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Exotic Quantifiers, Complexity Classes, and Complete Problems

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Abstract We define new complexity classes in the Blum–Shub–Smale theory of computation over the reals, in the spirit of the polynomial hierarchy, with the help of infinitesimal and generic quantifiers. Basic topological properties of semialgebraic sets like boundedness, closedness, compactness, as well as the continuity of semialgebraic functions are shown to be complete in these new classes. All attempts to classify the complexity of these problems in terms of the previously studied complexity classes have failed. We also obtain completeness results in the Turing model for the corresponding discrete problems. In this setting, it turns out that infinitesimal and generic quantifiers can be eliminated, so that the relevant complexity classes can be described in terms of the usual quantifiers only.

Keywords BSS computation · Complexity classes · Complete problems · Genericity · Infinitesimals

Mathematics Subject Classification (2000) 68Q15 · 68Q17

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1 Introduction

Complexity theory over the real numbers developed quickly after the foundational paper [6] by L. Blum, M. Shub, and S. Smale. Complexity classes other than $\mathsf{P}_{\mathbb{R}}$ and $\mathsf{NP}_{\mathbb{R}}$ were introduced (e.g., in [11, 14, 15]), completeness results were proven (e.g., in [11, 22, 29]), separations were obtained [14, 21], machine-independent characterizations of complexity classes were exhibited [8, 19, 23], ...

There are two points in this development which we would like to stress. First, all the considered complexity classes were natural versions over the real numbers of existing complexity classes in the classical setting. Second, the catalogue of completeness results is disappointingly small. For a given semialgebraic set $S \subseteq \mathbb{R}^n$, deciding whether a point in $\mathbb{R}^n$ belongs to S is $\mathsf{P}_{\mathbb{R}}$-complete [22], deciding whether S is nonempty (or nonconvex, or of dimension at least d for a given $d \in \mathbb{N}$) is $\mathsf{NP}_{\mathbb{R}}$-complete [6, 20, 29], and computing its Euler–Yao characteristic is $\mathsf{FP}_{\mathbb{R}}^{\#\mathsf{P}_{\mathbb{R}}}$-complete [11]. That is, essentially, all.

Yet, there are plenty of natural problems involving semialgebraic sets: computing local dimensions, deciding denseness, closedness, unboundedness, ... Consider, for instance, the latter. We can express that S is unbounded by

$$\forall K \in \mathbb{R}\, \exists x \in \mathbb{R}^n (x \in S \wedge \|x\| \geq K). \tag{1}$$

Properties describable with expressions like this one are common in classical complexity theory and in recursive function theory. Extending an idea by Kleene [26] for the latter, Stockmeyer introduced in [32] the polynomial time hierarchy which is built on top of NP and coNP in a natural way.[1] Recall a set S is in NP when there is a polynomial time decidable relation R such that, for every $x \in \{0,1\}^*$,

$$x \in S \iff \exists y \in \{0,1\}^{\mathrm{size}(x)^{\mathcal{O}(1)}} R(x, y).$$

The class coNP is defined replacing $\exists$ by $\forall$. Classes in the polynomial hierarchy are then defined by allowing the quantifiers $\exists$ and $\forall$ to alternate (with a bounded number of alternations). If there are k alternations of quantifiers, we obtain the classes Σ^{k+1} (if the first quantifier is $\exists$) and Π^{k+1} (if the first quantifier is $\forall$). Note that $\Sigma^1 = \mathsf{NP}$ and $\Pi^1 = \mathsf{coNP}$. The definition of these classes over $\mathbb{R}$ is straightforward [5, Chap. 21].

It thus follows from (1) that deciding unboundedness is in $\Pi^2_{\mathbb{R}}$, the universal second level of the polynomial hierarchy over $\mathbb{R}$. On the other hand, it is easy to prove that this problem is $\mathsf{NP}_{\mathbb{R}}$-hard. But we do not have completeness for any of these two classes.

A similar situation appears for deciding denseness. We can express that $S \subseteq \mathbb{R}^n$ is Euclidean dense by

$$\forall x \in \mathbb{R}^n\, \forall \varepsilon > 0\, \exists y \in \mathbb{R}^n (y \in S \wedge \|x - y\| \leq \varepsilon),$$

[1] Throughout this paper we use a subscript $\mathbb{R}$ to differentiate complexity classes over $\mathbb{R}$ from discrete complexity classes. To further emphasize this difference, we use sans serif to denote the latter.

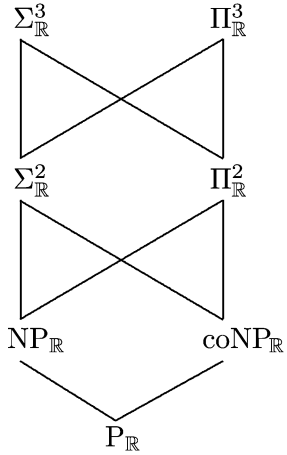

Fig. 1 Classes between $\mathsf{P}_{\mathbb{R}}$ and the third level of the polynomial hierarchy

thus showing that this problem is in $\Pi^2_{\mathbb{R}}$. But we cannot prove hardness in this class. Actually, we cannot even manage to prove $\mathsf{NP}_{\mathbb{R}}$-hardness or $\mathsf{coNP}_{\mathbb{R}}$-hardness. Yet a similar situation occurs with closedness, which is in $\Pi^3_{\mathbb{R}}$ since we express that S is closed by

$$\forall x \in \mathbb{R}^n\, \exists \varepsilon > 0\, \forall y \in \mathbb{R}^n (x \notin S \wedge \|x - y\| \le \varepsilon \Rightarrow y \notin S),$$

but the best hardness result we can prove is $\mathsf{coNP}_{\mathbb{R}}$-hardness. It would seem that the landscape of complexity classes between $\mathsf{P}_{\mathbb{R}}$ and the third level of the polynomial hierarchy (cf. Fig. 1) is not enough to capture the complexity of the problems above.

A main goal of this paper is to show that the two features we pointed out earlier namely, a theory uniquely based upon real versions of classical complexity classes, and a certain scarcity of completeness results, are not unrelated. We shall define a number of complexity classes lying in between the ones in Fig. 1 above. These new classes will allow us to determine the complexity of some of the problems we mentioned (and of others we didn't mention) or, in some cases, to decrease the gap between their lower and upper complexity bounds as we know them today.

A remarkable feature of these classes is that, as with the classes in the polynomial hierarchy, they are defined using quantifiers which act as operators on complexity classes. The properties of these operators naturally become an object of study for us. Thus, another goal of this paper is to provide some structural results for these operators.

We next define the operators we will deal with in this paper. We denote by $\mathbb{R}^\infty$ the disjoint union $\bigsqcup_{n\ge 0} \mathbb{R}^n$. If $x \in \mathbb{R}^n \subset \mathbb{R}^\infty$ we define its *size* to be $|x| = n$.

Our first new quantifier, H, captures the notion of "for all sufficiently small numbers" and defines an operator of complexity classes as follows.

Definition 1.1 Let $\mathcal{C}$ be a complexity class of decision problems. We say that a set A belongs to $\mathsf{H}\mathcal{C}$ if there exists $B \subseteq \mathbb{R} \times \mathbb{R}^\infty$, $B \in \mathcal{C}$, such that, for all $x \in \mathbb{R}^\infty$,

$$x \in A \iff \exists \mu > 0\, \forall \varepsilon \in (0, \mu)\, (\varepsilon, x) \in B.$$

The quantifiers $\forall^*$ and $\exists^*$ capture the notions of "for almost all points" and "for sufficiently many points" in a specific sense. They were first introduced by Koiran in [29].

Definition 1.2 Let $\mathcal{C}$ be a complexity class of decision problems. We say that a set A belongs to $\forall^*\mathcal{C}$ if there exist a polynomial p and a set $B \subseteq \mathbb{R}^\infty \times \mathbb{R}^\infty$, $B \in \mathcal{C}$, such that, for all $x \in \mathbb{R}^\infty$,

$$x \in A \iff \dim\{z \in \mathbb{R}^{p(|x|)} \mid (z, x) \notin B\} < p(|x|).$$

If $\mathcal{C}$ is a complexity class we denote by $\mathcal{C}^c$ the class of its complements, i.e., the class of all sets A such that $A^c \in \mathcal{C}$. We define $\exists^*\mathcal{C} = (\forall^*\mathcal{C}^c)^c$.

We note that A belongs to $\exists^*\mathcal{C}$ if and only if there exist a polynomial p and a set $B \subseteq \mathbb{R}^\infty \times \mathbb{R}^\infty$, $B \in \mathcal{C}$, such that, for all $x \in \mathbb{R}^\infty$,

$$x \in A \iff \dim\{z \in \mathbb{R}^{p(|x|)} \mid (z, x) \in B\} = p(|x|).$$

Using these operators we may define many new complexity classes. Denote the classes in the picture above by $\forall$ (for $\mathsf{coNP}_\mathbb{R}$), $\exists$ (for $\mathsf{NP}_\mathbb{R}$), $\forall\exists$ (for $\Pi^2_\mathbb{R}$), etc. Then, notations such as $\exists^*\forall$, $\mathsf{H}\forall$, or $\exists^*\mathsf{H}$ denote some of the newly created complexity classes in an obvious manner. To avoid a cumbersome notation, we also write H instead of $\mathsf{HP}_\mathbb{R}$. We call the classes defined this way *polynomial classes*.

If $\mathcal{C}$ is closed under (many-one) reductions, then so are $\mathsf{H}\mathcal{C}$, $\forall^*\mathcal{C}$, and $\exists^*\mathcal{C}$. Section 3 shows that all these newly defined classes possess complete problems. More importantly, Sections 4 to 7 exhibit a number of natural complete problems in these classes (and some in the already known classes $\forall$ and $\forall\exists$). Also in these sections, for some problems whose complexity remains open, we narrow the gap between their known upper and lower bounds. As we shall see, many of the membership proofs of these completeness results possess a simplicity that follows directly from the nature of our newly defined operators. However, some others of these membership proofs require trickier arguments (see Sects. 6.2–6.3).

Most of the problems considered in Sects. 4 to 7 deal with semialgebraic sets (as those mentioned before in this Introduction). But several others deal with piecewise rational functions $f : \mathbb{R}^n \to \mathbb{R}^m$, not necessarily total. Completeness results for this kind of functions are, to the best of our knowledge, new.

In Sect. 9 we deal with the relationship between polynomial classes and classical complexity theory. This is a recurrent theme in real complexity and has drawn the attention of researches in discrete complexity.[2] The basic idea is the following. Let S be a problem over $\mathbb{R}$ complete in a class $\mathcal{C}$. A natural restriction of S is $S^\mathbb{Z}$, the subset of S of those inputs describable over $\{0, 1\}^*$ (e.g., restricting coefficients of input polynomials to be integer). In general, proofs of completeness of a problem S in a class $\mathcal{C}$ use neither real constants nor iterated multiplications. Therefore, such a proof for S induces a completeness proof for $S^\mathbb{Z}$ in the class $\mathsf{BP}^0(\mathcal{C})$. This is the classical

[2] In the Foreword to [5], R. Karp writes "It is interesting to speculate as to whether the questions $\mathsf{P}_\mathbb{R} = \mathsf{NP}_\mathbb{R}$ and $\mathsf{P}_\mathbb{C} = \mathsf{NP}_\mathbb{C}$ are related to each other and to the classical P versus NP question."

complexity class obtained by restricting—for problems in $\mathcal{C}$—inputs to be in $\{0, 1\}^*$ and machines over $\mathbb{R}$ to be constant-free. In this way, all our completeness results induce completeness results in the classical setting. While some of the classes $\mathsf{BP}^0(\mathcal{C})$ may seem somehow arcane, others are quite natural (and have been considered for a good while) and yet some others become increasingly relevant due to the naturalness of the problems which turn out to be complete on them.

Besides exhibiting completeness, several results deal with structural aspects of the newly defined operators and classes. Among these are the inclusion

$$\exists^*\mathcal{C} \subseteq \exists\mathcal{C},$$

and, for any polynomial class $\mathcal{C}$, the equality of classical classes

$$\mathsf{BP}^0(\mathsf{H}\mathcal{C}) = \mathsf{BP}^0(\mathcal{C}).$$

This latter equality allows us to exhibit a number of problems featuring a remarkable property, namely, that while we do not know the problem S to be complete in a real complexity class $\mathcal{C}$ we can nevertheless prove that $S^{\mathbb{Z}}$ is complete in $\mathsf{BP}^0(\mathcal{C})$. We say that S has a *narrow gap for* $\mathcal{C}$. This is a purely structural notion of a narrowness in the gap between the best upper and lower bounds we may know for S.

Section 10 provides a summary, exhibiting both a list of problems and complexity bounds for them, and a diagram with an enhanced view of the universe between $\mathsf{P}_{\mathbb{R}}$ and the third level of the polynomial hierarchy. Finally, we remark that a similar classification has already been achieved in the so-called additive BSS model, without the need to introduce exotic quantifiers [12, 13].

2 Preliminaries

We assume some basic knowledge on real machines and complexity as presented, for instance, in [5, 6].

(1) We recall an *algebraic circuit* $\mathscr{C}$ over $\mathbb{R}$ is an acyclic directed graph where each node has indegree 0, 1, or 2. Nodes with indegree 0 are either labeled as *input nodes* or with elements of $\mathbb{R}$ (we shall call them *constant nodes*). Nodes with indegree 2 are labeled with the binary operators of $\mathbb{R}$, i.e., one of $\{+, \times, -, /\}$. They are called *arithmetic nodes*. Nodes with indegree 1 are either *sign nodes* or *output nodes*. All the output nodes have outdegree 0. Otherwise, there is no upper bound for the outdegree of the other kinds of nodes. Occasionally, the nodes of an algebraic circuit will be called *gates*.

An arithmetic node computes a function of its input values in an obvious manner. Sign nodes also compute a function, namely,

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

For an algebraic circuit $\mathscr{C}$, the *size* of $\mathscr{C}$ is the number of gates in $\mathscr{C}$. The *depth* of $\mathscr{C}$ is the length of the longest path from some input gate to some output gate.

To a circuit $\mathcal{C}$ with n input gates and m output gates is associated a function $f_{\mathcal{C}}:\mathbb{R}^n\to\mathbb{R}^m$. This function may not be total since divisions by zero may occur (in which case, by convention, $f_{\mathcal{C}}$ is not defined on its input).

We say that an algebraic circuit is a *decision* circuit if it has only one output gate whose parent is a sign gate. Thus, a decision circuit $\mathcal{C}$ with n input gates computes a function $f_{\mathcal{C}}:\mathbb{R}^n\to\{0,1\}$. The set *decided* by the circuit is

$$S_{\mathcal{C}}=\bigl\{x\in\mathbb{R}^n \mid f_{\mathcal{C}}(x)=1\bigr\}.$$

(2) Subsets of $\mathbb{R}^n$ decidable by algebraic circuits are known as *semialgebraic sets*. They are defined as those sets which can be written as a Boolean combination of solution sets of polynomial inequalities $\{x\in\mathbb{R}^n \mid f(x)\geq 0\}$.

Semialgebraic sets will be inputs to problems considered in this paper. They will either be given by a Boolean combination of polynomial equalities and inequalities or by a decision circuit. If not otherwise specified, we mean the first variant. In this case, polynomials are encoded with the so-called *dense encoding*, i.e., they are represented by the complete list of their coefficients (including zero coefficients).

Partial functions $f:\mathbb{R}^n\to\mathbb{R}^m$ computable by algebraic circuits are known as *piecewise rational*. These are the functions f for which there exists a semialgebraic partition $\mathbb{R}^n=S_0\cup S_1\cup\cdots\cup S_k$ and rational functions $g_i:S_i\to\mathbb{R}^m$, $i=1,\dots,k$, such that g_i is well defined on S_i and $f_{|S_i}=g_i$. Note that f is undefined on S_0. We will also consider piecewise rational functions as inputs to some problems. They will be encoded by algebraic circuits.

