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Arbitrage, linear programming and martingales in securities markets with bid-ask spreads

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Abstract. In a general, finite-dimensional securities market model with bid-ask spreads, we characterize absence of arbitrage opportunities both by linear programming and in terms of martingales. We first show that absence of arbitrage is equivalent to the existence of solutions to the linear programming problems that compute the minimum costs of super-replicating the feasible future cashflows. Via duality, we show that absence of arbitrage is also equivalent to the existence of *underlying frictionless (UF) state-prices*. We then show how to transform the *UF* state-prices into state-price densities, and use them to characterize absence of arbitrage opportunities in terms of existence of a securities market with zero bid-ask spreads whose price process lies inside the bid-ask spread. Finally, we argue that our results extend those of Naik (1995) and Jouini and Kallal (1995) to the case of intermediate dividend payments and positive bid-ask spreads on all assets.

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1. Introduction and summary

In this paper, we provide various characterizations of absence of arbitrage opportunities in a securities market model with bid-ask spreads on the asset prices. We do this in a simple framework in which the information flow over time can be described by an event-tree. Aside from this requirement, however, we do not impose other restrictions. In particular, we allow the bid-ask spreads to be positive on all assets, and we account for intertemporal dividends.

Our first characterization is based on a cost minimization argument. Given any intertemporal cashflow m, we say that a dynamic trading strategy super-replicates it if it generates an intertemporal cashflow at least as large as m. We consider then the problem of selecting the strategies that super-replicate m at the minimum initial cost. In our setting, the collection of these problems for all cashflows m is a family of linear programs that we employ to characterize absence of arbitrage from two perspectives. The first is a cost perspective: absence of arbitrage is characterized by the facts that (a) every cashflow is super-replicable at a finite minimum initial cost, (b) the cost of super-replicating a positive cashflow is positive, and (c) the cost of super-replicating a portfolio of cashflows is possibly lower than the cost of super-replicating its components separately. The second perspective is based on the properties of the cost-minimizing strategies. From this standpoint, in fact, only the minimum-cost super-replication of the null cashflow needs to be considered. Specifically, absence of arbitrage with positive bid-ask spreads is characterized by the fact that minimum-cost super-replication of $m = 0$ is achieved by the dynamic trading strategies that generate a null cashflow at zero cost. Interestingly, our result also shows that the existence of cost-minimizing super-replicating strategies for $m = 0$ guarantees that the same holds true for *any cashflow* m.

Our second characterization is based on the notion of *underlying frictionless (UF) state-prices*, and is related to the previous one via duality. In particular, we call *UF state-prices* the strictly positive vectors whose inner product with the cashflows generated by dynamically trading the assets with bid-ask spreads is non-positive. We argue that the *UF* state-prices are the strictly positive elements of the feasible set of the dual of the cost-minimization problem, and exploit this fact to show that absence of arbitrage opportunities is equivalent to the existence of *UF* state-prices. Moreover, we show that the minimum cost to super-replicate a cashflow m is the supremum of the values assigned to m by the *UF* state-prices. This allows us to interpret the *UF* state-prices as state-prices of securities markets with zero bid-ask spreads underlying those with positive bid-ask spreads, and to conclude that the minimum cost to super-replicate m is the supremum of the arbitrage values of m in such underlying markets.

Our last characterization is based on martingale arguments, and is related to the previous by the fact that it involves transforming the *UF* state-prices into *state-price deflators*. In the case of zero bid-ask spreads, a state-price deflator is a process that equates the asset prices to the discounted conditional expected values of the cumulative future dividends. For the case with positive bid-ask spreads, we show that absence of arbitrage is characterized by the existence of state-price deflators that place the discounted conditional expected value of the cumulative future dividends of each asset inside the bid-ask spread. The state-price deflators in this characterization are the *UF* state-prices normalized by the probability. In the zero bid-ask spreads case, absence of arbitrage is well-known to be equivalent to the existence of state-price deflators. Therefore, our characterization can be restated by saying that absence of arbitrage with positive bid-ask spreads is equivalent to the existence of asset prices that have zero bid-ask spread, lie inside the bid-ask spreads, and are arbitrage-free.

For a better comparison with the literature, we also consider the special case in which one asset has zero bid-ask spread, and distributes dividends strictly positive at the terminal date, and non-negative otherwise. We denote by V the intertemporal value of buying and holding this asset until the terminal date, reinvesting the dividends. We consider then the set of probabilities Q equivalent to the original one, and satisfying the following property: the conditional expected value computed under Q of the cumulative future dividends denominated in units of V lies inside the bid-ask spread. We show that the set of such probabilities Q is in one-to-one correspondence with the set of *UF* state-prices. In this special case, therefore, absence of arbitrage with positive bid-ask spreads is equivalent to existence of zero bid-ask spread prices that, once expressed in units of V, lie inside the bid-ask spreads and are equal to the conditional expected value computed under Q of the cumulative future dividends themselves in units of V. From this standpoint, the probabilities Q are then the equivalent martingale measures for these zero bid-ask spread prices. In this special case, moreover, the minimum cost to super-replicate a future payoff m can be expressed as the supremum over all these equivalent martingale measures Q of the expected value of m denominated in units of V.

After the early contributions of Garman and Ohlson (1981), Leland (1985), and Prisman (1986), the last decade has witnessed a mounting interest in the effects of introducing bid-ask spreads in the standard no-arbitrage model with frictionless markets. Merton (1990) and Boyle and Vorst (1992) have generalized the valuation-by-replication binomial option pricing model to the case of bid-ask spreads on the stock. Dermody and Rockafellar (1991), Bensaid et al. (1992) and Edirisinghe et al. (1993) have remarked that exact replication of the option's payoff may be unnecessarily costly, since there may exist strategies that dominate the payoff at a lower initial cost. Naik (1995), in an event-tree framework, and Jouini and Kallal (1995), in an infinite state-space environment, have exploited this observation to provide martingale-based characterizations of absence of arbitrage in securities markets with bid-ask spreads (see also Milne and Neave (1997) and Jouini and Kallal (1999)). However, both Naik (1995) and Jouini and Kallal (1995) assume that the assets pay no intermediate dividends, and that one asset is a pure discount bond with zero bid-ask spread. Jouini and Kallal (1995), moreover, restrict their analysis to self-financing trading. In this paper, we account for intermediate dividend payment, we do not require one asset to be a pure discount bond, and we allow for positive bid-ask spreads on all assets and for non-self-financing trading.[1]

The rest of the paper is structured as follows. In the next section, we set out the notation and definitions. We supply the characterizations of absence of arbitrage opportunities based on the existence of minimum-cost, super-replicating strategies and of underlying frictionless state-prices in Section 3. In Section 4 we discuss the martingale-based characterizations, both for the general case and for the special case in which one asset has zero bid-ask spread, and compare our results with Naik (1995) and Jouini and Kallal (1995). Section 5 concludes.

