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Sufficient Set of Integrability Conditions of an Orthonomic System

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Abstract Every orthonomic system of partial differential equations is known to possess a finite number of integrability conditions sufficient to ensure the validity of them all. Here we show that a redundancy-free sufficient set of integrability conditions can be constructed in a time proportional to the number of equations cubed.

Keywords Orthonomic system · Integrability conditions · Monomial ideal

Mathematics Subject Classification (2000) 35N10 · 12H20 · 52B20

1 Introduction

1. The Riquier–Janet theory [15] is perhaps the most intuitive way to study formal integrability of overdetermined systems of partial differential equations. This theory applies to orthonomic systems, i.e., systems resolved with respect to highest-rank derivatives. However, somewhat surprisingly, existing literature lacks an effective construction of a provably irredundant sufficient set of integrability conditions. The aim of this paper is to fill this gap.

In the special case of linear systems with constant coefficients, formal integrability can be established through the famous Buchberger algorithm [5]. This is so

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because such systems are equivalently described as ideals in the polynomial ring $\mathbb{R}[\partial/\partial x_1, \dots, \partial/\partial x_n]$, and all what is needed is computing the Gröbner basis. Analogues of integrability conditions are known as S-polynomials. The problem of minimizing the number of S-polynomials was already addressed by Buchberger [5, 6], with many later developments [1, 7, 9, 37]. The basic idea, expressed through syzygies, later migrated back to differential algebra (Boulier [4]) and Riquier theory (Reid's school [28, 30]). Within the syzygy approach one detects superfluous "critical pairs" (S-polynomials or integrability conditions) and removes them sequentially. The complete removal being deemed too expensive, methods to remove most of the redundancy were considered. In practical implementations such as Wittkopf's dissertation [38], detection of superfluous integrability conditions turns out to be nearly exhaustive (see examples in Sect. 8), but still consuming considerable time. In this paper, the emphasis is on effective description of the set of all nontrivial integrability conditions, and on efficient algorithms to construct it (at every step of the completion algorithm).

2. Present developments of formal integrability theory are, to a great extent, driven by computer algebra applications, especially solution of large systems of overdetermined PDEs connected with computation of symmetries, conservation laws, and other invariants of PDE [3, 13, 22]. Since input systems consisting of hundreds of equations are not uncommon, efficiency of the algorithms is an important issue.

Initially (Riquier [27], Janet [15]) the basic question was which coefficients of Taylor expansion of a solution could be chosen arbitrarily (parametric derivatives) and which were then uniquely determined by the system (principal derivatives). As is well known, hidden dependences between parametric derivatives lead to integrability (or compatibility) conditions. A system with or without unsatisfied integrability conditions is said to be *active* or *passive,* respectively. The procedure of augmenting an active system with its integrability conditions is called the *completion.* The augmented system is not necessarily passive, since new integrability conditions can emerge. However, repeated completion is guaranteed to stop after a finite number of steps under fairly general assumptions (Tresse [33]; see [25] for an overview).

Conventional wisdom says that computing integrability conditions amounts to taking cross-derivatives. But the notion of a cross-derivative applies only to orthonomic systems (ones resolved with respect to "highest" derivatives). Moreover, integrability conditions can depend substantially on the way the system is resolved (as opposed to the Cartan and Spencer geometric theory of involutivity [23] and the recent theory of Mayer brackets [17, 18]). On the other hand, if we accept all the unpleasant consequences, as we do in the present paper, we find ourselves placed in an environment tailored for easy and efficient implementation of reduction. Reduction is a procedure to compute a normal form modulo identifications following from the system.

3. As usual, this paper deals with the infinitely prolonged system Σ^∞ which consists of individual equations of the input system Σ differentiated with respect to every combination of independent variables. Under a suitable ranking of derivatives, every equation of Σ^∞ comes out resolved with respect to a principal derivative, which substantially simplifies the procedure of reduction with respect to Σ^∞. The main technical difficulty is that reduction is not unique unless the system Σ is passive, which is not guaranteed before the completion algorithm is finished. To circumvent this problem, we consider a triangular subsystem Σ' of Σ^∞ with the goal to show linear equivalence of Σ' and Σ^∞ under Definition 4.

Our main result says how to locate nonignorable integrability conditions in the monomial ideal(s) generated by the system. Namely, at each principal derivative u^k_μ, we determine the nontrivial integrability conditions from connected components of a subset $\mathcal{X}^k_\mu$ of the monomial ideal ordered by divisibility. It is easily seen that u^k_μ must be a cross-derivative in order to yield nontrivial integrability conditions. This observation immediately leads to a canonical construction of a sufficient set of integrability conditions (Construction 2), which is shown to be free of redundancies (Proposition 3). This set is computable in time proportional to the number of equations cubed (Algorithms 1, 2 and Remark 5).

The idea strongly resembles that of "subconnectedness" (see [19, 35, 36] and references therein) having roots in the same Gröbner basis theory [6]. What we actually find is a simple correspondence between nontrivial integrability conditions and subconnected components (and also an effective way to compute the latter). Thus, the present work can be viewed as a natural continuation of Buchberger's [6].

Considering the "staircase diagram" associated with the monomial ideal, our main result provides a geometric identification of the vertices where nontrivial integrability conditions reside. This opens the door to asking and answering various combinatorial questions, which is, however, out of the scope of the present paper.

It is often argued that orthonomicity essentially means linearity from the practical perspective because arbitrary nonlinearities can occur at later stages of completion. Let us, however, point out that for the integrability conditions to show up it is not necessary that the system be explicitly resolved. As a matter of fact, our results can be easily formulated for generic triangular polynomial systems [14]. The only substantial difficulty lies with computation of derivatives of implicit functions not satisfying the constant rank condition [26]. That said, we leave this issue to a further study.

An extended abstract of the previous version of this paper appeared in Proceedings of the GIFT 2006 conference [20]. Another abridged exposition will be made through the book [34].

2 Orthonomic Systems

In this section, we recall standard facts and fix our notation. For unexplained order-theoretic notions see [2, Chap. 1 §1–§3] or [12, Chap. 1 §1–§2].

We denote by $\mathcal{U} = \{u^1, \dots, u^l\}$ a set of dependent variables and by $\mathcal{X} = \{x_1, \dots, x_n\}$ a set of independent variables. Consider the free commutative monoid $\mathcal{X}^*$ over $\mathcal{X}$. An arbitrary element $\mu \in \mathcal{X}^*$ is of the form $\mu = x_1^{r_1} \cdots x_n^{r_n}$, $r_1 \geq 0, \dots, r_n \geq 0$, and will be called a *Janet monomial* [15]. A derivative $\partial^{r_1+\cdots+r_n} u^k / \partial x_1^{r_1} \cdots \partial x_n^{r_n}$ can be identified with a pair $(u^k, x_1^{r_1} \cdots x_n^{r_n}) \in \mathcal{U} \times \mathcal{X}^*$. It will be convenient to denote derivatives as u^k_μ, $\mu \in \mathcal{X}^*$. Dependent variables u^k can be identified with derivatives u^k_1 of order 0.

Elements of $\mathcal{X} \cup (\mathcal{U} \times \mathcal{X}^*)$ bijectively correspond to local coordinates on an appropriate infinite-dimensional jet space J^∞ [3, 22, 23, 31]. For the purpose of understanding the present paper it is sufficient to think of J^∞ as an infinite-dimensional space equipped with local coordinates indexed by elements of $\mathcal{X} \cup (\mathcal{U} \times \mathcal{X}^*)$. Smooth functions are defined as mappings $J^\infty \to \mathbb{R}$ that depend on only a finite number of the coordinates. The *total derivatives*

$$D_x = \frac{\partial}{\partial x} + \sum_{k,\mu} u^k_{\mu x} \frac{\partial}{\partial u^k_\mu}, \quad x \in \mathcal{X}$$

can be viewed as vector fields on J^∞ (or differentiations of the $\mathbb{R}$-algebra of smooth functions on J^∞). Observe that D_x acts on a derivative u^k_μ by multiplying the Janet monomial μ by x. As is well known, total derivatives commute. For every $\nu \in \mathcal{X}^*$, the corresponding composition of total derivatives is denoted by D_ν.

Definition 1 We denote by $\leq$ the relation of divisibility of monomials and by μ/ν the quotient of monomials $\mu, \nu \in \mathcal{X}^*$ whenever $\mu \geq \nu$. A derivative u^l_ν is said to be *lower* than the derivative u^k_μ, writing $u^l_\nu \leq u^k_\mu$, if $k = l$ and $\nu \leq \mu$. We say that u^l_ν is *strictly lower* than u^k_μ, writing $u^l_\nu < u^k_\mu$, if $u^l_\nu \leq u^k_\mu$ and $\nu \neq \mu$.

Essential in Riquier's theory is a suitable ordering of derivatives compatible with differentiation.

Definition 2 A *ranking* is a linear ordering $\preceq$ of the set $\mathcal{U} \times \mathcal{X}^*$ of derivatives such that

$$p \prec D_x p,$$
$$p \prec p' \quad \Longrightarrow \quad D_x p \prec D_x p'$$

for all $p, p' \in \mathcal{U} \times \mathcal{X}^*$ and every $x \in \mathcal{X}$.

