : X \to X$, we will denote the fixed point set by $\mathrm{Fix}(F, U) = \{x \in U \mid F(x) = x\}$.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. By π_i we will denote the projection on the ith coordinate, i.e., $\pi_i(x) = x_i$. Analogously for any multi-index $\alpha = (i_1, i_2, \dots, i_k) \in \mathbb{Z}_+^k$, we define $\pi_\alpha(x) = (x_{i_1}, x_{i_2}, \dots, x_{i_k})$. Sometimes the points in the phase space will have coordinates denoted by different letters, for example, $z = (\nu, x, y)$; then we will index the projection by the names of variables, i.e., $\pi_{(\nu,x)}(z) = (\nu, x)$ etc.

Definition 1 Let $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^n$ be C^1. Let $z_0 \in \mathrm{dom}(f)$. We say that z_0 is a *hyperbolic fixed point for* f iff $f(z_0) = z_0$ and $\mathrm{Sp}(Df(z_0)) \cap S^1 = \emptyset$, where $Df(z_0)$ is the derivative of f at z_0.

Definition 2 Consider a map $f : X \supset \mathrm{dom}(f) \to X$. Let $x \in X$. Any sequence $\{x_k\}_{k\in I}$, where $I \subset \mathbb{Z}$ is a set containing 0 and for any $l_1 < l_2 < l_3$ in $\mathbb{Z}$ if $l_1, l_3 \in I$, then $l_2 \in I$, such that

$$x_0 = x, \qquad f(x_i) = x_{i+1}, \quad \text{for } i, i+1 \in I$$

will be called an orbit through x. If $I = \mathbb{Z}_- \cup \{0\}$, then we will say that $\{x_k\}_{k\in I}$ is a full backward orbit through x.

Definition 3 Let X be a topological space and let the map $f : X \supset \mathrm{dom}(f) \to X$ be continuous.

Let $Z \subset \mathbb{R}^m$, $z_0 \in Z$, $Z \subset \mathrm{dom}(f)$. We define

$$W^s_Z(z_0, f) = \left\{ z \mid \forall n \geq 0\ f^n(z) \in Z,\ \lim_{n\to\infty} f^n(z) = z_0 \right\},$$

$$W^u_Z(z_0, f) = \Big\{z \mid \exists\{x_n\} \subset Z \text{ a full backward orbit through } z, \text{ such that } \lim_{n\to-\infty} x_n = z_0\Big\},$$

$$W^s(z_0, f) = \Big\{z \mid \lim_{n\to\infty} f^n(z) = z_0\Big\},$$

$$W^u(z_0, f) = \Big\{z \mid \exists\{x_n\} \text{ a full backward orbit through } z, \text{ such that } \lim_{n\to-\infty} x_n = z_0\Big\},$$

$$\mathrm{Inv}^+(Z, f) = \big\{z \mid \forall_{n\geq 0}\ f^n(z) \in Z\big\},$$

$$\mathrm{Inv}^-(Z, f) = \big\{z \mid \exists\{x_n\} \subset Z \text{ a full backward orbit through } z\big\},$$

$$\mathrm{Inv}(Z, f) = \mathrm{Inv}^+(Z, f) \cap \mathrm{Inv}^-(Z, f).$$

If f is known from the context, then we will usually drop it from the notation and use $W^s(z_0)$, $W^s_Z(z_0)$, etc. instead.

Definition 4 Let $P : J \times \mathbb{R}^n \to \mathbb{R}^n$, where $J \subset \mathbb{R}$. For each $\nu \in J$, we define $P_\nu = P(\nu, \cdot)$. We say that P_ν has *a period-doubling bifurcation* at (ν_0, x_0) iff there exists $V = [\nu_1, \nu_2] \times X \subset J \times \mathbb{R}^n$, such that the following conditions are satisfied:

- $(\nu_0, x_0) \in \operatorname{int} V$, $P_{\nu_0}(x_0) = x_0$.
- There exists a continuous function $x_{\mathrm{fp}} : [\nu_1, \nu_2] \to \operatorname{int} X$, such that

$$\mathrm{Fix}(P_\nu, X) = \big\{x_{\mathrm{fp}}(\nu)\big\}.$$

- There exist two continuous curves $c_i : [\nu_0, \nu_2] \to \operatorname{int} X$, $i = 1, 2$, such that for $\nu \in [\nu_0, \nu_2]$ there holds

$$c_1(\nu_0) = c_2(\nu_0) = x_{\mathrm{fp}}(\nu_0),$$
$$c_1(\nu) \neq c_2(\nu), \quad \nu \neq \nu_0,$$
$$P_\nu\big(c_1(\nu)\big) = c_2(\nu), \qquad P_\nu\big(c_2(\nu)\big) = c_1(\nu),$$
$$\mathrm{Fix}\big(P^2_\nu, X\big) = \big\{c_1(\nu), c_2(\nu), x_{\mathrm{fp}}(\nu)\big\}.$$

- The dynamics: for $\nu \leq \nu_0$

$$\mathrm{Inv}(X, P_\nu) = \big\{x_{\mathrm{fp}}(\nu)\big\}.$$

  For $\nu > \nu_0$, the maximal invariant set in X $\mathrm{Inv}(X, P_\nu)$ is equal to

$$\overline{W^u_X\big(x_{\mathrm{fp}}(\nu), P\big)} \cap \Big(\overline{W^s_X\big(c_1(\nu), P^2_\nu\big)} \cup \overline{W^s_X\big(c_2(\nu), P^2_\nu\big)}\Big)$$

  and is a one-dimensional connected manifold with boundary points $c_1(\nu)$, $c_2(\nu)$.

The objects appearing in the above definition are illustrated in Fig. 1.

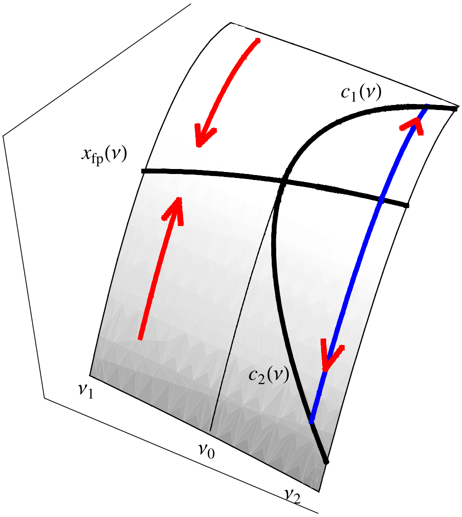


Fig. 1 (Color online) The dynamics in a small neighborhood of the bifurcation point as in Definition 4. The maximal invariant set for a certain parameter value $\nu > \nu_0$ is shown in *blue* (when in color). The *arrows* show the dynamics in a neighborhood of the bifurcation point

3 Derivation of the Conditions for the Occurrence of the Period-Doubling Bifurcation

The goal of this section is to present a set of conditions, which guarantee the existence of a period-doubling bifurcation for a given map, and which can be verified using rigorous numerics. The main tools used are the Lyapunov–Schmidt reduction [6] and the implicit function theorem.

Assume that we have a parameter dependent map $z \mapsto P(\nu, z)$, which apparently undergoes period-doubling bifurcation as the parameter ν changes. This means that a period-doubling bifurcation is numerically observed, but we do not assume that it exists. Let $z_{\mathrm{fp}}(\nu)$ be a fixed point curve for P. We assume that it is of the class C^k, with $k \geq 3$, and we can compute it and its all derivatives.

To prove the existence of the period-doubling bifurcation, we proceed as in [6]. First, we perform the Lyapunov–Schmidt reduction to obtain a function $G : \mathbb{R} \times \mathbb{R} \supset \mathrm{dom}(G) \to \mathbb{R}$, whose zeros correspond to the fixed points and period two points of P_ν then we try to describe the solution set for the equation $G(\nu, x) = 0$. Next, through some additional computation of eigenvalues, we will be able to decide about the hyperbolicity of bifurcating periodic orbits.

The basic steps of the Lyapunov–Schmidt reduction for P^2 are:

- We choose good coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. It is desirable to choose x in the approximate bifurcation direction (in the eigendirection corresponding to the -1 eigenvalue of DP_ν at the bifurcation point).
- Let $Z = [\nu_1, \nu_2] \times [x_1, x_2]$ and $Y \subset \mathbb{R}^{n-1}$ be such that the apparent bifurcation point (ν_0, x_0, y_0) belongs to the interior of $Z \times Y$.
- We need to show that there exists a function $y(\nu, x)$, defined on Z with values in Y, such that

$$y - \pi_y\big(P_\nu^2(x, y)\big) = 0 \quad \text{for } (\nu, x, y) \in Z \times Y \text{ iff } y = y(\nu, x). \tag{1}$$

- The bifurcation function $G : Z \to \mathbb{R}$ is defined by

$$G(\nu, x) = x - \pi_x\big(P_\nu^2\big(x, y(\nu, x)\big)\big). \tag{2}$$

Now, we have to find the solution set of the following equation:

$$G(\nu, x) = 0, \quad (\nu, x) \in Z. \tag{3}$$

Let $x_{\mathrm{fp}}(\nu) = \pi_x(z_{\mathrm{fp}}(\nu))$ be the x-coordinate of the fixed point curve. We assume that $[\nu_1, \nu_2] \subset \mathrm{dom}(x_{\mathrm{fp}})$ and $x_{\mathrm{fp}}([\nu_1, \nu_2]) \subset [x_1, x_2]$. Therefore, we have

$$G\big(\nu, x_{\mathrm{fp}}(\nu)\big) = 0. \tag{4}$$

The idea of solving (3) goes as follows: We introduce a new bifurcation function

$$g(\nu, x) = \frac{G(\nu, x)}{x - x_{\mathrm{fp}}(\nu)} \tag{5}$$

and then we solve equation $g(\nu, x) = 0$ by the implicit function theorem.

Observe that expression (5) defining $g(\nu, x)$ contains zero in the denominator. Moreover, usually the exact value of $x_{\mathrm{fp}}(\nu)$ is not known. Therefore, the formula (5) appears to be useless in rigorous computations. The next lemma will give us an integral representation of g, which will not contain any singularities and, therefore, it is well suited for rigorous numerics.

Lemma 1 *Assume $F : \mathbb{R}^n \to \mathbb{R}$ is C^1. Let $x, y \in \mathbb{R}^n$. Then*

$$F(x) - F(y) = \int_0^1 DF\big(t(x - y) + y\big)\,\mathrm{d}t \cdot (x - y).$$

Hence, we can define equivalently $g : [\nu_1, \nu_2] \to [x_1, x_2]$ by

$$g(\nu, x) = \int_0^1 \frac{\partial G}{\partial x}\big(\nu, t\big(x - x_{\mathrm{fp}}(\nu)\big) + x_{\mathrm{fp}}(\nu)\big)\,\mathrm{d}t. \tag{6}$$

We obtain

$$G(\nu, x) = \big(x - x_{\mathrm{fp}}(\nu)\big)g(\nu, x).$$

Therefore, we have to determine the solution set of the following equation:

$$g(\nu, x) = 0 \quad (\nu, x) \in Z, \tag{7}$$

where g is defined in (6).

In the case of a period-doubling bifurcation, we expect solutions of (7) to form a regular curve. The following lemma gives a set of conditions, which implies this fact.

Lemma 2 *Let $Z = [\nu_1, \nu_2] \times [x_1, x_2]$. Assume that $g : Z \to \mathbb{R}$ is a C^k-function, $k \geq 2$.*

Assume that

$$\frac{\partial^2 g}{\partial x^2}(Z) > 0, \tag{8}$$

$$\frac{\partial g}{\partial \nu}(Z) < 0, \tag{9}$$

$$g(\nu_1, x) > 0, \quad \textit{for } x \in [x_1, x_2], \tag{10}$$

$$g(\nu_2, x_1) > 0, \tag{11}$$

$$g(\nu_2, x_2) > 0, \tag{12}$$

$$g(\nu_2, x_0) < 0, \quad \textit{for some } x_0 \in (x_1, x_2). \tag{13}$$

Then there exist $\bar{x}_1, \bar{x}_2$, *such that* $x_1 < \bar{x}_1 < x_0 < \bar{x}_2 < x_2$ *and there exists a function* $\nu : [\bar{x}_1, \bar{x}_2] \to [\nu_1, \nu_2]$ *of class* C^k, *such that*

$$\{(\nu, x) \in Z \mid g(\nu, x) = 0\} = \{(\nu(x), x), x \in [\bar{x}_1, \bar{x}_2]\}.$$

Moreover, there exists $\bar{x}_0 \in (\bar{x}_1, \bar{x}_2)$ *such that*

$$\begin{aligned} \nu'(x) &> 0, \quad x \in (\bar{x}_0, \bar{x}_2), \\ \nu'(x) &< 0, \quad x \in (\bar{x}_1, \bar{x}_0), \\ \nu(x) &> \nu_1, \quad x \in [\bar{x}_1, \bar{x}_2], \\ \nu(\bar{x}_1) &= \nu(\bar{x}_2) = \nu_2. \end{aligned}$$

Proof Observe first that from condition (8) it follows that for any given $\nu \in [\nu_1, \nu_2]$ and any $c \in \mathbb{R}$ the equation

$$g(\nu, x) = c,$$

has at most two solutions in $[x_1, x_2]$.

From this observation and (11–13), it follows that there exist $\bar{x}_1$ and $\bar{x}_2$, such that

$$\begin{aligned} x_1 < \bar{x}_1 < x_0 &< \bar{x}_2 < x_2, \\ \{x \in [x_1, x_2] \mid g(\nu_2, x) = 0\} &= \{\bar{x}_1, \bar{x}_2\}, \\ g(\nu_2, x) &> 0, \quad \text{for } x < \bar{x}_1 \text{ or } x > \bar{x}_2, \\ g(\nu_2, x) &< 0, \quad \text{for } x \in (\bar{x}_1, \bar{x}_2). \end{aligned}$$

From the above conditions and conditions (9) and (10), it follows immediately that there exists a function $\nu : [\bar{x}_1, \bar{x}_2] \to [\nu_1, \nu_2]$, such that

$$\{(\nu, x) \in Z \mid g(\nu, x) = 0\} = \{(\nu(x), x), x \in [\bar{x}_1, \bar{x}_2]\}.$$

By the implicit function theorem, the function $\nu(x)$ is of class C^k.

There remains for us to show the existence of a unique minimum of $\nu(x)$ and its monotonicity properties.

Since $\nu(\bar{x}_1) = \nu(\bar{x}_2)$, there exists a critical point $\bar{x}_0 \in (\bar{x}_1, \bar{x}_2)$. Let $y \in (\bar{x}_1, \bar{x}_2)$ be any critical point of $\nu(x)$, i.e $\dot{\nu}(y) = 0$. We will show that $\ddot{\nu}(y) > 0$.

By differentiating twice equation $g(\nu(x), x) = 0$, we obtain

$$\frac{\partial^2 g}{\partial \nu^2}\big(\nu(x), x\big)\big(\dot{\nu}(x)\big)^2 + 2\frac{\partial^2 g}{\partial \nu \partial x}\big(\nu(x), x\big)\dot{\nu}(x) + \frac{\partial g}{\partial \nu}\big(\nu(x), x\big)\ddot{\nu}(x) + \frac{\partial^2 g}{\partial x^2}\big(\nu(x), x\big) = 0.$$

Therefore, for y, we have

$$0 = \frac{\partial g}{\partial \nu}\big(\nu(y), y\big)\ddot{\nu}(y) + \frac{\partial^2 g}{\partial x^2}\big(\nu(y), y\big),$$

$$\ddot{\nu}(y) = -\left(\frac{\partial g}{\partial \nu}\big(\nu(y), y\big)\right)^{-1} \frac{\partial^2 g}{\partial x^2}\big(\nu(y), y\big) > 0.$$

We see that all critical points are strong local minima. This implies that the set of critical points consists of just one point. □

The model for Lemma 2 is given by the function $g_1(\nu, x) = x^2 - \nu$ in the neighborhood of the point $(0, 0)$. By changing signs of ν and g, we obtain the following model functions: $g_2(\nu, x) = \nu + x^2$, $g_3(\nu, x) = \nu - x^2$ and $g_4(\nu, x) = -\nu - x^2$ for which we can state analogous lemmas.