2. The securities market model with bid-ask spreads: notation and definitions

We base our securities market model on a finite probability space (Ω, F, P), where $\Omega = \{\omega_1, \ldots, \omega_{s_T}\}$ is a finite set of states, $F = 2^{\Omega}$, and P is a strictly positive probability on $2^{\Omega}\backslash\{\emptyset\}$. The set of trading dates is $\mathbf{T} = \{0, 1, \ldots, T\}$, and the information flow shared by all investors is described by a filtration $\mathbb{F} = \{F_t\}_{t\in\mathbf{T}}$ of F, with $F_0 = \{\emptyset, \Omega\}$ and $F_T = F$. In this finite setting, the information flow can be described equivalently by a family $\mathbb{P} = \{P_t\}_{t\in\mathbf{T}}$ of partitions of Ω, with $P_0 = \{\Omega\}$, P_τ finer than P_t for $\tau > t$, and $P_T = \{\{\omega_1\}, \ldots, \{\omega_{s_T}\}\}$. Henceforth, we refer $\mathbb{P}$ as the event-tree associated with the filtration $\mathbb{F}$, to the generic cell f_k^t of the partition P_t as the generic time t node of the event-tree, to $s_t \equiv \mathrm{card}(P_t)$ as the number of time t nodes, and to $L = \sum_{t=0}^{T} s_t$ as the total number of nodes.

At each time t, markets open for trading in J assets, indexed by j. We assume that agents are price-takers, that unlimited short-sales with full use of proceeds are allowed, and that the assets are perfectly divisible. We consider however the presence of bid-ask spreads, and formalize this fact

[1] There is also a vast literature on the issues related to introducing bid-ask spreads in continuous-time models of securities markets. We refer to Cvitanic (1999) for a detailed survey of this literature.

by identifying the assets via triples $\left(S_j^A, S_j^B, d_j\right)$ of $\mathbb{F}$-adapted stochastic processes. The first component, S_j^A, represents the ex-dividend ask price of the j-th asset, i.e., the price that the investors pay to take a long position in asset j. The second component, S_j^B, represents the ex-dividend bid price, i.e., the price that the investors receive if they sell (possibly short) asset j. Finally, d_j represents the dividend flow paid by asset j. We assume that all prices and dividends are denominated in units of the same (numéraire) good. Without loss of generality, we let $d_j(0) = 0$ and $S_j^A(T) = S_j^B(T) = 0$ for all j. In words, the assets pay no initial dividends, and have zero ex-dividend ask and bid prices at the terminal date, so that $d_j(T)$ is interpreted as a liquidating dividend. We call $\left(S^A, S^B, d\right) = \left\{\left(S_j^A, S_j^B, d_j\right)\right\}_{j=1}^J$ *the price-dividend system of the securities market with bid-ask spreads*.

We model the dynamic trading strategies available to the investors as couples $\theta = \{\theta^A, \theta^B\}$ of $\Re^J$-valued, $\mathbb{F}$-predictable stochastic processes,[2] where $\theta_j^A(t+1)$ represents the number of units of asset j bought at time t, and $\theta_j^B(t+1)$ the number of units of asset j sold at time t. A basic requirement in a securities market with bid-ask spreads is that investors be prevented from buying the assets at the bid prices, or selling them at the ask prices. We formalize this requirement by deeming feasible only the dynamic trading strategies that are certainly non-negative. We denote by Θ the set of all feasible dynamic trading strategies.

We now define the *cashflow process* x_θ generated by a feasible dynamic trading strategy θ. To this end, we observe that the quantity $\sum_{\tau=1}^t \left[\theta^A(\tau) - \theta^B(\tau)\right]$ represents the net position held on the J assets before trading at time t. We have therefore

$$x_\theta(t) = \begin{cases} -\theta^A(1)\cdot S^A(0) + \theta^B(1)\cdot S^B(0) & t = 0 \\ d(t)\cdot \sum_{\tau=1}^{t}\left[\theta^A(\tau) - \theta^B(\tau)\right] & \\ -\left[\theta^A(t+1)\cdot S^A(t) - \theta^B(t+1)\cdot S^B(t)\right] & t = 1,\ldots,T-1 \\ d(T)\cdot \sum_{\tau=1}^T \left[\theta_j^A(\tau) - \theta_j^B(\tau)\right] & t = T. \end{cases} \tag{1}$$

Interpreted, $-x_\theta(0) = \theta^A(1)\cdot S^A(0) - \theta^B(1)\cdot S^B(0)$ represents the initial cost of θ, while $x_\theta(T)$ represents the dividends received at the final liquidation of θ. At the intermediate dates $t = 1, \ldots, T-1$, instead, the cashflow $x_\theta(t)$ consists of the difference of two components. The first, $d(t)\cdot \sum_{\tau=1}^{t}\left[\theta^A(\tau) - \theta^B(\tau)\right]$, represents the dividends obtained on the net positions held in the J assets before trading, while the second, $\theta^A(t+1)\cdot$

[2] That is, $\theta_j^A(t+1)$ and $\theta_j^B(t+1)$ are F_t-measurable for all $t = 0, 1, \ldots, T-1$.

$S^A(t) - \theta^B(t+1) \cdot S^B(t)$, represents the cost to update the positions in the J assets.

We denote by X_Θ the set of all cashflow processes generated by the feasible dynamic trading strategies via (1). In fact, X_Θ is a polyhedral convex cone in $\Re^L$, where L denotes the total number of nodes of the event-tree $\mathbb{P}$ that, in our finite-dimensional setting, is informationally equivalent to the filtration $\mathbb{F}$. To see this, interpret any feasible dynamic trading strategy $\theta = \{\theta^A, \theta^B\}$ as a column vector in $\Re_+^{2J(L-s_T)}$, with coordinates the realizations of θ on the $L - s_T$ nodes of $\mathbb{P}$ preceding the s_T terminal ones. Likewise, interpret the cashflow process x_θ generated by θ as a column vector in $\Re^L$, with coordinates the realizations of x_θ on the L nodes of $\mathbb{P}$. On comparing with (1), it is readily seen that $x_\theta = \mathcal{M}\theta$ for some $L \times 2J(L - s_T)$ matrix $\mathcal{M}$, which we refer to as the *payoff matrix* associated with the price-dividend system $\left(S^A, S^B, d\right)$.[3] Therefore, we have

$$X_\Theta = \left\{ x = \mathcal{M}\theta \;\middle|\; \theta \in \Theta \equiv \Re_+^{2J(L-s_T)} \right\}$$

which shows that X_Θ is indeed the polyhedral convex cone spanned by the columns of $\mathcal{M}$.

To define the arbitrage opportunities that may arise in our securities market with bid-ask spreads, we denote by $-c \in \Re^{2J(L-s_T)}$ the first row of the payoff matrix $\mathcal{M}$, that is,

$$-c = \left(-S_1^A(0), \ldots, -S_J^A(0), 0, \ldots, 0, S_1^B(0), \ldots, S_J^B(0), 0, \ldots, 0\right)$$

and by $\widetilde{\mathcal{M}}$ the submatrix of $\mathcal{M}$ formed by the remaining $L - 1$ rows, so that

$$\mathcal{M} = \begin{bmatrix} -c \\ \widetilde{\mathcal{M}} \end{bmatrix}.$$

For a feasible dynamic trading strategy θ, therefore, its initial cost is given by $-x_\theta(0) = c \cdot \theta$, while $\widetilde{\mathcal{M}}\theta$ represents its future cashflow, that is, the cashflow generated from time 1 on.