Again, $p \prec q$ means $p \preceq q$ and $p \neq q$, as usual. For a complete classification of rankings see [28, 29].

Obviously, we have the implication $q \leq p \Rightarrow q \preceq p$, but not the converse. It is an easy consequence of Dickson's lemma (see also Thomas [32]) that $\mathcal{U} \times \mathcal{X}^*$ is a well-ordered set under $\preceq$, in particular, every decreasing chain is finite. This enables transfinite induction through the set $\mathcal{U} \times \mathcal{X}^*$.

For a smooth function F let $\mathrm{LD}_\preceq F$ or simply $\mathrm{LD}\, F$ denote the *leading derivative* which is defined to be the $\preceq$-maximal element in the finite set $\operatorname{der} F$ of derivatives the function F depends on (hence, $\mathrm{LD}\, F$ exists if and only if $\operatorname{der} F$ is nonempty).

Now we take into consideration a system Σ of finitely many partial differential equations resolved with respect to leading derivatives

$$u^k_\mu = \Phi^k_\mu. \tag{1}$$

In particular, Φ^k_μ are smooth functions such that all derivatives $q \in \operatorname{der} \Phi^k_\mu$ satisfy $q \prec u^k_\mu$. By $\mathrm{LD}\, \Sigma$ we denote the set of all derivatives appearing on the left-hand side of at least one of (1).

The basic object of interest in formal integrability theory is the associated *infinitely prolonged* system Σ^∞ consisting of all possible differential consequences

$$u^k_{\mu\nu} = D_\nu \Phi^k_\mu, \quad \nu \in \mathcal{X}^*. \tag{2}$$

It easily follows from Definition 2 that (2) are resolved with respect to leading derivatives. Derivatives belonging to $\mathrm{LD}\,\Sigma^\infty$ are said to be *principal*. The other derivatives are said to be *parametric*. Thus, a derivative is principal if it either belongs to $\mathrm{LD}\,\Sigma$ or is a derivative of some derivative from $\mathrm{LD}\,\Sigma$.

Now, the input system (1) is assumed to be orthonomic in the following sense:

Definition 3 A system of equations Σ resolved with respect to leading derivatives is said to be

– *triangular*, if for every derivative $q \in \mathrm{LD}\,\Sigma$ there is exactly one equation with q appearing on its left-hand side
– *autoreduced*, if no principal derivative occurs on the right-hand side of any equation
– *orthonomic*, if it is triangular and autoreduced.

Of course, these properties depend on the choice of the ranking $\prec$.

In what follows, we shall need the notion of linear equivalence of possibly infinite systems of equations. (Let us remark that completion uses a weaker "algebraic" equivalence meaning linear equivalence combined with resolving a finite number of equations with respect to leading derivatives.)

Definition 4 Let Σ_1, Σ_2 be two (not necessarily finite) systems of partial differential equations. We say that Σ_2 *linearly follows* from Σ_1 if each equation from Σ_2 is a linear combination of a finite number of equations from Σ_1. Systems Σ_1, Σ_2 are said to be *linearly equivalent* if each linearly follows from the other.

The property of being triangular is usually lost in Σ^∞ (which is why integrability conditions occur). Having autoreduced right-hand sides of Σ is convenient, while for Σ^∞ no such property is needed.

Expositions of the Riquier–Janet theory now proceed by introducing division between multiplicative and nonmultiplicative variables [15]. Formalized by Gerdt and Blinkov [10, 11] (these works triggered an active thread of research in polynomial elimination theory), the so-called involutive divisions became a standard tool to prescribe a unique right-hand side to every principal derivative from $\mathrm{LD}\,\Sigma^\infty$.

3 Reduction Subsystem

Here we take another path to uniqueness. Namely, we consider an arbitrary triangular subsystem Σ' of Σ^∞ with $\mathrm{LD}\,\Sigma' = \mathrm{LD}\,\Sigma^\infty$ and call it a *reduction subsystem* of Σ^∞, since it provides us with a unique reduction (see below). All algorithms to follow actually refer only to a finite part of Σ'; hence, its infiniteness does not hamper computability.

Construction 1 Select a mapping $\gamma : \mathrm{LD}\,\Sigma^\infty \to \mathcal{X}^*$ such that for each $u^k_\mu \in \mathrm{LD}\,\Sigma^\infty$ we have $u^k_{\mu/\xi} \in \mathrm{LD}\,\Sigma$ for $\xi = \gamma(u^k_\mu)$. Hence, $\Phi^k_{\mu/\xi}$ exists, and we can put $\Psi^k_\mu = D_\xi \Phi^k_{\mu/\xi}$. The triangular system of equations

$$u^k_\mu = \Psi^k_\mu, \qquad u^k_\mu \in \mathrm{LD}\,\Sigma^\infty, \tag{3}$$

is the reduction subsystem sought.

Obviously, such a mapping γ always exists. The freedom of choice is measured by the number of elements in $\mathrm{LD}\,\Sigma$ that are lower than u^k_μ in the sense of Definition 1. Alternatively, we could select $\Gamma : \mathrm{LD}\,\Sigma^\infty \to \mathrm{LD}\,\Sigma$ such that $\Gamma(u^k_\mu) \le u^k_\mu$ for every $u^k_\mu \in \mathrm{LD}\,\Sigma^\infty$, with $\Gamma(u^k_\mu) = u^k_{\mu/\gamma(u^k_\mu)}$.

To implement such a selection, one can simply assign $\gamma(u^k_\mu)$ or $\Gamma(u^k_\mu)$ on the fly. If concerns arise that remembering all these assignments would consume too much space, a unique $\Gamma(u^k_\mu)$ can be determined, e.g., by means of an arbitrary fixed ordering of $\mathrm{LD}\,\Sigma$.

Given an expression F depending on a finite number of derivatives, one can apply equations of the reduction subsystem (3) as substitutions to obtain a linearly equivalent (Definition 4) expression SF without dependence on principal derivatives. In each step, the leading principal derivative $p = u^k_\mu$ the expression F actually depends on is substituted by the corresponding expression Ψ^k_μ from Construction 1. Such steps can be repeated while F depends on principal derivatives. There can be only a finite number of these steps since $\prec$ has the descending chain property. Hence, the reduction procedure is algorithmic.

Reduction S is an $\mathbb{R}$-algebra homomorphism $C^\infty(J^\infty) \to C^\infty(J^\infty)$ and satisfies $S \circ S = S$. If applying reduction S on the right-hand sides Ψ^k_μ of system (1) we obtain an autoreduced reduction system (see Definition 3), the right-hand sides Ψ^k_μ of which depend only on parametric derivatives; hence, we simply have $Su^k_\mu = \Psi^k_\mu$. The autoreduced system generates the same reduction as the unreduced one. Actually, the previous version of this paper (see the extended abstract [20]) depended on use of an autoreduced reduction subsystem. However, autoreduction is no longer necessary in practical implementations (see Remark 3).

Our main result below (Theorem 1) shows that reduction and total derivatives D_x on J^∞ satisfy

$$S \circ D_x \circ S = S \circ D_x.$$

The next example demonstrates that we have no such property until we know that Σ is passive.

Example 1 A simple example of an active system Σ with $SD_xSF \ne SD_xF$ is

$$u_x = f(u), \qquad u_y = g(u).$$

The ranking $\prec$ can be arbitrary. Let the reduction subsystem Σ' contain the equation

$$u_{xy} = D_y f$$

rather than its alternative $u_{xy} = D_x g$. For $F = u_y$ we obtain

$$SD_x Su_y = SD_x g = S\left(\frac{\partial g}{\partial u} u_x\right) = \frac{\partial g}{\partial u} f,$$

$$SD_x u_y = Su_{xy} = SD_y f = S\left(\frac{\partial f}{\partial u} u_y\right) = \frac{\partial f}{\partial u} g.$$

Observe that $SD_x Su_y = SD_x u_y$ is exactly the integrability condition

$$\frac{\partial f}{\partial u} g = \frac{\partial g}{\partial u} f$$

for the system Σ.

4 Integrability Conditions

Henceforth we fix a reduction subsystem Σ' of Σ^∞ such that $\operatorname{LD} \Sigma' = \operatorname{LD} \Sigma^\infty$ as in the preceding section. As above, S denotes the reduction with respect to Σ'. Integrability conditions investigated in this section measure the linear nonequivalence of various ways of prolongation.

Definition 5 For every principal derivative u^k_μ (i.e., $u^k_\mu \in \operatorname{LD} \Sigma^\infty$) we introduce the *principal subset* $\mathcal{X}^k_\mu$ as the set of all monomials $\xi \neq 1$ such that $u^k_{\mu/\xi} \in \operatorname{LD} \Sigma^\infty$. Thus, elements of $\mathcal{X}^k_\mu$ are principal derivatives strictly lower than u^k_μ (see Definition 1).