(3) The symbols H, $\exists^*$, and $\forall^*$ can be considered as logical quantifiers in the theory of the reals. If $\varphi(\varepsilon)$ is a formula with one free variable ε, and $\psi(x)$ is one with n free variables $x_1,\dots,x_n$, we define

$$\begin{aligned}
\mathsf{H}\varepsilon\varphi(\varepsilon) &\stackrel{\text{def}}{\equiv} \exists\mu>0\,\forall\varepsilon\in(0,\mu)\varphi(\varepsilon),\\
\forall^* x\,\psi(x) &\stackrel{\text{def}}{\equiv} \forall x_0\,\forall\varepsilon>0\,\exists x\bigl(\|x-x_0\|<\varepsilon\wedge\psi(x)\bigr),\\
\exists^* x\,\psi(x) &\stackrel{\text{def}}{\equiv} \exists x_0\,\exists\varepsilon>0\,\forall x\bigl(\|x-x_0\|<\varepsilon\Rightarrow\psi(x)\bigr).
\end{aligned}\tag{2}$$

The comments on these definitions, write $S=\{x\in\mathbb{R}^n \mid \psi(x) \text{ holds}\}$. The second line expresses that S is Euclidean dense in $\mathbb{R}^N$, which is equivalent to $\dim(\mathbb{R}^n-S)<n$. The third line expresses the fact that S is Zariski dense, which is equivalent to $\dim S=n$. (For the definition and some of the properties of the Zariski topology we refer to [24, Chap. I, §1].) Note that while H quantifies one real number ε, the quantifiers $\forall^*$ and $\exists^*$ refer to a vector $x=(x_1,\dots,x_n)$ of real numbers whose length n may be arbitrary.

The class $Q_1Q_2\dots Q_k$ with Q_i alternating between $\exists$ and $\forall$ is denoted by $\Sigma^k_{\mathbb{R}}$ when $Q_1=\exists$ and by $\Pi^k_{\mathbb{R}}$ when $Q_1=\forall$. Also, $\Sigma^0_{\mathbb{R}}=\Pi^0_{\mathbb{R}}=\mathsf{P}_{\mathbb{R}}$. The family of these classes is known as the *polynomial hierarchy* and its union is denoted by $\mathsf{PH}_{\mathbb{R}}$ (see [5, Chap. 21]).

By extension we will call *polynomial classes* all classes of the form $Q_1Q_2\dots Q_k$ with $k\geq 0$ (in the case $k=0$ we mean $\mathsf{P}_{\mathbb{R}}$) and $Q_i\in\{\exists,\forall,\exists^*,\forall^*,\mathsf{H}\}$. Note that if $\mathcal{C}$ is a polynomial class, then $\mathcal{C}\subseteq\mathsf{PH}_{\mathbb{R}}$.

(4) In the notation $\exists\mathcal{C}$ of Definition 1.2 the prefix $\exists$ refers to a block of existential quantifiers. We write $\exists^{[k]}\mathcal{C}$ to denote the fact that this block has exactly k quantifiers

(or, equivalently, that it is of the form $\exists x \in \mathbb{R}^k$). Similarly we define $\forall^{[k]}\mathcal{C}$. Thus, for instance, $\exists^{[1]}$ denotes a subclass of $\mathsf{NP}_{\mathbb{R}}$ where guesses are restricted to a single real number.

(5) We close this section by recalling a completeness result. Let $\text{DIM}_{\mathbb{R}}$ be the problem of, given a semialgebraic set S (given by a Boolean combination of polynomial equalities and inequalities) and a number $d \in \mathbb{N}$, deciding whether $\dim S \geq d$. In [29] Koiran proved that $\text{DIM}_{\mathbb{R}}$ is $\mathsf{NP}_{\mathbb{R}}$-complete.

3 Standard Complete Problems for Polynomial Classes

The *Circuit Evaluation problem* $\text{CEVAL}_{\mathbb{R}}$ consists of deciding, given a decision circuit $\mathcal{C}$ with n input gates and a point $a \in \mathbb{R}^n$, whether $a \in S_{\mathcal{C}}$. It was proved in [22] that $\text{CEVAL}_{\mathbb{R}}$ is $\mathsf{P}_{\mathbb{R}}$-complete (for parallel logarithmic time reductions). The proof of this result extends to yield complete problems in the classes considered thus far.

Let $Q_1, Q_2, \dots, Q_{p-1} \in \{\exists, \forall, \exists^*, \forall^*, \mathsf{H}\}$ and $Q_p \in \{\exists^*, \forall^*, \mathsf{H}\}$. We define $\text{STANDARD}(Q_1 Q_2 \dots Q_p)$ to be the problem of deciding, given a decision circuit $\mathcal{C}$ with $n_1 + n_2 + \cdots + n_p$ input gates, whether

$$Q_1 x_1 \in \mathbb{R}^{n_1} Q_2 x_2 \in \mathbb{R}^{n_2} \dots Q_p x_p \in \mathbb{R}^{n_p} \mathcal{C}(x_1, \dots, x_p) = 1. \tag{3}$$

Here $n_i = 1$ whenever $Q_i = \mathsf{H}$.

Similarly, for $Q_1, Q_2, \dots, Q_{p-1} \in \{\exists, \forall, \exists^*, \forall^*, \mathsf{H}\}$ and $Q_p = \exists$ or $Q_p = \forall$, we define $\text{STANDARD}(Q_1 Q_2 \dots Q_p)$ to be the problems of deciding, given a polynomial f in $n_1 + n_2 + \cdots + n_p$ variables (in dense encoding), whether

$$Q_1 x_1 \in \mathbb{R}^{n_1} Q_2 x_2 \in \mathbb{R}^{n_2} \dots \exists x_p \in \mathbb{R}^{n_p} f(x_1, \dots, x_p) = 0$$

and

$$Q_1 x_1 \in \mathbb{R}^{n_1} Q_2 x_2 \in \mathbb{R}^{n_2} \dots \forall x_p \in \mathbb{R}^{n_p} f(x_1, \dots, x_p) \neq 0,$$

respectively. From well-known arguments presented in [6, 22] the next result easily follows.

Proposition 3.1 *For all* $Q_1, Q_2, \dots, Q_p \in \{\exists, \forall, \exists^*, \forall^*, \mathsf{H}\}$ *the problem* $\text{STANDARD}(Q_1 Q_2 \dots Q_p)$ *is* $Q_1 Q_2 \dots Q_p$*-complete.*

Remark 3.2 One could define $\text{STANDARD}(Q_1 Q_2 \dots Q_p)$ as in (3) for $Q_p \in \{\exists, \forall\}$ as well, and Proposition 3.1 would still hold. We did not do so, to follow the usual practice of further reducing (in proofs of hardness) from circuits to polynomial equalities or inequalities whenever this is possible. In the case $Q_p \in \{\exists, \forall\}$ this is the case since one can add variables describing the values of the circuit nodes and quantify these variables with $\exists$ or $\forall$ (see the mentioned hardness proofs in [6, 22]).

The standard complete problems for the classes $\mathsf{P}_{\mathbb{R}}$, $\mathsf{NP}_{\mathbb{R}}$, etc., are precisely those introduced in [6, 22]. Taking $p = 0$ we have $\text{STANDARD}(\mathsf{P}_{\mathbb{R}}) = \text{CEVAL}_{\mathbb{R}}$. Also, the problem $\text{STANDARD}(\exists)$ consisting of deciding whether a real polynomial f has a real zero is what in the literature (see [5, 6, 11]) is denoted by $\text{FEAS}_{\mathbb{R}}$.

4 Piecewise Rational Functions

Besides semialgebraic sets, a natural input for machines over $\mathbb{R}$ are piecewise rational functions (given by algebraic circuits). These are not necessarily total functions. We say that $\mathcal{C}$ is *certified* to compute a total function when every division gate of $\mathcal{C}$ is preceded by a sign gate making sure that the denominator of the division is not zero. Note, however, that a circuit may compute a total function without being certified to do so. Denote by $\mathrm{Dom}(f_{\mathcal{C}})$ the subset of $\mathbb{R}^n$ where $f_{\mathcal{C}}$ is well defined.

Consider the following problems ($k > 0$):

TOTAL$_{\mathbb{R}}$ (*Totalness*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is total.
INJ$_{\mathbb{R}}$ (*Injectiveness*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is injective on its domain.
SURJ$_{\mathbb{R}}$ (*Surjectiveness*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is surjective.
LIPSCHITZ$_{\mathbb{R}}(k)$ (*Lipschitz-k*). Given a circuit $\mathcal{C}$, decide whether, $f_{\mathcal{C}}$ is Lipschitz-k on its domain, i.e., whether for all $x, y \in \mathrm{Dom}(f_{\mathcal{C}})$, $\|f(x) - f(y)\| \le k\|x - y\|$.

For these problems we have the following completeness results.

Proposition 4.1

(i) TOTAL$_{\mathbb{R}}$ *is* $\forall$-*complete.*
(ii) INJ$_{\mathbb{R}}$ *is* $\forall$-*complete.*
(iii) LIPSCHITZ$_{\mathbb{R}}(k)$ *is* $\forall$-*complete.*
(iv) SURJ$_{\mathbb{R}}$ *is* $\forall\exists$-*complete.*

Proof Given $x \in \mathbb{R}^n$ one can check in polynomial time whether $f_{\mathcal{C}}$ is well defined on x. This shows that TOTAL$_{\mathbb{R}} \in \mathsf{coNP}_{\mathbb{R}}$. To show hardness, let $f \in \mathbb{R}[X_1, \dots, X_n]$. We associate to f a circuit $\mathcal{C}$ computing, for $x \in \mathbb{R}^n$, $1/f(x)$. Clearly, $f \in \mathrm{FEAS}_{\mathbb{R}}$ if and only if $\mathcal{C} \notin$ TOTAL$_{\mathbb{R}}$. This proves (i).

The claimed memberships in parts (ii), (iii), and (iv) are obvious. For the hardness of INJ$_{\mathbb{R}}$, consider $f \in \mathbb{R}[X_1, \dots, X_n]$. We associate to f a circuit $\mathcal{C}$ with $n + 1$ input gates and $n + 1$ output gates computing the following:

$$\text{input } x \in \mathbb{R}^n, z \in \mathbb{R};$$
$$\text{if } f(x) = 0 \text{ then return } 0 \in \mathbb{R}^{n+1} \text{ else return } (x, z).$$

Clearly, $f \in \mathrm{FEAS}_{\mathbb{R}}$ if and only if $\mathcal{C} \notin$ INJ$_{\mathbb{R}}$. This proves (ii).

For hardness of LIPSCHITZ$_{\mathbb{R}}(k)$ we consider again $f \in \mathbb{R}[X_1, \dots, X_n]$. We associate to f a circuit $\mathcal{C}$ with $n + 1$ input gates and $n + 1$ output gates computing the following:

$$\text{input } x \in \mathbb{R}^n, z \in \mathbb{R};$$
$$\text{if } f(x) = 0 \text{ then return } (0, \mathrm{sgn}(z)) \in \mathbb{R}^{n+1} \text{ else return } k(x, z).$$

If $f \in \mathrm{FEAS}_{\mathbb{R}}$, then $f_{\mathcal{C}}$ is not continuous and, a fortiori, not Lipschitz-k. Otherwise, $f_{\mathcal{C}} = k\mathrm{Id}$ and, hence, $\mathcal{C} \in$ LIPSCHITZ$_{\mathbb{R}}(k)$. This proves (iii).

For hardness of $\mathrm{SURJ}_{\mathbb{R}}$ consider $f \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ and associate to it a circuit $\mathcal{C}$ computing the following function $F : \mathbb{R}^{n+r+1} \to \mathbb{R}^{n+1}$,

$$(x, y, z) \mapsto \begin{cases} (x, z) & \text{if } z \neq 0, \\ (x, 0) & \text{if } z = 0 \text{ and } f(x, y) = 0, \\ (x, 1) & \text{if } z = 0 \text{ and } f(x, y) \neq 0. \end{cases}$$

We have $\forall x \exists y f(x, y) = 0$ if and only if $f_{\mathcal{C}} = F$ is surjective. □

Remark 4.2 One can define versions of the problems $\mathrm{INJ}_{\mathbb{R}}$, $\mathrm{LIPSCHITZ}_{\mathbb{R}}(k)$, and $\mathrm{SURJ}_{\mathbb{R}}$ requiring $f_{\mathcal{C}}$ to be total. Or yet one requiring $\mathcal{C}$ to be division-free. It follows from the proof of Proposition 4.1 that these problems are also complete.

5 Quantifying Genericity

In this section we deal with complexity classes defined using the quantifiers $\forall^*$ and $\exists^*$. A motivating theme is a series of problems related to the notion of denseness. The first in the series are the following:

$\mathrm{EADH}_{\mathbb{R}}$ (*Euclidean Adherence*). Given a semialgebraic set S and a point x, decide whether x belongs to the Euclidean closure $\overline{S}$ of S.

$\mathrm{EDENSE}_{\mathbb{R}}$ (*Euclidean Denseness*). Given a decision circuit $\mathcal{C}$ with n input gates, decide whether $\overline{S_{\mathcal{C}}} = \mathbb{R}^n$.

$\mathrm{ZADH}_{\mathbb{R}}$ (*Zariski Adherence*). Given a semialgebraic set S and a point x, decide whether x belongs to the Zariski closure $\overline{S}^Z$ of S (recall $\overline{S}^Z$ is the smallest algebraic set containing S).

$\mathrm{ZDENSE}_{\mathbb{R}}$ (*Zariski Denseness*). Given a decision circuit $\mathcal{C}$ with n input gates, decide whether $\overline{S_{\mathcal{C}}}^Z = \mathbb{R}^n$.

Proposition 5.1 *Both* $\mathrm{EADH}_{\mathbb{R}}$ *and* $\mathrm{ZADH}_{\mathbb{R}}$ *are* $\exists$*-hard.*

Proof We reduce $\mathrm{FEAS}_{\mathbb{R}}$ to these problems. For $f \in \mathbb{R}[X_1, \ldots, X_n]$, let $S_f \subseteq \mathbb{R}^{n+1}$ be the semialgebraic set defined by

$$(x_0, x) \in S_f \quad \iff \quad f^h(x_0, x) = 0 \wedge x_0 \neq 0,$$

where $f^h = x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0)$ denotes the homogenization of f. Then $f \in \mathrm{FEAS}_{\mathbb{R}}$ if and only if $S_f \neq \emptyset$ and, if this is the case, $s = (0, \ldots, 0)$ is in the closure (Euclidean and, a fortiori, Zariski) of S_f. □

It is customary to express denseness in terms of adherence. For instance, for $S \subseteq \mathbb{R}^n$,

$$S \in \mathrm{EDENSE}_{\mathbb{R}} \quad \iff \quad \forall x \in \mathbb{R}^n (x, S) \in \mathrm{EADH}_{\mathbb{R}}$$

and similarly for the Zariski topology. Therefore, one would expect at least $\mathsf{NP}_{\mathbb{R}}$-hardness (if not $\Pi^2_{\mathbb{R}}$-completeness) for $\mathrm{EDENSE}_{\mathbb{R}}$ and $\mathrm{ZDENSE}_{\mathbb{R}}$. The following two results show a quite different situation.

Proposition 5.2 *The problem* EDENSE$_{\mathbb{R}}$ *is* $\forall^*$*-complete and the problem* ZDENSE$_{\mathbb{R}}$ *is* $\exists^*$*-complete.*

Proof For a circuit $\mathcal{C}$, $\mathcal{C} \in$ STANDARD($\exists^*$) if and only if $\mathcal{C} \in$ ZDENSE$_{\mathbb{R}}$ (compare the remarks following equation (2)). This shows the statement for ZDENSE$_{\mathbb{R}}$. For EDENSE$_{\mathbb{R}}$ we use the fact that a semialgebraic set S is Euclidean dense if and only if its complement S^{c} is not Zariski dense. □

Corollary 5.3 $\exists^* \subseteq \exists$ *and* $\forall^* \subseteq \forall$.

Proof The reduction in the $\mathsf{NP}_{\mathbb{R}}$-completeness of FEAS$_{\mathbb{R}}$ shown in [6] (which we mentioned as the basic argument in the proof of Proposition 3.1) proceeds as follows. Given an $\mathsf{NP}_{\mathbb{R}}$ problem L, it first reduces an arbitrary input z to a decision circuit $\mathcal{C}$ such that $z \in L$ if and only if $S_{\mathcal{C}} \neq \emptyset$. Then it reduces the circuit $\mathcal{C}$ (say, with n input nodes) to a polynomial f in $n+m$ variables satisfying that $\dim S_{\mathcal{C}} = \dim \mathcal{Z}(f)$ and $x \in S_{\mathcal{C}}$ if and only if $\exists y \in \mathbb{R}^m\, f(x,y) = 0$. Here $\mathcal{Z}(f)$ denotes the set of zeros of f.

To prove that $\exists^* \subseteq \exists$ we consider the following algorithm solving STANDARD($\exists^*$). Given a circuit $\mathcal{C}$, compute an f as in (the second part of) the reduction above. Then check whether $\dim(\mathcal{Z}(f)) \geq n$. The latter can be done in $\mathsf{NP}_{\mathbb{R}}$, see Section 2(5). □

Remark 5.4 (i) It follows from Proposition 5.2 and Corollary 5.3 that ZDENSE$_{\mathbb{R}}$ is $\exists$-hard if and only if $\exists = \exists^*$. Also, EDENSE$_{\mathbb{R}}$ is $\exists$-hard if and only if $\exists \subseteq \forall^*$.

