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Mean-variance hedging for interest rate models with stochastic volatility

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Abstract. The mean-variance hedging approach for pricing and hedging claims in incomplete markets was originally introduced for risky assets. The aim of this paper is to apply this approach to interest rate models in the presence of stochastic volatility, seen as a consequence of incomplete information. We fix a finite number of bonds such that the volatility matrix is invertible and provide an explicit formula for the density of the variance-optimal measure which is independent of the chosen times of maturity. Finally, we compute the mean-variance hedging strategy for a caplet and compare it with the optimal stategy according to the local risk minimizing approach.

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1. Introduction

The mean-variance hedging approach for pricing and hedging claims in incomplete markets was originally introduced for risky assets by several authors. Schweizer (2001) presents a general overview of the main results of mean-variance hedging theory and a complete bibliography.

A typical example of market incompleteness is given by stochastic volatility models. For risky assets, the mean-variance hedging criterion has been

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analyzed in models where the volatility follows a diffusion by Laurent and Pham (1999) and where the volatility jumps by Biagini, Guasoni and Pratelli (2000).

The aim of this paper is to apply the mean-variance hedging approach to interest rate models in the presence of stochastic volatility. Several stochastic volatility models for bonds have been proposed in the literature (Longstaff and Schwartz (1992), Chiarella and Kwon (2000)). Here a stochastic volatility model is seen as a model with *incomplete information* as in the approach introduced for risky assets by Föllmer and Schweizer (1991). In a Heath–Jarrow–Morton framework, we suppose that the forward rate volatility is affected by an additional source of randomness and is measurable with respect to a filtration larger than that available to the agent. In this setting the market is incomplete in spite of the fact that in principle an infinite number of bonds is available for trade. Since perfect replication is not possible, we compute the density of the variance-optimal measure in order to find the mean-variance optimal strategy for a given European option. We remark that we consider only self-financing portfolios consisting of a finite number of bonds as in the approach of Musiela and Rutkowski (1997).

2. The model

We introduce our basic model here. Our set of states is given by the product probability space $(\Omega \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$, where $(\Omega, \mathcal{F}^W, \mathcal{F}_t^W, P^W)$ and $(E, \mathcal{E}, \mathcal{E}_t, P^E)$ are two complete filtered probability spaces. In particular, all filtrations are assumed to satisfy the so-called *"usual hypothesis"*. We assume that W_t is a standard n-dimensional Brownian motion on $\Omega = \mathcal{C}([0, T], \mathbb{R})$, P^W is the Wiener measure and $\mathcal{F}_t^W$ is the P^W-augmentation of the filtration generated by W_t.

The space E represents an additional source of randomness which affects the market. In the terminology of Föllmer and Schweizer (1991), the market is now incomplete as a result of *incomplete information*: if the evolution of η had been known the market would be complete.

We suppose that there exists on E a square-integrable (eventually d-dimensional) martingale M_t endowed with the predictable representation property, i.e., for every square-integrable martingale N_t there exists a predictable process H_t such that $N_t = N_0 + \int_0^t H_s dM_s$.

We analyze the mean-variance hedging criterion in the case of interest rates models. The assets to be considered on the market are zero coupon bonds with different maturities. Following the notation of Björk (1998), we denote by $p(t, T)$ the price at time t of a bond maturing at time T, where, for every fixed T, the process $p(t, T)$ is an optional stochastic process such that $p(t, t) = 1$ for all t.

We assume that there exists a frictionless market for T-bonds for every $T > 0$ and that, for every fixed t, $p(t, T)$ is almost surely differentiable in the T-variable.

The *forward rate* $f(t, T)$ is defined as $f(t, T) = -\frac{\partial \log p(t, T)}{\partial T}$ and the *short rate* as $r_t = f(t, t)$. The *money market account* is given by the process $B_t = \exp\Big(\int_0^t r_s ds\Big)$.

Using to the Heath–Jarrow–Morton approach (see Heath, Jarrow and Morton (1992) for further details), we describe the forward rate dynamics. In this setting, $f(t, T)$ is represented by a process on the product probability space $(\Omega \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$ such that

$$df(t, T, \eta) = \alpha(t, T, \eta)dt + \sigma(t, T, \eta)dW_t \tag{1}$$

with initial condition $f(0, T, \eta) = f^*(0, T)$. We make the following assumptions.

(i) Equation (1) admits P^E-a.e. a unique strong solution with respect to the filtration $\mathcal{F}_t^W$. For example, it is sufficient that μ and σ are P^E-a.e. bounded.

(ii) The information available at time t is given by the filtration $\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{E}_t$.

(iii) There exists a predictable $\mathbb{R}^n$-valued process h_t such that the integral $\int h_s dW_s$ is well defined and

$$\alpha(t, T, \eta) = \sigma(t, T, \eta) \int_t^T \sigma(t, s, \eta)ds - \sigma(t, T, \eta)h_t(\eta) \tag{HJM}$$

for every $T \geq 0$. This condition is usually called the *Heath–Jarrow–Morton condition* on the drift. In general, it guarantees the existence of an equivalent martingale measure for $\frac{p(t, T)}{B_t}$ as long as $\mathcal{E}\Big(\int hdW\Big)$ is a uniformly integrable martingale. In the complete market case, it is even sufficient to characterize the unique martingale measure, but in our setting of incomplete information there exists an infinite number of them.