Now we can formulate a lemma based on the implicit function theorem addressing the assumptions implying intersection of curves, solving equation $G(\nu, x) = 0$, where G arises through the Lyapunov–Schmidt reduction in the context of a period-doubling bifurcation.

Lemma 3 *Let $Z = [\nu_1, \nu_2] \times [x_1, x_2]$. Assume that $G : Z \to \mathbb{R}$ is a C^k-function, with* $k \geq 3$.

Assume that there exists a C^k-function $x_{\text{fp}} : [\nu_1, \nu_2] \to (x_1, x_2)$, such that $G(\nu, x_{\text{fp}}(\nu)) = 0$ *for* $\nu \in [\nu_1, \nu_2]$.

Assume that

$$\frac{\partial^3 G}{\partial x^3}(Z) > 0, \tag{14}$$

$$\frac{\partial^2 G}{\partial x \partial \nu}(Z) + \frac{\partial^2 G}{\partial x^2}(Z)x'_{\text{fp}}\big([\nu_1, \nu_2]\big) \cdot [0, 1] < 0. \tag{15}$$

We assume that the following conditions are satisfied for some $x_1 \leq \delta_1 < x_{\text{fp}}(\nu_1) < \delta_2 \leq x_2$:

$$G\big(\nu_1, [\delta_2, x_2]\big) > 0, \qquad G\big(\nu_1, [x_1, \delta_1]\big) < 0, \tag{16}$$

$$\frac{\partial G}{\partial x}\big(\nu_1, [\delta_1, \delta_2]\big) > 0, \tag{17}$$

$$G(\nu_2, x_2) > 0, \qquad G(\nu_2, x_1) < 0, \tag{18}$$

$$\frac{\partial G}{\partial x}\big(\nu_2, x_{\text{fp}}(\nu_2)\big) < 0. \tag{19}$$

Then there exist $x_1 < \bar{x}_1 < \bar{x}_2 < x_2$, *such that* $x_{\text{fp}}(\nu_2) \in (\bar{x}_1, \bar{x}_2)$, *and a function* $\nu : [\bar{x}_1, \bar{x}_2] \to [\nu_1, \nu_2]$ *of class* C^{k-1}, *such that*

$$\begin{aligned}\big\{(\nu, x) \in Z \mid G(\nu, x) = 0\big\} &= C_{\text{fp}} \cup C_{\text{per}} \\ &= \big\{\big(\nu, x_{\text{fp}}(\nu)\big),\ \nu \in [\nu_1, \nu_2]\big\} \cup \big\{\big(\nu(x), x\big), x \in [\bar{x}_1, \bar{x}_2]\big\}\end{aligned} \tag{20}$$

and the intersection of curves C_{fp} *and* C_{per} *contains exactly one point.*

Moreover, there exists $\bar{x}_0 \in (\bar{x}_1, \bar{x}_2)$ *such that*

$$\begin{aligned} \nu'(x) &> 0, \quad x \in (\bar{x}_0, \bar{x}_2), \\ \nu'(x) &< 0, \quad x \in (\bar{x}_1, \bar{x}_0), \\ \nu(x) &> \nu_1, \quad x \in [\bar{x}_1, \bar{x}_2], \\ \nu(\bar{x}_1) &= \nu(\bar{x}_2) = \nu_2. \end{aligned}$$

Proof For the proof, we want to apply Lemma 2. To this end, we define g as in (6).

We start by showing that (14) and (15) imply that $\frac{\partial^2 g}{\partial x^2}(Z) > 0$ and $\frac{\partial g}{\partial \nu}(Z) < 0$, respectively.

We have

$$\frac{\partial^2 g}{\partial x^2}(\nu, x) = \int_0^1 \frac{\partial^3 G}{\partial x^3}\big(\nu, t\big(x - x_{\text{fp}}(\nu)\big) + x_{\text{fp}}(\nu)\big) t^2 \,\mathrm{d}t.$$

Hence, from (14), we obtain immediately that $\frac{\partial^2 g}{\partial x^2}(Z) > 0$.

$$\begin{aligned}\frac{\partial g}{\partial \nu}(\nu, x) &= \int_0^1 \bigg(\frac{\partial^2 G}{\partial x \partial \nu}\big(\nu, t\big(x - x_{\text{fp}}(\nu)\big) + x_{\text{fp}}(\nu)\big) \\ &\quad + \frac{\partial^2 G}{\partial x^2}\big(\nu, t\big(x - x_{\text{fp}}(\nu)\big) + x_{\text{fp}}(\nu)\big)(1 - t)x'_{\text{fp}}(\nu)\bigg) \,\mathrm{d}t \\ &\in \frac{\partial^2 G}{\partial x \partial \nu}(Z) + \frac{\partial^2 G}{\partial x^2}(Z) x'_{\text{fp}}\big([\nu_1, \nu_2]\big) \cdot [0, 1].\end{aligned}$$

This and (15) imply that $\frac{\partial g}{\partial \nu}(Z) < 0$.

To obtain condition (10), we need to split the interval $[x_1, x_2]$ into three parts $[x_1, \delta_1]$, $[\delta_1, \delta_2]$, and $[\delta_2, x_2]$, so that in the middle part we have the zero of $G(\nu_1, \cdot)$, and we need to use there the integral representation of g. On the remaining parts, it is enough to verify the signs of G. Hence, we see that conditions (16–17) imply (10).

The remaining assumptions in Lemma 2 follow easily from (18–19). Now, we use Lemma 2 to obtain the function $\nu(x)$ and the condition (20).

There remains for us to show that the curves C_{fp} and C_{per}, defined by (20), intersect in exactly one point. Observe that these curves intersect, because the curve C_{fp} cuts Z into two pieces and the end points of the second curve belong to different components, which follows directly from the fact that $x_{\text{fp}}(\nu_2) \in (\bar{x}_1, \bar{x}_2)$.

Now we turn to the question of the uniqueness of the intersection point.

Let $\alpha, \beta \in [\nu_1, \nu_2]$, $\alpha < \beta$. For $t \in [0,1]$, let $\nu_t = t\alpha + (1-t)\beta$ and $x_t = t x_{\text{fp}}(\alpha) + (1-t)x_{\text{fp}}(\beta)$. Observe that for each $t \in [0,1]$ the point (ν_t, x_t) belongs to Z. Let $\theta \in (\alpha, \beta)$ be such that $x'_{\text{fp}}(\theta) = \frac{x_{\text{fp}}(\alpha) - x_{\text{fp}}(\beta)}{\alpha - \beta}$. We have

$$
\begin{aligned}
&\frac{\partial G}{\partial x}\big(\alpha, x_{\text{fp}}(\alpha)\big) - \frac{\partial G}{\partial x}\big(\beta, x_{\text{fp}}(\beta)\big) \\
&\quad = \int_0^1 \left(\frac{\partial^2 G}{\partial x \partial \nu}(\nu_t, x_t)(\alpha - \beta) + \frac{\partial^2 G}{\partial x^2}(\nu_t, x_t)\big(x_{\text{fp}}(\alpha) - x_{\text{fp}}(\beta)\big) \right) \mathrm{d}t \\
&\quad = \left(\int_0^1 \left(\frac{\partial^2 G}{\partial x \partial \nu}(\nu_t, x_t) + \frac{\partial^2 G}{\partial x^2}(\nu_t, x_t) x'_{\text{fp}}(\theta) \right) \mathrm{d}t \right)(\alpha - \beta) \\
&\quad \in \left(\frac{\partial^2 G}{\partial x \partial \nu}(Z) + \frac{\partial^2 G}{\partial x^2}(Z) x'_{\text{fp}}\big([\nu_1, \nu_2]\big) \cdot [0,1] \right)(\alpha - \beta).
\end{aligned}
$$

Therefore, from the above computations and the assumption (15), it follows that the function $\nu \mapsto \frac{\partial G}{\partial x}(\nu, x_{\text{fp}}(\nu))$ is injective on $[\nu_1, \nu_2]$. Observe that from (6) it follows that if $(\nu, x_{\text{fp}}(\nu)) \in C_{\text{fp}} \cap C_{\text{per}}$, then $\frac{\partial G}{\partial x}(\nu, x_{\text{fp}}(\nu)) = 0$, so the intersection of C_{fp} and C_{per} contains at most one point. □

Observe that in the above lemma we cannot make the claim that the intersection point of the curves which solve equation $G(\nu, x) = 0$ is exactly in $(\nu(\bar{x}_0), \bar{x}_0)$. This can be easily seen in the following example. Let $G(\nu, x) = (x-1)(x^2 - \nu)$, $x_1 = -2$, $x_2 = 2$, $\nu_1 = -1$ and $\nu_2 = 1$. It is easy to see that all assumptions of Lemma 3 are satisfied, but the intersection of the curves $(\nu(x) = x^2, x)$ and $(\nu, x(\nu) = 1)$ is not $(0,0)$. On the other hand, in the context of a period-doubling bifurcation, the intersection point is $(\nu(\bar{x}_0), \bar{x}_0)$. But we cannot infer such a conclusion from Lemma 3 and we need to use the information about the dynamical origin of the function G. Now, we state the theorem which addresses this issue.

Theorem 4 *Let $P_\nu : \mathbb{R}^n \supset \operatorname{dom}(P_\nu) \to \mathbb{R}^n$, where $\nu \in I \subset \mathbb{R}$, be a one-parameter family of maps of class C^k $(k \geq 3)$, both with respect to the parameter ν and $x \in \mathbb{R}^n$.*

Let $Z = [\nu_1, \nu_2] \times [x_1, x_2]$ and $Y \subset \mathbb{R}^{n-1}$ be the closure of an open set, such that $[x_1, x_2] \times Y \subset \operatorname{dom}(P_\nu^2)$ for $\nu \in [\nu_1, \nu_2]$. Assume that

A1 *For any $(\nu, x) \in Z$ there exists a unique $y = y(\nu, x) \in \operatorname{int} Y$, such that $y - \pi_y(P_\nu^2(x, y)) = 0$. Moreover, we assume that $y : Z \to Y$ is C^k.*

A2 *There exists a C^k-function $x_{\text{fp}} : [\nu_1, \nu_2] \to (x_1, x_2)$, such that for $\nu \in [\nu_1, \nu_2]$ there holds*

$$\operatorname{Fix}\big(P_\nu, [x_1, x_2] \times Y\big) = \big\{\big(x_{\text{fp}}(\nu), y\big(\nu, x_{\text{fp}}(\nu)\big)\big)\big\}. \tag{21}$$

A3 *Let*

$$G(\nu, x) = x - \pi_x\big(P_\nu^2\big(x, y(\nu, x)\big)\big), \quad \text{for } (\nu, x) \in Z.$$

Assume that G *and* x_{fp} *satisfy assumptions of Lemma 3 and let* $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_0$ *and* $\nu : [\bar{x}_1, \bar{x}_2] \to [\nu_1, \nu_2]$ *be as in the assertion of Lemma 3.*

Then the fixed point set of P_ν^2 *for* $\nu \in [\nu_1, \nu_2]$, *i.e.*,

$$\big\{(\nu, x, y) \in Z \times Y \mid P_\nu^2(x, y) = (x, y)\big\}$$

is equal to the union of the fixed point set for P_ν

$$\mathrm{Per}_1 = \big\{\big(\nu, x_{\text{fp}}(\nu), y\big(\nu, x_{\text{fp}}(\nu)\big)\big) \mid \nu \in [\nu_1, \nu_2]\big\}$$

and the period-2 points set

$$\mathrm{Per}_2 = \big\{\big(\nu(x), x, y\big(\nu(x), x\big)\big) \mid x \in [\bar{x}_1, \bar{x}_2]\big\}.$$

For any point $(\nu(x), x, y(\nu(x), x)) \in \mathrm{Per}_2$ *the point* $(\nu(x), P_{\nu(x)}(x, y(\nu(x), x)))$ *also belongs to* Per_2, *and for* $\xi \in (\nu(\bar{x}_0), \nu_2]$ *the set* $\mathrm{Per}_2 \cap \{\nu = \xi\}$ *is a period-2 orbit for* P_ξ.

Sets Per_1 *and* Per_2 *have exactly one common point* (ν_b, z_b) *given by*

$$(\nu_b, z_b) = \big(\nu(\bar{x}_0), \big(\bar{x}_0, y\big(\nu(\bar{x}_0), \bar{x}_0\big)\big)\big).$$

Moreover, the projections of Per_1 *and* Per_2 *onto the* (ν, x)*-plane have exactly one common point* (ν_b, x_b) *given by*

$$(\nu_b, x_b) = \big(\nu(\bar{x}_0), \bar{x}_0\big).$$

Proof From the construction of the bifurcation function G and our assumptions, we immediately obtain that

$$\big\{(\nu, x, y) \in Z \times Y \mid P_\nu^2(x, y) = (x, y)\big\} = \mathrm{Per}_1 \cup \mathrm{Per}_2.$$

From Lemma 3, we know that projections onto the (ν, x)-plane of sets Per_1 and Per_2 intersect in exactly one point, say $(\bar{\nu}, \bar{x})$. Observe that the point $(\bar{\nu}, \bar{x}, y(\bar{\nu}, \bar{x}))$ belongs to the intersection of Per_1 and Per_2.

We need to show that $(\bar{\nu}, \bar{x}) = (\nu(\bar{x}_0), \bar{x}_0)$. We will show that the function $x \mapsto \nu(x)$ has a local extremum at $\bar{x}$. This will imply that $\bar{x} = \bar{x}_0$, because by Lemma 3 $\bar{x}_0$ is the only local extremum of $\nu(x)$.

We reason by contradiction. Let us assume that $\nu'(\bar{x}) \neq 0$. Let $U = U_\nu \times U_x \times U_y$, where $U_\nu \subset [\nu_1, \nu_2]$, $U_x \subset [x_1, x_2]$ and $U_y \subset Y$, be a neighborhood of $(\bar{\nu}, \bar{x}, y(\bar{\nu}, \bar{x}))$, such that

$$\begin{gathered} P_\nu(x, y) \in \mathrm{int}\big([x_1, x_2] \times Y\big), \quad \text{for } (\nu, x, y) \in U, \\ \nu(a) \neq \nu(b), \quad \text{for } a, b \in U_x \text{ and } a \neq b. \end{gathered} \tag{22}$$

Such U exists because $(\bar{x}, y(\bar{\nu}, \bar{x}))$ is a fixed point for $P_{\bar{\nu}}$ and $(\bar{x}, y(\bar{\nu}, \bar{x})) \in \mathrm{int}([x_1, x_2] \times Y)$.