(ii) The proof of hardness in Proposition 5.2 does not extend to semialgebraic sets defined via formulas (instead of circuits), since the usual way to pass from circuits to formulas adds variables (i.e., dimension of the ambient space) but preserves the dimension of the semialgebraic set.

(iii) It is easy to prove that $\exists^*\exists^* = \exists^*$ and $\forall^*\forall^* = \forall^*$.

(iv) We will extend Corollary 5.3 in Section 8 (see Theorem 8.2 therein).

A possible reason for the unexpected "low" complexity of EDENSE$_{\mathbb{R}}$ is the fact that we are dealing with absolute denseness, i.e., denseness in the ambient space. Consider the following two extensions of EDENSE$_{\mathbb{R}}$.

ERD$_{\mathbb{R}}$ (*Euclidean Relative Denseness*). Given semialgebraic sets S and V, decide whether S is included in $\overline{V}$.

LERD$_{\mathbb{R}}$ (*Linearly Restricted Euclidean Relative Denseness*). Given a semialgebraic set $V \subseteq \mathbb{R}^n$ and points $a_0, a_1, \ldots, a_k \in \mathbb{R}^n$, decide whether $a_0 + \langle a_1, \ldots, a_k\rangle$ is included in $\overline{V}$.

It is immediate that both ERD$_{\mathbb{R}}$ and LERD$_{\mathbb{R}}$ are in $\Pi^2_{\mathbb{R}}$. Note also that LERD$_{\mathbb{R}}$ is between EDENSE$_{\mathbb{R}}$ and ERD$_{\mathbb{R}}$. Indeed, EDENSE$_{\mathbb{R}}$ is a special case of LERD$_{\mathbb{R}}$ (take $k = n$, $a_0 = 0$, and $a_i = (0, \ldots, 1, \ldots, 0)$, the 1 on the ith place, for $i = 1, \ldots, n$) and LERD$_{\mathbb{R}}$ a special case of ERD$_{\mathbb{R}}$ (take $S = a_0 + \langle a_1, \ldots, a_k\rangle$). It is an open problem whether ERD$_{\mathbb{R}}$ is $\Pi^2_{\mathbb{R}}$-complete. For the intermediate problem LERD$_{\mathbb{R}}$ instead, a completeness result is easily shown.

Proposition 5.5 *The problem* LERD$_{\mathbb{R}}$ *is* $\forall^*\exists$*-complete.*

Proof Membership to $\forall^*\exists$ is easy. An input $(V, a_0, \ldots, a_k)$ is in $\text{LERD}_{\mathbb{R}}$ iff

$$\forall^* y_1, \ldots, y_k \, \forall^* \varepsilon \, \exists x \big(x \in V \wedge \| x - (a_0 + y_1 a_1 + \cdots + y_k a_k) \|^2 \leq \varepsilon^2 \big).$$

For showing hardness, we are going to reduce $\text{STANDARD}(\forall^*\exists)$ to $\text{LERD}_{\mathbb{R}}$. Consider $f(x, y) = \sum_\alpha f_\alpha(x) y^\alpha$ in the variables $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ with $\deg_Y(f) = d$ and define $V_f \subseteq \mathbb{R}^{n+m+1}$ as the set of points satisfying

$$f'(x, y, y_0) := \sum_\alpha f_\alpha(x) y_0^{d-|\alpha|} y^\alpha = 0 \wedge y_0 \neq 0.$$

Also, let $S_f \subseteq \mathbb{R}^{n+m+1}$ be the linear space $\{y_0 = 0, y = 0\}$ spanned by $a_0 = 0$, and the ith coordinate vector a_i for $i = 1, \ldots, n$. We claim that $\forall^* x \exists y f(x, y) = 0$ if and only if $S_f \subseteq \overline{V_f}$.

The "only if" part follows from the fact that, for all $\mathbf{x} \in \mathbb{R}^n$,

$$\exists y \; f(\mathbf{x}, y) = 0 \quad \Longrightarrow \quad (\mathbf{x}, 0) \in \overline{V_f \cap \{x = \mathbf{x}\}}.$$

This is shown as Proposition 5.1.

For the "if" part, assume that $\exists^* x \forall y f(x, y) \neq 0$. Then, there exist $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that for every x in the ball $B(\mathbf{x}, \varepsilon) \subset \mathbb{R}^n$ and every $y \in \mathbb{R}^m$, $f(x, y) \neq 0$.

If $S_f \subseteq \overline{V_f}$, then there exists a point $(x', y', y'_0) \in V_f$ such that $d(x', \mathbf{x}) < \varepsilon$. Since $(x', y', y'_0) \in V_f$, we have $y'_0 \neq 0$, and, taking $y_* = y'/y'_0$, we obtain

$$f(x', y_*) = \sum_\alpha f_\alpha(x') y_*^\alpha = \sum_\alpha f_\alpha(x') (y'_0)^{-|\alpha|} (y')^\alpha = (y')^{-d} f'(x', y', y'_0) = 0$$

a contradiction since $x' \in B(\mathbf{x}, \varepsilon)$. ☐

Corollary 5.6 *The problem* $\text{ERD}_{\mathbb{R}}$ *is in* $\forall\exists$ *and is* $\forall^*\exists$*-hard.*

Denseness problems also occur for piecewise rational functions. Consider the following:

IMAGEZDENSE$_{\mathbb{R}}$ (*Image Zariski Dense*). Given a circuit $\mathcal{C}$, decide whether the image of $f_{\mathcal{C}}$ is Zariski dense.
IMAGEEDENSE$_{\mathbb{R}}$ (*Image Euclidean Dense*). Given a circuit $\mathcal{C}$, decide whether the image of $f_{\mathcal{C}}$ is Euclidean dense.
DOMAINZDENSE$_{\mathbb{R}}$ (*Domain Zariski Dense*). Given a circuit $\mathcal{C}$, decide whether the domain of $f_{\mathcal{C}}$ is Zariski dense.
DOMAINEDENSE$_{\mathbb{R}}$ (*Domain Euclidean Dense*). Given a circuit $\mathcal{C}$, decide whether the domain of $f_{\mathcal{C}}$ is Euclidean dense.

Proposition 5.7

(i) IMAGEZDENSE$_{\mathbb{R}}$ *is* $\exists^*\exists$*-complete.*
(ii) IMAGEEDENSE$_{\mathbb{R}}$ *is* $\forall^*\exists$*-complete.*
(iii) DOMAINZDENSE$_{\mathbb{R}}$ *is* $\exists^*$*-complete.*
(iv) DOMAINEDENSE$_{\mathbb{R}}$ *is* $\forall^*$*-complete.*

Proof Membership is easy in all four cases. For showing hardness in (i) and (ii), consider a polynomial $f \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ and associate to it the circuit $\mathcal{C}$ computing the map

$$(x, y) \mapsto \begin{cases} x & \text{if } f(x, y) = 0, \\ 0 & \text{if } f(x, y) \neq 0. \end{cases}$$

Clearly, $f \in$ STANDARD($\forall^*\exists$) iff the image of this map is Euclidean dense in $\mathbb{R}^r$. In the same way, one shows that $f \in$ STANDARD($\exists^*\exists$) iff the image of $f_\mathcal{C}$ is Zariski dense in $\mathbb{R}^r$.

For (iii) and (iv) consider the map associating to a decision circuit $\mathcal{C}$ a circuit $\mathcal{C}'$ computing the function

$$x \mapsto \begin{cases} x & \text{if } f_\mathcal{C}(x) = 1, \\ 1/0 & \text{if } f_\mathcal{C}(x) \neq 1. \end{cases}$$

Then, $\mathcal{C} \in$ STANDARD($\exists^*$) iff $\mathcal{C}' \in$ DOMAINZDENSE$_\mathbb{R}$ and $\mathcal{C} \in$ STANDARD($\forall^*$) iff $\mathcal{C}' \in$ DOMAINEDENSE$_\mathbb{R}$. □

6 Quantifying Infinitesimals

We now deal with some complexity classes defined via the quantifier H. A first property of H, which will be repeatedly used in what follows, is some kind of symmetry which makes the operator H closed by complements.

Proposition 6.1 *For all formulas* $\varphi(\varepsilon)$, $\neg\mathsf{H}\varepsilon\varphi(\varepsilon) \Leftrightarrow \mathsf{H}\varepsilon\neg\varphi(\varepsilon)$.

Proof By definition, $\neg\mathsf{H}\varepsilon\varphi(\varepsilon) \Leftrightarrow \forall\mu > 0\, \exists\varepsilon \in (0, \mu)\neg\varphi(\varepsilon)$. And this happens if and only if 0 is an accumulation point of the set

$$S = \{\varepsilon \in (0, 1] \mid \neg\varphi(\varepsilon)\}.$$

But S is a semialgebraic subset of $\mathbb{R}$ and therefore has a finite number of connected components. It follows that $\neg\mathsf{H}\varepsilon\varphi(\varepsilon)$ if and only if $\exists\kappa > 0$ such that $(0, \kappa)$ is included in S, i.e.,

$$\exists\kappa > 0\, \forall\varepsilon \in (0, \kappa)\neg\varphi(\varepsilon).$$

We have thus proved $\neg\mathsf{H}\varepsilon\varphi(\varepsilon) \Leftrightarrow \mathsf{H}\varepsilon\neg\varphi(\varepsilon)$. □

Corollary 6.2

(i) $\exists\mathsf{H}\forall = \exists\forall$ *and* $\forall\mathsf{H}\exists = \forall\exists$.
(ii) $\exists\mathsf{H}\forall^* = \exists\forall^*$ *and* $\forall\mathsf{H}\exists^* = \forall\exists^*$.
(iii) $\exists^*\mathsf{H}\forall = \exists^*\forall$ *and* $\forall^*\mathsf{H}\exists = \forall^*\exists$.
(iv) $\exists^*\mathsf{H}\forall^* = \exists^*\forall^*$ *and* $\forall^*\mathsf{H}\exists^* = \forall^*\exists^*$.
(v) $\mathsf{H}\exists \subseteq \forall^{[1]}\exists$ *and* $\mathsf{H}\forall \subseteq \exists^{[1]}\forall$.

Proof The first equality in (i) is obvious. The second follows immediately from the first and Proposition 6.1.

Parts (ii)–(iv) follow in the same manner by noting that $\mathsf{H}\varepsilon$ is of the form $\exists\mu\,\forall^*\varepsilon$ or, alternatively, of the form $\exists^*\mu\,\forall\varepsilon$ or, yet, of the form $\exists^*\mu\,\forall^*\varepsilon$.

The second inclusion in part (v) is immediate from the definition of H. The first inclusion now follows from Proposition 6.1. □

Remark 6.3 (i) Note that, unlike for $\exists$, $\forall$, $\exists^*$, and $\forall^*$, the equality of operators $\mathsf{HH} = \mathsf{H}$ is not known to be true (and, most likely, isn't).

(ii) We believe that H is fundamentally simpler than the alternation of two quantifiers. A feature suggesting this is the fact that the standard algorithms for quantifier elimination applied to a sentence

$$\exists\mu\,\forall\varepsilon \in (0,\mu)\,\exists(x_1,\dots,x_n)\varphi(\varepsilon,x)$$

would have a much higher complexity than just applying quantifier elimination to

$$\exists(x_1,\dots,x_n)\varphi(\varepsilon,x)$$

and inspecting the resulting formula in ε. We will add more on this in Remark 9.9 below.

We now consider some problems whose complexity can be better understood in terms of classes of the form $\mathsf{H}\mathcal{C}$.

6.1 Local Topological Properties

We define:

UNBOUNDED$_{\mathbb{R}}$ (*Unboundedness*). Given a semialgebraic set S, is it unbounded?
LOCDIM$_{\mathbb{R}}$ (*Local Dimension*). Given a semialgebraic set $S \subseteq \mathbb{R}^n$, a point $x \in S$, and $d \in \mathbb{N}$, is $\dim_x S \geq d$?
ISOLATED$_{\mathbb{R}}$ (*Isolated*). Given a semialgebraic set $S \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, decide whether x is an isolated point of S.
EXISTISO$_{\mathbb{R}}$ (*Existence of isolated points*). Given a semialgebraic set $S \subseteq \mathbb{R}^n$, decide whether there exists a point x isolated in S.

Proposition 6.4 *The problem* UNBOUNDED$_{\mathbb{R}}$ *is* $\mathsf{H}\exists$*-complete.*

Proof Membership follows from the fact that, for a set S, S is unbounded if and only if

$$\exists\mu > 0\,\forall\varepsilon \in (0,\mu)\,\exists x \in \mathbb{R}^n(\varepsilon\|x\| \geq 1 \wedge x \in S).$$

For showing hardness, consider the auxiliary problem $\mathcal{L} \subseteq \mathbb{R}^\infty$ consisting of, given $g \in \mathbb{R}[\varepsilon, X_1,\dots,X_n]$, deciding whether

$$\exists\mu > 0\,\forall\varepsilon \in (0,\mu)\,\exists t \in (-1,1)^n g(\varepsilon,t_1,\dots,t_n) = 0.$$

We first reduce STANDARD(H∃) to $\mathcal{L}$. To do so, note that the existence of a root in $\mathbb{R}^n$ of a polynomial f is equivalent to the existence of a root in the open unit cube $(-1,1)^n$ for a suitable other polynomial. This is so since the mapping $\psi(\lambda) = \lambda/(1-\lambda^2)$ bijects $(-1,1)$ with $\mathbb{R}$. Therefore, for $f \in \mathbb{R}[Y, X_1, \dots, X_n]$,

$$\mathsf{H}\varepsilon \exists x \in \mathbb{R}^n f(\varepsilon, x_1, \dots, x_n) = 0 \iff \mathsf{H}\varepsilon \exists t \in (-1,1)^n g(\varepsilon, t_1, \dots, t_n) = 0,$$

where $d_i = \deg_{x_i} f$ and $g \in \mathbb{R}[Y, T_1, \dots, T_n]$ is given by

$$g(\varepsilon, t_1, \dots, t_n) := \big(1-t_1^2\big)^{d_1}\big(1-t_2^2\big)^{d_2} \cdots \big(1-t_n^2\big)^{d_n} f\big(\varepsilon, \psi(t_1), \dots, \psi(t_n)\big).$$

Note that we can construct g in time polynomial in the size of f (since we are representing f and g in the dense encoding, the divisions can be eliminated in polynomial time). So the mapping $f \mapsto g$ indeed reduces STANDARD(H∃) to $\mathcal{L}$.

We now reduce $\mathcal{L}$ to UNBOUNDED$_{\mathbb{R}}$. To do so, we associate to $g \in \mathbb{R}[Y, T_1, \dots, T_n]$ the semialgebraic set

$$S := \big\{(y,t) \in \mathbb{R} \times (-1,1)^n \mid h(y,t) = 0\big\},$$

where h is the polynomial defined by $h(Y,T) = Y^{2\deg_Y g} g(1/Y^2, T)$. Then $g \in \mathcal{L}$ if and only if S is unbounded. □

Corollary 6.5 *The problem* EADH$_{\mathbb{R}}$ *is* H∃*-complete.*

Proof Membership is easy: A point s is in the closure of S if and only if

$$\exists \mu > 0\, \forall \varepsilon \in (0,\mu)\, \exists x \in \mathbb{R}^n (x \in S \wedge \|s - x\| \le \varepsilon).$$

For hardness, we reduce UNBOUNDED$_{\mathbb{R}}$ to EADH$_{\mathbb{R}}$. To do so, recall that the *inversion* (with respect to the unit sphere) is the following homeomorphism:

$$i : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad x \mapsto \frac{x}{\|x\|^2}.$$

If f is a polynomial of degree d in n variables we define

$$f' := \|X\|^{2d} f\big(\|X\|^{-2} X\big).$$

Then $\mathcal{Z}(f') \setminus \{0\} = i(\mathcal{Z}(f) \setminus \{0\})$ and $\{x \in \mathbb{R}^n \setminus \{0\} \mid f'(x) > 0\} = i(\{x \in \mathbb{R}^n \setminus \{0\} \mid f(x) > 0\})$.