By Proposition 15.5 of Björk (1998), we obtain the bond price dynamics

$$\frac{dp(t, T)}{p(t, T)} = (r(t, \eta) + \frac{1}{2}\|S(t, T, \eta)\|^2 + A(t, T, \eta))dt + S(t, T, \eta)dW_t,$$

where

$$S_i(t,T,\eta) = -\int_t^T \sigma_i(t,s,\eta)ds,$$
$$A(t,T,\eta) = -\int_t^T \alpha(t,s,\eta)ds.$$

Since in principle an infinite number of bonds is available for trade, one can assume that the market is complete in spite of lack of information. This is not true since the future evolution of η cannot be predicted, not even through the observation of the entire term structure. For a rigorous proof of this fact, see the doctoral dissertation of Biagini (2001).

In this setting, the market would be complete if one had access to the filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_t^W \otimes \mathcal{E}$, which contains at any time all the information about the past and future evolution of η. In terms of conditions on the volatility matrix, by Proposition 4.3 by Björk (1997) we see that this fact reduces to the assumption of the existence of n maturity times $T_1, \dots, T_n$, where $n = \dim W_t$, such that the matrix of elements $[A_t]_{ji} = \int_{T_0}^{T_j} \sigma_i(t,s)ds$ has rank equal to n for every $t \in [0, T_0]$ and for P^E-almost every $\eta \in E$. We assume this as a standing hypothesis in the sequel. For instance, sufficient conditions implying the existence of such maturities are given by Proposition 5.5 and Theorem 5.6 of Björk, Kabanov and Runggaldier (1997).

3. The variance-optimal measure for interest rates

In this framework, we study the problem of an agent wishing to hedge a certain European option H expiring at time T_0 by using a self-financing portfolio composed of a finite number of bonds of convenient maturities and possibly of the market account B_t. In the sequel, for the sake of simplicity we will write $\frac{dQ}{dP}$ instead of $\frac{dQ}{dP}\Big|_{\mathcal{F}_{T_0}}$.

Definition 3.1. *A $\mathbb{R}^{n+1}$-valued predictable process (θ_0, θ), with $\theta = (\theta_1, \dots, \theta_n)$, is a self-financing strategy if it is integrable with respect to $(p(t,T_j))_{j=0,\dots,n}$ and the wealth process $V_t = \sum_{i=0}^{n} \theta_t^j p(t,T_j)$ satisfies $V_t = V_0 + \sum_{i=0}^{n} \int_0^t \theta_u^j dp(u,T_j)$. Since, under the numéraire $p(t,T_0)$, the discounted value of the portfolio is given by $\frac{V_t}{p(t,T_0)} = \frac{V_0}{p(0,T_0)} + \int_0^t \theta_u dX_u$, we assume that θ belongs to Θ.*

Since perfect replication is not possible, we look for a solution to the minimization problem

$$\min E\left[\left(H - V_{T_0}\right)^2\right].$$

Usually the money market account $B_t = \exp\left(\int_0^t r(s, \eta)ds\right)$ is used as discounting factor. Now the spot rate is stochastic so the choice of B_t as numéraire is unfortunate. In Sekine (1999), the impact of a stochastic interest rate is analyzed in a Markovian framework for the futures case. If the chosen discounting factor is a stochastic process, by Gouriéroux, Laurent and Pham (1998) we see that the minimization problem

$$\min E\left[(H - V_{T_0})^2\right]$$

is equivalent to

$$\min E^B\left[\left(\frac{H}{B_{T_0}} - \frac{V_{T_0}}{B_{T_0}}\right)^2\right],$$

where E^B is the expectation under the equivalent probability P^B with density

$$\frac{dP^B}{dP} = \frac{B_{T_0}^2}{E\left[B_{T_0}^2\right]}.$$

In order to avoid the computation of the new bond dynamics under P^B, we can choose as numéraire the bond $p(t, T_0)$ expiring at the same maturity time as H. We immediately have

$$\frac{dP^{T_0}}{dP} = \frac{p(T_0, T_0)^2}{E\left[p(T_0, T_0)^2\right]} = 1$$

or in other words $P^{T_0} \equiv P$.

We choose maturity times $T_1 < \cdots < T_n$ such that the matrix $\int_{T_0}^{T_j} \sigma_i(t, s)ds$ is P^E-a.e. invertible for every t and set $X_t^j = \frac{p(t, T_j)}{p(t, T_0)}$, $j = 1, \ldots, n$. We define

$$\Theta = \left\{\theta \in L(X) : \int \theta dX \in \mathcal{S}^2\right\},$$

where $\mathcal{S}^2$ is the space of square-integrable semimartingales and $L(X)$ is the set of integrable processes with respect to X_t. More precisely, we are not interested simply in a self-financing portfolio whose final value has minimal quadratic distance from H, but we are seeking a solution to the minimization problem

$$\min_{\substack{V_0 \in \mathbb{R} \\ \theta \in \Theta}} E\left[\left(H - V_0 - G_{T_0}(\theta)\right)^2\right], \tag{2}$$

where $G_t(\theta) = \int_0^t \theta_s dX_s$. The space of integrals $\mathcal{G} = G_{T_0}(\Theta)$ represents the self-financing strategies with initial value $V_0 = 0$. Hence, following Schweizer (2001), we impose on the underlying financial market the so-called *no-approximate profit condition*

$$1 \notin \bar{\mathcal{G}}$$

which represents a type of no-arbitrage condition. Here $\bar{\mathcal{G}}$ is the closure of $\mathcal{G}$ in L^2.