[3] The *payoff matrix* $\mathcal{M}$ is formally defined as follows. Denote by $S_j^A(f_k^t)$, $S_j^B(f_k^t)$ and $d_j(f_k^t)$ the realizations of $\left(S^A, S^B, d\right)$ on the L nodes of $\mathbb{P}$. For fixed $t \in \{0, \ldots, T-1\}$, $k \in \{1, \ldots, s_t\}$, $j \in \{1, \ldots, J\}$, let $L_t = \sum_{\tau \le t} s_\tau$, and consider the column vector in $\Re^L$ with components all equal to zero, except for the one with index $(L_{t-1} + k)$, which is set equal to $-S_j^A(f_k^t)$, and those with indices $(L_\tau + h)$, with $\tau = t+1, \ldots, T$ and $f_h^\tau \subset f_k^t$, which are set equal to $d_j(f_h^\tau)$. Let this vector be the $(J(L_{t-1} + k - 1) + j)$-th column of $\mathcal{M}$. Moreover, consider the column vector of $\Re^L$ with components all equal to zero, except for the one with index $(L_{t-1} + k)$, which is set equal to $S_j^B(f_k^t)$, and those with indices $(L_\tau + h)$, with $\tau = t+1, \ldots, T$ and $f_h^\tau \subset f_k^t$, which are set equal to $-d_j(f_h^\tau)$. Let this vector be the $(J(L - s_T) + J(L_{t-1} + k - 1) + j)$-th column of $\mathcal{M}$. All columns of $\mathcal{M}$ are obtained by letting t vary in $\{0, \ldots, T-1\}$, k in $\{1, \ldots, s_t\}$ and j in $\{1, \ldots, J\}$.

Definition 1. *A feasible dynamic trading strategy* θ *generates an arbitrage opportunity at the given price-dividend system* $\left(S^A, S^B, d\right)$ *if* $x_\theta = \mathcal{M}\theta > 0$.[4] *In particular, provided that* $\mathcal{M}\theta \geq 0$, θ *generates an arbitrage opportunity of:*

1. *the first type if* $\widetilde{\mathcal{M}}\theta > 0$*;*
2. *the second type if* $x_\theta(0) = -c \cdot \theta > 0$.

This definition extends the one standard in frictionless markets (see, for instance, Ingersoll (1987)) to a securities market with bid-ask spreads. Interpreted, an arbitrage opportunity of the first type allows an agent with zero income to consume a strictly positive amount in some node at some date $t > 0$, while maintaining his consumption level non-negative anyway. An arbitrage opportunity of the second type guarantees instead strictly positive consumption at 0, and non-negative consumption at all other times.

Absence of both types of arbitrage opportunities, a minimal requirement for the existence of equilibrium in any securities market populated by non-satiated agents, is formalized as follows.

Definition 2. *The price-dividend system* $\left(S^A, S^B, d\right)$ *is arbitrage-free if* $\mathcal{M}\theta = 0$ *for all* $\theta \in \Theta$ *such that* $\mathcal{M}\theta \geq 0$. *In words,* $\left(S^A, S^B, d\right)$ *is arbitrage-free if it does not admit arbitrage opportunities of the first or second type.*

It is readily seen that $S^A \geq S^B$ is necessary for $\left(S^A, S^B, d\right)$ to be arbitrage-free.[5] In words, the investors must pay at least as much to buy the assets as they receive from selling them.

In the characterizations of absence of arbitrage opportunities supplied in the next sections, we restrict our attention to securities markets that satisfy the following requirement.

Condition 1 (The internality condition). *For the price-dividend system* $\left(S^A, S^B, d\right)$*, there exist feasible dynamic trading strategies* θ *such that* $\widetilde{\mathcal{M}}\theta >> 0$.

In words, the internality condition requires the existence of dynamic trading strategies whose cashflows are certainly strictly positive from time 1 on. As we argue in the next section, this fact implies that *any* cashflow available after time 0 can be super-replicated by the cashflow generated by some

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[4] If $a, b \in \Re^L$, $a \geq b$ means $a - b \in \Re^L_+$, $a > b$ means $a - b \in \Re^L_+ \setminus \{\emptyset\}$, $a >> b$ means $a - b \in \Re^L_{++} \equiv \text{int}\left(\Re^L_+\right)$.

[5] If $S^A_j(f^t_k) < S^B_j(f^t_k)$ for some $j \in \{1, \ldots, J\}$, $t \in \{0, \ldots, T\}$, $k \in \{1, \ldots, s_t\}$, an arbitrage opportunity follows from contemporaneously buying and shorting asset j at time t in the node k, holding this position until T.

trading strategy. We exploit this feature in setting up our linear programming approach to absence of arbitrage in markets with bid-ask spreads.

We notice that the internality condition is a very mild requirement, satisfied in particular when one of the assets has strictly positive bid price process, and pays non-negative intermediate dividends and strictly positive terminal ones.[6] To the best of our knowledge, an asset that satisfies these requirements is present in all the characterizations of absence of arbitrage in markets with bid-ask spreads available in the literature.

Finally, we remark that the existence of second-type arbitrage opportunities implies the existence of first-type arbitrage opportunities if the internality condition holds.[7] Therefore, a price-dividend system $\left(S^A, S^B, d\right)$ that satisfies the internality condition is arbitrage-free as long as it is free of first-type arbitrage opportunities.

3. Absence of arbitrage opportunities and linear programming

In this section, we use linear programming to characterize the absence of arbitrage opportunities in the securities market with bid-ask spreads described in Section 2. Given any column vector $m \in \Re^{L-1}$, we consider the following parametric linear programming problem:

$$\min_{\theta \in \Re_+^{2J(L-s_T)}} \quad c \cdot \theta \qquad\qquad (\mathcal{P}|m|)$$
$$\text{s.t.} \quad \widetilde{\mathcal{M}}\theta \geq m.$$

To interpret this problem, recall that $c \cdot \theta$ is the initial cost of the feasible dynamic trading strategy θ, while $\widetilde{\mathcal{M}}\theta$ is the cashflow generated by θ from time 1 on. Moreover, observe that m can be interpreted as a generic cashflow available from time 1 on. The feasible set of problem $\mathcal{P}[m]$ then collects all the feasible dynamic trading strategies that super-replicate m, in the sense of generating a future cashflow at least as large as m. The solutions to $\mathcal{P}[m]$, therefore, are the feasible dynamic trading strategies that super-replicate m at the minimum initial cost.

Problem $\mathcal{P}[m]$ extends to a general finite-dimensional framework and to a general class of assets, similar problems analyzed in the literature. In

[6] To see this, let $S_i^B >> 0$, $d_i(t) \geq 0$ for $t = 1, \ldots, T-1$, and $d_i(T) >> 0$, and consider the dynamic trading strategy θ such that $\theta_i^A(1) = T$, $\theta_i^B(t+1) = 1$ for all $t = 1, \ldots, T-1$, and all the other components are equal to zero. Using (1) to compute the future cashflow generated by this strategy one has $x_\theta(t) = S_i^B(t) + (T-t+1)d_i(t)$, $t = 1, \ldots, T-1$, and $x_\theta(T) = d_i(T)$, from which the internality condition follows.

[7] To see this, let θ be such that $\widetilde{\mathcal{M}}\theta >> 0$, and θ' be a second-type arbitrage opportunity. For $\lambda = \max\left(0, -\frac{x_\theta(0)}{x_{\theta'}(0)}\right)$, then, $\theta + \lambda\theta'$ is a first-type arbitrage opportunity since $x_{\theta+\lambda\theta'}(0) \geq x_{\theta'}(0) > 0$ and $\widetilde{\mathcal{M}}\left(\theta + \lambda\theta'\right) \geq \widetilde{\mathcal{M}}\theta >> 0$.

particular, our approach extends both Dermody and Rockafellar (1991), who consider a non-stochastic term structure model with bid-ask spreads in which only buy-and-hold strategies are allowed, and Bensaid et al. (1992) and Edirisinghe et al. (1993), who analyze binomial models with a bond and a stock, and in which only the stock has a positive bid-ask spread. Our approach also extends the model of Naik (1995) in which, although the information structure is a general event-tree, one asset is a pure discount bond with zero bid-ask spread, and there are no intermediate dividends.[8]

Below we use first the minimum cost problem $\mathcal{P}[m]$, and then its dual $\mathcal{P}'[m]$, to provide alternative characterizations of absence of arbitrage in securities markets with bid-ask spreads.