Now let $S \subseteq \mathbb{R}^n$ be a semialgebraic set given by a Boolean combination of inequalities of the form $f(x) > 0$. Without loss of generality, $0 \notin S$. The set defined by the same Boolean combination of the inequalities $f'(x) > 0$ and the condition $x \neq 0$ is the image $i(S)$ of S and we have that S is unbounded if and only if 0 belongs to the closure of $i(S) \setminus \{0\}$. □

Corollary 6.6 *The problem* LOCDIM$_{\mathbb{R}}$ *is* H∃*-complete.*

Proof Membership follows from the equivalence

$$\dim_x S \geq d \iff \mathsf{H}\varepsilon \dim(S \cap B(x, \varepsilon)) \geq d$$

and the fact that $\text{DIM}_{\mathbb{R}} \in \mathsf{NP}_{\mathbb{R}}$. For hardness we reduce $\text{EADH}_{\mathbb{R}}$ to $\text{LOCDIM}_{\mathbb{R}}$. To do so, consider $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. If $x \in S$, then take $S' = \mathbb{R}^n$. Otherwise, let $S' = S \cup \{x\}$. Then, $x \in \overline{S} \Leftrightarrow \dim_x S' \geq 1$. □

Corollary 6.7 *The problem* $\text{ISOLATED}_{\mathbb{R}}$ *is* $\mathsf{H}\forall$*-complete.*

Proof Membership easily follows from the equivalence

$$x \text{ isolated in } S \iff x \in S \wedge \dim_x S < 1.$$

Hardness follows from the equivalence

$$x \in \overline{S} \iff x \in S \vee x \text{ not isolated in } S \cup \{x\}$$

which reduces $\text{EADH}_{\mathbb{R}}$ to the complement of $\text{ISOLATED}_{\mathbb{R}}$. □

Corollary 6.8 *The problem* $\text{EXISTISO}_{\mathbb{R}}$ *belongs to* $\exists\forall$ *and is* $\mathsf{H}\forall$*-hard.*

Proof $\text{EXISTISO}_{\mathbb{R}} \in \exists\mathsf{H}\forall = \exists\forall$. For hardness, we reduce $\text{ISOLATED}_{\mathbb{R}}$ to $\text{EXISTISO}_{\mathbb{R}}$. To do so, let $S \subseteq \mathbb{R}^n$ be semialgebraic and assume w.l.o.g. that $0_n \in S$ (here 0_n denotes the origin in $\mathbb{R}^n$). Define $S' \subset \mathbb{R}^{n+1}$ by

$$S' = \big((S - \{0_n\}) \times \mathbb{R}\big) \cup \{0_{n+1}\}.$$

If 0_n is an isolated point of S, then 0_{n+1} is an isolated point (actually the only one) of S'. Otherwise, S' has no isolated points. Since a description of S' can be computed in polynomial time from a description of S it follows that $\text{ISOLATED}_{\mathbb{R}} \preceq \text{EXISTISO}_{\mathbb{R}}$. □

6.2 Continuity

Complexity results for problems involving functions (instead of sets) and the quantifier H are also of interest. Consider the following problems:

$\text{CONT}_{\mathbb{R}}$ (*Continuity*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is total and continuous.
$\text{CONT}^{\text{DF}}_{\mathbb{R}}$ (*Continuity for Division-Free Circuits*). Given a division-free circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is continuous.
$\text{CONTPOINT}^{\text{DF}}_{\mathbb{R}}$ (*Continuity at a Point for Division-Free Circuits*). Given a division-free circuit $\mathcal{C}$ with n input gates and a point $x \in \mathbb{R}^n$, decide whether $f_{\mathcal{C}}$ is continuous at x.
$\text{LIPSCHITZ}_{\mathbb{R}}$ (*Lipschitz*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is Lipschitz on its domain, i.e., whether there exists $k > 0$ such that $f_{\mathcal{C}}$ is Lipschitz-k.

Our main results concerning these problems are the following four propositions.

Proposition 6.9 $\text{CONT}_{\mathbb{R}} \in \mathsf{H}^3\forall$ *and is* $\forall$*-hard.*

Proof The fact that f is total can be checked in $\mathsf{coNP}_{\mathbb{R}}$ by Proposition 4.1. For $r > 0$, let $\overline{B}(0,r)$ denote the closed ball of radius r and $f_r := f_{|\overline{B}(0,r)}$ the restriction of f to that ball. Then,

$$\begin{aligned} f_r \text{ is continuous} \quad &\Longleftrightarrow \quad f_r \text{ is uniformly continuous} \\ \Longleftrightarrow \quad &\mathsf{H}\varepsilon\mathsf{H}\delta\forall x, y \in \overline{B}(0,r)\big(\|x-y\|_\infty < \delta \Rightarrow \|f(x)-f(y)\|_\infty < \varepsilon\big). \end{aligned}$$

This last condition is in $\mathsf{H}^2\forall$. Since we have that

$$f \text{ is continuous } \Longleftrightarrow \mathsf{H}\rho f_{1/\rho} \text{ is continuous},$$

membership to $\mathsf{H}^3\forall$ follows. The $\forall$-hardness follows from the reduction in the proof of Proposition 4.1(iii). □

Proposition 6.10 $\text{LIPSCHITZ}_{\mathbb{R}} \in \mathsf{H}\forall$ *and it is* $\forall$*-hard.*

Proof Membership follows from (the proof of) Proposition 4.1(iii) (note that the algorithm given there is uniform in k). For hardness, the reduction in Proposition 4.1(iii) does the job again. □

A *straight-line program* (in short, SLP) is an algebraic circuit without sign nodes.

Lemma 6.11 *Let* $f \in \mathbb{R}[X_1,\dots,X_n]$ *be given by a division-free SLP of depth* d *with constants* $a_1,\dots,a_k \in \mathbb{R}$ *whose absolute value is bounded by* $b \geq 1$. *Let* $r \geq 1$. *Then, for all* $x, y \in \mathbb{R}^n$ *with* $\|x\|_\infty, \|y\|_\infty \leq r$,

$$|f(x) - f(y)| \leq C\|x-y\|_\infty,$$

where $C = (b+r)r^{2^d-1}2^{(d+1)2^d}$.

Proof Let $x, y \in \mathbb{R}^n$ such that $\|x\|_\infty, \|y\|_\infty \leq r$. The polynomial $F(Z) := f(y+Z) \in \mathbb{R}[Z_1,\dots,Z_n]$ is given by a division-free straight-line program of depth at most $d+1$ whose constants have absolute value at most $b+r$. Write

$$F(Z) = \sum_{|\alpha|=0}^{2^d} F_\alpha Z^\alpha,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we have, for $z \in \mathbb{R}^n$ such that $\|z\|_\infty \leq r$,

$$|F(z) - F(0)| \leq \sum_{\alpha \neq 0} |F_\alpha||z_1|^{\alpha_1}\cdots|z_n|^{\alpha_n} \leq \|z\|_\infty r^{2^d-1}\|F\|_1,$$

where $\|F\|_1 = \sum_{|\alpha|\leq 2^d} F_\alpha$. On the other hand, by [10, Lemma 4.16],

$$\log\|F\|_1 \leq (d+1)2^d \log(b+r)$$

(the statement there is for $a_i \in \mathbb{Z}$ but the proof carries over). Altogether we obtain

$$|f(x) - f(y)| = |F(x-y) - F(0)| \le r^{2^d-1} 2^{(d+1)2^d \log(b+r)} \|x-y\|_\infty$$

as claimed. □

Proposition 6.12 $\text{CONT}^{\text{DF}}_{\mathbb{R}} \in \mathsf{H}^2\forall$ *and it is* $\forall$*-hard.*

Proof Hardness for $\forall$ follows, one more time, from the reduction in Proposition 4.1(iii). So it suffices to show membership to $\mathsf{H}^2\forall$.

Let $f:\mathbb{R}^W \to \mathbb{R}^m$ be given by a division-free circuit $\mathcal{C}$ of depth d with constants $a_1, \ldots, a_k \in \mathbb{R}$. Note that

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \forall r > 0\, f_{|\overline{B}(0,r)} \text{ is uniformly continuous.}$$

Fix $r > 0$. Uniform continuity of $f_{|\overline{B}(0,r)}$ means that

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in \overline{B}(0,r) \big(\|x-y\|_\infty < \delta \Rightarrow \|f(x) - f(y)\|_\infty < \varepsilon\big).$$

We claim that this is in turn equivalent to

$$\mathsf{H}\varepsilon \forall x, y \in \overline{B}(0,r) \left(\|x-y\|_\infty < \frac{\varepsilon}{C} \Rightarrow \|f(x) - f(y)\|_\infty < \varepsilon\right), \tag{4}$$

where C is as in Lemma 6.11. To prove this claim, assume that $\varphi := f_{|\overline{B}(0,r)}$ is continuous. There is a semialgebraic partition $\overline{B}(0,r) = S_1 \cup \cdots \cup S_p$ and there are polynomials $f_1, \ldots, f_p$, computable by division-free straight-line programs of depth at most d and using constants from $\{a_1, \ldots, a_k\}$, such that $f_i = \varphi$ on S_i. By the continuity of φ we get $f_i = \varphi$ on $\overline{S_i}$. Let $x, y \in \overline{B}(0,r)$. Define the function $s : [0,1] \to \mathbb{R}^n$ given by $s(t) := tx + (1-t)y$. Denote by $[x,y]$ the image of s, which is a line segment. Finally, define $I_i := s^{-1}(S_i)$. This yields a semialgebraic partition of the interval

$$[0,1] = I_1 \cup \cdots \cup I_p.$$

Since the I_i are semialgebraic, there exist points $0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$ and integers $j(1), \ldots, j(N) \in \{1, \ldots, p\}$ such that, for $1 \le i \le N$,

$$s(t_{i-1}, t_i) \subseteq S_{j(i)}.$$

Put $x_i := s(t_i)$. Then $\{x_{i-1}, x_i\} \subseteq \overline{S_{j(i)}}$. By Lemma 6.11,

$$\|\varphi(x) - \varphi(y)\|_\infty \le \sum_{i=1}^N \|\varphi(x_i) - \varphi(x_{i-1})\|_\infty \le \sum_{i=1}^N C\|x_i - x_{i-1}\|_\infty = C\|x-y\|_\infty,$$

since $\varphi(x_i) = f_{j(i-1)}(x_i)$ and $\varphi(x_{i-1}) = f_{j(i-1)}(x_{i-1})$. This proves one implication in the claim. The converse is trivial.

Now note that condition (4) is of the type $\mathsf{H}\forall$. Hence, the continuity of f can be expressed as (take $r = 1/\rho$)

$$\mathsf{H}\rho\mathsf{H}\varepsilon\,\forall x, y\left(\|x\|_\infty \le \frac{1}{\rho} \wedge \|y\|_\infty \le \frac{1}{\rho} \wedge \|x-y\|_\infty \le \frac{\varepsilon}{C} \Rightarrow \|x-y\|_\infty \le \varepsilon\right).$$

An upper bound on C can be computed in polynomial time. This proves membership in $\mathsf{H}^2\forall$. □

Proposition 6.13 $\text{CONTPOINT}_{\mathbb{R}}^{\text{DF}}$ *is* $\mathsf{H}\forall$*-complete.*

Proof Let $\mathcal{C}$ be a division-free circuit with n input gates and $x \in \mathbb{R}^n$. Let $r = 2\|x\|_\infty$. Denote by $\varphi_{\mathcal{C}}$ the function computed by $\mathcal{C}$. We first show that checking whether $\varphi_{\mathcal{C}}$ is continuous at x can be decided in $\mathsf{H}\forall$.

Let d be the depth of $\mathcal{C}$, let $a_1, \dots, a_k \in \mathbb{R}$ be its constants, and $b \ge 1$ a bound for their absolute value. We claim that $\varphi_{\mathcal{C}}$ is continuous at x if and only if

$$\exists \mu > 0\, \forall \varepsilon \in (0, \mu)\, \forall y \in \mathbb{R}^n \left(\|x-y\|_\infty \le \frac{\varepsilon}{C} \Rightarrow \|\varphi_{\mathcal{C}}(x) - \varphi_{\mathcal{C}}(y)\|_\infty \le \varepsilon\right).$$

Here C is as in Lemma 6.11.

The "if" direction is obvious. For the "only if" direction note that there exists a semialgebraic partition $\mathbb{R}^n = S_1 \cup \cdots \cup S_p$ and polynomials $f_1, \dots, f_p \in \mathbb{R}[X_1, \dots, X_n]$, computable by division-free straight-line programs of depth at most d and using constants from $\{a_1, \dots, a_k\}$, such that the restriction of $\varphi_{\mathcal{C}}$ to S_i is f_i, for $i \le p$. Let $\mathcal{R} = \{i \le p \mid x \in \overline{S_i}\}$ and let $\mu > 0$ be such that $\mu/C \le r/2$ and, for all $i \notin \mathcal{R}$, $\text{dist}_\infty(x, S_i) > \mu/C$. Note that since $\varphi_{\mathcal{C}}$ is continuous at x, for all $i \in \mathcal{R}$, $f_i(x) = \varphi_{\mathcal{C}}(x)$.

Now let $\varepsilon \in (0, \mu)$ and $y \in \mathbb{R}^n$ such that $\|x-y\|_\infty \le \varepsilon/C$. Since $\varepsilon < \mu$ we have $\|x-y\|_\infty < \mu/C$ and, therefore, there exists $i \in \mathcal{R}$ such that $y \in S_i$. It follows that

$$\|\varphi_{\mathcal{C}}(x) - \varphi_{\mathcal{C}}(y)\|_\infty = \|f_i(x) - f_i(y)\|_\infty \le \varepsilon,$$

where the last inequality is a consequence of Lemma 6.11 (which we can apply since $\|y\|_\infty \le \mu/C + \|x\|_\infty \le r$). This proves the claim. Since C can be computed in polynomial time, membership of $\text{CONTPOINT}_{\mathbb{R}}^{\text{DF}}$ to $\mathsf{H}\forall$ follows.

For hardness, let $S \subseteq \mathbb{R}^n$ be semialgebraic and $x \in \mathbb{R}^n$. We define the function f on $\mathbb{R}^n$ by $f(y) := 1$ if $y \in S - \{x\}$ and $f(y) := 0$ otherwise. Clearly, f is continuous at x if and only if $x \notin \overline{S}$. The $\mathsf{H}\forall$-hardness follows from Corollary 6.5. □

Remark 6.14 (i) The usual definition of continuity easily yields $\text{CONT}_{\mathbb{R}} \in \forall\mathsf{H}\forall$. But $\text{CONT}_{\mathbb{R}}$ is not $\mathsf{H}^3\forall$-complete unless $\mathsf{H}^3\forall \subseteq \forall\mathsf{H}\forall$. The precise complexity of $\text{CONT}_{\mathbb{R}}$ remains to be determined. Note, we cannot even show $\mathsf{H}\forall$-hardness.

(ii) A result like Proposition 6.10 holds as well for a version of $\text{LIPSCHITZ}_{\mathbb{R}}$ requiring $f_{\mathcal{C}}$ to be total or $\mathcal{C}$ to be division-free (see Remark 4.2). In contrast, we do not know whether a version of $\text{CONT}_{\mathbb{R}}$ requiring $f_{\mathcal{C}}$ to be continuous on its domain is in $\mathsf{H}^3\forall$.

6.3 Basic Semialgebraic Sets

A *basic semialgebraic set* is the solution set of a system of polynomial equalities and inequalities. It thus has the form

$$S = \{f = 0, h_1 \geq 0, \dots, h_p \geq 0, g_1 > 0, \dots, g_q > 0\} \subseteq \mathbb{R}^n, \tag{5}$$

where we assumed there is only one equality for notational simplicity. (We can always reduce to this case by adding the squares of the equalities; actually we could even replace $f = 0$ by $f \geq 0, -f \geq 0$). Clearly, arbitrary semialgebraic sets can be written as finite unions of basic semialgebraic sets.

Now consider the following problems:

BASICCLOSED$_{\mathbb{R}}$ (*Closedness for Basic Semialgebraic Sets*). Given a basic semialgebraic set S, is it closed?

BASICCOMPACT$_{\mathbb{R}}$ (*Compactness for Basic Semialgebraic Sets*). Given a basic semialgebraic set S, is it compact?

Our last result in this section is the following.

Theorem 6.15 *The problems* BASICCLOSED$_{\mathbb{R}}$ *and* BASICCOMPACT$_{\mathbb{R}}$ *are* H∀-*complete.*

To prove membership, we will use two ideas. One is the notion of stereographic projection and the other is a characterization of closedness for basic semialgebraic sets (see Lemma 6.16 below).

Let SS^n denote the n-dimensional unit sphere. The *stereographic projection*

$$\pi : \mathbb{S}^n - \{(0, \dots, 0, 1)\} \xrightarrow{\sim} \mathbb{R}^n, \quad (x, t) \mapsto y,$$

given by the equations $y_i = x_i/(1 - t)$, is a homeomorphism. In the following we denote the "north pole" $(0, \dots, 0, 1)$ by $\mathcal{N}$.