Definition 6 If Σ includes an equation of the form $u^k_\mu = \Phi^k_\mu$ such that the principal subset $\mathcal{X}^k_\mu$ is nonempty and $\xi \in \mathcal{X}^k_\mu$, then the condition

$$\Phi^k_\mu = SD_\xi Su^k_{\mu/\xi} \tag{4}$$

is called an *integrability condition of the first kind* at the point u^k_μ.

For every pair $\xi, \eta \in \mathcal{X}^k_\mu$, the condition

$$SD_\xi Su^k_{\mu/\xi} = SD_\eta Su^k_{\mu/\eta} \tag{5}$$

is called an *integrability condition of the second kind* at the point u^k_μ.

Let us remind the reader that Σ is not necessarily a subset of Σ'. This explains why integrability conditions of the first kind are needed.

According to Definition 6, integrability conditions at the point u^k_μ are satisfied if any two possible ways of obtaining the value Su^k_μ lead to the same result. It is well known that all such integrability conditions follow from a finite subset. The main goal of this paper is to reestablish this result in an irredundant way.

The following definition is quite standard and reflects properties of the graph of the ordered principal subset.

Definition 7 For every principal derivative u^k_μ, consider the principal subset $\mathcal{X}^k_\mu$ ordered by the divisibility relation $\preceq$. Let $\approx$ denote the reflexive, symmetric, and transitive closure of the ordering $\preceq$.

Recall that the reflexive, symmetric, and transitive closure of a given relation is the smallest reflexive, symmetric, and transitive relation containing the given relation. The explicit description is given by $p \approx q$ if and only if there exists a finite sequence of monomials $z_1, \dots, z_{2s+1} \in \mathcal{X}^k_\mu$ such that $p = z_1$, $q = z_{2s+1}$ and $z_{2j-1} \preceq z_{2j}$ whereas $z_{2j} \succeq z_{2j+1}$ for every $j = 1, \dots, s$. Since $\approx$ is an equivalence relation, we obtain a partition $\mathcal{X}^k_\mu/\approx$ of the set $\mathcal{X}^k_\mu$ into equivalence classes $[x]_\approx$, $x \in \mathcal{X}^k_\mu$.

Definition 8 A pair of elements p, q of the principal subset $\mathcal{X}^k_\mu$ is said to be *connected* if $p \approx q$. Equivalence classes of $\mathcal{X}^k_\mu$ with respect to $\approx$ are called *connected components* of $\mathcal{X}^k_\mu$. The principal subset $\mathcal{X}^k_\mu$ is said to be *connected*, if it consists of a single connected component.

For a finite ordered set, connected components are just connected components of the corresponding (Hasse) diagram.

Let us remark that connectedness of the principal subset means subconnectedness in the sense of [6, 19, 35, 36].

Construction 2 For every $u^k_\mu \in \mathrm{LD}\,\Sigma$ with nonempty principal subset $\mathcal{X}^k_\mu$ choose one integrability condition of the first kind (4),

$$\Phi^k_\mu = SD_\xi S u^k_{\mu/\xi},$$

where $\xi \in \mathcal{X}^k_\mu$ is arbitrary. For every $u^k_\mu \in \mathrm{LD}\,\Sigma^\infty$ such that the principal subset $\mathcal{X}^k_\mu$ consists of s connected components $[\xi_1]_\approx, \dots, [\xi_s]_\approx$ with $s > 1$, choose arbitrary representatives $\xi_1, \dots, \xi_s$ of these components and consider integrability conditions of the second kind (5) in the form of a chain of equations

$$SD_{\xi_1} S u^k_{\mu/\xi_1} = SD_{\xi_2} S u^k_{\mu/\xi_2} = \cdots = SD_{\xi_s} S u^k_{\mu/\xi_s}. \tag{6}$$

The set of integrability conditions obtained in this way will be called a *sufficient set of integrability conditions*.

Clearly, each ξ of Construction 2 can be chosen so that $u^k_{\mu/\xi}$ is minimal in $\mathcal{X}^k_\mu$, and hence belongs to $\mathrm{LD}\,\Sigma$ so that we can replace $SD_\xi S u^k_{\mu/\xi}$ with $SD_\xi \Phi^k_{\mu/\xi}$ to obtain conventional integrability conditions in the sense of the following definition.

Definition 9 Integrability conditions of the form

$$\Phi^k_\mu = SD_\xi \Phi^k_{\mu/\xi} \quad \text{or} \quad SD_{\xi_1} \Phi^k_{\mu/\xi_1} = \cdots = SD_{\xi_s} \Phi^k_{\mu/\xi_s}$$

are said to be *conventional* if $u^k_{\mu/\xi}, u^k_{\mu/\xi_i} \in \mathrm{LD}\,\Sigma$.

Remark 1 Obviously, every integrability condition can become conventional at the expense of enlarging the system Σ by some reduced equations from Σ^∞.

Our immediate goal now is to show that satisfying the sufficient set from Construction 2 implies satisfying all the other integrability conditions of Definition 6. The following lemma is the key. Let $\operatorname{var} F$ denote the finite set of all variables (independent variables and derivatives) a smooth function F depends on.

Lemma 1 *Let x be an independent variable, $\sigma \in \mathcal{X}^*$ a monomial, and F a function of independent variables and parametric derivatives. Let $SD_\tau SD_x p = SD_{x\tau} p$ for every derivative $p \in \operatorname{var} F$ (i.e., $p \in \operatorname{der} F$) and every monomial $\tau \le \sigma$. Then $SD_\sigma SD_x F = SD_{x\sigma} F$.*

Proof We have

$$D_\sigma(FG) = \sum_{\rho\tau=\sigma} c_\sigma^{\rho\tau} D_\rho F \cdot D_\tau G,$$

for suitable constants $c_\sigma^{\rho\tau}$. Applying SD_σ to

$$D_x F = \sum_{q \in \operatorname{var} F} \frac{\partial F}{\partial q} D_x q, \qquad SD_x F = \sum_{q \in \operatorname{var} F} \frac{\partial F}{\partial q} SD_x q,$$

we get

$$SD_\sigma D_x F = \sum_{q \in \operatorname{var} F} \sum_{\rho\tau=\sigma} c_\sigma^{\rho\tau} SD_\rho \left(\frac{\partial F}{\partial q}\right) \cdot SD_{\tau x} q,$$

whereas

$$SD_\sigma SD_x F = \sum_{q \in \operatorname{var} F} \sum_{\rho\tau=\sigma} c_\sigma^{\rho\tau} SD_\rho \left(\frac{\partial F}{\partial q}\right) \cdot SD_\tau SD_x q.$$

These two expressions coincide since $SD_\tau SD_x q = SD_{\tau x} q$ holds by assumption for all q and $\tau \le \sigma$. □

Theorem 1 *Suppose that the reduction subsystem Σ' (see Sect. 3) satisfies some sufficient set of integrability conditions as in Construction 2. Then*

(i) *For all $u^k_\mu \in \operatorname{LD} \Sigma^\infty$ and all $\xi < \mu$ we have*

$$Su^k_\mu = SD_\xi S u^k_{\mu/\xi}. \tag{7}$$

(ii) *All integrability conditions in the sense of Definition 6 hold true (meaning that Σ is passive).*

(iii) *For every monomial ξ and every smooth function f on J^∞*

$$SD_\xi f = SD_\xi Sf$$

are satisfied.

(iv) *Systems Σ' and Σ^∞ are linearly equivalent* (*Definition* 4).

Proof To prove (i) we proceed by induction with respect to $p = u^k_\mu$. If $u^k_{\mu/\xi}$ exists and is parametric, then (7) is satisfied trivially since $Su^k_{\mu/\xi} = u^k_{\mu/\xi}$. It remains to deal with the case when $u^k_{\mu/\xi}$ exists and is principal, i.e., the case of $\xi \in \mathcal{X}^k_\mu$. To start with, we consider $p = u^k_\mu$ minimal with respect to the ordering $\prec$. Then (7) holds true in a trivial way, since $\mathcal{X}^k_\mu = \emptyset$ in that case.

To perform the induction step, let us consider an arbitrary derivative $p = u^k_\mu$ assuming validity of (7) for all $q \prec p$. We shall prove

$$SD_\sigma Su^k_{\mu/\sigma} = SD_\rho Su^k_{\mu/\rho} \tag{8}$$

for all $\sigma, \rho \in \mathcal{X}^k_\mu$. We start with the case of σ, ρ belonging to one connected component, i.e., $\sigma \approx \rho$. Obviously, this case can be reduced to $\rho < \sigma$ by the definition of $\approx$. But then it can be further reduced to the case when ρ is covered by σ, i.e., when $\rho < \sigma$ and there is no other monomial in between, which we shall denote by $\rho \lhd \sigma$. Indeed, $<$ is the transitive closure of $\lhd$ since $\mathcal{X}^k_\mu$ is always finite [12, Chap. 1 §2, Lemma 1]. However, $\rho \lhd \sigma$ obviously means that $\rho < \sigma$ and σ/ρ is a variable. Therefore, let x be an independent variable x such that $\rho = x\sigma$. To prove that $\sigma \equiv \rho$, we establish equalities

$$SD_\sigma Su^k_{\mu/\sigma} = SD_\sigma SD_x Su^k_{\mu/x\sigma} = SD_{x\sigma} Su^k_{\mu/x\sigma} = SD_\rho Su^k_{\mu/\rho}.$$

The first equality follows from $Su^k_{\mu/\sigma} = SD_x Su^k_{\mu/x\sigma}$, which is (7) for $\mu/\sigma \prec \mu$ and $\xi = x$, therefore holds by induction assumption.