3.1. Absence of arbitrage opportunities and minimum-cost super-replication

To present our characterization of absence of arbitrage based on the minimum-cost super-replication problem $\mathcal{P}[m]$, it is convenient to denote its feasible set by Θ_m, that is,

$$\Theta_m = \left\{\theta \in \Re_+^{2J(L-s_T)} \mid \widetilde{\mathcal{M}}\theta \geq m\right\}.$$

Under the internality condition, Θ_m is non-empty for any choice of m,[9] so that, for any future cashflow m, there exists a feasible dynamic trading strategy that super-replicates it.

Our first result characterizes absence of arbitrage opportunities both in terms of existence of optimal solutions to $\mathcal{P}[m]$, and in terms of properties of the value functional $\pi : \Re^{L-1} \to \Re \cup \{-\infty\}$ associated to $\mathcal{P}[m]$, defined as

$$\pi(m) = \inf\{c \cdot \theta \mid \theta \in \Theta_m\}.$$

In words, $\pi(m)$ is the minimum cost at which the future cashflow m can be super-replicated. For future reference, we observe that π is sub-additive and strictly positively homogeneous, that is, $\pi(m + m') \leq \pi(m) + \pi(m')$ and $\pi(\lambda m) = \lambda\pi(m)$ for any $\lambda > 0, m, m' \in \Re^{L-1}$.[10]

[8] See Cvitanic (1999) for a survey of the literature on the continuous-time counterpart of $\mathcal{P}[m]$.

[9] To see this, let θ be such that $\widetilde{\mathcal{M}}\theta >> 0$, and, given any m, let $\lambda \in \Re_+$ be such that $\lambda \min_{t>0,h} x_\theta(f_h^t) \geq \max_{t>0,h} m(f_h^t)$, with $m(f_h^t)$ the realization of m on f_h^t. Then $\lambda\theta \in \Theta_m$ since $x_{\lambda\theta}(f_k^t) = \lambda x_\theta(f_k^t) \geq \max_{t>0,h} m(f_h^t) \geq m(f_k^t)\ \forall\, t > 0, k$.

[10] Indeed, $\Theta_m + \Theta_{m'} \subset \Theta_{m+m'}$ for all $m, m' \in \Re^{L-1}$, so that $\pi(m+m') \leq c\cdot(\theta+\theta') \leq c\cdot\theta + c\cdot\theta'$ for all $\theta \in \Theta_m, \theta' \in \Theta_{m'}$, which implies $\pi(m+m') \leq \pi(m)+\pi(m')$. Moreover, $\pi(\lambda m) = \lambda\pi(m)$ for any $\lambda > 0$, $m \in \Re^{L-1}$ since $\Theta_{\lambda m} = \lambda\Theta_m$.

Theorem 1. *The following statements are equivalent for any price-dividend system* $\left(S^A, S^B, d\right)$ *that satisfies the internality condition.*

1. $\left(S^A, S^B, d\right)$ *is arbitrage-free.*
2. *The value functional* π *is real-valued, strictly positive, i.e.,* $\pi(m) > 0$ *for all* $m > 0$*, and satisfies* $\pi(0) = 0$.
3. *Problem* $\mathcal{P}[m]$ *admits optimal solutions for all* $m \in \Re^{L-1}$*. Moreover, if* θ^* *is an optimal solution to* $\mathcal{P}[0]$ *,then* $x_{\theta^*} = 0$.
4. *Problem* $\mathcal{P}[0]$ *admits optimal solutions* θ^**, all of which satisfy* $x_{\theta^*} = 0$.

Proof. $1 \to 2$. Under the internality condition, to show that π is real-valued we only need to show that $c \cdot \theta$ is bounded from below on Θ_m for all $m \in \Re^{L-1}$. To see this, given $m \in \Re^{L-1}$, let $\theta' \in \Theta_{-m}$, that is, $\widetilde{\mathcal{M}}\theta' \geq -m$. For any $\theta \in \Theta_m$, then, $x_{\theta+\theta'}(f_k^t) = x_\theta(f_k^t) + x_{\theta'}(f_k^t) \geq 0$ for all $t > 0, k$. Therefore, since by assumption there are no arbitrage opportunities, it must be the case that $x_{\theta+\theta'}(0) = x_\theta(0) + x_{\theta'}(0) \leq 0$, so that $c \cdot \theta = -x_\theta(0) \geq x_{\theta'}(0)$, which proves that π is indeed real-valued. To see that $\pi(m) > 0$ for all $m > 0$ and that $\pi(0) = 0$, we show first that $\pi(m) \geq 0$ for any $m \geq 0$. Indeed, given $m \geq 0$, we have $\widetilde{\mathcal{M}}\theta \geq m \geq 0$ for all $\theta \in \Theta_m$ and hence, since by the no-arbitrage assumption $\mathcal{M}\theta \geq 0$ implies $\mathcal{M}\theta = 0$, it must be the case that $-c \cdot \theta \leq 0$, so that $\pi(m) = \inf\{c \cdot \theta \mid \theta \in \Theta_m\} \geq 0$. We immediately get $\pi(0) = 0$ on noting that, since $\theta \equiv 0 \in \Theta_0$, we have $\pi(0) = \inf\{c \cdot \theta \mid \theta \in \Theta_0\} \geq 0$. To show that is π strictly positive, we observe that the linear objective function $c \cdot \theta$ is bounded from below on Θ_m if and only if there exists an optimal solution to $\mathcal{P}[m]$. Thus, for $m > 0$, there exists a strategy θ^* such that $c \cdot \theta^* = \inf\{c \cdot \theta \mid \theta \in \Theta_m\} = \pi(m)$ and $\widetilde{\mathcal{M}}\theta^* \geq m > 0$. But then $-\pi(m) = -c \cdot \theta^* < 0$, i.e., $\pi(m) > 0$, since otherwise $\mathcal{M}\theta^* > 0$, a contradiction to the no-arbitrage assumption.

$2 \to 3$. That there exists an optimal solution to $\mathcal{P}[m]$ for all $m \in M$ is an immediate consequence of the fact that π is real-valued, so that the linear objective function in $\mathcal{P}[m]$ is bounded from below on the polyhedron Θ_m. In particular, given any optimal solutions θ^* to $\mathcal{P}[0]$, we have $c \cdot \theta^* = \pi(0) = 0$ and $\widetilde{\mathcal{M}}\theta^* \geq 0$. Let, then, $m \equiv \widetilde{\mathcal{M}}\theta^*$ and suppose $m > 0$. Since $\theta^* \in \Theta_m$ and π is strictly positive, we have $c \cdot \theta^* \geq \pi(m) > 0$, a contradiction to $c \cdot \theta^* = 0$.

$3 \to 4$. Obvious.