For a polynomial $f \in \mathbb{R}[Y_1, \dots, Y_n]$ the inverse image $\pi^{-1}(\mathcal{Z}(f))$ in $\mathbb{S}^n \setminus \mathcal{N}$ of its zero set $\mathcal{Z}(f) \subseteq \mathbb{R}^n$ is given by $(1 - t)^{\deg f+1} f(\frac{1}{1-t}x) = 0$ together with the conditions $\|x\|^2 + t^2 = 1$ and $t < 1$. If, instead of $\mathcal{Z}(f)$, we consider the set $\{f > 0\}$ (or $\{f \geq 0\}$), its preimage in $\mathbb{S}^n \setminus \{\mathcal{N}\}$ is given by $\{(1 - t)^{\deg(f)+1} f(\frac{1}{1-t}x) > 0\}$ (or $\{(1 - t)^{\deg f+1} f(\frac{1}{1-t}x) \geq 0\}$), again with the extra conditions $\|x\|^2 + t^2 = 1$ and $t < 1$.

Note that if we exclude the latter condition "$t < 1$" we obtain the desired preimage plus the north pole $\mathcal{N}$. In particular, if $S \subseteq \mathbb{R}^n$ is a basic semialgebraic set, both $\pi^{-1}(S)$ and $\pi^{-1}(S) \cup \{\mathcal{N}\}$ are basic semialgebraic sets.

We now focus on characterizing closedness. Let S be a basic semialgebraic set given as in (5). Define

$$K^S := \{f = 0, h_1 \geq 0, \dots, h_p \geq 0\}$$

and, for $\varepsilon > 0$,

$$S_\varepsilon = \{f = 0, h_1 \geq 0, \dots, h_p \geq 0, g_1 \geq \varepsilon, \dots, g_q \geq \varepsilon\}.$$

Note that $S_\varepsilon \subseteq S_{\varepsilon'} \subseteq S$ for $0 < \varepsilon' < \varepsilon$ and that $S = \bigcup_{\varepsilon > 0} S_\varepsilon$.

Lemma 6.16 *Let S be a basic semialgebraic set with K^S bounded. Then*

$$S \text{ is closed} \quad \Longleftrightarrow \quad \exists \varepsilon > 0 \; S_\varepsilon = S.$$

Proof The "$\Leftarrow$" direction is trivial since S_ε is closed for all $\varepsilon \in \mathbb{R}$.

For the "$\Rightarrow$" direction, let $K^S = K_1 \cup K_2 \cup \cdots \cup K_t$ be the decomposition of K^S into connected components. Then

$$S = K^S \cap \{g_1 > 0, \ldots, g_q > 0\} = \bigcup_{\tau=1}^{t} S_\tau,$$

where $S_\tau := K_\tau \cap \{g_1 > 0, \ldots, g_q > 0\}$. Note that $S_\tau = S \cap K_\tau$. Hence S closed implies S_τ closed for all $\tau \le t$.

On the other hand, S_τ is open in K_τ. Since K_τ is connected we either have $S_\tau = \emptyset$ or $S_\tau = K_\tau$. Put $T := \{\tau \mid S_\tau = K_\tau\}$. Then $S = \bigcup_{\tau \in T} K_\tau$. Hence, for all $\tau \in T$ and all $x \in K_\tau$ we have $\min_i g_i(x) > 0$. Since K_τ is compact we conclude that

$$\varepsilon_\tau := \min_{x \in K_\tau} \min_{1 \le i \le r} g_i(x)$$

is positive. Then $\varepsilon := \min_\tau e_\tau$ is positive as well and we have $S_\varepsilon = S$. □

The proof of Theorem 6.15 follows from Lemmas 6.17 and 6.20 below.

Lemma 6.17 *The problems* BASICCLOSED$_\mathbb{R}$ *and* BASICCOMPACT$_\mathbb{R}$ *are in* H∀.

Proof We begin with BASICCLOSED$_\mathbb{R}$. Note that Lemma 6.16 shows that, for basic semialgebraic sets S with bounded K^S, S is closed $\Leftrightarrow$ $\mathsf{H}\varepsilon(S_\varepsilon = S)$ and the right-hand side is in H∀. So, it is enough to show we can reduce the general situation to one with bounded K^S. To do so, let

$$S = \{f = 0, h_1 \ge 0, \ldots, h_p \ge 0, g_1 > 0, \ldots, g_q > 0\} \subseteq \mathbb{R}^n.$$

Consider $\widetilde{S} := \pi^{-1}(S) \cup \{\mathcal{N}\}$ where π is the stereographic projection. Then, $\widetilde{S}$ is a basic semialgebraic set, it satisfies that $K^{\widetilde{S}}$ is bounded, and that

$$S \text{ is closed in } \mathbb{R}^n \quad \Longleftrightarrow \quad \widetilde{S} \text{ is closed in } \mathbb{R}^{n+1}.$$

This shows that BASICCLOSED$_\mathbb{R}$ $\in$ H∀. Membership of BASICCOMPACT$_\mathbb{R}$ to H∀ follows from the one of BASICCLOSED$_\mathbb{R}$ and that of UNBOUNDED$_\mathbb{R}$ to H∃ (Proposition 6.4). □

For the proof of hardness we need the following two auxiliary results.

Lemma 6.18 *Let $T \subseteq (0,\infty)\times(0,\infty)$ be a semialgebraic set given by a Boolean combination of inequalities of polynomials of degree strictly less than d and let $(0,0)\in\overline{T}$. Then there exists a sequence of points (t_ν,ε_ν) in T such that*

$$\lim_{\nu\to\infty}\frac{\varepsilon_\nu^d}{t_\nu}=0.$$

Proof We may assume without loss of generality that T is basic, hence given by inequalities $h_1\ge 0,\dots,h_p\ge 0, g_1>0,\dots,g_q>0, t>0, \varepsilon>0$. Moreover, since we study a local property at $(0,0)$ and $(0,0)\notin T$, we may assume without loss of generality that $q=0$ and, for all i, that $(0,0)$ is a point on the real algebraic curve $\mathcal{Z}(h_i)$, which is not isolated.

By [7, §9.4], $\mathcal{Z}(h_i)\cap B(0,\rho)$ is a disjoint union of its half-branches $C_{i1},\dots,C_{im_i}$ passing through $(0,0)$, for sufficiently small $\rho>0$. It is known that each $C_{i\mu}\setminus\{(0,0)\}$ is homeomorphic to the open interval $(0,1)$.

Without loss of generality, we may assume that $C_{i\mu}\cap\{\varepsilon=0\}$ is finite (otherwise, h_i vanishes on the line $\{\varepsilon=0\}$ and, by dividing h_i by an appropriate power of ε, we can remove this line from $\mathcal{Z}(h_i)$ without altering T). Similarly, we may assume that $C_{i\mu}\cap\{t=0\}$ is finite.

Thus we may choose ρ small enough so that $C_{i\mu}\cap\{\varepsilon=0\}=C_{i\mu}\cap\{t=0\}=\{(0,0)\}$ for all i,μ, and $C_{i\mu}\cap C_{j\nu}=\{(0,0)\}$ for all i,j,μ,ν such that $C_{i\mu}\neq C_{j\nu}$.

Without loss of generality, there exist $(t,\varepsilon)\in T\cap B(0,\rho)$ and $i\le p$ such that $h_i(t,\varepsilon)=0$ (otherwise, T would be a neighborhood of $(0,0)$ in $(0,\infty)^2$ and we were done). Hence, $(t,\varepsilon)\in C_{i\mu}$ for some $\mu\le m_i$. We have $C_{i\mu}\setminus\{(0,0)\}\subseteq\{\varepsilon>0\}$ since $C_{i\mu}\setminus\{(0,0)\}$ is connected and does not intersect the line $\{\varepsilon=0\}$. For the same reason, $C_{i\mu}\setminus\{(0,0)\}\subseteq\{t>0\}$.

We claim that

$$C_{i\mu}\setminus\{(0,0)\}\subseteq T. \tag{6}$$

Otherwise, there is a point $(t_1,\varepsilon_1)\in C_{i\mu}\cap\{t>0,\varepsilon>0\}$, which is not in T. The latter implies the existence of $j\neq i$ such that $h_j(t_1,\varepsilon_1)<0$. But $h_j(t,\varepsilon)\ge 0$ and $C_{i\mu}\setminus\{(0,0)\}$ is connected. Hence there exists a point (t_2,ε_2) in $C_{i\mu}\setminus\{(0,0)\}$ such that $h_j(t_2,\varepsilon_2)=0$. This in turn implies that $(t_2,\varepsilon_2)\in C_{j\nu}$ for some ν, hence $C_{i\mu}\cap C_{j\nu}\neq\{(0,0)\}$. On the other hand, we have $C_{i\mu}\neq C_{j\nu}$. This contradicts the choice of ρ and the claim is proved.

The half-branches of (real) algebraic curves can be described by means of the Puiseux series, see [4, §13] or [9]. Hence there exists a convergent real power series $\varphi(x)=\sum_{k\ge 1}a_k x^k$ and a positive integer N, called the *ramification index*, such that (after possibly decreasing ρ)

$$C_{i\mu}=\bigl\{\bigl(t,\varphi\bigl(t^{1/N}\bigr)\bigr)\mid 0\le t<\rho\bigr\}.$$

Moreover, it is known [4] that N can be bounded by the degree of the defining equation h_i, hence $N<d$.

Choose now a sequence $t_\nu>0$ converging to zero and put $\varepsilon_\nu:=\varphi(t_\nu^{1/N})$. By (6), the points (t_ν,ε_ν) lie in T and we have $\lim_{\nu\to\infty}\varepsilon_\nu^N/t_\nu=\lim_{\nu\to\infty}\varphi(t_\nu^{1/N})/t_\nu^{1/N}=a_1$. The assertion now follows from $N<d$. ☐

It will be convenient to use the notation $B_n := (-1, 1)^n$ for the open unit ball with respect to the maximum norm and to write $\partial B_n := \{x \in \mathbb{R}^n \mid \|x\|_\infty = 1\}$ for its boundary.

Lemma 6.19 *There exists a constant $c > 0$ with the following property. To $f \in \mathbb{R}[\varepsilon, X_1, \dots, X_n]$ of degree d and $N = (nd)^{cn}$ we assign the semialgebraic set*

$$S := \left\{(\varepsilon, x, y) \in (0, \infty) \times (-1, 1)^n \times \mathbb{R} \mid f(\varepsilon, x) = 0 \wedge y \prod_{k=1}^{n} \left(1 - x_k^2\right) = \varepsilon^N \right\}.$$

Then, for all f, we have

$$\mathsf{H}\varepsilon \forall x \in (-1, 1)^n \, f(\varepsilon, x) \neq 0 \quad \Longleftrightarrow \quad S \text{ is closed in } \mathbb{R}^{n+2}.$$

Proof For the direction "⇒" assume there exists $\mu > 0$ such that $f(\varepsilon, x) \neq 0$ for all $(\varepsilon, x) \in (0, \mu) \times B_n$. In order to show that S is closed, consider a sequence $(\varepsilon_\nu, x_\nu, y_\nu)$ in S converging to $(\overline{\varepsilon}, \overline{x}, \overline{y})$. Since $f(\varepsilon_\nu, x_\nu) = 0$, we have $\varepsilon_\nu \geq \mu$ for all ν and thus $\overline{\varepsilon} \geq \mu$. On the other hand, by taking the limit, we get $\overline{y} \prod_{k=1}^{n} (1 - \overline{x}_k^2) = \overline{\varepsilon}^N$. Since $\overline{\varepsilon} \neq 0$ we conclude that $\overline{x} \in B_n$. Therefore, the limit point $(\overline{\varepsilon}, \overline{x}, \overline{y})$ indeed lies in S.

For the direction "⇐" we assume that $\mathsf{H}\varepsilon \exists x \in B_n \, f(\varepsilon, x) = 0$. Then there exists a sequence $(\varepsilon_\nu, x_\nu) \in (0, \infty) \times B_n$ converging to some point $(0, \overline{x})$ such that $f(\varepsilon_\nu, x_\nu) = 0$ for all ν. We are going to show that the sequence (y_ν) defined by $y_\nu := \varepsilon_\nu^N \prod_{k=1}^{n} (1 - (x_\nu)_k^2)^{-1}$ converges to 0. Then the sequence $(\varepsilon_\nu, x_\nu, y_\nu)$ in S converges to the point $(0, \overline{x}, 0)$, which does not lie in S and, therefore, S is not closed.

If $\overline{x} \in B_n$, then it is clear that y_ν converges to 0. Assume now that $\overline{x} \in \partial B_n$. We consider the following semialgebraic set:

$$Z := \left\{(t, \varepsilon, x) \in (0, \infty) \times (0, \infty) \times B_n \mid f(\varepsilon, x) = 0, \ t = \prod_{k=1}^{n} \left(1 - x_k^2\right) \right\}$$

defined by a conjunction of polynomial inequalities of degree at most $\max\{2n, \deg f\}$. By assumption, we have $(0, 0, \overline{x}) \in \overline{Z}$. Consider now the image $T \subseteq (0, \infty) \times (0, \infty)$ of Z under the projection $(t, \varepsilon, x) \mapsto (t, \varepsilon)$. Then we have $(0, 0) \in \overline{T}$.

By efficient quantifier elimination, the projection T can be described by a Boolean combination of polynomial inequalities of degree at most $N = (n \deg f)^{cn}$, for some fixed $c > 0$, see [31, Part III].

We now apply Lemma 6.18 to obtain a sequence $(t_\nu, \varepsilon_\nu, x_\nu)$ in Z such that

$$\lim_{\nu \to \infty} \frac{\varepsilon_\nu^N}{t_\nu} = \lim_{\nu \to \infty} y_\nu = 0.$$

This completes the proof. □

Lemma 6.20 *The problems* BasicClosed$_\mathbb{R}$ *and* BasicCompact$_\mathbb{R}$ *are* $\mathsf{H}\forall$*-hard.*

Proof Lemma 6.19 allows us to reduce Standard($\mathsf{H}\forall$) to BasicClosed$_\mathbb{R}$. Indeed, a description of the set S in its statement can be obtained in polynomial time from a description of f. The exponent N is exponential in the size of f, so we should use the sparse representation for the polynomial $y\prod_{k=1}^n(1-x_k^2)=\varepsilon^N$. Alternatively, we may reduce the degree N by introducing the variables $z_1,\dots,z_{\log N}$ (we assume N is a power of 2) and replacing $y\prod_{k=1}^n(1-x_k^2)=\varepsilon^N$ by the equalities

$$z_1=\varepsilon^2, \qquad z_j=z_{j-1}^2 \quad (j=2,\dots,\log N), \qquad y\prod_{k=1}^n\bigl(1-x_k^2\bigr)=z_{\log N}.$$

This defines a basic semialgebraic set S' homeomorphic to S which is definable, with dense representation, in size polynomial in the size of f.

For showing hardness of BASICCOMPACT$_{\mathbb{R}}$ note that, for a given basic semialgebraic set $S\subseteq\mathbb{R}^n$, S is closed if and only if $\widetilde{S}:=\pi^{-1}(S)\cup\{\mathcal{N}\}$ is compact. Since a description of the basic semialgebraic set $\widetilde{S}$ can be obtained in polynomial time from such a description for S, we see that BASICCLOSED$_{\mathbb{R}}$ reduces to BASICCOMPACT$_{\mathbb{R}}$. □

Remark 6.21 A question naturally arising is whether Theorem 6.15 can be extended to characterize the complexity of deciding closedness for arbitrary semialgebraic sets. Lemma 6.20 immediately yields H∀-hardness for this problem. But the characterization in Lemma 6.16 does not extend to this case (for a counterexample, take $S\subset\mathbb{R}^2$ given by $S=\{x^2+y^2\le 1\}\cup\{y=0,x^2\le 100,x>1\}$ which is closed but different from S_ε for all $\varepsilon>0$). On the other hand, noting that S is closed if and only if

$$\forall x\,\exists\varepsilon>0\,\forall y(x\notin S\wedge\|x-y\|\le\varepsilon\Rightarrow y\notin S)$$

shows that the problem is in ∀H∀. While the gap between the best lower (H∀) and upper (∀H∀) bounds thus obtained for closedness is smaller than the one mentioned in Section 1 (i.e., ∀ against ∀∃∀) this is still an unsatisfactory situation.

We may also consider the problems of deciding, for an arbitrary semialgebraic set S, whether S is compact, or whether it is open. It is not difficult to see that both problems are polynomially equivalent to the problem of testing closedness. The gap between H∀ and ∀H∀ thus also being the best we can exhibit for these problems, we can say that the complexity of openness remains an open problem.