Let us take $v \in U_x$, such that $v \neq \bar{x}$. Then $(v, y(\nu(v), v))$ is not a fixed point for $P_{\nu(v)}$. Points $(v, y(\nu(v), v))$ and $P_{\nu(v)}(v, y(\nu(v), v))$ are different. Both belong to Z and are period-2 points for $P_{\nu(v)}$. Therefore, they both belong to Per_2 and

$$\nu\big(\pi_x P_{\nu(v)}\big(v, y\big(\nu(v), v\big)\big)\big) = \nu(v). \tag{23}$$

Observe that from the continuity, it follows that

$$\lim_{v \to \bar{x}} \pi_x P_{\nu(v)}\big(v, y\big(\nu(v), v\big)\big) = \pi_x P_{\nu(\bar{x})}\big(\bar{x}, y\big(\nu(\bar{x}), \bar{x}\big)\big) = \bar{x}.$$

From the above observation, it follows that for v sufficiently close to $\bar{x}$ the points v and $\pi_x P_{\nu(v)}(v, y(\nu(v), v))$ are in U_x. But in this situation, condition (23) contradicts (22). This proves that $\bar{x} = \bar{x}_0$.

It remains to prove that for any point $(\nu(x), x, y(\nu(x), x)) \in \mathrm{Per}_2$ the point $(\nu(x), P_{\nu(x)}(x, y(\nu(x), x)))$ also belongs to Per_2 and for $\xi \in (\nu(\bar{x}_0), \nu_2]$ the set $\mathrm{Per}_2 \cap \{\nu = \xi\}$ is a period-2 orbit for P_ξ.

By Lemma 3, Per_2 is the union of two curves $u, l : [\nu(\bar{x}_0), \nu_2] \to Z \times Y$, such that

$$u\big(\nu(\bar{x}_0)\big) = l\big(\nu(\bar{x}_0)\big) = \big(\bar{x}_0, y\big(\nu(\bar{x}_0), \bar{x}_0\big)\big), \tag{24}$$

$$\pi_x u(\nu_2) = \bar{x}_2, \qquad \pi_x l(\nu_2) = \bar{x}_1, \tag{25}$$

$$\frac{\mathrm{d}\pi_x(u(\nu))}{\mathrm{d}\nu} > 0, \qquad \frac{\mathrm{d}\pi_x(l(\nu))}{\mathrm{d}\nu} < 0. \tag{26}$$

Let us consider the set

$$K = \big\{\nu \in \big[\nu(\bar{x}_0), \nu_2\big] \,\big|\, \text{for } \nu(\bar{x}_0) \le \xi \le \nu \text{ holds } P_\xi\big(u(\xi)\big) = l(\xi)\big\}. \tag{27}$$

Observe that $\nu(\bar{x}_0) \in K$, because $l(\nu(\bar{x}_0)) = u(\nu(\bar{x}_0))$ is the fixed point for $P_{\nu(\bar{x}_0)} \in \mathrm{int}([x_1, x_2] \times Y)$. Hence, for $\xi > \nu(\bar{x}_0)$, ξ close to $\nu(\bar{x}_0)$ we have $l(\xi) \in \mathrm{int}([x_1, x_2] \times Y)$ and $P_\xi(u(\xi)) \in \mathrm{int}([x_1, x_2] \times Y)$. Therefore, $P_\xi(u(\xi)) = l(\xi)$ and $\xi \in K$, because $P_\xi(u(\xi))$ is a period-2 point in $\mathrm{int}([x_1, x_2] \times Y)$ different from $u(\xi)$. Observe that the same argument applies to $\xi = \sup K$. Therefore, $K = [\nu(\bar{x}_0), \nu_2]$. □

3.1 Hyperbolicity of Bifurcating Solutions

The Lyapunov–Schmidt projection does not give any direct information about the dynamical character of the bifurcating objects. The required information concerning the hyperbolicity is of course contained in the spectra of DP_ν and DP_ν^2 and its derivatives. Below, we present a lemma addressing this issue.

Lemma 5 *Assume that $P_\nu : \mathbb{R}^n \to \mathbb{R}^n$ for $\nu \in [\nu_1, \nu_2]$ satisfies all assumptions of Theorem 4 and in the sequel we will use all the notation introduced there.*

Let $z_{\mathrm{fp}}(\nu) = (x_{\mathrm{fp}}(\nu), y(\nu, x_{\mathrm{fp}}(\nu)))$.

Fixed points: *Assume that there exists $\epsilon > 0$, such that for all $\nu \in [\nu_1, \nu_2]$ there holds*

$$\mathrm{Sp}\big(DP_\nu\big(z_{\mathrm{fp}}(\nu)\big)\big) = A_\nu \cup B_\nu \cup \big\{\lambda(\nu)\big\},$$

where $\lambda(\nu) \in \mathbb{R}$ *has multiplicity one,* $A_\nu \subset \{\alpha \in \mathbb{C}, |\alpha| < 1 - \epsilon\}$ *and* $B_\nu \subset \{\beta \in \mathbb{C}, |\beta| > 1 + \epsilon\}$. *Moreover, we assume that*

$$\lambda(\nu_1) \in (-1, 1), \tag{28}$$

$$\lambda(\nu_2) < -1, \tag{29}$$

$$\frac{d\lambda}{d\nu}\big(z_{\text{fp}}(\nu)\big) < 0, \quad \nu \in [\nu_1, \nu_2]. \tag{30}$$

Then the fixed points for P_ν *on curve* $z_{\text{fp}}(\nu)$ *are hyperbolic for* $\nu \in [\nu_1, \nu_2] \setminus \{\nu(\bar{x}_0)\}$ *and*

$$\dim W^u\big(z_{\text{fp}}(\nu^-), P_{\nu^-}\big) + 1 = \dim W^u\big(z_{\text{fp}}(\nu^+), P_{\nu^+}\big)$$

for any $\nu_1 \le \nu^- < \nu(\bar{x}_0) < \nu^+ \le \nu_2$.

Period-2 points: *Assume that there exists* $\epsilon > 0$, *such that on the* Per_2 *curve* (*i.e., for* $x \in [\bar{x}_1, \bar{x}_2]$) *there holds*

$$\text{Sp}\big(DP^2_{\nu(x)}\big(x, y\big(\nu(x), x\big)\big)\big) = A_x \cup B_x \cup \big\{\gamma(x)\big\}$$

where $\gamma(x) \in \mathbb{R}$ *has multiplicity one,* $A_x \subset \{\alpha \in \mathbb{C}, |\alpha| < 1 - \epsilon\}$ *and* $B_x \subset \{\beta \in \mathbb{C}, |\beta| > 1 + \epsilon\}$. *Moreover, we assume that*

$$\frac{d^2\gamma}{dx^2}(x) < 0, \quad x \in [\bar{x}_1, \bar{x}_2], \tag{31}$$

$$0 < \gamma(\bar{x}_1) < 1. \tag{32}$$

Then for $x \in [\bar{x}_1, \bar{x}_2] \setminus \{\bar{x}_0\}$ *the period two points* $z_d(x) = (x, y(\nu(x), x))$ *for* $P_{\nu(x)}$ *are hyperbolic and*

$$\gamma(\bar{x}_0) = 1,$$

$$0 < \gamma(x) < 1,$$

$$\dim W^s\big(z_d(x), P^2_{\nu(x)}\big) = \dim W^s\big(z_{\text{fp}}(\nu^-), P_{\nu^-}\big)$$

for any $\nu_1 \le \nu^- < \nu(\bar{x}_0)$ *and* $x \in [\bar{x}_1, \bar{x}_2] \setminus \{\bar{x}_0\}$.

Proof Let us consider first the fixed points part. From conditions (28–30), it follows that there exists a unique $\bar{\nu} \in [\nu_1, \nu_2]$, such that $\lambda(\bar{\nu}) = -1$. We will show that $\bar{\nu} = \nu_b (= \nu(\bar{x}_0))$.

We reason by contradiction. Assume that $\lambda(\nu_b)) \neq -1$; then for some small interval I containing ν_b in its interior, all the fixed points $z_{\text{fp}}(\nu)$ will be hyperbolic. Therefore, there will be no period two points for P_ν for $\nu \in I$ in some small neighborhood of the curve $(\nu, z_{\text{fp}}(\nu))$. But this contradicts the fact (see Theorem 4) that (ν_b, z_b) is the intersection point of the fixed point branch with the period-two branch. Hence, $\lambda(\nu) = -1$ for $\nu \in [\nu_1, \nu_2]$ iff $\nu = \nu_b$. Moreover, by conditions (28, 30) $\lambda(\nu) < -1$ for $\nu_2 \ge \nu > \nu_b = \nu(\bar{x}_0)$ and $\lambda(\nu) \in (-1, 1)$ for $\nu_1 \le \nu < \nu_b$. This finishes the proof of the fixed points part of the lemma.

For the proof of the second part, observe that at the bifurcation point there holds

$$\lambda\big(\nu(\bar{x}_0)\big) = -1, \qquad \gamma(\bar{x}_0) = \lambda\big(\nu(\bar{x}_0)\big)^2 = 1. \tag{33}$$

We will argue that

$$\frac{d\gamma}{dx}(\bar{x}_0) = 0. \tag{34}$$

From Theorem 4, it follows that for $v_1 > \bar{x}_0$ and $v_1 - \bar{x}_0$ are sufficiently small. The point $z_2 = P_{\nu(v_1)}(v_1, y(\nu(v_1), v_1))$ also belongs to the curve Per_2 and its x-projection is converging to $\bar{x}_0$ for $v_1 \to \bar{x}_0$. Let $v_2 = \pi_x z_2$.

We have

$$v_2 < \bar{x}_0 < v_1, \quad \gamma(v_2) = \gamma(v_1). \tag{35}$$

This implies (34).

Condition (31) and (34) imply that at $\bar{x}_0$ the function $\gamma(x)$ achieves a maximum and since $\gamma(\bar{x}_1) = \gamma(\bar{x}_2)$ we see that

$$\gamma(x) \in (0,1) \quad \text{for } x \in [\bar{x}_1, \bar{x}_2] \setminus \{\bar{x}_0\}. \tag{36}$$

From the above observation, it follows that the number of eigenvalues inside the unit circle is the same for the fixed points for $\nu < \nu(\bar{x}_0)$ and the period-2 orbits. □

4 Continuation

To apply the tools described in Sect. 3 (in the part regarding the existence of the Lyapunov–Schmidt reduction), we need to prove the existence and uniqueness (locally) of a solution of the equation of the form $f(a, y) = 0$ for a given a, where $y \in \mathbb{R}^n$ and a is a parameter. Similarly, when continuing the fixed-point curve or period-2 point curve, we have to solve the existence and the local uniqueness of the solution of $x - P^i(a, x) = 0$, where a is the parameter. It turns out that both of the above mentioned tasks can be handled by the same tools.

In this section, we will discuss such tools; the first one consists of classical interval analysis tools: the interval Newton method [1, 17, 19] and the Krawczyk method [1, 12, 19], which can be seen as clever interval versions of the standard Newton method. These methods work very efficiently in the situation where the solution sought is well isolated from other solutions and it requires C^1-estimates only. The second approach, which is based on the implicit function theorem deals with the situation where we are close to the bifurcation point and, therefore, there are several solutions close to one another, as in the case of period-doubling, when we have the fixed point and period-two points in a small neighborhood.

4.1 Two Methods for Proving the Existence of Zeros for a Map

Let $A \subset \mathbb{R}^n$. By $[A]_I$, we will denote an interval enclosure of the set A, i.e., the smallest set of the form $[A]_I = [a_1, b_1] \times \cdots \times [a_n, b_n]$, such that $A \subset [A]_I$, where

$a_i, b_i \in \mathbb{R}^n \cup \{\pm\infty\}$. By an interval matrix M, we mean a subset of $\mathbb{R}^{n\times n}$ such that $[M]_I = M$. The inverse of an interval matrix M is defined by

$$M^{-1} = \left[C^{-1} : C \in M\right]_I$$

provided each $C \in M$ is invertible.

Theorem 6 (Interval Newton Method [1, 17, 19]) *Let $X \subset \mathbb{R}^n$ be a convex, compact set, let $f\colon X \to \mathbb{R}^n$ be smooth and fix a point $x \in X$. Let us denote by*

$$N(f, X, x) = \left[x - \left[Df(X)\right]_I^{-1} f(x)\right]_I \tag{37}$$

the Interval Newton Operator for a map f on the set X with fixed $x \in X$. Then

- *If $N(f, X, x) \subset \operatorname{int} X$ then the map f has a unique zero in X. Moreover, if x_* is this unique zero of f in X, then $x_* \in N(f, X, x)$.*
- *If $N(f, X, x) \cap X = \emptyset$, then the map f has no zeros in X.*

Theorem 7 (Interval Krawczyk Method [1, 12, 19]) *Let $X \subset \mathbb{R}^n$ be a convex, compact set, let $f\colon X \to \mathbb{R}^n$ be smooth and fix a point $x \in X$. Let $C \in \mathbb{R}^{n\times n}$ be an isomorphism. Let us denote by*

$$K(f, C, X, x) = \left[x - Cf(x) + \left(\mathrm{Id} - C \cdot \left[Df(X)\right]_I\right)(X - x)\right]_I \tag{38}$$

the Interval Krawczyk Operator for a map f on the set X with fixed $x \in X$ and matrix C. Then

- *If $K(f, C, X, x) \subset \operatorname{int} X$, then the map f has a unique zero in X. Moreover, if x_* is this unique zero of f in X, then $x_* \in K(f, C, X, x)$.*
- *If $K(f, C, X, x) \cap X = \emptyset$, then the map f has no zeros in X.*

4.2 Continuation Close to the Bifurcation Point

The next lemma gives us a tool for extending the curves of period-two points further from the bifurcation point. Schematically, the situation is shown in Fig. 2.

Lemma 8 *Let $X = [x_1, x_2]$. Assume $f_\nu\colon \mathbb{R} \times \mathbb{R}^{n-1} \supset X \times Y \to \mathbb{R}^n$, $\nu \in [\nu_1, \nu_2]$ is a C^k function with respect to both its argument and the parameter, with $k \geq 3$, such that*

1. *For $\nu \in [\nu_1, \nu_2]$, there exists a unique fixed point $(x_{\mathrm{fp}}(\nu), y_{\mathrm{fp}}(\nu))$ for f_ν in $X \times Y$.*
2. *For all $(\nu, x) \in [\nu_1, \nu_2] \times X$, there exists a unique $y(\nu, x) \in \operatorname{int} Y$ solving equation $y - \pi_y(f_\nu^2(x, y)) = 0$ and the map $y\colon [\nu_1, \nu_2] \times X \to Y$ is of class C^k.*
3. *The map $G(\nu, x) = x - \pi_x(f_\nu^2(x, y(\nu, x)))$ satisfies*

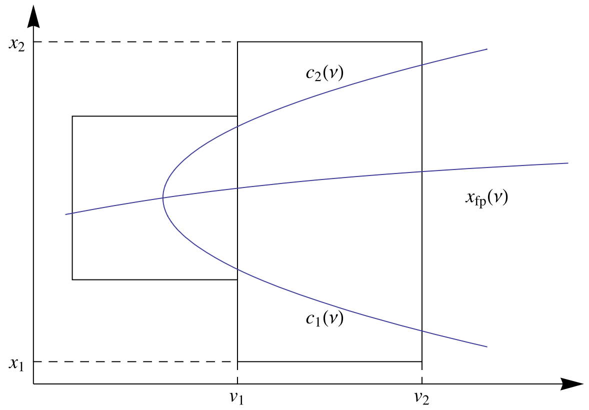
$$\frac{\partial^3 G}{\partial x^3}(\nu, x) > 0, \quad \textit{for } \nu \in [\nu_1, \nu_2],\ x \in X, \tag{39}$$

$$G(\nu, x_1) < 0, \qquad G(\nu, x_2) > 0, \quad \textit{for } \nu \in [\nu_1, \nu_2], \tag{40}$$

$$\forall \nu \in [\nu_1, \nu_2]\ \exists x_- \in \left(x_{\mathrm{fp}}(\nu), x_2\right) \quad G(\nu, x_-) < 0, \tag{41}$$

$$\forall \nu \in [\nu_1, \nu_2]\ \exists x_+ \in \left(x_1, x_{\mathrm{fp}}(\nu)\right) \quad G(\nu, x_+) > 0. \tag{42}$$

Fig. 2 Location of the objects which appear in Lemma 8



Then there exist two C^k curves $c_1, c_2 \colon [\nu_1, \nu_2] \to \mathbb{R}^n$ such that for $\nu \in [\nu_1, \nu_2]$ there holds $\pi_x(c_1(\nu)) < x_{\mathrm{fp}}(\nu) < \pi_x(c_2(\nu))$ and $c_i(\nu)$ is a period two point for f_ν, $i = 1, 2$.