$4 \to 1$. Suppose that $\mathcal{M}\theta \geq 0$ for some $\theta \in \Theta$, i.e. $-c \cdot \theta \geq 0$ and $\widetilde{\mathcal{M}}\theta \geq 0$, so that $\theta \in \Theta_0$. Since the optimal value of $\mathcal{P}[0]$ is 0, it must be the case that $c \cdot \theta \geq 0$ holds as well, which implies $c \cdot \theta = 0$ which, in turn, implies $\widetilde{\mathcal{M}}\theta = 0$, that is, $\mathcal{M}\theta = 0$. $\square$

Theorem 1 characterizes arbitrage-free securities markets with bid-ask spreads from two perspectives: the minimum cost of super-replication, and the trading strategies that attain it. From the minimum cost perspective, the equivalence of statements 1 and 2 shows that absence of arbitrage opportunities is characterized by the possibility of *super-replicating any future cashflow at a finite minimum initial cost*. The minimum cost is strictly positive for future cashflows that are certainly non-negative, and positive with strictly positive probability, while it is zero for the null future cashflow. As a consequence, the sub-additive value functional π is in fact sublinear, that is, for all $m \in \Re^{L-1}$ we have $\pi(\lambda m) = \lambda\pi(m)$ *for any non-negative* λ. Therefore, it may be cheaper to super-replicate a portfolio of cashflows than its components separately, and this is so because some positions in the J assets may cancel in super-replicating the portfolio, hence reducing the cost of dynamically rebalancing these positions. Also, we remark that the equivalence of statements 1 and 2 in our Theorem 1 extends the equivalence of statements (1) and (3) in Theorem 2 in Naik (1995) to the case in which a zero-coupon bond is not available, all assets are subject to bid-ask spreads, and intermediate dividends are accounted for.

From the perspective of the strategies that attain the minimum cost, absence of arbitrage opportunities is characterized by the fact that exact replication at zero cost is the optimal way to super-replicate the null future payoff, as witnessed by the equivalence of statements 1 and 4 in our Theorem 1. The equivalence of statements 3 and 4, moreover, shows that the optimality of exact replication at zero cost as the way to super-replicate the null future payoff guarantees the existence of minimum-cost super-replicating strategies for *any* future cashflow.

It is also interesting to compare our result with the case in which the bid-ask spreads are zero, that is, $S^A = S^B$. In this case, the cone X_Θ of cashflows generated by the feasible dynamic trading strategies is in fact a linear subspace of $\Re^L$. Therefore, the value functional π is actually linear on the projection of X_Θ on $\Re^{L-1}$, that is, on the set of future cashflows that can be exactly replicated. In turn, this implies that the minimum cost way to generate a future cashflow at least as large as one that can be exactly replicated is indeed exact replication, or otherwise arbitrage opportunities would arise.[11] In the case $S^A > S^B$, instead, there may very well exist future cashflows for which strict super-replication is cost-optimal even when exact replication is available. Formally, this means that, for some future cashflow m, problem $\mathcal{P}[m]$ may admit optimal solutions θ^* such that $\widetilde{\mathcal{M}}\theta^* > m$ even if $\widetilde{\mathcal{M}}\theta = m$ is feasible. Typically, such situations occur when the bid-ask spreads are so large as to make a super-replicating strategy, which usually requires a low volume of transactions, cheaper than the strategies that imposes a high volume of transactions to exactly replicate m.[12]

[11] If $S^A = S^B$, given $m \in \Re^{L-1}$ and θ feasible such that $\widetilde{\mathcal{M}}\theta = m$, the strategy $\theta^* - \theta$ would generate an arbitrage opportunity if θ^* was a solution to $\mathcal{P}[m]$ for which $\widetilde{\mathcal{M}}\theta^* > m$.

Finally, for a better comparison with the literature we provide a characterization of absence of second-type arbitrage opportunities based only on problem $\mathcal{P}[m]$. In particular, on inspecting the proof of Theorem 1, it is readily seen how the facts that π is real-valued, semi-positive (i.e., $\pi(m) \geq 0$ if $m \geq 0$) and $\pi(0) = 0$ are all direct consequences of the absence of second-type arbitrage opportunities. In turn, the requirement that π is real-valued constitutes, together with the linearity of the objective function and the fact that the feasible set is a polyhedron, a condition sufficient to guarantee that $\mathcal{P}[m]$ has solutions for any m. Finally, since by Definition 1 any arbitrage opportunity of the second type is generated by a strategy θ such that $c \cdot \theta < 0$ and $\widetilde{\mathcal{M}}\theta \geq 0$, and since the set Θ_0 of feasible programs for $\mathcal{P}[0]$ is a cone, the fact that $\mathcal{P}[0]$ admits solutions is readily seen to imply the absence of second-type arbitrage opportunities. We summarize these arguments as follows.

Corollary 1. *The following statements are equivalent for any price-dividend system* $\left(S^A, S^B, d\right)$ *that satisfies the internality condition.*

1. $\left(S^A, S^B, d\right)$ *is free of second-type arbitrage opportunities.*
2. *The value functional* π *is real-valued, semi-positive, i.e.,* $\pi(m) \geq 0$ *for all* $m \geq 0$, *and satisfies* $\pi(0) = 0$.
3. *The problem* $\mathcal{P}[m]$ *admits optimal solutions for any* $m \in M$.
4. *The problem* $\mathcal{P}[0]$ *admits optimal solutions.*

When the securities market is only assumed free of second-type arbitrage opportunities, therefore, any future cashflow can still be super-replicated at a finite minimum initial cost. Moreover, the value functional π is still sublinear, which shows that our Corollary 3 constitutes an extension of Theorems 3.1 and 4.1 in Dermody and Rockafellar (1991) to the case in which prices and dividends are stochastic, and dynamic trading is allowed.

3.2. Absence of arbitrage opportunities and underlying frictionless state-prices

It is well-known that, in the case $S^A = S^B$ of zero bid-ask spreads, absence of arbitrage opportunities is characterized by the existence of state-prices, i.e., strictly positive vectors with first coordinate equal to one, and *orthogonal* to the cashflows generated by the feasible dynamic trading strategies (see, e.g., Duffie (1996)). To provide a similar result for the case of positive bid-ask spreads, we first define a suitably extended notion of state-prices.

[12] See the examples in Bensaid et al.(1992), Edirisinghe et al. (1993), and Naik (1995). We remark, however, that the presence of bid-ask spreads per se does not rule out the optimality of exact replication (see, on this point, Theorems 3.2 and 3.3 in Bensaid et al. (1992)).

Definition 3. *We define the underlying frictionless state-price vectors (UF state-prices) for* (S^A, S^B, d) *as those vectors* $\psi \in \Re^L_{++}$ *with first coordinate* 1 *such that* $\psi \cdot x \leq 0$ *for all* $x \in X_\Theta$.

In contrast with the case of zero bid-ask spread, under positive bid-ask spreads the state-prices are only required to have a non-positive inner product with the cashflows generated by the feasible dynamic trading strategies.