7 The Classes H, H^k, and ∃*H

We now turn our attention to classes where H is in the innermost position, e.g., H and ∃*H. Consider the problem:

SOCS$_{\mathbb{R}}$(1) (*Smallest-Order Coefficient Sign*). Given a division-free straight-line program Γ in one input variable X, decide whether the smallest-order coefficient of f_Γ (the polynomial in X computed by Γ) is positive.

This problem is related to several well-studied problems. For instance, if one replaces the word "positive" by "zero" in the definition of SOCS$_{\mathbb{R}}$(1), we obtain the one-variable version of the problem SLP0$_{\mathbb{R}}$ of deciding whether the polynomial computed by a straight-line program Γ is identically zero. This is an archetype of problem solvable with randomization. The corresponding problem for constant-free straight-line programs is also called *Arithmetic Circuit Identity Testing* (ACIT), see [1, 25].

Proposition 7.1 *The problem* $\mathrm{SOCS}_{\mathbb{R}}(1)$ *is* H*-complete for Turing reductions.*

Proof Membership follows from the fact that $\Gamma \in \mathrm{SOCS}_{\mathbb{R}}(1)$ if and only if $\exists \mu > 0$ $\forall \varepsilon \in (0, \mu)\, f_\Gamma(\varepsilon) > 0$. The problem STANDARD($\mathsf{H}$) consisting of deciding whether, given a decision circuit $\mathcal{C}$ in a single variable X, $\mathsf{H}\varepsilon\, \mathcal{C}(\varepsilon) = 1$ is H-complete. We are going to Turing-reduce STANDARD(H) to $\mathrm{SOCS}_{\mathbb{R}}(1)$. Without loss of generality, we may assume that the circuit $\mathcal{C}$ is division-free. Recall that the node preceding the output node of $\mathcal{C}$ is a sign node. Now consider an algorithm performing the computation of $\mathcal{C}$ symbolically on an input variable X. When it reaches a sign node ν it queries $\mathrm{SOCS}_{\mathbb{R}}(1)$ with input the straight-line program corresponding to the arithmetic computations performed by $\mathcal{C}$ before reaching node ν (sign tests excluded). Once the oracle query is answered, the algorithm continues the computation of $\mathcal{C}$ taking that answer as outcome of the considered sign node. This is justified because the lowest nonzero coefficient determines the result when ε tends to 0.

The output of this algorithm is therefore 1 if and only if $\mathsf{H}\varepsilon\, \mathcal{C}(\varepsilon) = 1$. □

The next problem is related to a familiar notion in geometry. When, for a set $S \subset \mathbb{R}^n$ and a linear function $\ell: \mathbb{R}^n \to \mathbb{R}$, we have $S \cap \{\ell < 0\} = \emptyset$ and $\dim(\overline{S} \cap \{\ell = 0\}) = n - 1$ we say that S is *supported* by the hyperplane $\{\ell = 0\}$. The problem $\mathrm{LocSupp}_{\mathbb{R}}$ consists of deciding a local version of this notion.

$\mathrm{LocSupp}_{\mathbb{R}}$ (*Local Support*). Given a nonzero linear equation $\ell(x) = 0$ and a circuit $\mathcal{C}$ with n input nodes, decide whether there exists a point $x_0 \in \mathbb{R}^n$ and $\delta > 0$ such that $S_{\mathcal{C}} \cap \{\ell < 0\} \cap B(x_0, \delta) = \emptyset$ and

$$\dim\big(\overline{S_{\mathcal{C}} \cap \{\ell > 0\}} \cap \{\ell = 0\} \cap B(x_0, \delta)\big) = n - 1.$$

Proposition 7.2 *The problem* $\mathrm{LocSupp}_{\mathbb{R}}$ *is* $\exists^*\mathsf{H}$*-complete.*

Proof To prove the hardness, we define an auxiliary problem $\mathrm{STANDARD}^+(\exists^*\mathsf{H})$ consisting of deciding, given a circuit $\mathcal{C}$ with $n + 1$ input gates, whether

$$\exists^* x \in \mathbb{R}^n \mathsf{H}\varepsilon \big(\mathcal{C}(\varepsilon, x) = 1 \wedge \mathcal{C}(-\varepsilon, x) = 0\big).$$

By definition, $\mathrm{STANDARD}^+(\exists^*\mathsf{H}) \in \exists^*\mathsf{H}$. In addition, $\mathrm{STANDARD}^+(\exists^*\mathsf{H})$ is $\exists^*\mathsf{H}$-hard. Indeed, given a circuit $\mathcal{C}$ with $n + 1$ input variables $(\varepsilon, x_1, \dots, x_n)$ we can construct in polynomial time a circuit $\mathcal{C}^+$ with the same input nodes doing the following:

$$\text{if } \varepsilon < 0 \text{ return } 0, \text{ else return } \mathcal{C}(\varepsilon, x)$$

and, clearly, $\mathcal{C} \in \mathrm{STANDARD}(\exists^*\mathsf{H})$ if and only if $\mathcal{C}^+ \in \mathrm{STANDARD}^+(\exists^*\mathsf{H})$.

Now we claim that, for a circuit $\mathcal{C}$ with $n + 1$ input variables $(\varepsilon, x_1, \dots, x_n)$,

$$\mathcal{C} \in \mathrm{STANDARD}^+(\exists^*\mathsf{H}) \quad \Longleftrightarrow \quad (\{\varepsilon = 0\}, \mathcal{C}) \in \mathrm{LocSupp}_{\mathbb{R}}.$$

In order to see this, suppose that $\mathcal{C} \in \text{STANDARD}^+(\exists^*\mathsf{H})$. Then there exist $x \in \mathbb{R}^n$ and $\delta > 0$ such that, for all $y \in B(x, \delta)$, there exists $\mu_y > 0$ satisfying

$$S_{\mathcal{C}} \cap ((-\mu_y, 0) \times \{y\}) = \emptyset \quad \text{and} \quad ((0, \mu_y) \times \{y\}) \subseteq S_{\mathcal{C}}.$$

By the theorem on the cylindrical decomposition of semialgebraic sets (see [7, §2.3] or [3, §5.1]), we may assume that μ_y is a continuous function of y in a suitable closed ball $\overline{B(x', \delta')}$ contained in $B(x, \delta)$. By taking the minimum of μ_y over this closed ball, we may therefore assume that μ_y can be chosen independently of y. Hence, we obtain

$$\begin{aligned}
&\mathcal{C} \in \text{STANDARD}^+(\exists^*\mathsf{H}) \\
&\quad\iff \quad \exists x\, \exists \delta > 0 \exists \mu > 0 \big(S_{\mathcal{C}} \cap \big((-\mu, 0) \times B_{\mathbb{R}^n}(x, \delta)\big) = \emptyset \\
&\qquad\qquad \wedge (0, \mu) \times B_{\mathbb{R}^n}(x, \delta) \subseteq S_{\mathcal{C}}\big) \\
&\quad\iff \quad \exists x \exists \delta > 0 \big(S_{\mathcal{C}} \cap \{\varepsilon < 0\} \cap B_{\mathbb{R}^{n+1}}\big((0, x), \delta\big) = \emptyset \\
&\qquad\qquad \wedge \dim\big(\overline{S_{\mathcal{C}} \cap \{\varepsilon > 0\}} \cap \{\varepsilon = 0\} \cap B_{\mathbb{R}^{n+1}}\big((0, x), \delta\big)\big) = n - 1\big) \\
&\quad\iff \quad (\{\varepsilon = 0\}, \mathcal{C}) \in \text{LOCSUPP}_{\mathbb{R}}.
\end{aligned}$$

The $\exists^*\mathsf{H}$-hardness of $\text{LOCSUPP}_{\mathbb{R}}$ follows from the claim.

For showing membership, let $\ell(x) = \ell_1 x_1 + \ell_2 x_2 + \cdots + \ell_n x_n + c$ be a linear function such that, without loss of generality, $\ell_n \neq 0$, and let $\mathcal{C}$ be a circuit with n input nodes. A point $x \in \mathbb{R}^n$ is in $\{\ell = 0\}$ if and only if $x_n = \varphi(x_1, \ldots, x_{n-1}) = -(\ell_1 x_1 + \ell_2 x_2 + \cdots + \ell_{n-1} x_{n-1} + c)/\ell_n$. Therefore, by the reasoning above with ℓ taking the role of ε and $\overline{x} = (x_1, \ldots, x_{n-1})$, we have $(\ell, \mathcal{C}) \in \text{LOCSUPP}_{\mathbb{R}}$ if and only if

$$\exists^* \overline{x} \in \mathbb{R}^{n-1} \mathsf{H} \varepsilon \big(\mathcal{C}\big(\big(\overline{x}, \varphi(\overline{x})\big) + \varepsilon(\ell_1, \ldots, \ell_n)\big) = 1 \wedge \mathcal{C}\big(\big(\overline{x}, \varphi(\overline{x})\big) - \varepsilon(\ell_1, \ldots, \ell_n)\big) = 0\big)$$

and this shows membership. □

We noted in Remark 6.3 that, unlike for $\exists$, $\forall$, $\exists^*$, and $\forall^*$, the equality $\mathsf{HH} = \mathsf{H}$ is not known to be true. Denote by H^k the class $\mathsf{HH}\ldots\mathsf{H}$, k times. Proposition 7.1 readily extends to H^k. To do so, for a polynomial $f = \sum_\alpha f_\alpha X^\alpha$ in the variables $X_1, \ldots, X_k$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $X^\alpha = X_1^{\alpha_1} \cdots X_k^{\alpha_k}$, define its *smallest order coefficient* (with respect to the ordering $X_1 \succ X_2 \succ \cdots \succ X_k$) to be the coefficient $f_{\alpha_1^*}$ where α_i^* is defined by

$$\begin{aligned}
\alpha_k^* &= \min\{\beta \mid \exists \alpha_1, \ldots, \alpha_{k-1} f_{(\alpha_1, \ldots, \alpha_{k-1}, \beta)} \neq 0\}, \\
\alpha_{k-1}^* &= \min\{\beta \mid \exists \alpha_1, \ldots, \alpha_{k-2} f_{(\alpha_1, \ldots, \alpha_{k-2}, \beta, \alpha_k^*)} \neq 0\}, \\
&\ \ \vdots \\
\alpha_1^* &= \min\{\beta \mid f_{(\beta, \alpha_2^*, \ldots, \alpha_k^*)} \neq 0\}.
\end{aligned}$$

The k variables version of $\text{SOCS}_{\mathbb{R}}(1)$ is the following.

$\mathrm{SOCS}_{\mathbb{R}}(k)$ (*Smallest-Order Coefficient Sign, k Variables*). Given a division-free straight-line program Γ in k input variables $X_1, \dots, X_k$, decide whether the smallest-order coefficient of f_Γ is positive.

This notion of smallest-order coefficient is at the center of the work on ordered fields developed by Artin and Schreier [2] to solve Hilbert's 17th problem. Consider a (necessarily transcendental) ordered extension $K_1 = \mathbb{R}(\alpha_1)$ of $\mathbb{R}$. By replacing α_1 by $1/\alpha_1$ we may assume that α_1 is finite (in the sense that there exists $b \in \mathbb{R}$ such that $|\alpha_1| < b$). The completeness of $\mathbb{R}$ then implies that there exists $a_1 \in \mathbb{R}$ such that $a_1 - \alpha_1$ is an infinitesimal (i.e., $1/(a_1 - \alpha_1)$ is not finite). By replacing α_1 by $a_1 - \alpha_1$ we can assume that α_1 is an infinitesimal. Repeating k times this process we obtain a finitely generated ordered extension $K = \mathbb{R}(\alpha_1, \dots, \alpha_k)$ of $\mathbb{R}$ in which, for all $i \le k$, α_i is an infinitesimal with respect to $\mathbb{R}[\alpha_1, \dots, \alpha_{i-1}]$. We can denote this by writing $\alpha_1 \succ \alpha_2 \succ \cdots \succ \alpha_k$.

Comparing elements in K reduces to computing the sign of elements in this field, a task which itself reduces to computing signs of elements in $\mathbb{R}[\alpha_1, \dots, \alpha_k]$. If f is such an element, this can be done by looking at the coefficients of f: $f = 0$ if and only if all its coefficients are zero and, otherwise, $f > 0$ if and only if its smallest-order coefficient is positive. If f is given explicitly, its sign can then be trivially computed. But this is not so if f is given by a (division-free) straight-line program. In this case we have already remarked that deciding whether $f = 0$ is precisely the problem $\mathrm{SLP0}_{\mathbb{R}}$ and that this problem can be solved using randomization. We now observe that to deciding whether $f > 0$ amounts to deciding whether $f \in \mathrm{SOCS}_{\mathbb{R}}(k)$.

The following result is proved as Proposition 7.1.

Proposition 7.3 *The problem* $\mathrm{SOCS}_{\mathbb{R}}(k)$ *is* H^k*-complete for Turing reductions.*

Remark 7.4 Let $\mathrm{SOCS}_{\mathbb{R}}(*)$ be the union of $\mathrm{SOCS}_{\mathbb{R}}(k)$ for $k \ge 1$. Similarly, let $\mathsf{H}^\bullet$ be the class resulting from allowing a polynomial time machine to use the quantifier H (in the same way $\mathsf{PAT}_{\mathbb{R}}$ is defined by allowing a polynomial time machine to use the quantifiers $\exists$ and $\forall$ [15]). Then, $\mathrm{SOCS}_{\mathbb{R}}(*)$ is $\mathsf{H}^\bullet$-complete and the hierarchy

$$\mathsf{P}_{\mathbb{R}} \subseteq \mathsf{H} \subseteq \mathsf{H}^2 \subseteq \mathsf{H}^3 \subseteq \cdots \subseteq \mathsf{H}^\bullet$$

collapses if and only if $\mathrm{SOCS}_{\mathbb{R}}(*) \in \mathsf{H}^k$ for some $k \ge 1$.

8 Some Inclusions of Complexity Classes

Koiran [29] describes an efficient method to express generic quantifiers $\exists^*$ using instead existential quantifiers. We briefly recall this method in the following.

Recall that $\mathcal{F}_{\mathbb{R}}$ denotes the set of first-order formulas over the language of the theory of ordered fields with constant symbols for real numbers. Let $F(u, a) \in \mathcal{F}_{\mathbb{R}}$ be a formula with free variables $u \in \mathbb{R}^s$ (viewed as parameters) and let $a \in \mathbb{R}^k$ (viewed as instances). Let $\widetilde{F}(u, y_1, \dots, y_{k+s+2})$ denote the following formula derived from F:

$$\exists a \in \mathbb{R}^k \; \exists \epsilon > 0 \; \bigwedge_{i=1}^{k+s+2} F(u, a + \epsilon y_i). \tag{7}$$

Hereby, each variable y_i is in $\mathbb{R}^k$. Let $W(F)$ denote the set of *witness sequences for F*, that is, the set of points $y = (y_1, \dots, y_{k+s+2}) \in \mathbb{R}^{k(k+s+2)}$ satisfying the property

$$\forall u \in \mathbb{R}^s \big(\exists^* a \in \mathbb{R}^k \ F(u,a) \Leftrightarrow \widetilde{F}(u, y_1, \dots, y_{k+s+2})\big). \tag{8}$$

Koiran [29] proved the following result.

Theorem 8.1

(i) *$W(F)$ is Zariski dense in $\mathbb{R}^{k(k+s+2)}$, for any $F(u,a) \in \mathcal{F}_{\mathbb{R}}$.*
(ii) *Suppose that F is in prenex form with free variables $u \in \mathbb{R}^s, a \in \mathbb{R}^k$, and n bounded variables, w alternating quantifier blocks, and m atomic predicates given by polynomials of degree at most $d \geq 2$ with integer coefficients of bit size at most ℓ. Then a point in $W(F)$ can be computed by a straight-line program of length $(k+s+n)^{\mathcal{O}(w)} \log(md) + \mathcal{O}(\log \ell)$, which is division-free, has 1 as its only constant, and no inputs.*

This theorem implies the following inclusion of complexity classes.