Moreover, if for some $\nu_0 \in \{\nu_1, \nu_2\}$ there holds $f_{\nu_0}(c_1(\nu_0)) = c_2(\nu_0)$ or $f_{\nu_0}(c_2(\nu_0)) = c_1(\nu_0)$, then for all $\nu \in [\nu_1, \nu_2]$

$$f_\nu\big(c_1(\nu)\big) = c_2(\nu), \qquad f_\nu\big(c_2(\nu)\big) = c_1(\nu). \tag{43}$$

Proof The second assumption and (39) imply that for a fixed ν the map f_ν^2 has at most three fixed points in $X \times Y$. From the first assumption, we know that f_ν has a unique fixed point $(x_{\mathrm{fp}}(\nu), y_{\mathrm{fp}}(\nu))$ in $X \times Y$. Therefore, any zero of $G(\nu, \cdot)$ which is different from $(\nu, x_{\mathrm{fp}}(\nu))$ corresponds to a period-two point of f_ν. From the continuity of G and from (40–42), it follows that G has one zero in each of the intervals $x^1_{\mathrm{per}}(\nu) \in (x_1, x_{\mathrm{fp}}(\nu))$ and $x^2_{\mathrm{per}}(\nu) \in (x_{\mathrm{fp}}(\nu), x_2)$. It is easy to see that the functions x^i_{per} are continuous for $i = 1, 2$. We set $c_i(\nu) = (x^i_{\mathrm{per}}(\nu), y(\nu, x^i_{\mathrm{per}}(\nu))$.

We will show the smoothness of x^i_{per}, which together with the assumption that $y(x, \nu)$ is C^k implies the smoothness of c_i. It is enough to show that

$$\frac{\partial G}{\partial x}\big(\nu, x^i_{\mathrm{per}}(\nu)\big) \neq 0,$$

because then we can apply the implicit function theorem to obtain the required differentiability. Let us fix $\nu \in [\nu_1, \nu_2]$. Observe that condition (39) implies that for any fixed ν the function $x \mapsto \frac{\partial G}{\partial x}(\nu, x)$ has at most two zeros in $[x_1, x_2]$. From the remaining assumptions, it is clear that on the intervals $[x_1, x_{\mathrm{fp}}(\nu)]$ and $[x_{\mathrm{fp}}(\nu), x_2]$ the function $x \mapsto G(\nu, x)$ has a strictly positive maximum and a strictly negative minimum, respectively. Therefore, these extremal points are zeros of $\frac{\partial G}{\partial x}(\nu, x)$ and obviously they are different from the points $x^i_{\mathrm{per}}(\nu)$, which are zeros of $G(\nu, \cdot)$. Hence, we have shown that $\frac{\partial G}{\partial x}(\nu, x^i_{\mathrm{per}}(\nu)) \neq 0$.

Assume that $\nu_0 = \nu_1$ (the other case is analogous). From the implicit function theorem, it follows that for some $\nu' > \nu_1$ condition (43) is satisfied for $\nu_1 \le \nu < \nu'$. Let ν_m be the supremum of such $\nu' \le \nu_2$. It is easy to see that $\nu_m = \nu_2$, because at ν_m

it is also satisfied by continuity and the implicit function theorem allows us to extend the range of ν satisfying (43) to the right if $\nu_m < \nu_2$. □

5 Extracting the Dynamical Information from the Lyapunov–Schmidt Reduction

As was mentioned already in Sect. 3.1, the Lyapunov–Schmidt projection does not give us any direct information about the dynamics of bifurcating solutions regarding the invariant manifolds of the bifurcating objects as required by Definition 4. In this section, following the ideas of de Oliveira and Hale [9, 21], we show that the information obtained from the Lyapunov–Schmidt reduction and the spectrum of the bifurcating fixed point curve is enough to say precisely what are the dynamics in the neighborhood of the bifurcation point.

Our argument follows the ideas from [6, Chap. 9, Theorems 3.1 and 4.2], where an analogous problem was considered for fixed points for ODEs and periodic orbits for periodically forced ODEs. The notion of the central manifold [11] plays a crucial role in this proof.

Theorem 9 *Let $P_\nu : \mathbb{R}^n \to \mathbb{R}^n$ for $\nu \in [\nu_1, \nu_2]$ be a C^k-map ($k \geq 3$) both with respect to ν and its arguments. Assume that on the set $K = [\nu_1, \nu_2] \times [x_1, x_2] \times Y$, where $Y \subset \mathbb{R}^{n-1}$ is the closure of an open set, we are able to perform the Lyapunov–Schmidt reduction and verify the assumptions of Theorem 4. Let (ν_b, z_b) be the bifurcation point and $z_{\text{fp}}(\nu) = (x_{\text{fp}}(\nu), y(\nu, x_{\text{fp}}(\nu))$ be the fixed point curve for P_ν in K.*

Let v be the eigenvector of $DP_{\nu_b}(z_b)$ corresponding to the eigenvalue -1. We assume that $\pi_x v \neq 0$.

Assume that there exists $\epsilon > 0$, such that for all $\nu \in [\nu_1, \nu_2]$ there holds

$$\operatorname{Sp}\big(DP_\nu\big(z_{\text{fp}}(\nu)\big)\big) = A_\nu \cup B_\nu \cup \big\{\lambda(\nu)\big\},$$

where $\lambda(\nu) \in \mathbb{R}$ has multiplicity one, $A_\nu \subset \{\alpha \in \mathbb{C}, |\alpha| < 1 - \epsilon\}$ and $B_\nu \subset \{\beta \in \mathbb{C}, |\beta| > 1 + \epsilon\}$. Moreover, we assume that

$$\lambda(\nu_1) \in (-1, 1),$$
$$\lambda(\nu_2) < -1.$$

Then the map P has a period-doubling bifurcation at $(\nu_b, x_b, y(\nu_b, x_b))$.

Proof Let $\nu : [\bar{x}_1, \bar{x}_2] \to [\nu_1, \nu_2]$ be the function from assumption A3 of Theorem 4 (in fact of Lemma 3), which is assumed to be satisfied. In the notation used in Theorem 4, we have $(\nu_b, z_b) = (\nu(\bar{x}_0), (\bar{x}_0, y(\nu(\bar{x}_0), \bar{x}_0)))$. Let $c_1(\nu)$ and $c_2(\nu)$ be, respectively, the lower and upper branches of the graph of the function $x \to \nu(x)$ giving period-2 points; see Fig. 3.

Let us define a map $H : K \to \mathbb{R} \times \mathbb{R}^n$ by

$$H(\nu, z) = \big(\nu, P(\nu, z)\big).$$

Consider the spectrum of $DH(\nu_b, z_b)$. It is easy to see that $+1$ is an eigenvalue of $DH(\nu_b, z_b)$ of multiplicity one, $\lambda = -1$ has also multiplicity one and all other eigenvalues are off the unit circle.

We apply the center manifold theorem [7, 10, 11] to H in the neighborhood of (ν_b, z_b). Therefore, there exists a neighborhood M of (ν_b, z_b) and a two-dimensional center manifold $W^c \subset M$ such that

$$\begin{aligned} &\forall(\nu, z) \in W^c \quad \text{if } H^i(\nu, z) \in M \quad \text{then } H(\nu, z) \in W_c, \quad \text{for } i = -1, 1, \\ &\text{Inv}(M, H) \subset W^c, \end{aligned}$$

W^c is tangent at (ν_b, z_b) to the subspace spanned by vectors $\{(1, 0), (0, v)\} \subset \mathbb{R} \times \mathbb{R}^n$. Observe that from our assumption about v, i.e., $\pi_x(v) \neq 0$, it follows that we can use on W^c in the neighborhood of (ν_b, z_b) the same coordinates (ν, x) as in the Lyapunov–Schmidt reduction. There exists a neighborhood of (ν_b, z_b) denoted by $U = [\tilde{\nu}_1, \tilde{\nu_2}] \times [\tilde{x}_1, \tilde{x}_2] \times \tilde{Y} \subset M \cap K$ and C^k-functions $h : [\tilde{\nu}_1, \tilde{\nu_2}] \times [\tilde{x}_1, \tilde{x}_2] \to \tilde{Y}$ and $f : [\tilde{\nu}_1, \tilde{\nu_2}] \times [\tilde{x}_1, \tilde{x}_2] \to \mathbb{R}$ satisfying

$$\begin{aligned} W^c &= \{(\nu, x, h(\nu, x))\}, \\ P(\nu, x, h(\nu, x)) &= (f(\nu, x), h(\nu, f(\nu, x))), \\ \text{Inv}(U, H) &\subset W^c. \end{aligned} \tag{44}$$

Let us stress that the dynamics of P_ν in W^c is one-dimensional, namely that of $x \mapsto f(\nu, x)$.

From the Lyapunov–Schmidt reduction, we know that a point $(\nu, x, y) \in U$ has period one or two with respect to map H iff $y = y(\nu, x)$ and $G(\nu, x) = 0$. Let $N = [\tilde{\nu}_1, \tilde{\nu_2}] \times [\tilde{x}_1, \tilde{x}_2]$. If U is chosen to be sufficiently close to the bifurcation point, then the set $N \setminus \{(\nu, x) \mid G(\nu, x) = 0\}$ has four connected components; see Fig. 3. Namely,

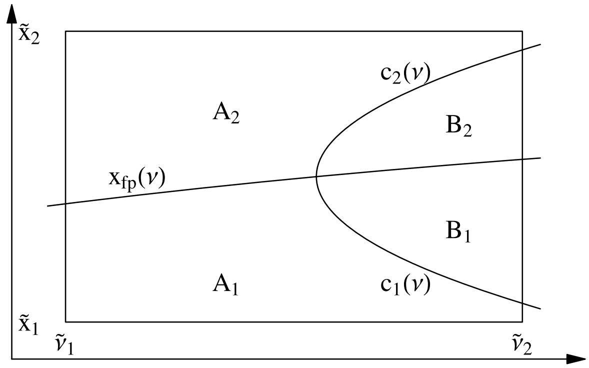
$$\begin{aligned} A_1 &= \{(\nu, x) \in N \mid ((\nu \le \nu_b) \text{ and } (x < x_{\text{fp}}(\nu))) \text{ or} \\ &\qquad ((\nu > \nu_b) \text{ and } (x < c_1(\nu)))\}, \\ A_2 &= \{(\nu, x) \in N \mid ((\nu \le \nu_b) \text{ and } (x > x_{\text{fp}}(\nu))) \text{ or} \\ &\qquad ((\nu > \nu_b) \text{ and } (x > c_2(\nu)))\}, \\ B_1 &= \{(\nu, x) \in N \mid (\nu > \nu_b) \text{ and } (x_{\text{fp}}(\nu) > x > c_1(\nu))\}, \\ B_2 &= \{(\nu, x) \in N \mid (\nu > \nu_b) \text{ and } (x_{\text{fp}}(\nu) < x < c_2(\nu))\}. \end{aligned}$$

We also require that

$$\tilde{x}_1 < c_1(\tilde{\nu}_2) < c_2(\tilde{\nu}_2) < \tilde{x}_2. \tag{45}$$

Such a neighborhood N exists, since we assumed that Theorem 4 applies. Therefore, the zeroes of the function G in the neighborhood of the bifurcation point are two intersecting curves, as presented in Fig. 3.

Fig. 3 Location of sets A_i, B_i with respect to zeros of G



On each of these components, the function $d(\nu, x) = x - f(\nu, f(\nu, x))$ must have a constant sign. Observe that on A_2 we have

$$x - f\big(\nu, f(\nu, x)\big) > 0, \quad \text{for } (\nu, x) \in A_2 \tag{46}$$

because $z_{fp}(\tilde{\nu}_1)$ is attracting on W^c and we consider the second iterate. Analogously, we obtain

$$x - f\big(\nu, f(\nu, x)\big) < 0, \quad \text{for } (\nu, x) \in A_1. \tag{47}$$

For the component B_2, we have

$$x - f\big(\nu, f(\nu, x)\big) < 0, \quad \text{for } (\nu, x) \in B_2,$$

because $x_{fp}(\bar{\nu}_2)$ is repelling on W^c and we consider the second iterate. Analogously,

$$x - f\big(\nu, f(\nu, x)\big) > 0, \quad \text{for } (\nu, x) \in B_1.$$

For a subset $Z \subset N$ by Z_ν, we will denote $Z_\nu = \{x : (\nu, x) \in N\}$. Observe that for each $\nu \in [\tilde{\nu}_1, \tilde{\nu}_2]$ there holds

$$f_\nu^2\big((A_i)_\nu\big) \cap [\tilde{x}_1, \tilde{x}_2] \subset (A_i)_\nu, \qquad f_\nu^2\big((B_i)_\nu\big) \cap [\tilde{x}_1, \tilde{x}_2] \subset (B_i)_\nu \quad i = 1, 2. \tag{48}$$

For the proof of (48), observe that map $l(\nu, x) = (\nu, f(\nu, x))$ maps connected components of $N \setminus \{G(\nu, x) = 0\}$ into connected components, i.e., for any $S \in \{A_1, A_2, B_1, B_2\}$ there exists $T = T(S) \in \{A_1, A_2, B_1, B_2\}$, such that

$$\big(l(S) \cap N\big) \subset T(S), \tag{49}$$

because

$$l\big(G^{-1}(0) \cap N\big) \cap N = G^{-1}(0) \cap N = l^{-1}\big(G^{-1}(0) \cap N\big) \cap N.$$

Observe that the relevant eigenvalue of DP_ν at $z_{fp}(\nu)$ describing the dynamics on W^c is $\lambda(\nu)$, which is real and since we consider the second iterate we see that in the neighborhood of the fixed point curve we have points mapped into the same component. This together with (49) proves (48).