To prove the second equality we apply Lemma 1 to $F = Su^k_{\mu/x\sigma}$. Let us verify the assumptions. Consider an arbitrary monomial $\tau \le \sigma$ and $q \in \operatorname{var} F$. Then $q \prec u^k_{\mu/x\sigma}$, whence $D_{x\tau} q \prec D_{x\tau} u^k_{\mu/x\sigma} = u^k_{\mu\tau/\sigma} \preceq u^k_{\mu\sigma/\sigma} = u^k_\mu$. By induction assumption, (7) holds with u^k_μ replaced with $D_{x\tau} q$ and ξ with x, meaning that $SD_{x\tau} q = SD_x SD_\tau q$. Assumptions of Lemma 1 being thus verified, the second equality follows. The third equality is obvious. Therefore, (8) holds for arbitrary ρ, σ belonging to one connected component of $\mathcal{X}^k_\mu$.

We are left with the case when σ, ρ belong to different components. But if the set $\mathfrak{I}$ of integrability conditions is sufficient in the sense of Construction 2, as it is supposed to be, then $\mathfrak{I}$ contains an integrability condition $SD_\sigma Su^k_{\mu/\sigma'} = SD_\rho Su^k_{\mu/\rho'}$ with some other $\rho' \approx \rho$ and $\sigma' \approx \sigma$ from the same components, and then (8) holds for σ', ρ' by assumption and then for σ, ρ by transitivity.

This means that we have the same value $SD_\sigma Su^k_{\mu/\sigma} = SD_\rho Su^k_{\mu/\rho}$ for all $\sigma, \rho \in \mathcal{X}^k_\mu$. To establish (7), it remains to show that this common value is also equal to Su^k_μ. If $u^k_\mu \notin \mathrm{LD}\,\Sigma$, then $Su^k_\mu = SD_\xi Su^k_{\mu/\xi}$ for some $\xi \in \mathcal{X}^k_\mu$ by construction of the reduction system Σ' (Construction 1). If $u^k_\mu \in \mathrm{LD}\,\Sigma$ and $\mathcal{X}^k_\mu = \emptyset$, then (7) is void. If $u^k_\mu \in \mathrm{LD}\,\Sigma$ and $\mathcal{X}^k_\mu \ne \emptyset$ and $Su^k_\mu = \Phi^k_\mu$, then the sufficient system involves an integrability condition of the first kind $\Phi^k_\mu = SD_\xi Su^k_{\mu/\xi}$ for some $\xi \in \mathcal{X}^k_\mu$. Finally, if $u^k_\mu \in \mathrm{LD}\,\Sigma$

and $\mathcal{X}_\mu^k \neq \emptyset$ and $Su_\mu^k \neq \Phi_\mu^k$, then (7) follows from the same Construction 1 again. Thus, statement (i) is proved.

Statement (ii) follows immediately from (i) or (8). Statement (iii) holds for all functions f if it holds for all derivatives u_ν^k, and then it follows from (i). Finally, by (ii) every equation from system Σ^∞ becomes an identity when reduced with respect to S. This means that systems Σ^∞ and Σ' are linearly equivalent. Hence statement (iv). □

Theorem 1 implies that Reid's standard form [24] of a passive system Σ can be easily obtained from the reduction system Σ'.

Construction 3 Let Σ' be a reduction subsystem satisfying some sufficient set of integrability as in Construction 2. Let H be the set of principal derivatives $\mathrm{LD}\,\Sigma' = \mathrm{LD}\,\Sigma^\infty$ ordered by divisibility $\le$ and M be the subset of minimal elements in H. Consider the subsystem $\Sigma'_M \subset \Sigma'$ determined by $\mathrm{LD}\,\Sigma'_M = M$. Denote by Σ° the orthonomic system obtained when applying reduction to right-hand sides of equations in Σ'_M.

As an immediate consequence of Theorem 1(iv) we obtain the following proposition.

Proposition 1 *The system Σ° is the standard form of Σ as defined in* [24]. *Passive orthonomic systems Σ_1 and Σ_2 have linearly equivalent prolongations Σ_1^∞ and Σ_2^∞ if and only if the standard forms Σ_1° and Σ_2° coincide.*

Remark 2 Operators SD_xS on the full jet space J^∞ are $\mathbb{R}$-linear and satisfy the Leibniz rule, hence they are vector fields. Moreover, they commute, since $[SD_xS, SD_yS]f = SD_xSD_ySf - SD_ySD_xSf = SD_xD_ySf - SD_yD_xSf = 0$ by Theorem 1(iii). Hence, the full jet space J^∞ equipped with vector fields SD_xS, $x \in \mathcal{X}$, is a diffiety in the sense of [3]. Now, denoting $\mathcal{E}_{\Sigma^\infty}$ and $\mathcal{E}_{\Sigma'}$ the submanifolds in J^∞ determined by Σ^∞ and Σ', respectively, we have $C^\infty\mathcal{E}_{\Sigma^\infty} = C^\infty\mathcal{E}_{\Sigma'} \cong C^\infty J^\infty/\operatorname{Ker} S \cong SC^\infty J^\infty$ by Theorem 1(iv). Hence, SD_xS induce well defined operators on the manifold $\mathcal{E}_{\Sigma^\infty}$, turning it into a diffiety.

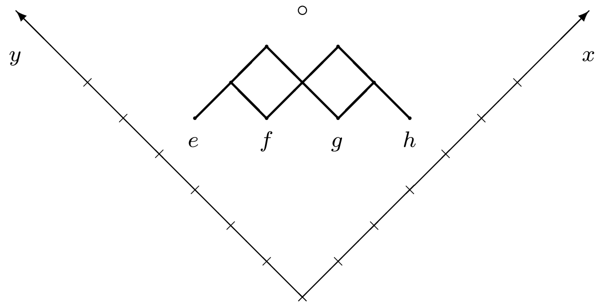
In an attempt to convey the sense of the method we conclude this section with a simple example in dimension two. For less trivial examples see Sect. 8.

Example 2 Consider the following system Σ:

$$u_{xyyyy} = e, \qquad u_{xxyyy} = f, \qquad u_{xxxyy} = g, \qquad u_{xxxxy} = h,$$

where e, f, g, h are arbitrary functions of parametric derivatives. Figure 1 shows the ordered set $\mathrm{LD}\,\Sigma^\infty$ of principal derivatives placed within a coordinate system. Symbols e, f, g, h denote the four generating derivatives u_{xyyyy}, u_{xxyyy}, u_{xxxyy}, $u_{xxxxy} \in \mathrm{LD}\,\Sigma$, respectively. The circle at $u_{xxxxyyyy}$ denotes a typical principal derivative. Thick lines show the corresponding principal subset $\mathcal{X}_{xxxxyyyy}$, which is obviously connected. Actually, one easily sees that *all* principal subsets are connected

Fig. 1 A typical principal subset (Example 2)



except

$$\mathcal{X}_{xxxxyy} = \{u_{xxxxy}, u_{xxxyy}\},$$
$$\mathcal{X}_{xxxyyy} = \{u_{xxxyy}, u_{xxyyy}\},$$
$$\mathcal{X}_{xxyyyy} = \{u_{xxyyy}, u_{xyyyy}\},$$

which consist of two isolated points each. Correspondingly, each of the derivatives u_{xxxxyy}, u_{xxxyyy}, u_{xxyyyy} harbors one nontrivial integrability condition. They are, respectively,

$$SD_x e = SD_y f, \qquad SD_x f = SD_y g, \quad \text{and} \quad SD_x g = SD_y h.$$

5 Cross-Derivatives

In this section, we find an alternative description of the sufficient set suitable for effective implementation.

Construction 2 leaves us with the problem of finding all principal derivatives with disconnected principal subset. Indeed, a nontrivial integrability condition of the second kind at a point u^k_μ exists if and only if there are at least two distinct connected components in $\mathcal{X}^k_\mu$. To extend this line of argument further, let us consider different possible descriptions of the quotient sets $\mathcal{X}^k_\mu/{\approx}$.

Let B be a subset of $\mathcal{X}^k_\mu$ such that every element of $\mathcal{X}^k_\mu$ is connected to an element of B in the sense of Definition 8. Then, obviously, every connected component intersects with B. In particular, the quotient set $\mathcal{X}^k_\mu/{\approx}$ is the same as the quotient set $B/{\approx_B}$, where $\approx_B$ is the equivalence relation on B inherited from the relation $\approx$ on $\mathcal{X}^k_\mu$. There are two natural choices for B, which lead to two alternative descriptions of $\mathcal{X}^k_\mu/{\approx}$:

(a) $\min \mathcal{X}^k_\mu =$ the subset of minimal elements in $\mathcal{X}^k_\mu$.
(b) $\max \mathcal{X}^k_\mu =$ the subset of maximal elements in $\mathcal{X}^k_\mu$.