Theorem 8.2 *Let $\mathcal{C}$ be a polynomial class. Then $\exists^*\mathcal{C} \subseteq \exists\mathcal{C}$ and $\forall^*\mathcal{C} \subseteq \forall\mathcal{C}$.*

Proof It suffices to prove that $\exists^*\mathcal{C} \subseteq \exists\mathcal{C}$. Let the polynomial class $\mathcal{C}$ be defined by the sequence of quantifiers $Q_1, \dots, Q_p$, where $Q_i \in \{\exists, \forall, \exists^*, \forall^*, \mathsf{H}\}$. It is sufficient to show that the standard complete problem STANDARD($\exists^*\mathcal{C}$) belongs to $\exists\mathcal{C}$. We prove only the case $Q_p \in \{\exists^*, \forall^*, \mathsf{H}\}$ (the case $Q_p \in \{\exists, \forall\}$ being simpler). Thus, STANDARD($\exists^*\mathcal{C}$) is the problem of deciding, given a circuit $\mathfrak{C}$ with $k + n_1 + \cdots + n_p$ input gates and constants $u \in \mathbb{R}^s$, whether $\exists^* a \in \mathbb{R}^k F(u,a)$, where $F(u,a)$ denotes the formula

$$Q_1 x_1 \in \mathbb{R}^{n_1} \dots Q_p x_p \in \mathbb{R}^{n_p} \mathfrak{C}(a, x_1, \dots, x_p, u) = 1.$$

According to Theorem 8.1, a witness sequence $\widetilde{y} = (\widetilde{y}_1, \dots, \widetilde{y}_{k+s+2}) \in \mathbb{R}^{k(k+s+2)}$ in $W(F)$ can be computed by a constant-free, division-free, straight-line program of length polynomial in size($\mathfrak{C}$) without input gates. From (7) and (8), we see that $\exists^* a \in \mathbb{R}^k F(u,a)$ is equivalent to

$$\exists a \in \mathbb{R}^k \, \exists \epsilon > 0 \bigwedge_{i=1}^{k+s+2} F(u, a + \epsilon \widetilde{y}_i).$$

We next show that the problem to decide $\bigwedge_{i=1}^{k+s+2} F(u, a+\epsilon\widetilde{y}_i)$ for given u, a, and ϵ is in the class $\mathcal{C}$.

This can be shown by induction on p. Suppose first that $p = 1$, that is, $F(u,a)$ is in the class Q_1. Then, introducing additional variables $x_1^{(i)}$ for $1 \leq i \leq k+s+2$, we see that $\bigwedge_{i=1}^{k+s+2} F(u, a+\epsilon\widetilde{y}_i)$, i.e.,

$$\bigwedge_{i=1}^{k+s+2} Q_1 x_1 \in \mathbb{R}^{n_1} \ \mathfrak{C}(a + \epsilon\widetilde{y}_i, x_1, u)$$

is equivalent to

$$Q_1 x_1^{(1)} \in \mathbb{R}^{n_1} \dots Q_1 x_1^{(k+s+2)} \in \mathbb{R}^{n_1} \bigwedge_{i=1}^{k+s+2} \mathcal{C}\big(a + \epsilon \widetilde{y}_i, x_1^{(i)}, u\big)$$

(if $Q_1 = \mathsf{H}$ we do not even need to introduce additional variables). Since $\widetilde{y}_i$ is computed in time polynomial in size($\mathcal{C}$), the computation of $\mathcal{C}(u, a + \epsilon\widetilde{y}_i, x_1^{(i)})$ is also done in time polynomial in size($\mathcal{C}$). Hence $\bigwedge_{i=1}^{k+s+2} F(u, a + \epsilon\widetilde{y}_i)$ can be decided in the class Q_1.

The induction step can be settled similarly, which concludes the proof. □

Corollary 8.3

(i) *We have* $\exists^*\forall^* \subseteq \exists\forall^* \subseteq \exists\forall$ *and* $\exists^*\forall^* \subseteq \exists^*\forall \subseteq \exists\forall$.
(ii) *We have* $\exists^*\exists = \exists$. *In particular,* IMAGEZDENSE$_{\mathbb{R}}$ *is* $\exists$*-complete.*

Proof This follows immediately from Corollary 5.3 and Theorem 8.2. □

The next observation will be of great use in the next section.

Proposition 8.4 *We have* $\exists \subseteq \mathsf{H}^2\exists^*$ *and* $\forall \subseteq \mathsf{H}^2\forall^*$.

Proof It suffices to prove the first statement. To do so, let $f \in \mathbb{R}[X_1, \dots .X_n]$. Then

$$\begin{aligned} \exists x f(x) = 0 \quad &\Longleftrightarrow \quad \mathsf{H}\delta\exists x\big(\|x\|^2 \le \delta^{-1} \wedge f(x) = 0\big) \\ &\Longleftrightarrow \quad \mathsf{H}\delta\mathsf{H}\varepsilon\exists x\big(\|x\|^2 \le \delta^{-1} \wedge f(x)^2 < \varepsilon\big) \\ &\Longleftrightarrow \quad \mathsf{H}\delta\mathsf{H}\varepsilon\exists^* x\big(\|x\|^2 < \delta^{-1} \wedge f(x)^2 < \varepsilon\big) \end{aligned}$$

the second equivalence by the compactness of closed balls. This shows that STANDARD($\exists$) can be solved in $\mathsf{H}^2\exists^*$. □

9 Exotic Quantifiers in the Discrete Setting

It is common to restrict the input polynomials in the problems considered so far to polynomials with integer coefficients (represented in binary), or to constant-free circuits (i.e., circuits which use only 0 and 1 as values associated to their constant nodes). The resulting problems can be encoded in a finite alphabet and studied in the classical Turing setting. In general, if L denotes a problem defined over $\mathbb{R}$ or $\mathbb{C}$, we denote its restriction to integer inputs by $L^{\mathbb{Z}}$. This way, the discrete problems ISOLATED$_{\mathbb{R}}^{\mathbb{Z}}$, SURJ$_{\mathbb{R}}^{\mathbb{Z}}$, CONT$_{\mathbb{R}}^{\mathbb{Z}}$, etc., are well defined.

Another natural restriction (considered, e.g., in [17, 27, 28]), now for real machines, is the requirement that no constants other than 0 and 1 appear in the machine program. Complexity classes arising by considering such a constant-free machines are indicated by a superscript 0 as in $\mathsf{P}_{\mathbb{R}}^0$, $\mathsf{NP}_{\mathbb{R}}^0$, etc.

The simultaneous consideration of both these restrictions leads to the notion of a constant-free Boolean part.

Definition 9.1 Let $\mathcal{C}$ be a complexity class over $\mathbb{R}$. The *Boolean part* of $\mathcal{C}$ is the discrete complexity class

$$\mathsf{BP}(\mathcal{C}) = \big\{S \cap \{0,1\}^\infty \mid S \in \mathcal{C}\big\}.$$

We denote by $\mathcal{C}^0$ the subclass of $\mathcal{C}$ obtained by requiring all the considered machines over $\mathbb{R}$ to be constant-free. The *constant-free Boolean part* of $\mathcal{C}$ is defined as $\mathsf{BP}^0(\mathcal{C}) := \mathsf{BP}(\mathcal{C}^0)$.

Some of the classes $\mathsf{BP}^0(\mathcal{C})$ do contain natural complete problems. This raises the issue of characterizing these classes in terms of already known discrete complexity classes. Unfortunately, there are not many real complexity classes $\mathcal{C}$ for which $\mathsf{BP}^0(\mathcal{C})$ is completely characterized in such terms. The only such result that we know is $\mathsf{BP}^0(\mathsf{PAR}_\mathbb{R}) = \mathsf{PSPACE}$, proved in [16]. An obvious solution (which may be the only one) is to define new discrete complexity classes in terms of Boolean parts. In this way we define the classes $\mathsf{PR} := \mathsf{BP}^0(\mathsf{P}_\mathbb{R})$, $\mathsf{NPR} := \mathsf{BP}^0(\mathsf{NP}_\mathbb{R})$, and $\mathsf{coNPR} = \mathsf{coBP}^0(\mathsf{NP}_\mathbb{R}) = \mathsf{BP}^0(\mathsf{coNP}_\mathbb{R})$.

While never explicited as a complexity class (to the best of our knowledge) the computational resources behind PR have been around for quite a while. A constant-free machine over $\mathbb{R}$ restricted to binary inputs is, in essence, a (unit-cost) Random Access Machine (RAM). Therefore, PR is the class of subsets of $\{0,1\}^*$ decidable by a RAM in polynomial time. In [1] it was shown that PR is contained in the counting hierarchy and some empirical evidence pointing towards $\mathsf{P} \neq \mathsf{PR}$ was collected.

The class NPR naturally occurs when considering the existential theory of the reals (ETR), over the language $\{\{0,1\}, +, -, \times, \leq\}$. This is a discrete decisional problem which is NPR-complete (see Corollary 9.4 below).

The main result of this section is the following.

Theorem 9.2 *Let $\mathcal{C}$ be a polynomial class. Then*

$$\mathsf{BP}^0(\mathsf{H}\mathcal{C}) = \mathsf{BP}^0(\mathcal{C}).$$

From this theorem, Corollary 5.3, and Proposition 8.4 the following immediately follows.

Corollary 9.3

(i) *For all $k \geq 1$,* $\mathsf{BP}^0(\mathsf{H}^k) = \mathsf{PR}$.
(ii) *For all $k \geq 1$,* $\mathsf{BP}^0(\exists^*) = \mathsf{BP}^0(\mathsf{H}^k\exists^*) = \mathsf{BP}^0(\mathsf{H}^k\exists) = \mathsf{BP}^0(\exists) = \mathsf{NPR}$.

Corollary 9.4

(i) *For all $k \geq 1$, the problem* $\text{SOCS}_\mathbb{R}(k)^\mathbb{Z}$ *is* PR*-complete.*
(ii) *The discrete versions of the following problems are* NPR*-complete*: $\text{FEAS}_\mathbb{R}$, $\text{DIM}(d)$, $\text{EADH}_\mathbb{R}$, $\text{ZDENSE}_\mathbb{R}$, $\text{UNBOUNDED}_\mathbb{R}$, $\text{LOCDIM}_\mathbb{R}$, $\text{IMAGEZDENSE}_\mathbb{R}$, $\text{DOMAINZDENSE}_\mathbb{R}$. *The same is true for the discrete problem* ETR.
(iii) *The discrete versions of the following problems are* coNPR*-complete*: $\text{EDENSE}_\mathbb{R}$, $\text{ISOLATED}_\mathbb{R}$, $\text{BASICCLOSED}_\mathbb{R}$, $\text{BASICCOMPACT}_\mathbb{R}$. $\text{TOTAL}_\mathbb{R}$, $\text{INJ}_\mathbb{R}$, $\text{DOMAINEDENSE}_\mathbb{R}$, $\text{CONT}_\mathbb{R}$, $\text{CONT}_\mathbb{R}^{\text{DF}}$, $\text{CONTPOINT}_\mathbb{R}^{\text{DF}}$, $\text{LIPSCHITZ}_\mathbb{R}(k)$, $\text{LIPSCHITZ}_\mathbb{R}$.

Proof The claimed memberships follow from the definition of BP^0, Corollary 9.3, and a cursory look to the membership proofs for their real versions which show that the involved algorithms are constant-free.

For proving the hardness we first remark that, for any polynomial class $\mathcal{C}$, the problem $\text{STANDARD}(\mathcal{C})^{\mathbb{Z}}$ is hard for $\mathsf{BP}^0(\mathcal{C})$. This follows by inspecting the original reduction for $\text{CEVAL}_{\mathbb{R}}$ as given in [22] and noting that, when restricted to binary inputs, it can be performed by a Turing machine in polynomial time. Since this reduction is extended to arbitrary polynomial classes by adding quantifiers, our remark follows. We next note that the reductions shown in this paper for all the problems above can also be performed by a Turing machine in polynomial time when restricted to binary inputs. This finishes the proof. □

Thus, based on Theorem 9.2, we obtain in Corollary 9.4 the completeness for the discrete problems $\text{CONT}_{\mathbb{R}}^{\mathbb{Z}}$, $\text{CONT}_{\mathbb{R}}^{\text{DF}}, \mathbb{Z}$, and $\text{LIPSCHITZ}_{\mathbb{R}}^{\mathbb{Z}}$, even though we do not have completeness results for the corresponding real problems. This suggests that we are not far away from completeness and this situation deserves a proper name.

Definition 9.5 We say that a problem S has a *narrow gap for the class* $\mathcal{C}$ when S is $\mathcal{C}$-hard and there is a complexity class $\mathcal{C} \subseteq \mathcal{D}$ satisfying that $S \in \mathcal{D}$ and $\mathsf{BP}^0(\mathcal{C}) = \mathsf{BP}^0(\mathcal{D})$.

We turn now to the proof of Theorem 9.2, which uses a few facts from various sources.

The *separation* $\text{sep}(h)$ of a nonzero univariate polynomial $h \in \mathbb{C}[Y]$ is defined as the minimal distance between two distinct complex roots of h, or ∞ if h does not have two distinct roots. We denote by $\|h\|$ the Euclidean norm of the coefficient vector of h.

A proof of the following lower bound on the separation can be found in [30].

Lemma 9.6 *Let $h \in \mathbb{Z}[Y]$ be a nonconstant integer polynomial of degree D. Then*

$$\text{sep}(h) \geq \frac{1}{D^{(D+2)/2}\|h\|^{D-1}}.$$

The easy proof of the next lemma is left to the reader.

Lemma 9.7 *Let $\mathscr{C}$ be a division-free and constant-free algebraic decision circuit of size N in n variables. There exist $K \leq N2^N$ polynomials $g_1, \ldots, g_K$ of degree at most 2^N and coefficient bit-size at most $\mathcal{O}(2^N)$ such that $S_{\mathscr{C}} = G(x_1, \ldots, x_n)$, where G is a Boolean combination of equalities and inequalities of $g_1, \ldots, g_K$.*

Let $\mathcal{C}$ be a polynomial class and $\text{STANDARD}(\mathcal{C})$ its standard complete problem as defined in Section 3. The standard problem $\text{STANDARD}^{\mathbb{Z}}(\mathcal{C}) := \text{STANDARD}(\mathcal{C})^{\mathbb{Z}}$ is obtained by requiring that the circuit $\mathscr{C}$ (or the polynomial f) given as input in $\text{STANDARD}(\mathcal{C})$ has no real constants (respectively, has integer coefficients). The reductions in Proposition 3.1 show that $\text{STANDARD}^{\mathbb{Z}}(\mathcal{C})$ is $\mathsf{BP}^0(\mathcal{C})$-complete.

Proof of Theorem 9.2 Let $\mathcal{C} = Q_1 Q_2 \dots Q_w$ where $Q_i \in \{\exists, \forall, \exists^*, \forall^*, \mathsf{H}\}$ for $i \le w$. Assume that $Q_w \in \{\exists^*, \forall^*, \mathsf{H}\}$. In this case, an input for the problem $\text{STANDARD}^{\mathbb{Z}}(\mathsf{H}Q_1 Q_2 \dots Q_w)$ is a constant-free algebraic decision circuit $\mathcal{C}$ and this input is in $\text{STANDARD}^{\mathbb{Z}}(\mathsf{H}Q_1 Q_2 \dots Q_w)$ if and only if

$$\mathsf{H}\varepsilon Q_1 \overline{x_1} Q_2 \overline{x_2} \dots Q_w \overline{x_w} (\varepsilon, \overline{x_1}, \overline{x_2}, \dots, \overline{x_w}) \in S_{\mathcal{C}}.$$

Here $\overline{x_i} \in \mathbb{R}^{n_i}$ for some $n_i \ge 1$.

The problem $\text{STANDARD}^{\mathbb{Z}}(\mathsf{H}Q_1 Q_2 \dots Q_w)$ is $\mathsf{BP}^0(\mathsf{H}\mathcal{C})$-complete. It is therefore sufficient to prove that this problem belongs to the class $\mathsf{BP}^0(\mathcal{C})$.

Let N be the size of $\mathcal{C}$. By Lemma 9.7, $S_{\mathcal{C}} = G(\varepsilon, \overline{x_1}, \overline{x_2}, \dots, \overline{x_w})$ where G is a Boolean combination of equalities and inequalities of polynomials $g_1, \dots, g_K$ where $K \le N2^N$ and the degree and coefficient bit-size of these polynomials is at most $\mathcal{O}(2^N)$. Now consider the formula

$$Q_1 \overline{x_1} Q_2 \overline{x_2} \dots Q_w \overline{x_w} G(\varepsilon, \overline{x_1}, \overline{x_2}, \dots, \overline{x_w})$$

with free variable ϵ.

We may replace the generic quantifiers (or H) by the usual quantifiers as in (2). Then, by a well-known result on the efficient quantifier elimination over the reals [31, Part III], this formula is equivalent to a quantifier-free formula in disjunctive normal form

$$\bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} (h_{ij} \Delta_{ij} 0), \tag{9}$$

with $\sum_{i=1}^{I} J_i \le 2^{N^{\mathcal{O}(1)}}$ atomic predicates involving (nonzero) polynomials h_{ij} of degree at most $2^{N^{\mathcal{O}(1)}}$ and integer coefficients of bit size at most $2^{N^{\mathcal{O}(1)}}$.