From the above considerations, we obtain for $\nu \le \nu_b$

$$x_{\text{fp}}(\nu) < f\big(\nu, f(\nu, x)\big) < x, \quad \text{for } x_{\text{fp}}(\nu) < x \le \tilde{x}_2,$$
$$x_{\text{fp}}(\nu) > f\big(\nu, f(\nu, x)\big) > x, \quad \text{for } x_{\text{fp}}\nu) > x \ge \tilde{x}_1.$$

The above conditions, (44) and nonexistence of other period-two points in U imply that

$$\text{Inv}(\pi_{x,y} U, P_\nu) = \big\{z_{\text{fp}}(\nu)\big\}, \quad \text{for } \nu \in [\tilde{\nu}, \nu_b].$$

For $\nu \in (\nu_b, \tilde{\nu}_2]$, we have

$$x_{\text{fp}}(\nu) < x < f\big(\nu, f(\nu, x)\big) < c_2(\nu), \quad \text{for } x_{\text{fp}}(\nu) < x < c_2(\nu),$$
$$x_{\text{fp}}(\nu) > x > f\big(\nu, f(\nu, x)\big) > c_1(\nu), \quad \text{for } x_{\text{fp}}(\nu) > x > c_1(\nu),$$
$$c_2(\nu) < f\big(\nu, f(\nu, x)\big) < x, \quad \text{for } x > c_2(\nu),$$
$$c_1(\nu) > f\big(\nu, f(\nu, x)\big) > x, \quad \text{for } x < c_1(\nu).$$

The above conditions, conditions (44–45) and nonexistence of other period two points in U imply that for $\nu \in (\nu_b, \tilde{\nu}_2]$

$$\text{Inv}(\pi_{x,y} U, P_\nu) = \big\{\big(x, h(\nu, x)\big) \mid x \in \big[c_1(\nu), c_2(\nu)\big]\big\}. \qquad \square$$

We would like to stress here that contrary to all previous theorems and lemmas, in the proof of the above theorem we prove the statements about the invariant manifold of bifurcating orbits from Definition 4 on some set U, whose size we do not control, whereas it is given by the range of the existence of the central manifold. In principle, this range can be inferred from the proof of the center manifold theorem, but it will be an interesting task to develop a computable approach, which will allow to rigorously prove these facts on the whole set V. Such a task will require explicit estimates of the central manifold in the region very close to the bifurcation as well as some other tools, maybe of the Conley index type [15], further away from the bifurcation.

6 Application to the Rössler System

Consider the autonomous ODE in $\mathbb{R}^3$ called the Rössler system [23]

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + by, \\ \dot{z} = b + z(x - a). \end{cases} \tag{50}$$

The classical parameter values (considered by Rössler) are $a = 5.7$ and $b = 0.2$. For the remainder of this paper, we fix $b = 0.2$.

The system (50) has been extensively studied in the literature and is treated as one of the classical examples of systems generating chaotic attractors. Yet, the number of rigorous results concerning it is very small. In Fig. 4, we show a numerically obtained bifurcation diagram for periodic orbits on section $x = 0$ with $b = 0.2$ and a

as parameter. We see that when the parameter a increases from 2 to 5.7, one observes a cascade of period-doubling bifurcations. In Fig. 5, we show some periodic orbits for different values of a. Our goal in this section is to validate the part of the bifurcation diagram in Fig. 4 containing two first period doublings using the approach introduced in the previous sections.

Let us list the few known rigorous results about (50). Pilarczyk (see [22] and references therein) gave a computer assisted proof of the following facts: for $a = 2.2$, there exists a periodic orbit; for $a = 3.1$, there exist two periodic orbits. From his proof, however, one cannot infer any information about the dynamical character of these orbits. He constructs suitable isolating neighborhoods, which have an index

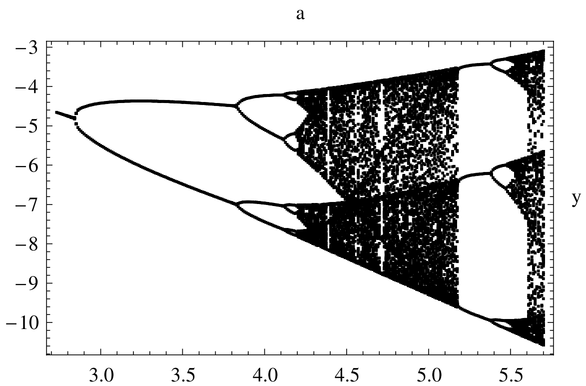


Fig. 4 Bifurcation diagram for the Rössler system

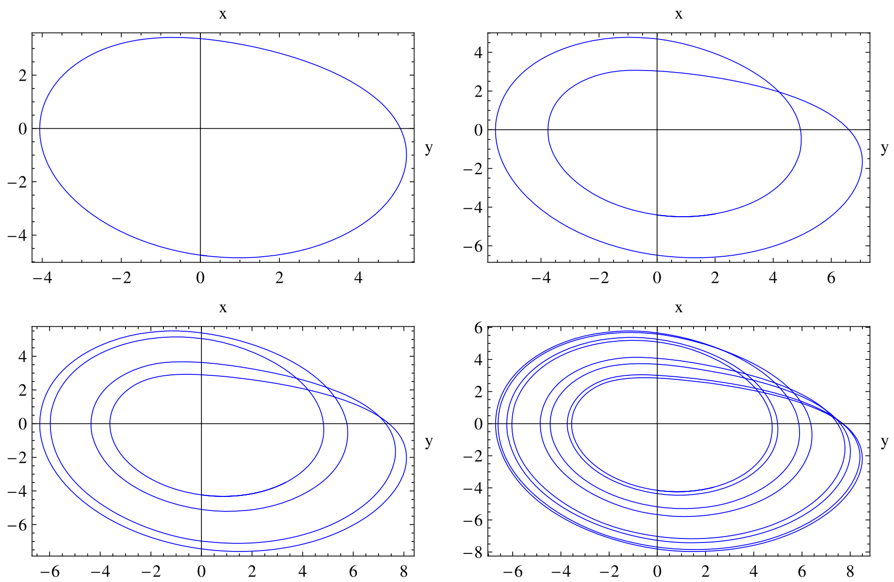


Fig. 5 Periodic orbits corresponding to fixed point, period two point, period four point, and period eight point for the Poincaré map of the Rössler system (50). Parameter values are $a = 2.8$, $a = 3.5$, $a = 4$, and $a = 4.2$

of an attracting or a hyperbolic orbit with one unstable direction, but no such claim can be made about the periodic orbit proved to exist. In fact, we do not even know whether this orbit is unique.

Finally, for the classical parameter values $b = 0.2$ and $a = 5.7$, the system is chaotic [31] in the following sense: a suitable Poincaré map has an invariant set S and the dynamics on S for the second iterate of the Poincaré map is semiconjugated to the full shift map on two symbols.

Before proceeding any further, we need to introduce some notation. Let $\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, x' > 0\}$ be a Poincaré section. Since for $u \in \Pi$, the first coordinate is equal to zero we will use the remaining two coordinates $u = (y, z)$ to represent a point on Π. For a fixed parameter value $a > 0$ by $P_a = (P_{a,y}, P_{a,z})\colon \Pi \to \Pi$, we will denote the corresponding Poincaré return map. By P, we will denote the map defined by $P(a, y, z) = (a, P_{a,y}(y, z), P_{a,z}(y, z))$.

Apparently, the first period-doubling bifurcation is observed for $a \approx 2.832445$ and the second one for $a \approx 3.837358$. In the remainder of this section, we discuss the computer assisted proof of the existence of both these bifurcations. In our presentation, we will discuss the first one more in detail, while for the second one, we will just state relevant lemmas and estimates.

Let $u_0 = (y_0, z_0)$ be an approximate fixed point for P_{a_0}, i.e., we set

$$u_0 = (y_0, z_0) = (-4.7946653021070986256, 0.052488098609082899093) \quad (51)$$

and put

$$M = \begin{bmatrix} 0.99999765967819775891 & -0.95820959262174687 51 \\ 0.0021634782474835700244 & -0.28606708410382636343 \end{bmatrix}. \quad (52)$$

The columns of M are normalized approximate eigenvectors of $DP_{a_0}(u_0)$, where first column corresponds to the eigenvalue close to -1 and the second one to the eigenvalue close to zero. On the section Π, we choose new coordinates $(\tilde{y}, \tilde{z}) = M^{-1}((y, z) - u_0)$ and since in the sequel, we will use only the new coordinates we and will drop the tilde.

Define

$$\begin{aligned} A &= [a_1, a_2] = [2.83244, 2.832446], \\ Y &= [y_1, y_2] = 1.3107 \cdot [-1, 1] \times 10^{-3}, \\ Z &= [z_1, z_2] = 1.3107 \cdot [-1, 1] \times 10^{-4}. \end{aligned} \quad (53)$$

Now our goal is to present the proof of the following theorem.

Theorem 10 *The map P_a has a period-doubling bifurcation at some point $(a, y, z) \in \operatorname{int}(A \times Y \times Z)$.*

Remark 11 The existence of a period-doubling bifurcation is a local phenomenon. In fact, the sets A, Y, Z can be chosen to be smaller which speeds up the proof (13 minutes versus 87 minutes), namely we were able to prove the existence of a period-doubling bifurcation in the set

$$\begin{aligned}A &= [a_1, a_2] = [2.83244, 2.832445028],\\ Y &= [y_1, y_2] = [-1, 1] \cdot 10^{-4},\\ Z &= [z_1, z_2] = [-1, 1] \cdot 10^{-5}.\end{aligned}$$

However, the choice of a larger set facilitates the proof of the existence of the connecting branch of period-two points between first and second period-doubling bifurcation, because decreasing a_2 results in the eigenvalue of period-two points to be very close to 1, which makes it very difficult to rigorously continue it.

6.1 The Existence of the Fixed-Point Curve

Lemma 12 *There exists a function* $(y_{\text{fp}}, z_{\text{fp}})\colon A \to Y \times Z$ *of class* C^∞ *such that for* $(a, y, z) \in A \times Y \times Z$ *there holds*

$$P_a(y, z) = (y, z) \quad \textit{iff } (y, z) = \big(y_{\text{fp}}(a), z_{\text{fp}}(a)\big)$$

and

$$y'_{\text{fp}}(A) \subset [-1.3336825610133946629, \ -1.3275439332565022177]. \tag{54}$$

Proof The proof, which is computer assisted, consists of two parts. In the first one, we prove the existence of the fixed point curve, and in the second part we establish the estimate (54).

For the first part, we use the Interval Newton Method (Theorem 6) and the C^1-Lohner algorithm [30] to prove that for $a \in A$ there exists a unique fixed point $(y_{\text{fp}}(a), z_{\text{fp}}(a))$ for P_a in $Y \times Z$. In the computations, we insert the whole set $A \times Y \times Z$ as an initial condition in our routine, which computes the Interval Newton Operator and obtain that for all $a \in A$ the fixed point $(y_{\text{fp}}(a), z_{\text{fp}}(a))$ belongs to the set

$$\begin{aligned}N &:= N(\mathrm{Id} - P_a, Y \times Z)\\ &= \begin{bmatrix}[-2.8383789385970495 59, 3.2727784971172813446] \times 10^{-5}\\ [-4.8121450471307824034, 4.2979575521536656939] \times 10^{-6}\end{bmatrix}^{\mathrm{T}}.\end{aligned} \tag{55}$$

To obtain (54), we apply the C^1-Lohner algorithm [30] to the system

$$\begin{cases}\dot{x} = -y - z,\\ \dot{y} = x + by,\\ \dot{z} = b + z(x - a),\\ \dot{a} = 0,\end{cases} \tag{56}$$

with $b = 0.2$ in order to compute a bound for $y'_{\text{fp}}(A)$. Differentiating

$$P\big(a, y_{\text{fp}}(a), z_{\text{fp}}(a)\big) = \big(a, y_{\text{fp}}(a), z_{\text{fp}}(a)\big)$$

with respect to a, we obtain

$$y'_{\text{fp}} = \frac{\frac{\partial P_{a,y}}{\partial a}(1 - \frac{\partial P_{a,z}}{\partial z}) + \frac{\partial P_{a,z}}{\partial a} \cdot \frac{\partial P_{a,y}}{\partial z}}{(\frac{\partial P_{a,y}}{\partial y} - 1)(\frac{\partial P_{a,z}}{\partial z} - 1) - \frac{\partial P_{a,y}}{\partial z}\frac{\partial P_{a,z}}{\partial y}} \tag{57}$$

where the partial derivatives of P are evaluated at $(y_{\text{fp}}(a), z_{\text{fp}}(a))$.

We use the set $A \times N$, where N is defined in (55) as the initial condition in our routine which computes partial derivatives of P and after substituting them in (57) we obtain a bound for y'_{fp} as in (54).

We used the Taylor method of order 14 and the time step equal to 0.02 to integrate the system (50) in $\mathbb{R}^3$ for the first part of the proof and the order 10 and the time step 0.01 when we integrate the extended system (56) in the second part. □

Lemma 13 *The eigenvalues $\lambda_1, \lambda_2 : A \to \mathbb{R}$ of $DP_a(y_{\text{fp}}(a), z_{\text{fp}}(a))$ are given by*

$$\lambda_1(a) = \frac{1}{2}\left(\frac{\partial P_{a,y}}{\partial y} + \frac{\partial P_{a,z}}{\partial z} - s(a)\right),$$

$$\lambda_2(a) = \frac{1}{2}\left(\frac{\partial P_{a,y}}{\partial y} + \frac{\partial P_{a,z}}{\partial z} + s(a)\right),$$

$$s(a) = \sqrt{\left(\frac{\partial P_{a,y}}{\partial y} - \frac{\partial P_{a,z}}{\partial z}\right)^2 + 4\frac{\partial P_{a,y}}{\partial z}\frac{\partial P_{a,z}}{\partial y}}$$

where the partial derivatives of P are evaluated at $(y_{\text{fp}}(a), z_{\text{fp}}(a))$. Let $v(a)$ be the normalized eigenvector corresponding to eigenvalue $\lambda_1(a)$. Then

$$\lambda_1(a_1) \in [-0.99997819449145786 13, -0.99995489192177511 31],$$

$$\lambda_1(a_2) \in [-1.00000064581599335217, -1.00000064581598072628],$$

$$\lambda_2(A) \subset [-0.00135332613671033420 71, 0.00135303783404876719 34],$$

$$\lambda'_1(A) \subset [-0.70107900728585614836, -0.62770519734197127715],$$

$$v_y(A) \subset \pm[0.99728887963031764841, 1.0027184248801992439]$$

where v_y denotes the y coordinate of v.

Proof We leave the derivation of formulas for λ_1, λ_2 to the reader. We used the C^1-Lohner algorithm applied to the system (50) in order to compute bounds for $\lambda_1(a_1)$ and $\lambda_1(a_2)$. Since the parameter a_2 has been chosen to be very close to the bifurcation parameter, we find difficulties with the verification of condition $\lambda_1(a_2) < -1$ in computations performed in interval arithmetics based on double precision (52-bit mantissa) boundary value type. In our computations, we used interval arithmetics based on float numbers with 150-bit mantissa (MPFR [16] and GMP [8] packages).