Recall now that X_Θ is the convex cone generated by the columns of the payoff matrix $\mathcal{M}$, and $\mathcal{M} = \begin{bmatrix} -c \\ \widetilde{\mathcal{M}} \end{bmatrix}$. Hence, the set Ψ of *UF* state-prices for (S^A, S^B, d) takes the following form:

$$\begin{aligned} \Psi &= \left\{ \begin{pmatrix} 1 \\ \widetilde{\psi} \end{pmatrix} \in \Re^L \;\middle|\; \widetilde{\psi} \in \Re^{L-1}_{++},\ (1, \widetilde{\psi}^T)\mathcal{M} \leq 0 \right\} \\ &= \left\{ \begin{pmatrix} 1 \\ \widetilde{\psi} \end{pmatrix} \in \Re^L \;\middle|\; \widetilde{\psi} \in \Re^{L-1}_{++},\ \widetilde{\psi}^T \widetilde{\mathcal{M}} \leq c \right\}. \end{aligned} \tag{2}$$

Consider then the dual $\mathcal{P}'[m]$ of problem $\mathcal{P}[m]$, that is,

$$\begin{aligned} &\max_{\phi \in \Re^L_+} \ \phi \cdot m \\ &\text{s.t.} \quad \phi^T \widetilde{\mathcal{M}} \leq c. \end{aligned} \tag{$\mathcal{P}'[m]$}$$

On comparing the representation of Ψ in (2) with the feasible set $\Phi \equiv \left\{ \phi \in \Re^{L-1}_+ \;\middle|\; \phi^T \widetilde{\mathcal{M}} \leq c \right\}$ of $\mathcal{P}'[m]$, it is immediately seen that the existence of *UF* state-prices for (S^A, S^B, d) is equivalent to the existence of strictly positive vectors in Φ. We exploit this fact to characterize absence of arbitrage opportunities in terms of *UF* state-prices for (S^A, S^B, d), and to establish interesting relations between *UF* state-prices and the value functional π of problem $\mathcal{P}[m]$.

Theorem 2. *Under the internality condition, absence of arbitrage opportunities is equivalent to the existence of UF state-price vectors for* (S^A, S^B, d). *Moreover, if* (S^A, S^B, d) *is arbitrage-free, then,*

$$\Psi = \left\{ \begin{pmatrix} 1 \\ \widetilde{\psi} \end{pmatrix} \in \Re^L \;\middle|\; \widetilde{\psi} \in \Re^{L-1}_{++},\ \widetilde{\psi} \cdot m \leq \pi(m) \ \ \forall\, m \in \Re^{L-1} \right\}, \tag{3}$$

$$\pi(m) = \sup_{(1, \widetilde{\psi}^T) \in \Psi} \widetilde{\psi} \cdot m \quad \forall\, m \in \Re^{L-1}. \tag{4}$$

Proof. We assume first that (S^A, S^B, d) is arbitrage-free, and show that then Φ contains a strictly positive vector. To show this, given any $t \geq 1$ and k, consider the (primal) linear programming problem $\mathcal{P}[\mathbf{1}_{f^t_k}]$, where $\mathbf{1}_{f^t_k}$

is a vector whose $(L_{t-1}+k)$-th entry is 1 ($L_{t-1} = \sum_{\tau \leq t-1} s_\tau$ and $s_\tau \equiv$ card(P_τ)), while all other entries are zeroes. By statement 2 in Theorem 1, the value $\pi(\mathbf{1}_{f_k^t})$ of the objective function at any optimal solution is strictly positive. Therefore, by the duality theorem of linear programming (see, for instance, Luenberger (1973)) the dual problem $\mathcal{P}'[\mathbf{1}_{f_k^t}]$ admits an optimal solution, say $\phi^{(t,k)}$, such that $\phi^{(t,k)} \cdot \mathbf{1}_{f_k^t} = \pi(\mathbf{1}_{f_k^t}) > 0$. Then, letting $\widetilde{\phi} \equiv \sum_{t,k} \frac{1}{L-1}\phi^{(t,k)}$, we see that $\widetilde{\phi} >> 0$ since each of its components is a sum of non-negative terms with at least one of them different from zero. Moreover, $\widetilde{\phi} \in \Phi$ since Φ is manifestly a convex set.

Conversely, to prove that the existence of *UF* state-price vectors implies the absence of arbitrage opportunities, we first argue that, under the internality condition and the assumption that $\Psi \neq \emptyset$, the (obviously non-empty and closed) set Φ is compact. Under the internality condition, indeed, there exists a feasible dynamic trading strategy θ such that $\widetilde{\mathcal{M}}\theta >> 0$, so that $0 \leq \phi^T(\widetilde{\mathcal{M}}\theta) \leq c \cdot \theta$ for all $\phi \in \Phi$, which shows that Φ is bounded. From the continuity of the linear objective function $\phi \cdot m$, therefore, $\mathcal{P}'[m]$ has an optimal solution for any m. By the duality theorem of linear programming, the primal problem $\mathcal{P}[m]$ then has solutions for any m, so that the value functional π is real-valued. Since the objective function of the dual problem $\mathcal{P}'[0]$ is identically zero on Φ, clearly $\pi(0) = 0$. Moreover, the existence of a strictly positive element of Φ guarantees that the optimal value of the objective function of the dual problem $\mathcal{P}'[m]$ is strictly positive whenever $m > 0$, which implies that π is strictly positive. This shows that statement 2 in Theorem 1 holds and hence proves the equivalence between $\Psi \neq \emptyset$ and absence of arbitrage opportunities.

Finally, (3) is based on the fact that, for any $\psi \in \Psi$ such that $\widetilde{\psi} \cdot m \leq \pi(m)\, \forall\, m \in \Re^{L-1}$, we have $\left(\widetilde{\psi}\right)^T \widetilde{\mathcal{M}}\theta \leq \pi(\widetilde{\mathcal{M}}\theta) \leq c{\cdot}\theta$ for any θ, and hence $\left(\widetilde{\psi}\right)^T \widetilde{\mathcal{M}} \leq c$, while (4) follows upon observing that $(1, \alpha\widetilde{\psi}^T + (1-\alpha)\phi^T) \in \Psi$ for any $\phi \in \Phi$, $(1, \widetilde{\psi}) \in \Psi$, $0 < \alpha < 1$, and that $(\alpha\widetilde{\psi} + (1-\alpha)\phi) \cdot m$ converges to $\phi \cdot m$ as α goes to zero. □

Relations (3) and (4) in Theorem 2 allow us to better explain and motivate the term *underlying frictionless* used for the state-prices in Ψ. To this end, observe first that $\psi = (1, \widetilde{\psi}^T) \in \Psi$ can be interpreted as a state-price vector in a securities market with zero bid-ask spreads, and $\widetilde{\psi} \cdot m$ as the arbitrage value of the future cashflow m in that case. To see why, denote by $\psi(f_k^t)$ the coordinates of ψ,[13] and consider the securities market with zero bid-ask

[13] That is, $\psi(f_k^t)$ denotes the coordinate of ψ corresponding to the node f_k^t of the event-tree $\mathbb{P}$.

spreads in which the ex-dividend prices of the J assets are as follows:

$$S_j(f_k^t) = \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \frac{\psi(f_h^\tau)}{\psi(f_k^t)} d_j(f_h^\tau), \quad k = 1, \dots, s_t,\ t = 0, \dots, T-1. \tag{5}$$

If in the payoff matrix $\mathcal{M}$ we set $S_j^A(f_k^t) = S_j^B(f_k^t) = S_j(f_k^t)$, it is readily checked that ψ becomes *orthogonal* to the columns of $\mathcal{M}$, hence, to the linear space generated by $\mathcal{M}$. This means that ψ is in fact a state-price vector for the securities market with zero bid-ask spreads in which the realizations of the ex-dividend price process are governed by (5). Therefore, $\widetilde{\psi} \cdot m$ is the arbitrage value of m in this market with zero bid-ask spreads. Now interpret the minimum cost $\pi(m)$ for super-replicating m under positive bid-ask spreads as the value of m, in the sense of the maximum price at which an investor would be willing to take a long position in an asset with future cashflow m. The results in Theorem 2 can now be interpreted as follows. First, (3) shows that the elements of Ψ are the state-prices that, in a securities market with zero bid-ask spreads, would assign to the future payoffs an arbitrage value lying below (hence, underlying) the value assigned to that cashflow in the case of positive bid-ask spreads. Moreover, (4) shows that the value of m under positive bid-ask spreads, $\pi(m)$, is the supremum of the arbitrage values assigned to m in the case of zero bid-ask spreads. These results, furthermore, show that the *UF* state-prices defined here constitute the finite-dimensional counterparts of the *underlying frictionless linear pricing rules* introduced by Jouini and Kallal (1995, 1999) to characterize absence of multiperiod free lunches in infinite-dimensional models of securities markets with positive bid-ask spreads and other frictions.