Let $L^{(k)}$ denote the set of all monomials μ such that $u^k_\mu \in \mathrm{LD}\,\Sigma$. Assuming $L^{(k)}$ ordered by divisibility $\preceq$, let $M^{(k)} = \min L^{(k)}$ denote the set of all minimal elements in $L^{(k)}$. Obviously, the subset $\min \mathcal{X}_\mu^k$ coincides with the intersection $\mathcal{X}_\mu^k \cap M^{(k)}$. Define a reflexive and symmetric relation $\uparrow$ on $\min \mathcal{X}_\mu^k$ by $p \uparrow q$ if $\operatorname{lcm}(p,q) \in \mathcal{X}_\mu^k$, i.e., if $\operatorname{lcm}(p,q)$ is a proper divisor of u_μ^k.

Elements of $\max \mathcal{X}_\mu^k$ are quotients μ/x with $x \in \mathcal{X}$ an independent variable such that the derivative $u_{\mu/x}^k$ is principal. To simplify reasoning, we identify $\max \mathcal{X}_\mu^k$ with a subset of $\mathcal{X}$. Define a reflexive and symmetric relation $\downarrow$ on $\max \mathcal{X}_\mu^k \subseteq \mathcal{X}$ by $x \downarrow y$ if there exists $\sigma \in L^{(k)}$ (equivalently, $\sigma \in M^{(k)}$) and $\sigma \le \mu/x, \mu/y$. The same relation $\downarrow$ can be defined by $x \downarrow y$ if $x = y$ or the derivative $u_{\mu/xy}^k$ is principal as well.

We have the following obvious lemma.

Lemma 2 *The inherited equivalence relation $\approx_{\min \mathcal{X}_\mu^k}$ coincides with the transitive closure $\uparrow^*$ of $\uparrow$. The inherited equivalence relation $\approx_{\max \mathcal{X}_\mu^k}$ coincides with the transitive closure $\downarrow^*$ of $\downarrow$.*

Corollary 1 *We have the obvious induced bijections*

$$\mathcal{X}_\mu^k/{\approx} \leftrightarrow \min \mathcal{X}_\mu^k/{\uparrow^*} \leftrightarrow \max \mathcal{X}_\mu^k/{\downarrow^*}.$$

Thus, we have obtained two "extremal" descriptions of the quotients $\mathcal{X}_\mu^k/{\approx}$. The former is essentially through the concept of subconnectedness and is well-known to be computationally hard. The latter description is what we actually use below.

Proposition 2 *Let u_μ^k be a principal derivative such that the principal subset $\mathcal{X}_\mu^k$ contains nonequivalent elements $\sigma \not\approx \tau$. Then $\mu = \operatorname{lcm}(\sigma,\tau)$. Elements σ, τ can be chosen to lie in $\min \mathcal{X}_\mu^k$.*

Proof Since $\sigma, \tau \in \mathcal{X}_\mu^k$, we have $\operatorname{lcm}(\sigma,\tau) \le \mu$. If $\operatorname{lcm}(\sigma,\tau) < \mu$, then obviously $\sigma \approx \tau$, contradicting the assumptions. The last statement follows from the fact that every connected component intersects with $\min \mathcal{X}_\mu^k$. □

Now we introduce cross-derivatives as the "least common derivatives."

Definition 10 A *cross-derivative* is a derivative $u_{\mathrm{LCM}(\sigma,\tau)}^k$, where $u_\sigma^k, u_\tau^k \in \mathrm{LD}\,\Sigma$ and σ, τ do not divide one another.

By Proposition 2, nontrivial integrability conditions of the second kind can be found only at cross derivatives. Hence the well-known result that the number of such integrability conditions is always finite and less or equal to $\frac{1}{2}p(p-1)$, where p is the number of equations in the system Σ.

Of course, a cross-derivative gives rise to integrability conditions (6) if and only if it satisfies the following nontriviality condition:

Definition 11 A cross-derivative u_μ^k is said to be *trivial* if the principal subset $\mathcal{X}_\mu^k$ is connected. Otherwise it is called *nontrivial*.

Example 3 Generalizing Example 2, consider an arbitrary system of r equations in two dimensions such that $\text{LD}\,\Sigma$ consists of incomparable derivatives (with respect to $\le$ of Definition 1). It is an easy exercise to show that of the $r(r-1)/2$ cross-derivatives only $r-1$ are nontrivial.

Let us close this section with some remarks concerning visualization of the relation $\downarrow$. The monoid $\mathcal{X}^*$ can be visualized as the n-dimensional grid $\mathbb{N}^n \subset \mathbb{R}^n$, where $\mathbb{N} = \{0, 1, 2, \dots\}$, via the correspondence $x_1^{r_1} \cdots x_n^{r_n} \leftrightarrow (r_1, \dots, r_n)$. Given a monomial $\mu \in \mathcal{X}^*$ the *cone* generated by μ is defined to be $C(\mu) = \{\mu\nu \mid \nu \in \mathcal{X}^*\}$. A union of cones in $\mathcal{X}^*$ is called a *monomial ideal* (see, e.g., [21]). For each k, we have

$$\left\{\sigma \in \mathcal{X}^* \middle| u_\sigma^k \in \text{LD}\,\Sigma^\infty\right\} = \bigcup_{u_\mu^k \in \text{LD}\,\Sigma} C(\mu).$$

Hence, to every infinitely prolonged system Σ^∞ there corresponds a collection of monomial ideals, one for each k, consisting of principal derivatives u_μ^k with one and the same k.

Monomial ideals are usually visualized by staircase diagrams in $\mathbb{R}^n$. In $\mathbb{R}^n$, every point $(z_1, \dots, z_n) \in \mathbb{N}^n$ generates the *corner*

$$C(z_1, \dots, z_n) = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \,\middle|\, z_i \le x_i \text{ for all } i\right\}.$$

A union of corners, which is an unbounded orthogonal (usually non-convex) polytope with vertices in integer points $\mathbb{N}^n \subset \mathbb{R}^n$, is called a *staircase diagram*. An oriented edge between two integer points $p = (z_1, \dots, z_i, \dots, z_n)$ and $q = (z_1, \dots, z_i + 1, \dots, z_n)$ is called a *direction* from p to q. A square bounded by four adjacent edges is called a *tile*. An *xy-tile* is a tile parallel to the xy-plane. Two of the bounding directions end in a common point, which will be called the *vertex* of the tile. Now, on the staircase diagram u_μ^k lies in, $\max \mathcal{X}_\mu^k$ can be seen as the set of all directions that lead to μ. Two distinct directions $x, y \in \max \mathcal{X}$ then satisfy $x \downarrow y$ if and only if the staircase diagram contains the xy-tile with the vertex μ.

6 The Algorithm

Before proceeding to more substantial examples, let us finally present the procedure to find a sufficient set of integrability conditions. Below the symbol # denotes the number of elements in a finite set and $\text{var}(\rho) = \{x \in \mathcal{X} \mid x \text{ divides } \rho\}$ for $\rho \in \mathcal{X}^*$ is the set of all variables to occur in a monomial. For clarity, we present two separate algorithms, one for integrability conditions of each kind. Both algorithms share partitioning of the system Σ into subsystems $\Sigma^{(k)}$ such that $\text{LD}\,\Sigma^{(k)}$ contains derivatives u_μ^k of u^k. Sets $\text{LD}\,\Sigma^{(k)}$ being denoted by $L^{(k)}$, their minimal elements are then collected in subsets $M^{(k)} \subseteq L^{(k)}$. Of course, the algorithms can share the k loop.

Algorithm 1 to compute integrability conditions of the first kind is very simple. To each non-minimal element $u_\sigma^k \in L^{(k)} \setminus M^{(k)}$ there corresponds exactly one integrability condition of the first kind.

Algorithm 1 Integrability conditions of the first kind

Input: Σ.
Output: the set $\mathrm{IC}_\Sigma^{(1)}$ of integrability conditions of the first kind.