Since the eigenvalue $\lambda_1(a)$ of $DP_a(y_{\text{fp}}(a), z_{\text{fp}}(a))$ is given by an explicit formula, one can express $\lambda'_1(a)$ in terms of first and second order partial derivatives of P. We

Table 1 Parameters of the C^1–C^2-Lohner algorithms

	Order	Step
$\lambda_2(A), \lambda_1'(A)$	10	0.03
$\lambda_1(a_1)$	10	0.1
$\lambda_1(a_2)$–150-bit precision	14	0.05

obtain

$$\lambda_1'(a) = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,y}}{\partial y} + \frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,z}}{\partial z} - s'(a)\right),$$

$$s'(a) = \frac{1}{s(a)}\left(\frac{\partial P_{a,y}}{\partial y} - \frac{\partial P_{a,z}}{\partial z}\right)\left(\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,y}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,z}}{\partial z}\right) + \frac{2}{s(a)}\left(\left(\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,y}}{\partial z}\right)\frac{\partial P_{a,z}}{\partial y} + \left(\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,z}}{\partial y}\right)\frac{\partial P_{a,y}}{\partial z}\right),$$

where the symbols $\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,y}}{\partial z}$ and $\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,z}}{\partial y}$ should be understood as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\frac{\partial P_{a,z}}{\partial y}\big(y_{\mathrm{fp}}(a), z_{\mathrm{fp}}(a)\big) &= \frac{\partial^2 P_{a,z}}{\partial a \partial y}\big(y_{\mathrm{fp}}(a), z_{\mathrm{fp}}(a)\big) \\ &\quad + \frac{\partial^2 P_{a,z}}{\partial y^2}\big(y_{\mathrm{fp}}(a), z_{\mathrm{fp}}(a)\big) y_{\mathrm{fp}}'(a) \\ &\quad + \frac{\partial^2 P_{a,z}}{\partial y \partial z}\big(y_{\mathrm{fp}}(a), z_{\mathrm{fp}}(a)\big) z_{\mathrm{fp}}'(a)\end{aligned}$$

and $y_{\mathrm{fp}}'(a)$ and $z_{\mathrm{fp}}'(a)$ can be computed as in (57). Next, we applied the C^2-Lohner algorithm [29] to the extended system (56) in order to compute a bound for the first and the second order partial derivatives of P and in consequence a bound for $\lambda_1'(A)$.

We inserted $A \times N$, where N is defined in (55), as the initial condition in our routine, which computes the partial derivatives of the Poincaré map up to second order. In these computations, we simultaneously computed bounds for $\lambda_2(A)$ and $\lambda_1'(A)$. The parameter settings of the Taylor method used in the computations are listed in Table 1. □

6.2 The Existence of Lyapunov–Schmidt Reduction

Lemma 14 *For all $(a, y) \in A \times Y$, there exists a unique $z = z(a, y) \in Z$ such that*

$$P_{a,z}^2(y, z) = z \quad \text{iff } z = z(a, y) \tag{58}$$

and the map $z\colon A \times Y \to Z$ is smooth of class C^∞. Moreover, the map $G\colon A \times Y \to \mathbb{R}$ defined by

$$G(a, y) = y - P_{a,y}^2\big(y, z(a, y)\big)$$

satisfies

$$\frac{\partial^3 G}{\partial y^3}(A \times Y) \subset [1.8296823158090675943, 7.2204769494502958338], \tag{59}$$

$$\frac{\partial^2 G}{\partial y^2}(A \times Y) \subset [-0.2084557586786322414, 0.2080871792788867581]. \tag{60}$$

Proof Let us fix$(a, y) \in A \times Y$ and define a function $V_{a,y}\colon Z \to \mathbb{R}$ by $V_{a,y}(z) = z - P^2_{a,z}(y, z)$. The computer assisted proof of this lemma consists of the following steps:

- We divide uniformly the interval A onto 30 parts. For each subinterval $\bar{A}$ in this covering, we proceed as follows.
- We verified that for all $(a, y) \in \bar{A} \times Y$ the function $V_{a,y}$ has exactly one zero in Z. Notice that Z is centered at zero—see (53). Put

$$\begin{aligned}\bar{Z} = N(V_{\bar{A},Y}, Z, 0) &= 0 - \left(1 - \frac{\partial}{\partial z} P^2_{\bar{A},z}(Y \times Z)\right)^{-1} \left(0 - P^2_{\bar{A},z}(Y, 0)\right) \\ &= P^2_{\bar{A},z}(Y, 0)\left(1 - \frac{\partial}{\partial z} P^2_{\bar{A},z}(Y \times Z)\right)^{-1}.\end{aligned}$$

  Using the Interval Newton Method (Theorem 6), we obtain that if $\bar{Z} \subset Z$ then for each $(a, y) \in \bar{A} \times Y$ there exists unique $z(a, y) \in \bar{Z}$ such that $V_{a,y}(z(a, y)) = 0$. This defines the unique map $z\colon \bar{A} \times Y \to Z$ which is smooth by implicit function theorem and which satisfies (58). The subdivision of the set A onto 30 equal parts is the minimal one for which we were able to obtain inclusion $\bar{Z} \subset Z$ for each $\bar{A}$ in the subdivision and such that $\frac{\partial^3 G}{\partial y^3}(A \times Y) > 0$; see below.
- Let $\bar{Z}$ denote a bound for $z(\bar{A}, Y)$ resulting from the previous step. Differentiating $z(a, y) - P^2_{a,z}(y, z(a, y)) = 0$ with respect to y, we obtain

$$\begin{aligned}
\left(1 - \frac{\partial P_{a,z}}{\partial z}\right)\frac{\partial z}{\partial y} &= \frac{\partial P_{a,z}}{\partial y}, \\
\left(1 - \frac{\partial P_{a,z}}{\partial z}\right)\frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 P_{a,z}}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2 + 2\frac{\partial^2 P_{a,z}}{\partial y \partial z}\frac{\partial z}{\partial y} + \frac{\partial^2 P_{a,z}}{\partial y^2}, \\
\left(1 - \frac{\partial P_{a,z}}{\partial z}\right)\frac{\partial^3 z}{\partial y^3} &= \frac{\partial^3 P_{a,z}}{\partial z^3}\left(\frac{\partial z}{\partial y}\right)^3 + 3\frac{\partial^3 P_{a,z}}{\partial y \partial z^2}\left(\frac{\partial z}{\partial y}\right)^2 \\
&\quad + 3\left(\frac{\partial^2 z}{\partial y^2}\frac{\partial^2 P_{a,z}}{\partial z^2} + \frac{\partial^3 P_{a,z}}{\partial y^2 \partial z}\right)\frac{\partial z}{\partial y} \\
&\quad + 3\frac{\partial^2 z}{\partial y^2}\frac{\partial^2 P_{a,z}}{\partial y \partial z} + \frac{\partial^3 P_{a,z}}{\partial y^3}.
\end{aligned}$$

  We see that we can compute all the partial derivatives of $z(a, y)$ as a function of the partial derivatives of P. Hence, the partial derivatives of $G(a, y) = y - P^2_{a,y}(y, z(a, y))$ can be expressed in terms of the partial derivatives of P.

Using the C^3-Lohner algorithm [29] applied to the system (50) with a range of parameter values $\bar{A}$ and an initial condition $Y \times \bar{Z}$, we computed bounds of partial derivatives of the Poincaré map P up to third order and an estimation for $\frac{\partial^3 G}{\partial y^3}(\bar{A} \times Y)$ and $\frac{\partial^2 G}{\partial y^2}(\bar{A} \times Y)$. The estimates (59) and (60) are an interval enclosures of the estimates obtained in each of 30 steps.

We used 6th order Taylor method with the time step 0.04, both to verify the existence of $z(a, y)$ and to compute higher order partial derivatives of P. □

6.3 The Existence of a Period-Doubling Bifurcation for P

Lemma 15 *For $A = [a_1, a_2]$, $Y = [y_1, y_2]$ the following estimations hold true*

$$\frac{\partial G}{\partial y}\big(a_2, y_{fp}(a_2)\big) \in [-1.2916325, -1.2916323] \times 10^{-6}, \tag{61}$$

$$G(a_2, y_1) \in [-1.15, -1.07] \times 10^{-13}, \tag{62}$$

$$G(a_2, y_2) \in [5.2, 5.21] \times 10^{-12}, \tag{63}$$

$$\frac{\partial^2 G}{\partial y \partial a}(A \times Y) \subset [-2.421398492231531, -0.278863623843693], \tag{64}$$

$$\frac{\partial G}{\partial y}\big(\{a_1\} \times Y\big) \subset [0.83, 16.87] \times 10^{-6}. \tag{65}$$

Proof The estimations have been obtained using C^0–C^1–C^2-Lohner algorithms [29, 30] applied to the systems (50) and (56). The verification of conditions (61–63) required computations in interval arithmetics based on 150-bit mantissa floating points.

The settings of C^0–C^2-Lohner methods for the above computations are listed in Table 2. □

Proof of Theorem 10 The assertion follows from Theorems 4, 9, and numerical Lemmas 12, 13, 14, and 15.

Indeed, assumptions of Theorem 4 has been verified in

- A1–Lemma 14,
- A2–from Lemma 12 there exists a fixed point curve $(y_{fp}, z_{fp})\colon A \to (y_1, y_2)$ and from Lemma 14 it has form as desired in A2,

Table 2 Parameters of the C^0–C^2-Lohner algorithms

	Order	Step	Grid	Remarks
$\frac{\partial G}{\partial y}(a_2, y_{fp}(a_2))$	14	0.05	–	150-bit mantissa
$G(a_2, y_1)$	14	0.05	–	150-bit mantissa
$G(a_2, y_2)$	14	0.05	–	150-bit mantissa
$\frac{\partial^2 G}{\partial y \partial a}(A \times Y)$	6	0.05	5×30	Integration of (56)
$\frac{\partial G}{\partial y}(\{a_1\} \times Y)$	10	0.05	1×16000	Nonequal parts

- A3–0 $\notin \frac{\partial^3 G}{\partial y^3}(A \times Y)$ because of (59).

  From (60), (64), and (54), it follows that $0 \notin \frac{\partial^2 G}{\partial a \partial y}(A \times Y) + \frac{\partial^2 G}{\partial y^2} y'_{\mathrm{fp}}(A) \cdot [0, 1]$. Lemma 15 guarantees the remaining assumptions of Lemma 3 with $[\delta_1, \delta_2] = [y_1, y_2]$. Finally, from Lemma 13, we see that the assumptions about the spectrum of $DP_a(A)$ and an eigenvector $v(a)$ as desired in Theorem 9 are satisfied. □

6.4 The Existence of a Second Period-Doubling Bifurcation

In Sect. 6.3, we gave a computer assisted proof that for some parameter value $\bar{a}_1 \in [2.83244, 2.832446]$ a period-doubling bifurcation occurs for $P_{\bar{a}_1}$. In this section, we use similar arguments in order to prove that $P^2_{\bar{a}_2}$ has a period-doubling bifurcation for some $\bar{a}_2 \in [3.83735812, 3.837358168411]$.

Since the arguments used to prove the existence of a second period-doubling bifurcation are the same as for the first period-doubling bifurcation, we omit the details and we present only the sets and the necessary estimates.

Define

$$
\begin{aligned}
A_2 &= [a_3, a_4] = [3.83735812, 3.837358168411],\\
Y_2 &= [y_3, y_4] = [-1.1, 1.1] \times 10^{-6},\\
Z_2 &= \frac{1}{3} Y_2,\\
u_2 &= (-4.5003284169596655673, 0.043136987520848421584),\\
M_2 &= \begin{bmatrix} 0.99999908059259889903 & 0.82277742767392003653 \\ 0.0013560287448822982113 & -0.56836370794614177182 \end{bmatrix}.
\end{aligned}
$$

The point u_2 is an approximate period two point for parameter value a_4, and the columns of matrix M_2 are normalized eigenvectors of $DP^2_{a_4}$, where the first column corresponds to eigenvalue close to -1.

On the Poincaré section Π, we will use coordinates $(y, z) = M_2^{-1}(u - u_2)$, where u denotes a point in cartesian coordinates. In this subsection, we will use only these coordinates.

Theorem 16 *The Poincaré map P^2_a has a period doubling bifurcation at some point $(\bar{a}_2, \bar{y}_2, \bar{z}_2) \in \mathrm{int}(A_2 \times Y_2 \times Z_2)$.*

The proof is a consequence of the following lemmas (proved with computer assistance)

Lemma 17 *There exist a function $(y_{\mathrm{per}}, z_{\mathrm{per}})\colon A_2 \to Y_2 \times Z_2$ smooth of class C^∞ such that for $(a, y, z) \in A_2 \times Y_2 \times Z_2$ there hold*

$$P^2_a(y, z) = (y, z) \quad \textit{iff } (y, z) = \bigl(y_{\mathrm{per}}(a), z_{\mathrm{per}}(a)\bigr)$$

and

$$y'_{\mathrm{per}}(A_2) \subset [-0.36435039423614490328, -0.36419313389173590956].$$

Lemma 18 *Let $\lambda_1, \lambda_2 : A \to \mathbb{R}$ be the eigenvalues of $DP_a^2(y_{\mathrm{per}}(a), z_{\mathrm{per}}(a))$ defined by similar formulas as in Lemma 13. Let $v(a)$ be the normalized eigenvector corresponding to the eigenvalue $\lambda_1(a)$. Then*

$$\begin{aligned}
\lambda_1(a_3) &\in [-0.99999992011934590863, -0.99999992005484927837],\\
\lambda_1(a_4) &\in [-1.0000000000149573159618, -1.0000000000149573159615],\\
\lambda_2(A_2) &\subset [-7.7304566166653588839, 7.7302177026359675856] \times 10^{-5},\\
\lambda_1'(A_2) &\subset [-1.6554066232416912996, -1.6460891324715511974],\\
\pi_y v(A_2) &\subset \pm[0.99984657385734598822, 1.0001534381577519284].
\end{aligned}$$

Lemma 19 *For all $(a, y) \in A_2 \times Y_2$, there exists a unique $z = z(a, y) \in Z_2$ such that*

$$P^4_{a,z}(y, z) = z \quad \textit{iff } z = z(a, y)$$

and the map $z\colon A_2 \times Y_2 \to Z_2$ is smooth of class C^∞. Moreover, the map $G\colon A_2 \times Y_2 \to \mathbb{R}$ defined by

$$G(a, y) = y - P^4_{a,y}\big(y, z(a, y)\big)$$

satisfies

$$\begin{aligned}
\frac{\partial^3 G}{\partial y^3}(A \times Y) &\subset [11.780861336872181511, 22.544626008881969881],\\
\frac{\partial^2 G}{\partial y^2}(A \times Y) &\subset [-0.12474597648618415136, 0.12474408945310766494].
\end{aligned}$$

Lemma 20 *The following estimations hold true*

$$\begin{aligned}
\frac{\partial G}{\partial y}\big(a_4, y_{\mathrm{per}}(a_4)\big) &\in [-2.992, -2.991] \times 10^{-12},\\
G(a_4, y_3) &\in [-5.13, -5.11] \times 10^{-19},\\
G(a_4, y_4) &\in [5.21, 5.22] \times 10^{-19},\\
\frac{\partial^2 G}{\partial y \partial a}(A_2 \times Y_2) &\subset [-4.354355790892265432, -2.244876361570084633],\\
\frac{\partial G}{\partial y}\big(\{a_3\} \times Y_2\big) &\subset [0.99, 30.98] \times 10^{-8}.
\end{aligned}$$

Parameter settings of computations involved in proofs of the above lemmas are listed in Table 3.

Table 3 Parameters of the C^0–C^3-Lohner algorithms in the proof of the existence of second period doubling bifurcation

	Order	Step	Grid	Remarks
$\frac{\partial^3 G}{\partial y^3}(A_2 \times Y_2)$	10	0.04	150×1	–
$\frac{\partial^2 G}{\partial y^2}(A_2 \times Y_2)$	10	0.04	150×1	–
$\frac{\partial G}{\partial y}(a_4, y_{per}(a_4))$	25	0.05	–	150-bit mantissa
$G(a_4, y_3), G(a_4, y_4)$	25	0.05	–	150-bit mantissa
$\frac{\partial^2 G}{\partial y \partial a}(A_2 \times Y_2)$	10	0.05	150×1	Integration of (56)
$\frac{\partial G}{\partial y}(\{a_3\} \times Y_2)$	14	0.05	$1 \times 10\,000$	–

7 Continuation of Bifurcation Diagram

In the previous sections, we proved that the map P_a has period-doubling bifurcations for parameter values $\bar{a}_1 \in [a_1, a_2] = A$ and $\bar{a}_2 \in [a_3, a_4] = A_2$ in sets $Y \times Z$ and $Y_2 \times Z_2$, respectively.