Problem (2) admits a unique solution (V_0, θ) for all $H \in L^2$ under the hypothesis that $G_{T_0}(\Theta)$ is closed (see Gouriéroux, Laurent and Pham (1998), and Rheinländer and Schweizer (1997), for the proof). In this case, θ is called the *mean-variance optimal* strategy and V_0 the *approximation price*; they can be computed in terms of the so-called *variance-optimal measure* (Schweizer (1996), Rheinländer and Schweizer (1997)).

We denote by $\mathcal{M}_s^2(T_1, \ldots, T_n)$ and $\mathcal{M}_e^2(T_1, \ldots, T_n)$ respectively the set of *square-integrable signed martingale measures* and the set of *square integrable equivalent martingale measures* for $X_t^j = \dfrac{p(t, T_j)}{p(t, T_0)}$, $j = 1, \ldots, n$.

Definition 3.2. *The variance-optimal measure $\tilde{P}^0$ is the element of $\mathcal{M}_s^2(T_1, \ldots, T_n)$ of minimal norm, where, for every $Q \in \mathcal{M}_s^2(T_1, \ldots, T_n)$,*

$$\left\| \frac{dQ}{dP} \right\|^2 = E\left[\left(\frac{dQ}{dP}\right)^2\right].$$

If $\mathcal{M}_s^2(T_1, \ldots, T_n)$ is nonempty, then $\tilde{P}$ always exists as it is the minimizer of the norm on a convex set, and is actually an equivalent martingale measure if X_t has continuous paths (Delbaen and Schachermayer (1996)). Apparently, Definition 3.2 of $\widetilde{P}^0$ depends on the chosen maturities $T_1, \ldots, T_n$. Theorem 3.3 provides an explicit expression for the density of the variance-optimal martingale measure and shows that it is actually invariant under a change of the set of maturities if the matrix $\int_{T_0}^{T_j} \sigma_i(t, s) ds$ is P^E-a.e. invertible for every t. Its proof follows from Lemma 7.3 contained in the Appendix.

Theorem 3.3. *Let H, K be two predictable processes such that the exponential martingales $\mathcal{E}\left(\int_0^{\cdot}(h_s(\eta)+S(s,T_0,\eta))dW_s+\int_0^{\cdot}K_s dM_s\right)$ and $\mathcal{E}\left(\int_0^{\cdot}(h_s(\eta)+S(s,T_0,\eta)+H_s)d\widehat{W}_s\right)$ are square-integrable. Then*

$$\frac{d\widetilde{P}^0}{dP}=\mathcal{E}\left(\int_0^{\cdot}(h_s(\eta)+S(s,T_0,\eta))dW_s+\int_0^{\cdot}K_s dM_s\right)_{T_0} \tag{3}$$

or, equivalently,

$$\frac{d\widetilde{P}^0}{dP}=\frac{\mathcal{E}\left(-\int_0^{\cdot}\beta_s dX_s\right)_{T_0}}{E\left[\mathcal{E}\left(-\int_0^{\cdot}\beta_s dX_s\right)_{T_0}\right]},$$

where H, K are solutions of Equation (9) of Lemma 7.3 and $\beta_s^j = -\frac{p(s,T_0)}{p(s,T_j)}\sum_i (h_s^i(\eta)+S^i(s,T_0,\eta))+H_s^i)[A_s^{-1}]_{ij}$.

In particular, if $\sigma(t,T,\eta,\omega)=\sigma(t,T,\eta)$, by Biagini, Guasoni and Pratelli (2000) we find that the density of $\widetilde{P}$ has the form

$$\frac{d\widetilde{P}}{dP}=\mathcal{E}\left(\int_0^{\cdot}\lambda_s dW_s\right)_{T_0}\frac{\exp\left(-\int_0^{T_0}\|\lambda_s\|^2 ds\right)}{E\left[\exp\left(-\int_0^{T_0}\|\lambda_s\|^2 ds\right)\right]}, \tag{4}$$

where $\lambda_t = h_t(\eta)+S(t,T_0,\eta)$.

4. Examples

Here we provide some examples in which the stochastic volatility is a consequence of incomplete information and show how to construct the additional probability space $(E,\mathcal{E},\mathcal{E},P^E)$ and the martingale M_t on E with the representation property.

The Heath–Jarrow–Morton condition on the drift allows us to model only the forward rate volatility $\sigma(t,T,\eta)$ and $h_t(\eta)$. Without loss of generality, we assume that $h_t(\eta)$ is affected by the same behavior as $\sigma(t,T)$ and we model only the volatility $\sigma(t,T)$.

Example 4.1. First we consider the case when $\dim W_t = 1$ and

$$\sigma(t,T)=\sigma_0 I_{\{t<\eta,t\le T\}}+\sigma_1 I_{\{t\ge\eta,t\le T\}},$$

where $\sigma_0,\sigma_1\in\mathbb{R}^+$ and η is a totally inaccessible stopping time. Here we set $E=\mathbb{R}^+$ and $\mathcal{E}_t=\mathcal{B}([0,t])\vee(t,+\infty]$; a fundamental martingale is given by $M_t = I_{\{t \geq \eta\}} - a_t$, where a_t is the compensator of the process $I_{\{t \geq \eta\}}$ associated to η.