$\mathrm{IC}_\Sigma^{(1)} := \emptyset$
for all k **do**
  $L^{(k)} := \{\nu \in \mathcal{X}^* \mid u_\nu^k \in \mathrm{LD}\,\Sigma\}$
  $M^{(k)} :=$ the set of minimal elements in $L^{(k)}$ with respect to $\leq$
  for $\mu \in L^{(k)} \setminus M^{(k)}$ **do**
    select arbitrary $\nu \in M^{(k)}$ such that $\nu < \mu$
    adjoin expression $\Phi_\mu^k - D_{\mu/\nu}\Phi_\nu^k$ to $\mathrm{IC}_\Sigma^{(1)}$
  end for
end for
return $\mathrm{IC}_\Sigma^{(1)}$

Algorithm 2 Integrability conditions of the second kind

Input: Σ. *Output*: the set $\mathrm{IC}_\Sigma^{(2)}$ of integrability conditions of the second kind

1: $\mathrm{IC}_\Sigma^{(2)} := \emptyset$
2: **for all** k **do**
3:   let $L^{(k)}$, $M^{(k)}$ be those of Algorithm 1
4:   $C^{(k)} := \{\mathrm{lcm}(\nu_1, \nu_2) \mid \{\nu_1, \nu_2\} \subseteq M^{(k)}$ a two-element subset$\}$
5:   **for** $\mu \in C^{(k)}$ **do**
6:     $\mathfrak{M} = \{\mathrm{var}(\mu/\sigma) \mid \sigma \in M^{(k)}$ and $\sigma < \mu\}$
7:     $\mathfrak{N} :=$ the set of connected components of the hypergraph $(\bigcup \mathfrak{M}, \mathfrak{M})$
8:     **if** $\#\mathfrak{N} > 1$ **then**
9:       select arbitrary $\sigma_1, \ldots, \sigma_s \in L^{(k)}$ such that $\mathrm{var}(\mu/\sigma_i)$ is a subset of the ith connected component,
10:       adjoin $D_{\mu/\sigma_2}\Phi_{\sigma_2}^k - D_{\mu/\sigma_1}\Phi_{\sigma_1}^k, \ldots, D_{\mu/\sigma_s}\Phi_{\sigma_s}^k - D_{\mu/\sigma_1}\Phi_{\sigma_1}^k$ to $\mathrm{IC}_\Sigma^{(2)}$
11:     **end if**
12:   **end for**
13: **end for**
14: **return** $\mathrm{IC}_\Sigma^{(2)}$

Before explaining Algorithm 2 to compute integrability conditions of the second kind, let us consider various descriptions offered by Corollary 1. There is no upper bound for the size $\#\min \mathcal{X}_\mu^k$ since the number of equations in the system Σ can be arbitrary, whereas $\#\max \mathcal{X}_\mu^k$ is bounded by the number of independent variables $\#\mathcal{X}$. During completion, $\#\min \mathcal{X}_\mu^k$ grows as new equations are added to the system, while $\#\max \mathcal{X}_\mu^k$ remains essentially stable. This is why Algorithm 2 uses the relation $\downarrow$ on $\max \mathcal{X}_\mu^k$ rather than the relation $\uparrow$ on $\min \mathcal{X}_\mu^k$.

Now, Algorithm 2 works as follows. On line 4 cross-derivatives are computed, using only minimal derivatives collected in $M^{(k)}$. This is justified by Proposition 2 as trivial cross-derivatives do not contribute to $\mathrm{IC}_\Sigma^{(2)}$.

To explain lines 6 and 7, observe that the reflexive and symmetric relation $\downarrow$ is conveniently represented as the union of relations $\downarrow_\sigma$, where σ runs through $M^{(k)}$ and $\downarrow_\sigma$ is defined by $x \downarrow_\sigma y$ if $x = y$ or $\sigma \le \mu/x, \mu/y$. Obviously, each $\downarrow_\sigma$ is an equivalence relation. Moreover, the associated partition of $\mathcal{X} \subseteq \max \mathcal{X}^k_\mu$ has only one nontrivial class, apart from one-element sets, and this class can be identified with $\operatorname{var}(\mu/\sigma)$. Hence, the partition $\mathfrak{N}$ corresponding to the transitive hull $\downarrow^*$ is the least partition such that every $\operatorname{var}(\mu/\sigma)$ is a subset of some class of $\mathfrak{N}$. Computing such a least partition amounts to joining all incident subsets. Alternatively, $\mathfrak{N}$ can be described as the set of connected components of the hypergraph $(\max \mathcal{X}^k_\mu, \mathfrak{M})$, where the set of hyperedges is $\mathfrak{M} = \{\operatorname{var}(\mu/\sigma) \mid \sigma \text{ in } M^{(k)}\}$ and the set of vertices is $\max \mathcal{X}^k_\mu = \bigcup \mathfrak{M}$. Known algorithms [16] are capable of labeling connected components of a (hyper)graph in expected time linear in the number of vertices. The time complexity of Algorithm 2 is estimated in Remark 5 below. An obvious time-saving modification is to interlace lines 6 and 7 and break as soon as $\mathfrak{N} = \mathfrak{X}$ and consists of one connected component. The arbitrary selections made on line 6 of Algorithm 1 and lines 10, 11 of Algorithm 2 exhaust the entire freedom of choice of conventional integrability conditions (Definition 9) relative to Remark 1.

Let us discuss the whole completion procedure now. If all the integrability conditions of Theorem 1 are satisfied (i.e., if they reduce to identities as explained in Sect. 3), then the system is passive and no further steps are needed. Otherwise let $\bar{\Sigma}$ denote the extended system obtained by resolving the non-identical integrability conditions with respect to maximal derivatives (under the same ranking) and adjoining them to Σ (incrementally, see below). The system $\bar{\Sigma}$ is afterwards subject to the same procedure of selecting a reduction subsystem $\bar{\Sigma}'$ of the infinite prolongation $\bar{\Sigma}^\infty$, identifying the integrability conditions, etc. Obviously, we then have $\bar{S} \circ S = \bar{S} = S \circ \bar{S}$ and $\mathcal{E}_{\Sigma'} \subseteq \mathcal{E}_{\bar{\Sigma}'}$ (see Remark 2).

Of course, the time spent on redundancy elimination is only one factor, and not the most important one. Far greater savings can be achieved by an appropriate choice of the completion strategy (cf. Buchberger [5, 6]). It is well known that adding one integrability condition at a time is better than adding all of them at once. The number of newly emerging integrability conditions (of both kinds) that result from extending Σ to $\bar{\Sigma}$ is at most $\# M^{(k)}$ per one added $\mu \in \bar{M}^{(k)} \setminus M^{(k)}$. Nevertheless, yet unresolved integrability conditions of the second kind can trivialize and the objective is to avoid computing the integrability conditions that will trivialize. Prospects are higher if linear combinations of integrability conditions are taken into consideration, as is the case with the Faugère algorithm [8]. Faugère's empirically optimal "normal strategy" is to select minimal order leading derivatives. Hopes are that our work can provide a theoretical foundation for optimal selection. The idea of early integration [39] is also worth further study.

Of course, being a part of a completion algorithm, the implementation should keep track of the integrability conditions already satisfied in previous steps. But do the once satisfied integrability conditions of Σ continue to hold under the new reduction $\bar{S}$? Recall the inductive proof of (7) (the core statement of Theorem 1 the others follow from) for the system $\bar{\Sigma}$. Assume (7_Σ), i.e., $Su^k_\mu = SD_\xi Su^k_{\mu/\xi}$, at u^k_μ, while for all $u^l_\nu \prec u^k_\mu$ assume ($7_{\bar{\Sigma}}$), i.e., $\bar{S}u^l_\nu = \bar{S}D_\xi \bar{S}u^l_{\nu/\xi}$. By ($7_{\bar{\Sigma}}$) we have $\bar{S}D_\xi \bar{S}F = \bar{S}D_\xi F$ for arbitrary function F of derivatives that precede u^k_μ, in particular, for $F = Su^k_{\mu/\xi}$. But then $\bar{S}D_\xi \bar{S}u^k_{\mu/\xi} = \bar{S}D_\xi \bar{S}Su^k_{\mu/\xi} = \bar{S}D_\xi Su^k_{\mu/\xi} = \bar{S}SD_\xi Su^k_{\mu/\xi} = \bar{S}Su^k_\mu = \bar{S}u^k_\mu$. Thus, we have ($7_{\bar{\Sigma}}$) at u^k_μ as well.

An important question is whether the completion algorithm eventually stops. An affirmative answer easily follows from the Dickson lemma, since new integrability conditions can reside only at points u^k_μ outside the monomial ideals generated by $\mathrm{LD}\,\Sigma^{(k)}$.

We close this section with a remark on autoreduction.

Remark 3 Certain grounds exist for maintaining the reduction subsystem non-autoreduced. For example, let the input system Σ contain two equations $u_\mu = Fu$, $u_\nu = Gu$, where F, G are linear differential operators with *constant* coefficients. Let, moreover, μ, ν be relatively prime. Then the corresponding integrability condition of the second kind at $u_{\mu\nu}$ is nontrivial, yet automatically satisfied: $D_\nu Fu = FGu = GFu = D_\mu Gu$ on $\mathcal{E}^\infty_\Sigma$. (In polynomial elimination theory, this case is covered by the so-called first Buchberger criterion [5].) Now, the crucial identity $FGu = GFu$ (which only appears in expanded form) is much easier to check before applying any reductions for $u_\sigma \in \operatorname{var} FGu = \operatorname{var} GFu$. To a lesser extent this is the case even if F, G have non-constant coefficients etc.

7 Irredundancy

In this section, we prove that Construction 2 produces no redundant integrability condition. By a redundant condition we usually mean one that can be safely omitted from the checklist, since it is satisfied automatically whenever the others are. To put it more formally, observe that the essentials of Construction 2 depend only on the set $P = \mathrm{LD}\,\Sigma$ of derivatives u^k_μ on the left-hand side of the input system (1), while functions Φ^k_ξ on the right-hand side play the role of parameters. Let us therefore consider the whole class $\mathfrak{S}_P$ of orthonomic systems Σ with fixed set $P = \mathrm{LD}\,\Sigma$, parametrized by arbitrary functions Φ^k_ξ subject only to requirements of orthonomicity. Obviously, Construction 2 provides a set of integrability conditions applicable to every member of the class $\mathfrak{S}_P$.