Our goal now is to connect these bifurcations with the curve of period two points. More precisely, we prove the following result.

Theorem 21 *There exists a continuous curve*

$$(y_{per}, z_{per}) \colon (\bar{a}_1, \bar{a}_2] \to \mathbb{R}^2$$

of period two points for P_a. Moreover,

$$\big(y_{per}(a), z_{per}(a)\big), P_a\big(y_{per}(a), z_{per}(a)\big) \in Y \times Z \quad \text{for } \bar{a}_1 < a \le a_2,$$

$$\big(y_{per}(a), z_{per}(a)\big), P_a^2\big(y_{per}(a), z_{per}(a)\big) \in Y_2 \times Z_2 \quad \text{for } a_3 \le a \le \bar{a}_2.$$

Therefore, the curve (y_{per}, z_{per}) connects the two bifurcation points for $a = \bar{a}_1$ and $a = \bar{a}_2$.

The proof of the existence of a branch of period two points for P consists of the following steps:

1. The existence of a continuous curve of period two points on intervals $(\bar{a}_1, a_2]$ and $[a_3, \bar{a}_2]$ is a consequence of Theorem 10 and Theorem 16, respectively.
2. For parameter values slightly above a_2, $a_2 < a \le \tilde{a}$, with $\tilde{a} - a_2$ small, we extend this curve using Lemma 8, which requires some C^3 estimates (hence it is demanding computationally).
3. For parameters far from a_2 up to a_3, i.e., $\tilde{a} < a \le a_3$, we verify the existence of period two point curves using the Krawczyk method (Theorem 7), which requires only C^1 estimates.

4. Since we use different methods for proving the existence of segments of period two points curve over some intervals in $[\bar{a}_1, \bar{a}_2]$, it is necessary to verify that these segments can be glued to produce a continuous curve.

At first, it appears that step 2, requiring costly C^3 computations, is not necessary, because in step 3 we can consider also points close to a_2 using C^1 computations. But it turns out that while in principle it is possible, this approach may require very large computation times, because the hyperbolicity is very weak there due to the fact that one eigenvalue of $P^2_{a_2}$ is very close to 1.

To deal with this problem, we used Lemma 8 to prove that for parameter values slightly above a_2 there exists a continuous branch of period two points. Algorithm 1 is designed to verify assumptions of Lemma 8. In Lemma 22, we prove its correctness.

Definition 5 Let $U \subset \mathbb{R}^n$ be a bounded set. We say that $\mathcal{G} \subset 2^{\mathbb{R}^n}$ is a grid of U if

1. $\mathcal{G}$ is a finite set and each $G \in \mathcal{G}$ is a closed set
2. $U \subset \bigcup_{G\in\mathcal{G}} G$.

In our algorithms, which will be presented below, we always use grids consisting of interval sets, i.e., sets which are Cartesian products of intervals, most of the time uniform grids, which are defined as follows.

Definition 6 Let $Y = \prod_{i=1}^n Y_i$, where $Y_i = [a_i, b_i]$ for $a_i \leq b_i$ and let $(g_1, \ldots, g_n) \in \mathbb{Z}_+$. We define *a (uniform)* $g_1 \times g_2 \times \cdots \times g_n$*-grid for* Y denoted by $\mathcal{G}(g_1, \ldots, g_n, Y)$ as follows.

For any $(j_1, \ldots, j_n) \in \mathbb{Z}_+$, such that $j_i \leq g_i$, we set

$$g_{j_1,\ldots,j_n} = \prod_{i=1}\left[a_i, a_i + j_i \cdot \frac{b_i - a_i}{g_i}\right]. \tag{66}$$

Then $\mathcal{G}(g1_, \ldots, g_n, Y)$ is a collection of all $g_{j_1,\ldots,j_n}$.

Lemma 22 *If Algorithm 1 is called with its arguments* $\nu_1, \nu_2, g_1, g_2, g_3, g_x, t, X \times Y$ *and* f_ν *and it does not exit with an exception then the assumptions of Lemma 8 are satisfied for* f_ν, $\nu \in [\nu_1, \nu_2]$ *on* $X \times Y$.

Proof The assumption about the existence of the fixed-point curve is verified in lines 15–19 since for all $\nu \in [\nu_1, \nu_2]$ the Interval Newton Operator satisfies the assumptions of Theorem 6.

The existence of the Lyapunov–Schmidt reduction together with condition (39) is verified in lines 2–8. In lines 4–6, we see that $y(\nu, x)$ which solves equation $y - \pi_y(f^2_\nu(x, y))$ is unique for fixed ν. Therefore, by the implicit function theorem, $y(\nu, x)$ is smooth and we can compute map G and its partial derivatives.

In lines 9–10, we verify that $G(\nu_2, \min(X)) < 0$ and $G(\nu_2, \max(X)) > 0$. Since $\frac{\partial G}{\partial \nu}(\nu, \min(X)) > 0$ and $\frac{\partial G}{\partial \nu}(\nu, \max(X)) < 0$ (lines 11–14), we see that for $\nu \in [\nu_1, \nu_2]$ there holds $G(\nu, \min(X)) < 0$ and $G(\nu, \max(X)) > 0$. Therefore, (40) holds true.

Algorithm 1: verification of assumptions of Lemma 8

Data: $[\nu_1, \nu_2]$ - an interval, g_1, g_2, g_3, g_x - integers, $t \in (0, 1)$ - a floating-point number, $X \times Y$ - a convex, compact set, f_ν - a parameterized family of maps

Result: If the algorithm stops and does not exit with an exception, then the assumptions of Lemma 8 are satisfied

1 **begin**
2 $\mathcal{G}_1 \longleftarrow g_1 \times g_x$-grid for $[\nu_1, \nu_2] \times X$;
3 **foreach** $\bar{\nu} \times \bar{X} \in \mathcal{G}_1$ **do**
4 $y(\bar{\nu}, \bar{X}) \longleftarrow$ IntervalNewtonOperator$(\mathrm{Id}_Y - \pi_Y \circ f^2_{\bar{\nu}}(\bar{X}, \cdot), Y, \mathrm{center}(Y))$;
5 **if** ***not*** $y(\bar{\nu}, \bar{X}) \subset \mathrm{int}\, Y$ **then**
6 **exit**: Lyapunov–Schmidt reduction not verified;
7 **if** ***not*** $\frac{\partial^3 G}{\partial x^3}(\bar{\nu}, \bar{X}) > 0$ **then**
8 **exit**: condition (39) is not satisfied;
9 **if** *(**not** $G(\nu_2, \min(X)) < 0$) **or** (**not** $G(\nu_2, \max(X)) > 0$)* **then**
10 **exit**: condition (40) is not satisfied;
11 $\mathcal{G}_2 \longleftarrow g_2$-grid for $[\nu_1, \nu_2]$;
12 **foreach** $\bar{\nu} \in \mathcal{G}_2$ **do**
13 **if** *(**not** $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_1) > 0$) **or** (**not** $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_2) < 0$)* **then**
14 **exit**: condition (40) is not satisfied;
15 $\mathcal{G}_3 \longleftarrow g_3$-grid for $[\nu_1, \nu_2]$;
16 **foreach** $\bar{\nu} \in \mathcal{G}_3$ **do**
17 $(\bar{X}, \bar{Y}) \longleftarrow$ IntervalNewtonOperator$(\mathrm{Id} - f_{\bar{\nu}}, X \times Y, \mathrm{center}(X \times Y))$;
18 **if** ***not*** $(\bar{X}, \bar{Y}) \subset \mathrm{int}(X \times Y)$ **then**
19 **exit**: fixed points curve not verified;
20 $x_+ \longleftarrow (1 - t)\min(X) + t \min(\bar{X})$;
21 $x_- \longleftarrow (1 - t)\max(X) + t \max(\bar{X})$;
22 **if** *(**not** $G(\min(\bar{\nu}), x_+) > 0$) **or** (**not** $G(\min(\bar{\nu}), x_-) < 0$)* **then**
23 **exit**: condition (41) or (42) is not satisfied;
24 **if** *(**not** $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_+) < 0$) **or** (**not** $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_-) > 0$)* **then**
25 **exit**: condition (41) or (42) is not satisfied;
26 **end**

Finally, in lines 15–25, we verify conditions (41–42). Again, we verify that for an element of grid $\bar{\nu}$ it there holds $G(\min(\bar{\nu}), x_+) > 0$ and $G(\min(\bar{\nu}), x_-) < 0$. This together with $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_+) > 0$ and $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_-) < 0$ proves (41–42). □

As was mentioned earlier, Algorithm 1 is used to prove the existence of a period-two points curve for parameter values slightly above the first bifurcation, i.e., for parameters close to a_2 where G has three solutions close to one another. For these

Algorithm 2: verification the existence of period two points branch.

Data: $[a_*, a^*]$ - an interval, $Y_i \times Z_i$, $i = 1, 2, 3, 4$ - convex, compact sets, g - an integer

Result: if the algorithm stops and does not exit with an exception then there exists a continuous branch of period two points for P_a for parameter values $a \in [a_*, a^*]$

begin
- $\mathcal{G} \longleftarrow g$-grid for $[a_*, a^*]$;
- **foreach** $\bar{a} \in \mathcal{G}$ **do**
  - $a \longleftarrow \text{center}(\bar{a})$;
  - $u_1 = (y_1, z_1) \longleftarrow$ find approximate period two point for P_a^2 using standard Newton method;
  - $u_2 = (y_2, z_2) \longleftarrow \bar{P}_a(y_1, z_1)$;
  - $u_3 = (y_3, z_3) \longleftarrow \bar{P}_a(y_2, z_2)$;
  - $u_4 = (y_4, z_4) \longleftarrow \bar{P}_a(y_3, z_3)$;
  - $C \longleftarrow$ compute approximate value of $DF_a(u_1, u_2, u_3, u_4)$;
  - **if** C *is singular* **then**
    - $C \longleftarrow \text{Id}$;
  - $U \longleftarrow (u_1 + Y_1 \times Z_1, u_2 + Y_2 \times Z_2, u_3 + Y_3 \times Z_3, u_4 + Y_4 \times Z_4)$;
  - $u \longleftarrow (u_1, u_2, u_3, u_4)$;
  - $K_{\bar{a}} = (k_{1,\bar{a}}, k_{2,\bar{a}}, k_{3,\bar{a}}, k_{4,\bar{a}}) \longleftarrow$ IntervalKrawczykOperator$(F_{\bar{a}}, C^{-1}, U, u)$;
  - **if** ***not*** $K_{\bar{a}} \subset \operatorname{int} U$ **then**
    - **exit**: cannot verify the existence of period two point;
  - **if** $k_{1,\bar{a}} \cap k_{3,\bar{a}} \neq \emptyset$ **then**
    - **exit**: the unique fixed point for $P_{\bar{a}}^2$ in $k_{1,\bar{a}}$ is not necessary period two point for $P_{\bar{a}}$;
- $B \longleftarrow \bigcup_{\bar{a} \in \mathcal{G}} \bar{a} \times k_{1,\bar{a}}$;
- **if** B *is not connected* **then**
  - **exit**: cannot verify if branch of fixed point curve is continuous on interval $[a_*, a^*]$;

end

parameter values, we found difficulties with verifying the existence of period-two points curve using C^1-computations only.

For parameter values away from the bifurcation, where all eigenvalues of periodic orbit are well separated from the unit circle, we use Algorithm 2 based on the Newton interval method and the Krawczyk method, both requiring only C^1-computations, which verifies the existence of only one branch of period two points for P_a.

Before we present this algorithm, we need to introduce some notations. Let $\bar{\Pi} = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ be a Poincaré section for (50) and $\overline{P}_a : \bar{\Pi} \to \bar{\Pi}$ be the corresponding Poincaré map for a system with parameter value a. Notice that the trajectory can intersect $\bar{\Pi}$ at a point $(y,z)\in\bar{\Pi}$ for which $x'=-y-z$ is positive or negative (if it is equal to zero the Poincaré map is not defined). Hence, we have $\bar{P}_a^2|_{\Pi}=P_a$, where P_a is the Poincaré map for the section $\Pi=\{(x,y,z)\in\mathbb{R}^3 \mid x=0, x'>0\}$ and, therefore, period two points for P_a correspond to period four points for $\bar{P}_a$. Let us define a map $F_a:\bar{\Pi}^4\to\mathbb{R}^8$ by

$$F_a\begin{bmatrix}(y_1,z_1)\\(y_2,z_2)\\(y_3,z_3)\\(y_4,z_4)\end{bmatrix}=\begin{bmatrix}(y_2,z_2)-\bar{P}_a(y_1,z_1)\\(y_3,z_3)-\bar{P}_a(y_2,z_2)\\(y_4,z_4)-\bar{P}_a(y_3,z_3)\\(y_1,z_1)-\bar{P}_a(y_4,z_4)\end{bmatrix}.$$

Algorithm 2 was used to verify the existence of a continuous branch of period two points for P_a for a belonging to some interval. Sets $X_i\times Y_i$ give the size of the neighborhood around a candidate periodic orbit on section $\overline{\Pi}$. Lines 4 to 8 constitute a heuristic part and their task is to find a good candidate.

Lemma 23 *If Algorithm 2 is called with its arguments* $[a_*,a^*]$, $Y_i\times Z_i$, $i=1,2,3,4$, *and* g *and does not exit with an exception then there exists a continuous curve* $(y_{\text{per}},z_{\text{per}})\colon[a_*,a^*]\to\Pi$ *such that* $(y_{\text{per}}(a),z_{\text{per}}(a))$ *is a period two point for* P_a.

Proof The existence of a fixed point for P_a^2 for all $a\in[a_*,a^*]$ is verified in lines 12–16. Lines 17–18 guarantee that this is a period two point for P_a, in fact, a unique one in U.

Uniqueness implies continuity on each $\bar{a}\in\mathcal{G}$ and due to uniqueness and connectedness of the set B defined in line 19 we see that they agree on boundaries of $\bar{a}$. □

Proof of Theorem 21 The existence of a continuous curve of period two points on intervals $(\bar{a}_1,a_2]$ and $[a_3,\bar{a}_2]$ is a consequence of Theorem 10 and Theorem 16, respectively. Let $a_s=2.8329$. For the parameter values $[a_2,a_s]$, we verify the existence of a period-two points branch using Algorithm 1 and for parameter values $a\in[a_s,a_3]$ we use Algorithm 2.

We have run Algorithm 1 five times with parameters listed in Table 4 (in each case the map is P_a). Since in each case the algorithm had stopped and did not exit with an exception, we conclude that in each interval of parameters listed in Table 4 there exist two continuous curves $c_1(a)$, and $c_2(a)$ of period two points. Sets $X_i\times Y_i$ listed in Table 4 are chosen so that

$$Y\times Z\subset X_1\times Y_1\subset\cdots\subset X_5\times Y_5, \tag{67}$$

where $Y\times Z$ is the set used in the proof of the existence of first period doubling bifurcation. Observe also that since we know that for a_2 holds $P_{a_2}(c_1(a_2))=c_2(a_2)$ and $c_1(a_2),c_2(a_2)\in Y\times Z\subset X_1\times Y_1$ from Lemma 8 we obtain that $\{c_1(a),c_2(a)\}$ is a period two orbit for P_a, for $a\in[a_2,a_s]$, i.e., the whole interval of parameters covered by intervals listed in the first columns in Table 4. The uniqueness of a period two orbit together with condition (67) implies that the curves are continuous on $(\bar{a}_1,a_s]$.