More generically η can be assumed to be a Markov process η_t in continuous time with a finite set of states I. This example models the situation in which the volatility has multiple jumps occurring at independent random times.

Example 4.2. If in Example 4.1 the volatility assumes values after the jump according to a general probability distribution, then there is no finite set of martingales with the predictable representation property. Following Jacod and Shiryaev (1987), we can replace M_t by the compensated integer-valued random measure $\mu - \nu$ associated to η_t, which has the predictable representation property with respect to the smallest filtration under which μ is optional.

Example 4.3. Lastly η_t can be given by a diffusion process

$$df(t,T) = \alpha(t,T,\eta_t)dt + \sigma(t,T,\eta_t)dW_t^1$$
$$d\eta_t = F(t,T,\eta_t)dt + G(t,T,\eta_t)dW_t^2,$$

where W_t^1 can be correlated with W_t^2. This example has been studied in the case of risky assets by using dynamic programming techniques in Laurent and Pham (1999). Clearly, $M_t = W_t^2$ here.

5. Mean-variance hedging for a call option

As in Biagini and Guasoni (2002), we now assume that $\sigma(t,T,\omega,\eta) = \sigma(t,T,\eta)$. We remark that in this particular case the variance-optimal density is given by (4). We now compute the mean-variance optimal strategy for a call option expiring at time T_0 on a T_1-bond ($T_0 < T_1$) by exploiting the explicit characterization for the density of the variance-optimal measure provided by Theorem 3.3. Let $T_1 < T_2 < \cdots < T_n$ be maturities such that the matrix $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is P^E-a.e. invertible for every t. If there exists at least an equivalent martingale measure for $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$, $j = 1,\ldots,n$, and the space of integrands $G_{T_0}(\Theta)$ is closed, then the variance-optimal strategy for the call option $H = (P(T_0,T_1) - K)^+$ is given by the following

Proposition 5.1. *If $p(T_0,T_1)$ is square-integrable with respect to $\tilde{P}$, then the components θ^j of the variance-optimal strategy are given in the following feedback form:*

for $j = 1$

$$\theta_t^1 = \xi_t^1 + \beta_t^1 \left(\xi_t^1 \frac{p(t,T_1)}{p(t,T_0)} - K\xi_t^0 - \tilde{E}^0 \left[(p(T_0,T_1) - K)^+ \right] - \int_0^t \theta_s dX_s \right);$$

for $j > 1$

$$\theta_t^j = \beta_t^j \left(\xi_t^1 \frac{p(t,T_1)}{p(t,T_0)} - K\xi_t^0 - \tilde{E}^0 \left[(p(T_0,T_1) - K)^+ \right] - \int_0^t \theta_s dX_s \right),$$

where $\xi_t^0 = -K\tilde{E}^0 \left[1_A | \mathcal{F}_{t-} \right]$, $\tilde{\xi}_t^1 = \tilde{E}^1 \left[1_A | \mathcal{F}_{t-} \right]$ *with* $A = \{p(T_0,T_1) \geq K\}$ *and* $\beta_t^j = -\dfrac{p(t,T_0)}{p(t,T_j)} \sum_i (h_t^i(\eta) + S^i(t,T_0,\eta))[A_t^{-1}]_{ij}$.

Proof. We need to compute all terms in the implicit characterization of the mean-variance optimal strategy given in Theorem 6 of Rheinländer and Schweizer (1997).

By Schweizer (1996), it follows that $c = \tilde{E}^0 \left[(p(T_0,T_1) - K)^+ \right]$. By Theorem 3.3 we obtain

$$\frac{\tilde{\zeta}_t^j}{\tilde{Z}_t} = -\beta_t^j = \frac{p(t,T_0)}{p(t,T_j)} \sum_i (h_t^i + S^i(t,T_0,\eta))[A_t^{-1}]_{ij}.$$

In order to compute ξ_t^j, note that, with respect to the “enlarged” filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{E}_{T_0}$ which contains all information about η, H can be perfectly replicated by the self-financing portfolio $\tilde{E}^0 \left[H \,|\tilde{\mathcal{F}}_t \right]$. If $\tilde{\xi}_t^j$ denotes the portfolio component with respect to $p(t,T_j)$, then by the standard theory of complete markets we obtain $\tilde{\xi}_t^0 = -K\tilde{E}^0 \left[1_A \,|\tilde{\mathcal{F}}_t \right]$, $\tilde{\xi}_t^1 = \tilde{E}^1 \left[1_A \,|\tilde{\mathcal{F}}_t \right]$ and $\tilde{\xi}_t^j = 0$ for every $j \neq 0, 1$. Then ξ_t^i is given by the predictable projection of $\tilde{\xi}_t^i$ with respect to $\mathcal{F}_t$, i.e., $\xi_t^0 = -K\tilde{E}^0 \left[1_A | \mathcal{F}_{t-} \right]$, $\xi_t^1 = \tilde{E}^1 \left[1_A | \mathcal{F}_{t-} \right]$ and $\xi_t^i = 0$ for every $i \neq 0, 1$. For further details, see Biagini (2001).