The polynomial $h := \prod_{i,j} h_{ij}$ has degree at most $2^{N^{\mathcal{O}(1)}} 2^{N^{\mathcal{O}(1)}} = 2^{N^{\mathcal{O}(1)}}$ and satisfies $\log \|h\| \le 2^{N^{\mathcal{O}(1)}}$. By Lemma 9.6, the separation $\mu := \text{sep}(h)$ of h satisfies $\mu \ge 2^{-2^{N^{\mathcal{O}(1)}}}$.

Let $S \subseteq \mathbb{R}$ be the semialgebraic set defined by formula (9). Note that every connected component of S, which is not a point, has length at least μ, and the same is true for the complement $\mathbb{R} - S$. Therefore, the following algorithm works in $\mathsf{BP}^0(\mathcal{C})$ and solves $\text{STANDARD}^{\mathbb{Z}}(\mathsf{H}Q_1 Q_2 \dots Q_w)$:

> **input** $\mathcal{C}$
>
> compute an upper bound $U := 2^{2^{N^{\mathcal{O}(1)}}}$ on μ^{-1}
>
> **if** $Q_1 \overline{x_1} Q_2 \overline{x_2} \dots Q_w \overline{x_w} G(1/(2U), \overline{x_1}, \overline{x_2}, \dots, \overline{x_w})$ **then** `accept`
>
> **else** `reject`.

This proves that $\mathsf{BP}^0(\mathsf{H}\mathcal{C}) = \mathsf{BP}^0(\mathcal{C})$ in the case that $Q_w \in \{\exists^*, \forall^*, \mathsf{H}\}$. The other cases are simpler. □

We finish with some comments and remarks following from Theorem 9.2.

Remark 9.8 We have just seen that, for all $k \geq 1$, $\mathsf{BP}^0(\mathsf{H}^k) = \mathsf{PR}$ and thus $\mathrm{SOCS}_{\mathbb{R}}(k)^{\mathbb{Z}} \in \mathsf{PR}$. We now note that, in contrast, we do not know the equality $\mathsf{BP}^0(\mathsf{H}^\bullet) = \mathsf{PR}$—or, equivalently, the membership $\mathrm{SOCS}_{\mathbb{R}}(*)^{\mathbb{Z}} \in \mathsf{PR}$—to hold.

Remark 9.9 We suggested in Remark 6.3(ii) that we believe that H is fundamentally simpler than the alternation of two quantifiers. In some aspects, it is even simpler than a single quantifier. Indeed, consider the problem of deciding whether, given a decision circuit $\mathcal{C}$ with n input gates, there exists $x \in \{0,1\}^n$ such that $x \in S_{\mathcal{C}}$. This problem is complete in the class $\mathsf{DNP}_{\mathbb{R}}$ which captures the complexity of problems where nondeterminism restricted to $\{0,1\}$ suffices (e.g., the real versions of the traveling salesman problem or the knapsack problem) [18]. It also belongs to $\mathsf{BP}^0(\exists^{[1]})$ (see Section 2(4)) since we can guess a real number $z \in [0,1]$ such that the first n bits of its binary expansion encode the candidate $x \in \{0,1\}^n$.

On the other hand, we believe that it is unlikely that the discrete version of problems in $\mathsf{DNP}_{\mathbb{R}}$ (many of them known to be NP-complete) can be solved in PR, which would be the case if $\exists^{[1]} \subseteq \mathsf{H}$ since $\mathsf{BP}^0(\mathsf{H}) = \mathsf{PR}$.

10 Summary

In this section we try to give a summary of our main results “at a glance.” First, we consider the landscape of complexity classes in the lower levels of $\mathsf{PH}_{\mathbb{R}}$ emerging from the previous sections. This is done in Fig. 2. Here all upward lines mean inclusion. In addition, a dashed line means that the Boolean parts of the two classes coincide. Note that not all possible classes below $\Sigma^3_{\mathbb{R}}$ or $\Pi^3_{\mathbb{R}}$ are in the diagram. We restricted attention to those which have played a visible role in our development (e.g., because of having natural complete problems).

Boxes enclosing groups of complexity classes do not have a very formal meaning. They are rather meant to convey the informal idea that some classes are “close enough” to be clustered together (for instance, because of having the same constant-free Boolean part).

Next we summarize complexity results for a number of natural problems over $\mathbb{R}$. Recall:

$\mathrm{FEAS}_{\mathbb{R}}$ (*Polynomial Feasibility*). Given a polynomial $f \in \mathbb{R}[X_1,\dots,X_n]$, decide whether there exists $x \in \mathbb{R}^n$ such that $f(x) = 0$.

$\mathrm{DIM}_{\mathbb{R}}(d)$ (*Semialgebraic Dimension*). Given a semialgebraic set S and $d \in \mathbb{N}$, decide whether $\dim S \geq d$.

$\mathrm{CONVEX}_{\mathbb{R}}$ (*Convexity*). Given a semialgebraic set S, decide whether S is convex.

EULER–YAO (*Euler–Yao Characteristic*). Given a semialgebraic set S, decide whether it is empty and if not, compute its Euler–Yao characteristic $\chi^*(S)$.

$\mathrm{EADH}_{\mathbb{R}}$ (*Euclidean Adherence*). Given a semialgebraic set S and a point x, decide whether x belongs to the Euclidean closure $\overline{S}$ of S.

$\mathrm{EDENSE}_{\mathbb{R}}$ (*Euclidean Denseness*). Given a decision circuit $\mathcal{C}$ with n input gates, decide whether $\overline{S_{\mathcal{C}}} = \mathbb{R}^n$.

$\mathrm{ERD}_{\mathbb{R}}$ (*Euclidean Relative Denseness*). Given semialgebraic sets S and V, decide whether S is included in $\overline{V}$.

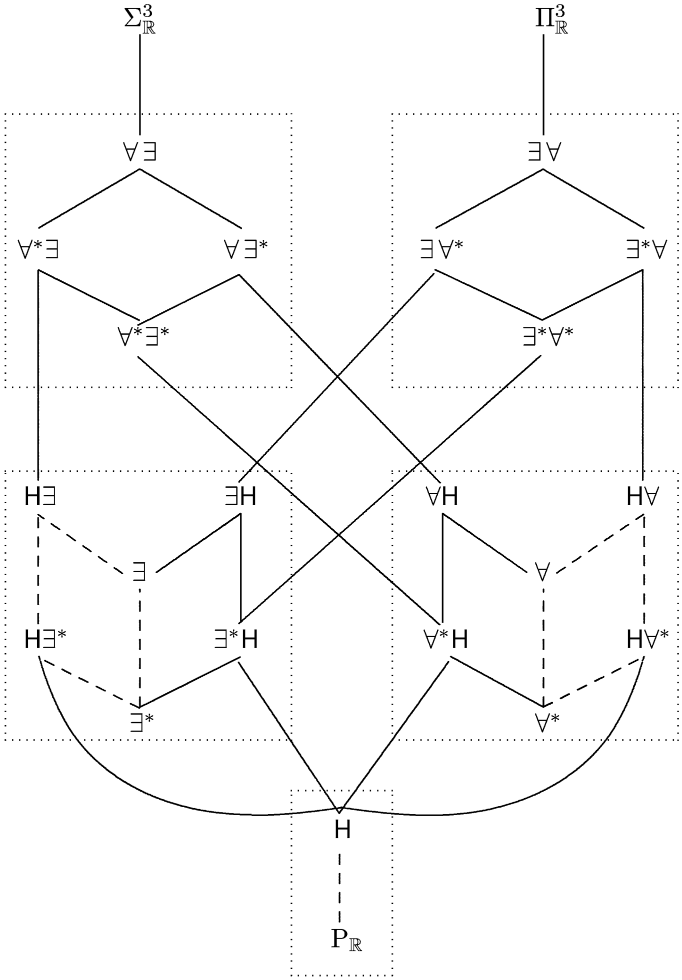

Fig. 2 A landscape of complexity classes in the lower levels of $\mathsf{PH}_{\mathbb{R}}$

LERD$_{\mathbb{R}}$ (*Linearly Restricted Euclidean Relative Denseness*). Given a semialgebraic set $V \subseteq \mathbb{R}^n$ and points $a_0, a_1, \ldots, a_k \in \mathbb{R}^n$, decide whether $a_0 + \langle a_1, \ldots, a_k \rangle$ is included in $\overline{V}$.

ZADH$_{\mathbb{R}}$ (*Zariski Adherence*). Given a semialgebraic set S and a point x, decide whether x belongs to the Zariski closure $\overline{S}^Z$ of S.

ZDENSE$_{\mathbb{R}}$ (*Zariski Denseness*). Given a decision circuit $\mathcal{C}$ with n input gates, decide whether $\overline{S_{\mathcal{C}}}^Z = \mathbb{R}^n$.

UNBOUNDED$_{\mathbb{R}}$ (*Unboundedness*). Given a semialgebraic set S, is it unbounded?

LOCDIM$_{\mathbb{R}}$ (*Local Dimension*). Given a semialgebraic set $S \subseteq \mathbb{R}^n$, a point $x \in S$, and $d \in \mathbb{N}$, is $\dim_x S \geq d$?

ISOLATED$_{\mathbb{R}}$ (*Isolated*). Given a semialgebraic set $S \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, decide whether x is an isolated point of S.

Table 1 Previously known complexity classes

Problem	Complete in
$\text{CEVAL}_{\mathbb{R}}$	$\mathsf{P}_{\mathbb{R}}$
$\text{FEAS}_{\mathbb{R}}$	$\exists$
$\text{DIM}(d)$	$\exists$
$\text{CONVEX}_{\mathbb{R}}$	$\forall$
EULER–YAO	$\mathsf{FP}_{\mathbb{R}}^{\#\mathsf{P}_{\mathbb{R}}}$

$\text{EXISTISO}_{\mathbb{R}}$ (*Existence of Isolated Points*). Given a semialgebraic set $S \subseteq \mathbb{R}^n$, decide whether there exists a point x isolated in S.

$\text{BASICCLOSED}_{\mathbb{R}}$ (*Closedness for Basic Semialgebraic Sets*). Given a basic semialgebraic set S, is it closed?

$\text{BASICCOMPACT}_{\mathbb{R}}$ (*Compactness for Basic Semialgebraic Sets*). Given a basic semialgebraic set S, is it compact?

$\text{SOCS}_{\mathbb{R}}(k)$ (*Smallest-Order Coefficient Sign, k Variables*). Given a division-free straight-line program Γ in k input variables $X_1, \dots, X_k$, decide whether the smallest-order coefficient (with respect to the ordering $X_1 \succ X_2 \succ \cdots \succ X_k$) of f_Γ (the polynomial in X computed by Γ) is positive.

$\text{LOCSUPP}_{\mathbb{R}}$ (*Local Support*). Given a circuit $\mathcal{C}$ with n input nodes and a linear equation $\ell(x) = 0$, decide whether there exists $x_0 \in \mathbb{R}^n$ and $\delta > 0$ such that $S_{\mathcal{C}} \cap \{\ell < 0\} \cap B(x_0, \delta) = \emptyset$ and $\dim(\overline{S_{\mathcal{C}}} \cap \{\ell = 0\} \cap B(x_0, \delta)) = n - 1$.

$\text{TOTAL}_{\mathbb{R}}$ (*Totalness*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is total.

$\text{INJ}_{\mathbb{R}}$ (*Injectiveness*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is injective.

$\text{SURJ}_{\mathbb{R}}$ (*Surjectiveness*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is surjective.

$\text{IMAGEZDENSE}_{\mathbb{R}}$ (*Image Zariski Dense*). Given a circuit $\mathcal{C}$, decide whether the image of $f_{\mathcal{C}}$ is Zariski dense.

$\text{IMAGEEDENSE}_{\mathbb{R}}$ (*Image Euclidean Dense*). Given a circuit $\mathcal{C}$, decide whether the image of $f_{\mathcal{C}}$ is Euclidean dense.

$\text{DOMAINZDENSE}_{\mathbb{R}}$ (*Domain Zariski Dense*). Given a circuit $\mathcal{C}$, decide whether the domain of $f_{\mathcal{C}}$ is Zariski dense.

$\text{DOMAINEDENSE}_{\mathbb{R}}$ (*Domain Euclidean Dense*). Given a circuit $\mathcal{C}$, decide whether the domain of $f_{\mathcal{C}}$ is Euclidean dense.

$\text{CONT}_{\mathbb{R}}$ (*Continuity*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is continuous.

$\text{CONT}_{\mathbb{R}}^{\text{DF}}$ (*Continuity for Division-Free Circuits*). Given a division-free circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is continuous.

$\text{CONTPOINT}_{\mathbb{R}}^{\text{DF}}$ (*Continuity at a Point for Division-Free Circuits*). Given a division-free circuit $\mathcal{C}$ with n input gates and $x \in \mathbb{R}^n$, decide whether $f_{\mathcal{C}}$ is continuous at x.

$\text{LIPSCHITZ}_{\mathbb{R}}(k)$ (*Lipschitz-k*). Given a circuit $\mathcal{C}$, and $k > 0$, decide whether $f_{\mathcal{C}}$ is Lipschitz-k, i.e., whether, for all $x, y \in \mathbb{R}^n$, $\|f(x) - f(y)\| \le k\|x - y\|$.

$\text{LIPSCHITZ}_{\mathbb{R}}$ (*Lipschitz*). Given a circuit $\mathcal{C}$, decide whether $f_{\mathcal{C}}$ is Lipschitz, i.e., whether there exists $k > 0$ such that $f_{\mathcal{C}}$ is Lipschitz-k.

Table 1 shows the main previously known results (we emphasize on completeness) for the problems in the list above.

Table 2 does the same for the results shown in this paper.

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Table 2 Summary results of this paper

Problem	Complete in	Lower bound	Upper bound	Discrete version complete in
$\text{SOCS}_{\mathbb{R}}(k)$	H^k			PR
$\text{ZDENSE}_{\mathbb{R}}$	$\exists^*$			NPR
$\text{DOMAINZDENSE}_{\mathbb{R}}$	$\exists^*$			NPR
$\text{EDENSE}_{\mathbb{R}}$	$\forall^*$			coNPR
$\text{DOMAINEDENSE}_{\mathbb{R}}$	$\forall^*$			coNPR
$\text{IMAGEZDENSE}_{\mathbb{R}}$	$\exists$			NPR
$\text{TOTAL}_{\mathbb{R}}$	$\forall$			coNPR
$\text{INJ}_{\mathbb{R}}$	$\forall$			coNPR
$\text{LIPSCHITZ}_{\mathbb{R}}(k)$	$\forall$			coNPR
$\text{ZADH}_{\mathbb{R}}$		$\exists$	?	
$\text{EADH}_{\mathbb{R}}$	$\mathsf{H}\exists$			NPR
$\text{UNBOUNDED}_{\mathbb{R}}$	$\mathsf{H}\exists$			NPR
$\text{LOCDIM}_{\mathbb{R}}$	$\mathsf{H}\exists$			NPR
$\text{ISOLATED}_{\mathbb{R}}$	$\mathsf{H}\forall$			coNPR
$\text{CONT}_{\mathbb{R}}$		$\forall$	$\mathsf{H}^3\forall$	coNPR
$\text{CONT}_{\mathbb{R}}^{\text{DF}}$		$\forall$	$\mathsf{H}^2\forall$	coNPR
$\text{CONTPOINT}_{\mathbb{R}}^{\text{DF}}$	$\mathsf{H}\forall$			coNPR
$\text{LIPSCHITZ}_{\mathbb{R}}$		$\forall$	$\mathsf{H}\forall$	coNPR
$\text{LOCSUPP}_{\mathbb{R}}$	$\exists^*\mathsf{H}$			$\mathsf{BP}^0(\exists^*\mathsf{H})$
$\text{EXISTISO}_{\mathbb{R}}$		$\mathsf{H}\forall$	$\exists\forall$	
$\text{BASICCLOSED}_{\mathbb{R}}$	$\mathsf{H}\forall$			coNPR
$\text{BASICCOMPACT}_{\mathbb{R}}$	$\mathsf{H}\forall$			coNPR
$\text{LERD}_{\mathbb{R}}$	$\forall^*\exists$			$\mathsf{BP}^0(\forall^*\exists)$
$\text{IMAGEEDENSE}_{\mathbb{R}}$	$\forall^*\exists$			$\mathsf{BP}^0(\forall^*\exists)$
$\text{ERD}_{\mathbb{R}}$		$\forall^*\exists$	$\forall\exists$	
$\text{SURJ}_{\mathbb{R}}$	$\forall\exists$			$\mathsf{BP}^0(\forall\exists)$

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