For a further comparison with the literature, we use the dual problem $\mathcal{P}'[m]$ to provide a characterization of absence of second-type arbitrage opportunities only. To this end, we define a *semi-positive UF state-price vector* for $\left(S^A, S^B, d\right)$ as any vector in $\Re_+^L$ with first coordinate equal to 1 and with a non-positive inner product with all $x \in X_\Theta$. A *semi-positive UF* state-price vector, therefore, satisfies the same properties as a *UF* state-price vector, except that only weak positivity is required. The following result is then an immediate consequence of the duality theorem of linear programming, and of the fact that the existence of semi-positive *UF* state-prices is manifestly equivalent to Φ being non-empty.

Corollary 2. *Under the internality condition, absence of second-type arbitrage opportunities is equivalent to the existence of semi-positive UF state-price vectors for* $\left(S^A, S^B, d\right)$.

On comparing with Dermody and Rockafellar (1991), it is readily seen that their model is a special case of ours with no uncertainty, and in which only buy-and-hold strategies are allowed. Therefore, our notion of semi-positive *UF* state-price vector coincides with their notion of *current term structure packet*. This is why our Theorem 1 and Corollary 2 constitute generalizations, to the stochastic case in which dynamic trading is allowed, of their Theorems 4.5 and 3.2 respectively.

4. Absence of arbitrage opportunities and martingales

We now provide a characterization of the price-dividend systems $\left(S^A, S^B, d\right)$ that are arbitrage-free based on martingale processes. To better compare our results with the existing literature, we discuss separately the general case in which the bid-ask spread is positive for all assets, and the special case in which one asset has zero bid-ask spread.

4.1. The general case

To establish our results, given any $\mathbb{F}$-adapted and strictly positive process ξ, we define the $\Re^J$-valued and $\mathbb{F}$-adapted process S^ξ as follows:

$$S^\xi(t) = \begin{cases} \dfrac{1}{\xi(t)} E\left[\sum_{\tau=t+1}^{T} \xi(\tau) d(\tau) \,\middle|\, F_t \right] & t = 0, \ldots, T-1 \\ 0 & t = T. \end{cases} \tag{6}$$

We interpret the components $S_j^\xi(t)$ of $S^\xi(t)$ as ex-dividend prices for the J assets in a securities market with zero bid-ask spreads, but otherwise identical to that with positive bid-ask spreads.[14] We denote by $\left(S^\xi, d\right)$ the price-dividend system of this securities market with zero bid-ask spreads, and observe that the process ξ is, by construction, a *state-price deflator* for $\left(S^\xi, d\right)$ (see Duffie (1996)). We recall then that, in the case of zero bid-ask spreads, the existence of *state-price deflators* is a necessary and sufficient condition for absence of arbitrage opportunities (Duffie (1996), Theorem 2C). This fact allows us to assert that (a) $\left(S^\xi, d\right)$ is an arbitrage-free price-dividend system with zero bid-ask spreads, and (b) for any arbitrage-free price-dividend system (S, d) with zero bid-ask spreads, but otherwise identical to that with positive bid-ask spreads, there exists an $\mathbb{F}$-adapted and strictly positive process ξ such that $S = S^\xi$, with S^ξ defined as in (6). We exploit these facts in the following characterizations of absence of arbitrage opportunities in securities markets with positive bid-ask spreads.

[14] That is, a securities market with the same probability space, same information structure, same number of assets, and same dividend process of the securities market with positive bid-ask spreads.

Theorem 3. *The following statements are equivalent for any price-dividend system* $\left(S^A, S^B, d\right)$ *that satisfies the internality condition.*

1. $\left(S^A, S^B, d\right)$ *is arbitrage-free.*
2. *There exists an* $\mathbb{F}$*-adapted and strictly positive process* ξ *such that the process* S^ξ *defined in (6) satisfies* $S^B \leq S^\xi \leq S^A$.
3. *There exists an arbitrage-free price-dividend system* (S, d) *with zero bid-ask spreads such that* $S^B \leq S \leq S^A$.

Proof. As explained above, the equivalence of statements 2 and 3 follows directly from Theorem 2C in Duffie (1996). Therefore, we only need to establish the equivalence of 1 and 2. To this end, suppose first that $\left(S^A, S^B, d\right)$ is arbitrage-free so that, by Theorem 2 in this paper, it admits *UF* state-prices. Then, given any *UF* state-price vector ψ, let ξ be the $\mathbb{F}$-adapted process with the following realizations on the nodes f_k^t of the event-tree $\mathbb{P}$ informationally equivalent to the filtration $\mathbb{F}$:

$$\xi(f_k^t) = \frac{\psi(f_k^t)}{P(f_k^t)}, \quad \forall k, t. \tag{7}$$

In the above expression, $\psi(f_k^t)$ denotes the $(L_{t-1} + k)$-th coordinate of the *UF* state-price vector ψ, that is, the coordinate corresponding to the node f_k^t of $\mathbb{P}$, and $P(f_k^t)$ denotes the probability of that node. The process ξ defined in this way is strictly positive since the *UF* state-prices are strictly positive by definition, and the probability P is strictly positive on $2^\Omega/\{\emptyset\}$ by assumption. We show now that, if this process ξ is used in (6) to define S^ξ, then $S^B \leq S^\xi \leq S^A$. To see this, we exploit the characterization of the set Ψ of *UF* state-prices supplied in (2), and the construction of the payoff matrix $\mathcal{M}$ described in Section 2, to observe that $\psi \in \Re^L_{++}$ with first coordinate 1 is a *UF* state-price vector for $\left(S^A, S^B, d\right)$ if and only if it satisfies the following set of inequalities:

$$\begin{gathered}\psi(f_k^t) S_j^B(f_k^t) \leq \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \psi(f_h^\tau) d_j(f_h^\tau) \leq \psi(f_k^t) S_j^A(f_k^t), \\ j = 1, \ldots, J, \quad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1.\end{gathered} \tag{8}$$

In fact, system (8) is equivalent to

$$\begin{gathered}S_j^B(f_k^t) \leq \left[\frac{\psi(f_k^t)}{P(f_k^t)}\right]^{-1} \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \frac{P(f_h^\tau)}{P(f_k^t)} \frac{\psi(f_h^\tau)}{P(f_h^\tau)} d_j(f_h^\tau) \leq S_j^A(f_k^t), \\ j = 1, \ldots, J, \quad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1;\end{gathered}$$

exploiting (7), we see that this implies that

$$S_j^B(f_k^t) \leq \frac{1}{\xi(f_k^t)} \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \frac{P(f_h^\tau)}{P(f_k^t)} \xi(f_h^\tau) d_j(f_h^\tau) \leq S_j^A(f_k^t),$$
$$j = 1, \ldots, J, \quad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1.$$