Furthermore, by applying these results and the change of numéraire technique introduced in Geman, El Karoui and Rochet (1995) to $\tilde{V}_t = \tilde{E}^0 \left[(p(T_0,T_1) - K)^+ \middle| \mathcal{F}_t \right]$, we obtain

$$\tilde{V}_{t-} = p(t,T_1)\tilde{E}^1 \left[1_A | \mathcal{F}_{t-} \right] - Kp(t,T_0)\tilde{E}^0 \left[1_A | \mathcal{F}_{t-} \right],$$

where $A = \{p(T_0,T_1) \geq K\}$. $\square$

We remark that the mean-variance optimal strategy depends on the number of bonds which is equal to (dim $W_t + 1$).

We apply these results in order to price and hedge the caplet $H = \delta\left(\frac{1-p(T_0,T_1)}{\delta p(T_0,T_1)} - R\right)^+$ in this framework of incomplete information by using the mean-variance hedging approach. We refer to Björk (1997) for all definitions and properties concerning caplets.

Since the caplet is settled in arrears, we consider H as a T_1-option and we choose $p(t, T_1)$ as discounting factor. The approximation price of H is equal to $\tilde{E}^1[H]$, where the expectation is calculated under the variance-optimal measure with $p(t, T_1)$ as numéraire. The caplet can be written as

$$H = \frac{R^*}{p(T_0,T_1)}\left(\frac{1}{R^*} - p(T_0,T_1)\right)^+,$$

where $R^* = 1 + \delta R$. The approximation price is given by

$$\tilde{E}^1[H] = R^*\tilde{E}^1\left[\frac{1}{p(T_0,T_1)}\left(\frac{1}{R^*} - p(T_0,T_1)\right)^+\right]. \tag{5}$$

Since

$$\left.\frac{d\widetilde{P}^1}{d\widetilde{P}^0}\right|_{\mathcal{F}_{T_0}} = \frac{p(T_0,T_1)}{p(T_0,T_0)} \cdot \frac{p(0,T_0)}{p(0,T_1)}$$

we can exploit in (5) the change of numéraire technique obtaining

$$\tilde{E}^1[H] = R^*\tilde{E}^0\left[\left(\frac{1}{R^*} - p(T_0,T_1)\right)^+\right].$$

Note that H has the same approximation price as R^* put options $K = \left(\frac{1}{R^*} - p(T_0,T_1)\right)^+$ on $p(t, T_1)$ expiring at time T_0.

Remark 5.2. Since H is actually $\mathcal{F}_{T_0}$-measurable, a natural question is whether the mean-variance optimal strategy up to time T_0 for the T_1-option H coincides with (R^* times) that for the T_0-put option K as in the complete market case (see Björk (1997)). The answer is negative as expected since $\Theta_{T_0} \subseteq \Theta_{T_1}$.

We fix $(n+1)$ bonds $p(t,T_1), p(t,T_2), \ldots, p(t,T_{n+1})$ such that the matrix $[B_t]_{ji} = \int_{T_1}^{T_j} \sigma_i(t,s)ds$ is invertible for every t P^E-almost everywhere and put $Y_t^j = \frac{p(t,T_j)}{p(t,T_1)}$, $j = 2, \ldots, n+1$. Note that, in order to compute the two mean-variance hedging strategies, we need to use the same assets for both. Consequently, we cannot choose $p(t, T_0)$ since it is not defined

after $t = T_0$. All computations are made with $p(t, T_1)$ as numéraire. We set $\tilde{V}_t = p(t, T_1)\tilde{E}^1\left[H|\mathcal{F}_t\right]$. Recall that $\tilde{V}_t = R^* p(t, T_1)\tilde{E}^1\left[\frac{K}{p(T_0, T_1)}\middle|\mathcal{F}_t\right]$.
By Proposition 5.1, we obtain:
for $j > 2$, the optimal components for H as T_1-option are given by

$$\theta_t^{j,1} = \beta_t^j\left(\tilde{V}_{t-} - \tilde{V}_0 - \int_0^t \theta_s^1 dY_s\right),$$

where $\beta_t^j = -\frac{p(t, T_1)}{p(t, T_j)}\sum_i (h_t^i + S(t, T_0, \eta))[B_t^{-1}]_{ij}$;
for $j > 2$, the optimal components for the T_0-option R^*K are given by

$$\theta_t^{j,0} = (\beta_t^j + \gamma_t^1)\left(\tilde{V}_{t-} - \tilde{V}_0 - \int_0^t \theta_s^0 dY_s\right),$$

where γ_t^1 is the solution of the equation

$$\frac{dP^{T_1}}{dP} = \frac{p(T_0, T_1)^2}{E\left[p(T_0, T_1)^2\right]} = \mathcal{E}\left(\int_0^\cdot \gamma_s^1 dW_s + \int_0^\cdot \gamma_s^2 dM_s\right)_{T_0}$$

since the use of $p(t, T_1)$ as discounting factor for a T_0-option compels us to work under the probability P^{T_1}.

We can easily conclude that the two strategies do not coincide up to time $t = T_0$ unless $\gamma_t^1 = 0$, which is not the case in general.