Table 4 Parameters of Algorithm 1

$[\nu_1, \nu_2]$	g_1	g_2	g_3	g_x	t	$(X_i \times Y_i) \times 10^{-4}$
$[a_2, 2.8325]$	8	70	600	150	0.88	$[-99.022, 97.95] \times [-4, 4]$
$[2.8325, 2.8326]$	14	80	300	270	0.7	$[-166.9, 163.9] \times [-4, 4]$
$[2.8326, 2.8327]$	16	60	150	340	0.6	$[-214.59, 209.62] \times [-4, 4]$
$[2.8327, 2.8328]$	17	50	120	340	0.5	$[-253.8, 246.8] \times [-8, 8]$
$[2.8328, 2.8329]$	20	50	100	350	0.5	$[-287.757, 279] \times [-8, 8]$

One can see that the total number of initial values for which we need compute third order derivatives of G,which is equal to the sum of $g_1 g_x$ over all rows in Table 4, is equal to 23200. The total time of computation of this step is 10 hours on the Pentium IV, 3 GHz processor.

We have run Algorithm 2 with 74 different arguments listed in Table 5. We have chosen the parameters of the Algorithm 2 so that such 74 intervals $[(a_*)_i, (a^*)_i]$, $i = 1, \dots, 74$ cover the interval $[a_s, a_3]$. Notice also that for parameters a closer to a_s we need larger values of g since the hyperbolicity close to a_2 is very weak. The total number of subintervals used to cover interval $[a_s, a_3]$ is 614450. In fact, this is the longest part of the numerical proof. The total time of computation of this step is 53 hours on the Pentium IV, 3 GHz processor. Since in each case Algorithm 2 stops and does not exit with an exception, we conclude that on each subinterval $[(a_*)_i, (a^*)_i]$ there exists a continuous branch of period two points. We need to show that these curves glue continuously at a_i^*'s. In fact, this algorithm returns an upper bound for this period two points branch which is of the form

$$B = \bigcup_{i=1}^{74} B_i \tag{68}$$

where the B_i's are defined in line 19 of Algorithm 2. We verified that B is connected—this together with an information that for fixed $a \in [a_s, a_3]$ there exists a unique period two point $(y_{\text{per}}(a), z_{\text{per}}(a))$ such that $(a, y_{\text{per}}(a), z_{\text{per}}(a)) \in B$ implies that the curve $(y_{\text{per}}(a), z_{\text{per}}(a))$ is continuous on $[a_s, a_3]$.

There remains for us to show the continuity of a period two point branch for parameter values $a = a_s$ and $a = a_3$. For $a = a_s$, we know that there exist period two points $c_1(a_s), c_2(a_s)$ which belongs to the last set listed in Table 4, i.e.,

$$c_1(a_s), c_2(a_s) \in W_1 = u_0 + M \cdot \big([-287.757, 279] \times [-8, 8]\big) \times 10^{-4}$$

where u_0 and M define coordinate system close to the first period doubling bifurcation and are defined in (51–52). On the other hand, the estimation for period two point resulting from the Krawczyk method used in the Algorithm 2 is

$$\begin{aligned}
W_2 &= \big(W_2^1, W_2^2\big),\\
W_2^1 &= [-4.766805178829389255\,7, -4.766783274392596858\,6],\\
W_2^2 &= [0.052543190547910088861, 0.052543238016254205369].
\end{aligned}$$

Table 5 Parameters of Algorithm 2. The initial set U is defined in the first line of the table

i	$U=[-1,1]\cdot(1\,100, 3, 1\,000, 3\,000, 1\,100, 3, 1\,000, 3\,000)\times 10^{-8}$		
	$[a_*, a^*]$	g	$Y_1\times Z_1\times Y_2\times Z_2\times Y_3\times Z_3\times Y_4\times Z_4$
1	[2.8329, 2.83291]	14 000	U
2	[2.83291, 2.83292]	13 500	U
3	[2.83292, 2.83293]	12 200	$3U$
4	[2.83293, 2.83294]	11 250	$3U$
5	[2.83294, 2.83295]	10 400	$3U$
6	[2.83295, 2.83296]	9 650	$3U$
7	[2.83296, 2.83297]	9 050	$3U$
8	[2.83297, 2.83298]	8 500	$3U$
9	[2.83298, 2.83299]	8 000	$3U$
10	[2.83299, 2.833]	7 550	$3U$
11	[2.833, 2.83301]	7 200	$3U$
12	[2.83301, 2.83302]	6 800	$3U$
13	[2.83302, 2.83303]	6 500	$3U$
14	[2.83303, 2.83304]	6 200	$3U$
15	[2.83304, 2.83305]	6 000	$3U$
16	[2.83305, 2.83306]	5 700	$3U$
17	[2.83306, 2.83307]	5 500	$3U$
18	[2.83307, 2.83308]	5 300	$3U$
19	[2.83308, 2.83309]	5 100	$3U$
20	[2.83309, 2.8331]	4 900	$3U$
21	[2.8331, 2.83311]	4 800	$3U$
22	[2.83311, 2.83312]	4 600	$3U$
23	[2.83312, 2.83313]	4 450	$3U$
24	[2.83313, 2.83314]	4 300	$3U$
25	[2.83314, 2.83315]	4 150	$3U$
26	[2.83315, 2.83316]	4 050	$3U$
27	[2.83316, 2.83317]	3 950	$3U$
28	[2.83317, 2.83318]	3 850	$3U$
29	[2.83318, 2.83319]	3 750	$3U$
30	[2.83319, 2.8332]	3 650	$3U$
31	[2.8332, 2.8333]	36 000	$3U$
32	[2.8333, 2.8334]	29 000	$3U$
33	[2.8334, 2.8335]	24 000	$3U$
34	[2.8335, 2.8336]	20 000	$3U$
35	[2.8336, 2.8337]	18 000	$3U$
36	[2.8337, 2.8338]	16 000	$3U$
37	[2.8338, 2.8339]	14 000	$3U$
38	[2.8339, 2.834]	13 000	$3U$
39	[2.834, 2.8345]	59 000	$3U$
40	[2.8345, 2.835]	42 000	$3U$

Table 5 (*Continued*)

i	$U = [-1, 1] \cdot (1100, 3, 1000, 3000, 1100, 3, 1000, 3000) \times 10^{-8}$		
	$[a_*, a^*]$	g	$Y_1 \times Z_1 \times Y_2 \times Z_2 \times Y_3 \times Z_3 \times Y_4 \times Z_4$
41	[2.835, 2.8355]	17 500	$15U$
42	[2.8355, 2.836]	11 000	$15U$
43	[2.836, 2.8365]	8 000	$15U$
44	[2.8365, 2.837]	6 200	$15U$
45	[2.837, 2.8372]	2 800	$30U$
46	[2.8372, 2.8375]	3 600	$30U$
47	[2.8375, 2.838]	5 000	$30U$
48	[2.838, 2.8385]	3 600	$30U$
49	[2.8385, 2.839]	2 900	$30U$
50	[2.839, 2.8395]	2 400	$30U$
51	[2.8395, 2.84]	2 100	$30U$
52	[2.84, 2.841]	3 700	$30U$
53	[2.841, 2.842]	3 000	$30U$
54	[2.842, 2.843]	2 500	$30U$
55	[2.843, 2.844]	2 200	$30U$
56	[2.844, 2.845]	2 000	$30U$
57	[2.845, 2.846]	1 700	$30U$
58	[2.846, 2.848]	3 100	$30U$
59	[2.848, 2.85]	2 600	$30U$
60	[2.85, 2.86]	11 000	$30U$
61	[2.86, 2.87]	6 500	$30U$
62	[2.87, 2.88]	4 700	$30U$
63	[2.88, 2.89]	3 700	$30U$
64	[2.89, 2.9]	3 100	$30U$
65	[2.9, 2.95]	4 400	$150U$
66	[2.95, 3]	2 500	$150U$
67	[3, 3.1]	3 500	$150U$
68	[3.1, 3.2]	2 500	$150U$
69	[3.2, 3.3]	2 100	$150U$
70	[3.3, 3.4]	1 800	$150U$
71	[3.4, 3.5]	1 600	$150U$
72	[3.5, 3.6]	1 400	$150U$
73	[3.6, 3.7]	1 500	$150U$
74	$[3.7, a_3]$	2 400	$150U$

One can verify that $W_2 \subset W_1$ which obviously means that a period two point $(y_{\text{per}}(a_s), z_{\text{per}}(a_s)) \in W_2$ resulting from the Krawczyk method and Algorithm 2 is one of the points $c_1(a_s), c_2(a_s)$ resulting from Algorithm 1. Hence, the curve of period two points is continuous at $a = a_s$.

In a similar way, we verified continuity at $a = a_3$. From the Krawczyk method used in Algorithm 2, we know that $(y_{\text{per}}(a_3), z_{\text{per}}(a_3))$ is a unique period two point in the set

$$W_3 = (W_3^1, W_3^2),$$
$$W_3^1 = [-4.50101168206074131460, -4.4996232549240025023],$$
$$W_3^2 = [0.0431342339336403327030, 0.04314029068165581293 2].$$

On the other hand from Theorem 16, we know that for $a = a_3$ any period-two point belongs to the set

$$W_4 = u_2 + M_2 \cdot (Y_2 \times Z_2)$$

where M_2, u_2, Y_2, Z_2 define the set on which we verify the existence of a second period-doubling bifurcation. One can verify that $W_4 \subset W_3$ which proves that the branch of period two points is continuous at $a = a_3$. □

7.1 Technical Data

In order to compute the Poincaré maps P and P^2 with their partial derivatives, we used the interval arithmetic [17, 24], the set algebra and the C^r-Lohner algorithm [29] developed at the Jagiellonian University by the CAPD group [5]. The C++ source files of the program with an instruction how it should be compiled and run are available at [26].

All computations were performed with the Pentium IV, 3 GHz processor and 2 GB RAM under the Ubuntu Intrepid Ibex linux with gcc-4.3.1 and MS Windows XP Professional with gcc-3.4.4. The computations took approximately 3 days. The main time-consuming part (over 53 hours) is the verification of the existence of a connecting branch of period-two points between the first and second bifurcations.

The program is also available to compile and run in a multithreaded version for multiprocessor computers. In that case, the computations are as much faster, as there are processors free to use the program. We successfully compiled and ran the program on a computer with 8 processors Quad-Core AMD Opteron(tm) 8354, each 2.2 GHz. The program ran 2 hours on this computer.

References

1. G. Alefeld, Inclusion methods for systems of nonlinear equations—the interval Newton method and modifications, in *Topics in Validated Computations*, ed. by J. Herzberger (Elsevier, Amsterdam, 1994), pp. 7–26.
2. A. Andronov, Mathematical Problems of self-oscillation theory, in *I All-Union Conference on Oscillations*, November 1931, GTTI, Moscow-Leningrad 1933, pp. 32–71.
3. V. Arnold, V. Afraimovich, Y. Ilyashenko, L. Silnikov, Theory of bifurcations, in *Dynamical Systems*, ed. by V. Arnold. Encyclopedia of Mathematical Sciences, vol. 5 (Springer, New York, 1994).
4. M. Benedicks, L. Carleson, The dynamics of the Hénon map, *Ann. Math. 2* **133**(1), 73–169 (1991).
5. CAPD–Computer Assisted Proofs in Dynamics group, a C++ package for rigorous numerics, http://capd.wsb-nlu.edu.pl.
6. S.-N. Chow, J. Hale, *Methods of Bifurcation Theory* (Springer, New York, 1982).
7. G. Iooss, *Bifurcation of Maps and Applications* (North-Holland, Amsterdam, 1979).
8. GNU Multiple Precision Arithmetic Library, http://gmplib.org.

9. J. Hale, Stability from the bifurcation function, in *Differential Equations*, ed. by S. Ahmed, M.B. Keener, A.C. Lazer, Proc. Eighth Fall Conf., Oklahoma State Univ. Stillwater Okla, 1979 (Academic Press, San Diego, 1980), pp. 23–30.
10. M.W. Hirsch, C.C. Pugh, M. Shub, *Invariant Manifolds*. Lecture Notes in Mathematics, vol. 583 (Springer, Berlin, 1997).
11. A. Kelley, The stable, Center-stable, center-unstable, unstable manifolds, in *Transversal Mappings and Flows*, ed. by R. Abraham, J. Robbin (Benjamin, Elmsford, 1967).
12. R. Krawczyk, Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschanken, *Computing* **4**, 187–201 (1969).
13. Y. Kuzniecov, *Elements of Applied Bifurcation Theory*. Applied Mathematical Sciences, vol. 112 (Springer, Berlin, 1994).
14. R.J. Lohner, Computation of guaranteed enclosures for the solutions of ordinary initial and boundary value problems, in *Computational Ordinary Differential Equations*, ed. by J.R. Cash, I. Gladwell (Clarendon Press, Oxford, 1992).
15. K. Mischaikow, M. Mrozek, The Conley index theory, in *Handbook of Dynamical Systems III: Towards Applications*, ed. by B. Fiedler, G. Iooss, N. Kopell (Elsevier, Amsterdam, 2002), pp. 393–460.
16. A C library for multiple-precision floating-point computations with correct rounding, http://www.mpfr.org.
17. R.E. Moore, *Interval Analysis* (Prentice Hall, Englewood Cliffs, 1966).
18. L. Mora, M. Viana, Abundance of strange attractors, *Acta Math.* **171**(1), 1–71 (1993).
19. A. Neumeier, *Interval Methods for Systems of Equations* (Cambridge University Press, Cambridge, 1990).
20. N.S. Nedialkov, K.R. Jackson, An interval Hermite-Obreschkoff method for computing rigorous bounds on the solution of an initial value problem for an ordinary differential equation, in *Developments in Reliable Computing*, Budapest, 1998 (Kluwer Academic, Dordrecht, 1999), pp. 289–310.
21. J.C.F. de Oliveira, J. Hale, Dynamic behavior from bifurcation equations, *Tôhoku Math. J.* **32**(4), 577–592 (1980).
22. P. Pilarczyk, Topological numerical approach to the existence of periodic trajectories, in *ODE's, Discrete and Continuous Dynamical Systems, A Supplement Volume: Dynamical Systems and Differential Equations* (2003), pp. 701–708.
23. O.E. Rössler, An equation for continuous chaos, *Phys. Lett. A* **57**(5), 397–398 (1976).
24. The IEEE Standard for Binary Floating-Point Arithmetics, ANSI-IEEE Std 754 (1985).
25. R. Thom, *Stabilité Structurelle et Morphogenése* (Benjamin, Elmsford, 1972).
26. D. Wilczak, http://www.ii.uj.edu.pl/~wilczak. A reference for auxiliary materials.
27. Q. Wang, L.-S. Young, Strange attractors with one direction of instability, *Commun. Math. Phys.* **218**(1), 1–97 (2001).
28. Q. Wang, L.-S. Young, From invariant curves to strange attractors, *Commun. Math. Phys.* **225**(2), 275–304 (2002).
29. D. Wilczak, P. Zgliczyński, C^r-Lohner algorithm, available at http://www.ii.uj.edu.pl/~wilczak.
30. P. Zgliczyński, C^1-Lohner algorithm, *Found. Comput. Math.* **2**, 429–465 (2002).
31. P. Zgliczyński, Computer assisted proof of chaos in the Hénon map and in the Rössler equations, *Nonlinearity* **10**(1), 243–252 (1997).