Definition 12 Consider the class $\mathfrak{S}_P$ of orthonomic systems Σ with a fixed set $P = \mathrm{LD}\,\Sigma$. Let $\mathcal{I}$ be a set of integrability conditions of $\mathfrak{S}_P$. An integrability condition $I \in \mathcal{I}$ is said to be *redundant* if it is satisfied for every choice of right-hand sides Φ^k_ξ for which all the other integrability conditions $\mathcal{I} \setminus \{I\}$ are satisfied. The set $\mathcal{I}$ is said to be irredundant if it contains no redundant integrability condition.

A chain (6) is to be considered as a sequence of $s - 1$ integrability conditions, so that each equality sign determines a separate integrability condition.

Remark 4 Definition 12 implicitly refers to some functional space $\mathcal{S}$ to choose the right-hand sides Φ^k_ξ from. The proof of Proposition 3 below only requires that $\mathcal{S}$ contains all polynomials in the independent variables.

Proposition 3 *The set $\mathcal{I}$ of integrability conditions resulting from Construction 2 is irredundant.*

Proof To start with, we assume that all integrability conditions from $\mathcal{I}$ are conventional (see Definition 9). Let $\Phi^k_\mu = SD_{\mu/\xi} S\Phi^k_\xi \in \mathcal{I}$ be such an integrability condition of the first kind. By assigning $\Phi^k_\mu = 1$ and $\Phi^l_\sigma = 0$ for all $u^l_\sigma \in \mathrm{LD}\,\Sigma \setminus \{u^k_\mu\}$, we obtain an orthonomic system obviously satisfying all integrability conditions except $\Phi^k_\mu = 1 \neq 0 = SD_{\mu/\xi} S\Phi^k_\xi$.

Similarly, consider an arbitrary conventional integrability condition of the second kind from $\mathcal{I}$, say

$$SD_{\mu/\xi_1}\Phi^k_{\xi_1} = SD_{\mu/\xi_2}\Phi^k_{\xi_2} = \cdots = SD_{\mu/\xi_s}\Phi^k_{\xi_s} \tag{9}$$

at $u^k_\mu \in \mathrm{LD}\,\Sigma^\infty$. Let $[\xi_1], \ldots, [\xi_s]$ denote the corresponding equivalence classes in $\min \mathcal{X}^k_\mu$. Let $1 \le r < s$ be an arbitrary integer and I denote the rth integrability condition in the chain, i.e., $SD_{\mu/\xi_r}\Phi^k_{\xi_r} = SD_{\mu/\xi_{r+1}}\Phi^k_{\xi_{r+1}}$. By Construction 2, $\mathcal{I}$ contains no more than one integrability condition of the first kind of the form $\Phi^k_\mu = SD_{\mu/\sigma}\Phi^k_\sigma$. If such a σ exists, let Ξ denote $[\xi_1] \cup \cdots \cup [\xi_r]$ or $[\xi_{r+1}] \cup \cdots \cup [\xi_s]$ whichever contains σ. If no such σ exists, then let Ξ be one (arbitrarily chosen) of these two sets.

For every monomial $\sigma = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{X}^*$ we introduce the function of independent variables

$$F_\sigma(x_1, \ldots, x_n) = \frac{x_1^{a_1} \cdots x_n^{a_n}}{a_1! \cdots a_n!},$$

which obviously satisfies

$$D_\tau F_\sigma = \begin{cases} F_{\sigma/\tau} & \text{if } \tau \le \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Turning back to our proof, for every $u^l_\sigma \in \mathrm{LD}\,\Sigma$ we assign Φ^l_σ according to the following simple rule:

$$\Phi^l_\sigma = \begin{cases} F_{\mu/\sigma} & \text{if } l = k \text{ and } \sigma \in \{\mu\} \cup \Xi, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Consider an arbitrary integrability condition $\Phi^k_\nu = SD_{\nu/\sigma}\Phi^k_\sigma$, $\sigma < \nu$, of the first kind from $\mathcal{I}$. The only possibility how the left-hand side can be nonzero is when

$$\text{(A)} \quad \sigma \in \{\mu\} \cup \Xi, \qquad \nu \le \mu,$$

and then it equals $D_{\nu/\sigma}\Phi^k_\sigma = D_{\nu/\sigma}F_{\mu/\sigma} = F_{\mu/\nu}$. The only possibility how the right-hand side can be nonzero is when

$$\text{(B)} \quad \nu \in \{\mu\} \cup \Xi,$$

and then it equals the same $F_{\mu/\nu}$. It remains to be checked that conditions (A) and (B) are equivalent. Before that we observe that the inequality $\sigma < \nu$ implies

$$\text{(C)} \quad \text{if } \sigma, \nu < \mu, \quad \text{then } \sigma, \nu \text{ both or neither lie in } \Xi.$$

Indeed, under these conditions we have $\sigma \approx \nu$ in $\mathcal{X}^k_\mu$.

Let (A) hold true. The case $\nu = \mu$ being trivial, consider $\nu < \mu$. Then also $\sigma < \mu$ and therefore $\sigma \in \Xi$. But then $\nu \in \Xi$ by (C), giving (B). Conversely, let (B) hold true. If $\nu = \mu$, then $\sigma \in \Xi$ since Ξ was chosen that way. Otherwise $\nu \in \Xi$, but then $\sigma \in \Xi$ by (C) again. Therefore, (A) is true. Thus, we have proved the equivalence (A) $\Leftrightarrow$ (B) and hence validity of all integrability conditions of the first kind.

Now consider an integrability condition of the second kind from $\mathcal{I}$, at some u^l_ν. In the case of $u^l_\nu = u^k_\mu$, the integrability condition is (9). However, $SD_{\mu/\xi}\Phi^k_\xi$ equals 1 if $\xi \in \Xi$ and 0 otherwise, hence all equalities (9) hold except for I. Thus, we are left with the case of an integrability condition $SD_{\nu/\sigma_1}\Phi^k_{\sigma_1} = \cdots = SD_{\nu/\sigma_t}\Phi^k_{\sigma_t}$ from $\mathcal{I}$, at some $u^l_\nu \ne u^k_\mu$. By (11) and (10), the values $SD_{\nu/\sigma_i}\Phi^k_{\sigma_i}$ to be compared at u^l_ν are all zero except when

$$(\mathrm{A}_i) \quad l = k, \qquad \sigma_i < \nu \le \mu, \quad \sigma_i \in \Xi,$$

and then they are $D_{\nu/\sigma_i}\Phi^k_{\sigma_i} = D_{\nu/\sigma_i}F_{\mu/\sigma_i} = F_{\mu/\nu}$ independently of i. Let us show that conditions (A_i) are equivalent. However, if one of (A_i) holds, then $l = k$ and $\nu \le \mu$, hence $\nu < \mu$ (otherwise $u^l_\nu = u^k_\mu$). We have $\sigma_j < \nu$ for all j by Construction 2, hence $\sigma_1 \approx \cdots \approx \sigma_t$ in $\mathcal{X}^k_\mu$ (although not in $\mathcal{X}^k_\nu$). Therefore, all σ_j belong to Ξ. Equivalence of conditions (A_i) is thereby established.

Thus, we have proved the proposition in case of conventional integrability conditions. But since all Φ^l_σ assigned during the proof were functions of independent variables only, we have simply $SD_\nu\Phi^l_\sigma = D_\nu\Phi^l_\sigma$ for the reduction S of any principal derivative. This means that every integrability condition can be identified with a conventional integrability condition. Hence, the proposition holds for general integrability conditions as well. $\square$

It remains to compare our definition of redundancy with that used by other authors, notably Rust [28, 30]. Consider the free abelian algebra $\mathcal{A}^k_\mu$ over the set of abstract generators of the form $D_{\mu/\xi}\Phi^k_\xi$. The total derivatives D_x act upon the generators, hence upon the whole algebra, in a natural way. An integrability condition can be viewed as a difference $D_{\mu/\xi}\Phi^k_\xi - D_{\mu/\eta}\Phi^k_\eta \in \mathcal{A}^k_\mu$ of two generators. Given a finite set $\mathcal{I} \subset \mathcal{A}^k_\mu$ of integrability conditions, another integrability condition $I = D_{\mu/\xi}\Phi^k_\xi - D_{\mu/\eta}\Phi^k_\eta$ is said to be *syzygy redundant* if monomials $\mu_i \le \mu$, integrability conditions $I_i = D_{\mu_i/\xi_i}\Phi^k_{\xi_i} - D_{\mu_i/\eta_i}\Phi^k_{\eta_i} \in \mathcal{I}$, and integers $c_i \in \mathbb{Z}$ exist such that

$$I = \sum_i c_i D_{\mu/\mu_i} I_i = \sum_i c_i \left(D_{\mu/\xi_i}\Phi^k_{\xi_i} - D_{\mu/\eta_i}\Phi^k_{\eta_i}\right) \tag{12}$$

holds in $\mathcal{A}^k_\mu$.