In order to compute an approximation strategy for the caplet, we can proceed as in the complete market case (see Björk (1997)). We find the variance-optimal portfolio for the T_0-put option $K = \left(\frac{1}{R^*} - p(T_0, T_1)\right)^+$ and invest the final value V_{T_0} in $p(t, T_1)$ from time $t = T_0$ to T_1. As shown in Remark 5.2, this strategy is not the optimal one for H. Nevertheless, this method remains of some interest since the strategy can be computed in terms of the "natural" assets $p(t, T_0)$ and $p(t, T_1)$ and the approximation price $\tilde{E}^0[K]$ for K coincides with the approximation price $\tilde{E}^1[H]$ for H.

6. A comparison with the local risk minimizing approach

An alternative approach to pricing and hedging a contingent claim in the incomplete market case is local risk minimization (for a complete treatment of the subject, see Schweizer (1999)). The main difference with respect to mean-variance hedging is that a local risk minimizing strategy perfectly replicates the value of a given option, but it is not self-financing. More precisely, suppose we want to hedge a T_0-option H by using a portfolio based on a finite number of bonds $p(t, T_0), p(t, T_1), \dots, p(t, T_n)$ such that the matrix $\int_{T_0}^{T_j} \sigma_i(t, s)ds$ is P^E-a.e. invertible for every t. As in the previous sections, we set $X_t^j = \frac{p(t, T_j)}{p(t, T_0)}$, $j = 1, \dots, n$. By exploiting the approach of Biagini and Pratelli (1999), we have the following

Definition 6.1. *An* L^2-strategy *is a triple* (θ, θ^0, V) *such that* $\theta \in \Theta = \left\{\theta \in L(X) : \int \theta dX \in \mathcal{S}^2\right\}$, θ^0 *is a real predictable process and the* value process V_t *is a square-integrable stochastic process whose left limit is equal to* $\frac{V_{t-}}{p(t, T_0)} = \theta_t \cdot X_t + \theta_t^0$ *for* $0 \le t \le T_0$. *The (cumulative)* cost process *is defined by* $C_t = \frac{V_t}{p(t, T_0)} - \sum_{j=0}^{n} \int_0^t \theta_s^j dX_s$, $0 \le t \le T_0$.

By Definition 6.1, we get that jumps in the portfolio coincide with the jumps in the cost process. In a self-financing portfolio, the cost is constant.

Definition 6.2. *Let* $H \in L^2(\mathcal{F}_{T_0}, P)$ *be a contingent claim. An* L^2*-strategy* (θ, θ^0, V) *with* $V_{T_0} = H \; P - a.s.$ *is called* pseudo-locally risk-minimizing *or* pseudo-optimal *for* H *if the cost process* C_t *is a* P*-martingale and is strongly orthogonal to the martingale part of* X.

We remark that the optimal strategy is invariant under a change of numéraire (for more details, see Biagini and Pratelli (1999)).

By Definition 6.2, it immediately follows that a contingent claim $H \in L^2(\mathcal{F}_{T_0}, P)$ admits a pseudo-optimal strategy if and only if H can be written as

$$H = H_0 + \int_0^{T_0} \xi_u dX_u + L_{T_0}, \tag{6}$$

where $H_0 \in L^2(\mathcal{F}_{T_0}, P)$, $\xi \in \Theta$ and L is a square-integrable martingale strongly P-orthogonal to the martingale part of X. Equation (7) is usually referred to in the literature as the *Föllmer–Schweizer decomposition* of H. It is connected to a suitably chosen martingale measure, the so-called *minimal martingale measure*.

Definition 6.3. $\widehat{P}^0 \in \mathcal{M}_e^2(T_1, \dots, T_n)$ *is the* minimal measure *(with respect to* $p(t, T_0)$ *as numéraire) if any locally square-integrable local martingale which is orthogonal to the martingale part of* X *under* P *remains a local martingale under* $\widehat{P}^0$.

By Definition 6.3 it follows immediately that the pseudo-optimal portfolio $\widehat{V}(\phi)$ is a local $\widehat{P}^0$-martingale and we obtain

$$\widehat{V}_t(\phi) = p(t, T_0)\widehat{E}^0 \left[H | \mathcal{F}_t\right].$$

The optimal portfolio is a true martingale if $\widehat{Z}_t = \widehat{E}^0 \left[\frac{d\widehat{P}^0}{dP} \middle| \mathcal{F}_t \right]$ is itself a square-integrable martingale. By exploiting the results of Theorem 3.3, we find that

$$\frac{d\widehat{P}^0}{dP} = \mathcal{E}\left(\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta)) dW_s \right)_{T_0}$$

defines the density of the minimal measure as long as the Doleans Exponential $\mathcal{E}\left(\int_0^{\cdot}(h_s(\eta) + S(s, T_0, \eta))dW_s\right)$ is a uniformly integrable martingale.

The pseudo-optimal strategy for a call option in the presence of incomplete information has been computed in Biagini and Pratelli (1999) for the risky assets case. Their results can be easily extended to the interest rate case. By Theorem 5.1 of Biagini and Pratelli (1999), we see that the pseudo-optimal portfolio is given by

$$\widehat{V}_t(\phi) = p(t, T_0)\widehat{E}^0 \left[H | \mathcal{F}_t \right] = p(t, T_1)\widehat{E}^1 \left[I_A | \mathcal{F}_t \right] - K p(t, T_0)\widehat{E}^0 \left[I_A | \mathcal{F}_t \right]$$

and the optimal strategy components are $\theta_t^0 = -K\widehat{E}^0 \left[1_A | \mathcal{F}_{t-} \right]$, $\theta_t^1 = \widehat{E}^1 \left[1_A | \mathcal{F}_{t-} \right]$ and $\theta_t^j = 0$ for all $j = 2, \ldots, n$. Here $\widehat{E}^1$ denotes the expectation taken under the minimal measure $\widehat{P}^1$ with respect to the numéraire $p(t, T_1)$. Note that, in the local risk minimization case, the pseudo-optimal strategy depends only on two assets in spite of the dimension of the driving Brownian motion. In contrast, the mean-variance optimal strategy is based on $(n + 1)$ bonds, where $n = \dim W_t$.

We apply these results in order to compute the local risk minimizing strategy for a caplet. In the notation of the previous section, the pseudo-optimal portfolio for the caplet $H = \dfrac{R^*}{p(T_0, T_1)} \left(\dfrac{1}{R^*} - p(T_0, T_1) \right)^+$ is given by $\widehat{V}_t = p(t, T_1)\widehat{E}^1 \left[H | \mathcal{F}_t \right]$ which, for $t \leq T_0$, coincides with the optimal portfolio for the T_0-put option $K = \left(\dfrac{1}{R^*} - p(T_0, T_1) \right)^+$ since, by Theorem 3.2 of Biagini and Pratelli (1999),

$$\frac{d\widehat{P}^1}{d\widehat{P}^0} = p(T_0, T_1) \frac{p(0, T_0)}{p(0, T_1)}.$$

For $t > T_0$, $\dfrac{\widehat{V}_t}{p(t, T_1)} = \widehat{E} \left[H | \mathcal{F}_t \right] = H$ since H is $\mathcal{F}_{T_0}$-measurable. In other words, the pseudo-optimal components are constant after $t = T_0$. Consequently, in the local risk-minimization case the strategies for the T_1-option H and for the T_0-option K coincide up to time T_0 and we can act exactly as in the complete market case. The key is that in this approach we perfectly replicate the option value in spite of approximating it as in the mean-variance hedging criterion.

7. Appendix

Here we simply sketch how to find an explicit characterization of $\widetilde{P}^0$ in order to solve the mean-variance hedging problem in the interest-rate case. In Biagini (2001), the details of all computations are provided.

In order to obtain an explicit formula for the variance-optimal measure, we first characterize the martingale measures for $\left(\frac{p(t,T)}{p(t,T_0)}\right)_{t\in[0,T_0]}$ for every $T > 0$.

Lemma 7.1. *Let Z_t be a local martingale with $Z_0 = 1$. The following conditions are equivalent.*

(a) $Z_t \frac{p(t,T)}{p(t,T_0)}$ *is a local martingale for every $T > 0$.*

(b) $Z_t = \mathcal{E}\left(\int_0^\cdot (h_s + S(s,T_0,\eta))dW_s\right)_t (1 + \int_0^t k_s dM_s)$ *for some predictable process k_s such that the integral $\int_0^t k_s dM_s$ is a local martingale.*

Proof. The pair (W_t, M_t) has the representation property on $(\Omega \times E, \mathcal{F} \otimes \mathcal{E}, P^W \otimes P^E)$; hence there exist predictable processes λ_t and k_t such that

$$Z_t = 1 + \int_0^t \lambda_s dW_s + \int_0^t k_s dM_s$$

(see, for example, Biagini, Guasoni and Pratelli (2000)). By applying Itô's formula, we see that the process $Z_t \frac{p(t,T)}{p(t,T_0)}$ is a local martingale if and only if the process λ_t solves the following equation for every $T > 0$:

$$Z_{t-} \sum_i (h_t^i(\eta) + S_i(t,T_0,\eta)) \int_{T_0}^T \sigma_i(t,s,\eta)ds - \lambda_t \int_{T_0}^T \sigma_i(t,s,\eta)ds = 0. \tag{7}$$

Since we assume that there exist $T_1, \ldots, T_n$ such that the matrix $\int_{T_0}^{T_j} \sigma_i(t,s)ds$ is P^E-a.e. invertible for every t, it follows immediately that

$$\lambda_t^i = (h_t^i(\eta) + S_i(t,T_0,\eta))$$

for $i = 1, \ldots, n$. □

By Equation (7) it follows immediately that the set $\mathcal{M}_s^2(T_1, \ldots, T_n)$ of martingale measures for $\frac{p(t,T_j)}{p(t,T_0)}$, $j = 1, \ldots, n$, coincides with the set $\mathcal{M}_s^2(T)$ of martingale measures for $\frac{p(t,T)}{p(t,T_0)}$, $T \geq 0$. We bring them together it in the following

Proposition 7.2. (i) *If* $Q \in \mathcal{M}_s^2(T_1, \dots, T_n)$*, then*

$$\frac{dQ}{dP} = \mathcal{E}\left(\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s\right)_{T_0} (1 + \int_0^{T_0} k_s dM_s)$$

for some predictable process k_t *such that the above expression is square-integrable.*

(ii) *If* $Q \in \mathcal{M}_e^2(T_1, \dots, T_n)$*, then*

$$\frac{dQ}{dP} = \mathcal{E}\left(\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s\right)_{T_0} \mathcal{E}\left(\int_0^{\cdot} k_s dM_s\right)_{T_0}$$

for some predictable process k_t *such that* $k_t \cdot \Delta M_t > -1$ *and* $\mathcal{E}\left(\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s + \int_0^{\cdot} k_s dM_s\right)_t$ *is a square-integrable martingale.*

We define the two process $\widehat{W}_t$ and W_t^* as follows:

$$\widehat{W}_t = W_t - \int_0^t (h_s(\eta) + S(s, T_0, \eta))ds,$$
$$W_t^* = W_t - 2\int_0^t (h_s(\eta) + S(s, T_0, \eta))ds.$$

Lemma 7.3 is quite technical, but together with Proposition 7.2 it gives us an explicit expression for the density of the variance-optimal measure. Its proof is formally analogous to that of Lemma 1.15 of Biagini, Guasoni and Pratelli (2000).