It is clear that if I is syzygy redundant, then it is also redundant in the sense of Definition 12 for all choices of $\mathcal{S}$ (see Remark 4). Hence the sufficient set of integrability conditions resulting from Construction 2, proved to be irredundant when $\mathcal{S}$ contains all polynomials in independent variables, is also syzygy irredundant.

8 Examples

The first two examples compare our algorithms to Algorithm 9 from Wittkopf's dissertation [38]. Wittkopf's algorithm removes nearly all of the syzygy redundancy. Experiments with randomly generated monomial ideals showed that Wittkopf's algorithm can miss r redundant integrability conditions in case of ideals with $4r$ generators, but such instances are rather rare. A surprise was that Wittkopf's algorithm could be substantially slower than ours.

Example 4 Consider a system Σ of the form

$$u_{xyz} = f_1, \qquad u_{xxz} = f_2, \qquad u_{yyz} = f_3, \qquad u_{xxyy} = f_4.$$

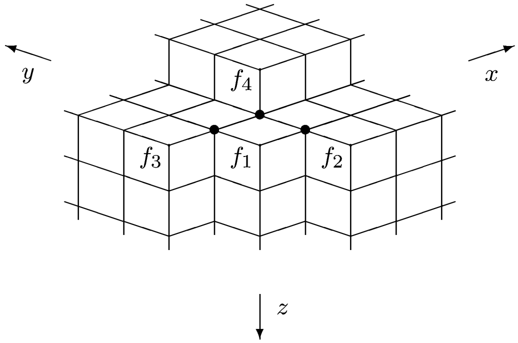
We summarize the work of Algorithm 2 in Table 1.

The first column lists all possible cross-derivatives μ. Columns 2–5 correspond to the four derivatives u_σ from LD Σ. These four columns list variables the monomial μ/σ depends on whenever σ divides μ, and contain an empty space when μ/σ is not a monomial. The sixth column contains the least partition of $\max \mathcal{X}_\mu$ generated by the sets occurring in columns 2–5 ($\max \mathcal{X}_\mu$ is the union of these sets). By results of Section 5 this partition corresponds to the equivalence relation $\approx$ on $\max \mathcal{X}_\mu$ inherited from $\mathcal{X}_\mu$. Algorithm 2 also says how to choose the integrability conditions (we omit the reduction symbol S). In the first and second row, the only possibility is that given in the last column (when $\mu = x^2yz$ or $\mu = xy^2z$, each connected component of $\mathcal{X}_\mu$ contains a single σ). Contrary to that, one of the connected components of $\mathcal{X}_{xxyyz}$ contains three monomials σ, namely xyz, x^2z, y^2z. Hence, apart from $D_{xy} f_1 = D_z f_4$ shown in the table, there are two other equivalent ways to write the third integrability condition: $D_{yy} f_2 = D_z f_4$ and $D_{xx} f_3 = D_z f_4$.

Table 1 Summary of Example 4

μ	var(μ/σ), $\sigma =$				$\max \mathcal{X}_\mu/\approx$	IC$^{(2)}$
	xyz	x^2z	y^2z	x^2y^2		
x^2yz	x	y			$\{x\}, \{y\}$	$D_x f_1 = D_y f_2$
xy^2z	y		x		$\{x\}, \{y\}$	$D_y f_1 = D_x f_3$
x^2y^2z	xy	y	x	z	$\{x, y\}, \{z\}$	$D_{xy} f_1 = D_z f_4$

The example can be visualized (see the end of Sect. 5) as follows:



The reader may wish to locate the tile that induces the equivalence relation $x \approx y$ at $\mu = x^2y^2z$.

Example 4 is one of the simplest where Algorithm 9 from Wittkopf's dissertation [38] misses one redundant integrability condition. However, Wittkopf's algorithm depends on a choice of what is called compatible ranking [38, Definition 10] of syzygies, which itself depends on a choice of a ranking for Σ (which we fix to be $x \prec y \prec z$) and a permutation of the set Σ. In Example 4, Wittkopf's algorithm has a very favorable ratio $\frac{11}{12}$ of correct answers in the set of all $4! = 24$ permutations of Σ. This ratio can be less favorable in other examples.

Example 5 Consider a system Σ of the form

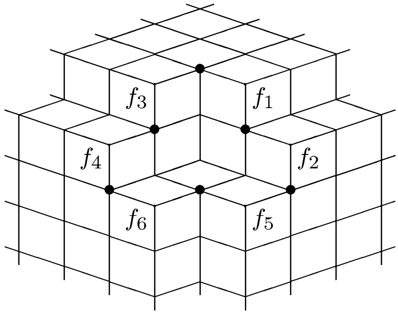
$$u_{xxy} = f_1, \qquad u_{xxz} = f_2, \qquad u_{xyy} = f_3,$$
$$u_{xzz} = f_4, \qquad u_{yyz} = f_5, \qquad u_{yzz} = f_6.$$

We summarize the work of Algorithm 2 in Table 2.

Table 2 Summary of Example 5

μ	$\mathrm{var}(\mu/\sigma),\ \sigma =$						$\max \mathcal{X}_\mu/\approx$	$\mathrm{IC}^{(2)}$
	x^2y	x^2z	xy^2	y^2z	xz^2	yz^2		
x^2yz	z	y					$\{y\}, \{z\}$	$D_z f_1 = D_y f_2$
xy^2z			z	x			$\{x\}, \{z\}$	$D_z f_3 = D_x f_4$
xyz^2					y	x	$\{x\}, \{y\}$	$D_y f_5 = D_x f_6$
x^2y^2	y		x				$\{x\}, \{y\}$	$D_y f_1 = D_x f_3$
x^2z^2		z			x		$\{x\}, \{z\}$	$D_z f_2 = D_x f_5$
y^2z^2				z		y	$\{y\}, \{z\}$	$D_z f_4 = D_y f_6$
x^2y^2z	yz	y	xz	x			$\{x, y, z\}$	
x^2yz^2	z	yz			xy	x	$\{x, y, z\}$	
xy^2z^2			z	xz	y	xy	$\{x, y, z\}$	

For explanation of the table see Example 4. The last column shows the unique integrability condition in each of the first six rows and none in the remaining three. The corresponding diagram is



Wittkopf's Algorithm 9 gives an incorrect number of integrability conditions (seven) in 540 of the full number $6! = 720$ of permutations of Σ. Thus, the ratio of correct answers is only $1/4$ now.

Remark 5 In randomly generated examples, Wittkopf's Algorithm 9 ran substantially longer than ours on the same data. Both algorithms take advantage of the partitioning $\Sigma = \bigcup \Sigma^{(k)}$. Let us therefore attempt a comparison in the case of one dependent variable (so that there is no k loop).

The then outer loop of Wittkopf's algorithm runs over the syzygy system $\mathcal{S}$ which has $O(r^2)$ elements, where $r = \#\Sigma$. At each run, the subset $\mathcal{S}' \subseteq \mathcal{S}$ of already executed (accepted or rejected) syzygies is incremented. Processing elements $s' \in \mathcal{S}'$ in Step 3.1 costs $\#\mathcal{S}'$ time units. Processing pairs of elements $s', s'' \in \mathcal{S}'$ in Step 3.2 costs between $\#\mathcal{S}'$ and $(\#\mathcal{S}')^2$ time units (s' and s'' are not independent). This suggests running time of at least $O(r^4)$.

In our Algorithm 2, $L^{(k)}$ as well as $M^{(k)}$ have $O(r)$ elements. The main loop 5–12 runs over $C^{(k)}$, which has $O(r^2)$ elements. At each run, building $\mathfrak{M}$ on line 6 requires time proportional to $\#M^{(k)}$. Obtaining connected components of the hypergraph $\mathfrak{M}$ on line 7 requires time proportional to $\#M^{(k)} + \#\mathcal{X}$, where typically $\#M^{(k)} \geq \#\mathcal{X}$. This means $O(r^3)$ running time.

Finally, we give an example where integrability conditions ordered by divisibility form a chain. It is easy to show that $n - 1$ is the maximal length of such a chain in the case of n variables.

Example 6 Let $n > 2$ be arbitrary. Consider the following n Janet monomials in n variables:

$$x_2 x_3 x_4 \cdots x_n,$$

$$x_1^2 x_3 x_4 \cdots x_n,$$

$$x_1^2 x_2^2 x_4 \cdots x_n,$$

$$\dots,$$
$$x_1^2 x_2^2 x_3^2 \cdots x_{n-1}^2.$$

The cross-derivatives

$$x_1^2 x_2 x_3 x_4 \cdots x_n,$$
$$x_1^2 x_2^2 x_3 x_4 \cdots x_n,$$
$$x_1^2 x_2^2 x_3^2 x_4 \cdots x_n,$$
$$\dots,$$
$$x_1^2 x_2^2 x_3^2 \cdots x_{n-1}^2 x_n$$

are all nontrivial and form a chain of length $n-1$.

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