Lemma 7.3. *Let* H, K *be two predictable stochastic processes whose stochastic integrals* $\int_0^t H_s dW_s^*$ *and* $\int_0^t K_s dM_s$ *are defined. The following conditions are equivalent:*

$$\exp\left(\int_0^T \|(h_s(\eta) + S(s, T_0, \eta))\|^2 ds\right) = c\,\frac{\mathcal{E}\left(\int_0^{\cdot} H_s dW_s^*\right)_T}{\mathcal{E}\left(\int_0^{\cdot} K_s dM_s\right)_T} \qquad (8)$$

$$\mathcal{E}\left(\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta))dW_s + \int_0^{\cdot} K_s dM_s\right)_T = $$
$$= c\,\mathcal{E}\left(\int_0^{\cdot} (h_s(\eta) + S(s, T_0, \eta) + H_s)d\widehat{W}_s\right)_T, \qquad (9)$$

where c *is the same constant in both equations.*

We obtain the proof of Theorem 3.3 as follows. Equation (9) completely characterizes the variance-optimal measure $\tilde{P}^0$ since, by Schweizer (1996), it is the unique martingale measure which can be written in the form $\mathcal{E}\left(-\int \beta dX\right)$, where $X_t^j = \dfrac{p(t,T_j)}{p(t,T_0)}$. By using the equivalence stated in Lemma 7.3 we solve Equation (8) instead: since a solution (H, K) always exists because of the representation property of (W_t, M_t) on $(\Omega \times E, \mathcal{F} \otimes \mathcal{E}, P^W \otimes P^E)$, we obtain Equation (3) for the density of the variance-optimal measure.

References

Biagini, F. (2001): Quadratic hedging approaches for interest rate models with stochastic volatility. Tesi di Perfezionamento. Scuola Normale Superiore di Pisa, Pisa

Biagini, F., Guasoni, P. (2002): Mean-variance hedging with random volatility jumps. *Stochastic Analysis and Applications* **20**. To appear

Biagini, F., Guasoni, P., Pratelli, M. (2000): Mean-variance hedging for stochastic volatility models. *Mathematical Finance* **10**, 109–123

Biagini, F., Pratelli, M. (1999): Local risk minimization and numéraire. *Journal of Applied Probability* **36**, 1126–1139

Björk, T. (1997): Interest rate theory. In: Runggaldier, W.J. (ed.): Financial mathematics (Lecture Notes in Mathematics, **1656**), Springer, Berlin, pp. 53–122

Björk, T. (1998): Arbitrage theory in continuous time. Oxford University Press, Oxford

Björk, T., Kabanov, Y., Runggaldier, W. (1997): Bond market structure in the presence of marked point processes. *Mathematical Finance* **7**, 211–239

Chiarella, C., Kwon, O.K. (2000): A class of Heath-Jarrow-Morton term structure models with stochastic volatility. Research Paper **34**. Quantitative Finance Research Group, University of Technology Sydney, Sydney

Föllmer, H., Schweizer, M. (1991): Hedging of contingent claims under incomplete information. In: Elliot, R.J., Davis, M.H.A. (eds.): Applied stochastic analysis. Gordon and Breach, New York, pp. 389–414

Geman, H., El Karoui, N., Rochet, J.C. (1995): Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability* **32**, 443–458

Gouriéroux, L., Laurent, J.P., Pham, H. (1998): Mean-variance hedging and numéraire. *Mathematical Finance* **8**, 179–200

Heath, D., Jarrow, R., Morton, A. (1992): Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica* **60**, 77–105

Jacod, J., Shiryaev, N.A. (1987): Limit theorems for stochastic processes. Springer, Berlin

Laurent, J.P., Pham, H. (1999): Dynamic programming and mean-variance hedging. *Finance and Stochastics* **3(1)**, 83–110

Longstaff, F.A., Schwartz, E.S. (1992): A two-factor interest-rate model and contingent claims valuation. *The Journal of Fixed Income* **3**, 16–23

Musiela, M., Rutkowski, M.(1997): Martingale methods in financial modelling. Springer, Berlin

Rheinländer, T., Schweizer, M. (1997): On L^2-projections on a space of stochastic integrals. *The Annals of Probability* **25**, 1810–1831

Sekine, J. (1999): Mean-variance hedging in continuous-time with stochastic interest rate. *Stochastics and Stochastic Reports* **67**, 1–16

Schweizer, M. (1996): Approximation pricing and the variance-optimal martingale measure. *The Annals of Probability* **24**, 206–236

Schweizer, M. (2000): A guided tour through quadratic hedging approaches. In: Jouini, E. et al. (eds.): Option pricing, interest rates and risk management. Cambridge University Press, Cambridge, pp. 538–574