Observe now that the quantity that lies between $S_j^B(f_k^t)$ and $S_j^A(f_k^t)$ in the above relation is a realization of the random variable $\frac{1}{\xi(t)} E\left[\sum_{\tau=t+1}^{T} \xi(\tau) d_j(\tau) \,\middle|\, F_t\right]$. Therefore, absence of arbitrage opportunities implies that the components $S_j^\xi(t)$, $j = 1, \ldots, J$, of the process S^ξ obtained by using in (6) the process ξ defined via (7) satisfy

$$S_j^B(t) \leq S_j^\xi(t) \leq S_j^A(t), \qquad j = 1, \ldots, J, \quad t = 0, \ldots, T-1. \tag{9}$$

Relation (9), together with the assumption $S_j^B(T) = S_j^A(T) = 0$ for all j, and the fact that by construction $S_j^\xi(T) = 0$ for all j, allows us to conclude that 1→2. To establish the converse implication, given an $\mathbb{F}$-adapted and strictly positive process ξ such that S^ξ defined in (6) satisfies $S^B \leq S^\xi \leq S^A$, let ψ be the vector in $\Re^L$ with the following coordinates:

$$\psi(f_k^t) = \frac{P(f_k^t)\xi(f_k^t)}{\xi(0)}, \quad \forall k, t.$$

Such a ψ is clearly strictly positive and with first coordinate equal to 1. Moreover, by working backwards along the lines used to establish 1→2, it is readily seen that such a ψ satisfies (8) and hence is a *UF* state-price vector for (S^A, S^B, d). Therefore, (S^A, S^B, d) admits *UF* state-prices so that, by Theorem 2, it is indeed arbitrage-free. □

The characterization of arbitrage-free price-dividend systems with bid-ask spreads supplied in Theorem 3 has the following interesting consequences. First, our result shows that *any* arbitrage-free price-dividend system with bid-ask spreads can be obtained from an arbitrage-free price-dividend system (S, d) with zero bid-ask spread by substituting S with bid and ask prices S^A, S^B that leave S in the middle. Conversely, whenever the price process S of an arbitrage-free price-dividend system (S, d) with zero bid-ask spreads is replaced by bid and ask prices S^A, S^B that leave S in the middle, and the dividends d are left unaltered, the resulting price-dividend system with bid-ask spreads is arbitrage-free.

4.2. A special case

For a better comparison with the literature, we now consider the case in which Condition 1, the internality condition, is replaced with the following requirement.

Condition 2. *One of the assets in* $\left(S^A, S^B, d\right)$*, say asset* 1*, has null bid-ask spread, strictly positive price process, and pays non-negative intermediate dividends and strictly positive terminal ones. Formally,* $S_1^A = S_1^B \equiv S_1 \gg 0$*,* $d_1(t) \geq 0$ *for* $t = 1, \ldots, T-1$*, and* $d_1(T) \gg 0$.

Since Condition 2 implies the internality condition,[15] the following is in fact a special case of the framework considered so far.

We now denote by $\widehat{\theta} = (\widehat{\theta}^A, \widehat{\theta}^B)$ the feasible dynamic trading strategy that is required to buy one share of asset 1 at time 0, to reinvest the dividends in asset 1 itself, and to leave the other assets inactive. Formally, $\widehat{\theta}$ satisfies $\widehat{\theta}_1^A(1) = 1$, $\widehat{\theta}_1^A(t) = [S_1(t)]^{-1} d_1(t) \sum_{\tau=1}^{t} \widehat{\theta}_1^A(\tau)$, and all other components equal to zero. Using (1) to compute the future cashflow generated by $\widehat{\theta}$, we see that $x_{\widehat{\theta}}(t) = 0$ for $t = 1, \ldots, T-1$, and $x_{\widehat{\theta}}(T) = d_1(T) \sum_{t=1}^{T} \widehat{\theta}_1^A(t) \gg 0$, that is, $\widehat{\theta}$ is a self-financing strategy with strictly positive terminal payoff. We then define the *value process* of $\widehat{\theta}$ to be the following $\mathbb{F}$-adapted process V:

$$V(t) = \begin{cases} S_1(t) \sum\limits_{\tau=1}^{t+1} \widehat{\theta}_1^A(\tau) & t = 0, \ldots, T-1 \\ x_{\widehat{\theta}}(T) & t = T. \end{cases} \tag{10}$$

Since $S_1 >> 0$ and $\sum\limits_{\tau=1}^{t+1} \widehat{\theta}_1^A(\tau) \geq 1$ for all $t = 0, \ldots, T-1$, the value process V is strictly positive. The following result establishes a property of the value process V that becomes useful in our last characterization of absence of arbitrage.

Lemma 1. *If* $\left(S^A, S^B, d\right)$ *satisfies Condition 2, then any UF state-price vector* $\psi \in \Psi$ *satisfies*[16]

$$\psi(f_k^t) V(f_k^t) = \sum_{\omega \in f_k^t} \psi(\{\omega\}) V(\{\omega\}), \quad \begin{matrix} k = 1, \ldots, s_t \\ t = 0, \ldots, T-1. \end{matrix} \tag{11}$$

[15] To see this, simply replace $S_i^B(t)$ and $d_i(t)$ with $S_1(t)$ and $d_1(t)$ in the argument in footnote 6.

[16] In (11), $\{\omega\}$ represents the generic terminal node of the event-tree $\mathbb{P}$ informationally equivalent to $\mathbb{F}$.

Proof. We first observe that, by backward induction, establishing (11) is equivalent to showing that $\forall\, \psi \in \Psi$ we have

$$\psi(f_k^t)V(f_k^t) = \sum_{f_h^{t+1} \subset f_k^t} \psi(f_h^{t+1})V(f_h^{t+1}), \quad \begin{array}{l} k = 1, \ldots, s_t \\ t = 0, \ldots, T-1. \end{array} \tag{12}$$

To establish (12), we recall that all $\psi \in \Psi$ satisfy (8) in Theorem 3. This, together with the fact that $S_1^A = S_1^B \equiv S_1$ under Condition 2, shows that all $\psi \in \Psi$ satisfy

$$\psi(f_k^t)S_1(f_k^t) = \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \psi(f_h^\tau)d_1(f_h^\tau), \quad \begin{array}{l} k = 1, \ldots, s_t \\ t = 0, \ldots, T-1, \end{array}$$

which is equivalent to (see Girotto and Ortu, 1996)

$$\psi(f_k^t)S_1(f_k^t) = \sum_{f_h^{t+1} \subset f_k^t} \psi(f_h^{t+1}) \left[S_1(f_h^{t+1}) + d_1(f_h^{t+1})\right], \qquad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1. \tag{13}$$

Recall now that $\widehat{\theta} = (\widehat{\theta}^A, \widehat{\theta}^B)$ is the dynamic trading strategy such that $\widehat{\theta}_1^A(1) = 1$, $\widehat{\theta}_1^A(t) = [S_1(t)]^{-1}\, d_1(t) \sum_{\tau=1}^{t} \widehat{\theta}_1^A(\tau)$, and all other components are zero. We then multiply both sides of (13) by $\sum_{f_k^t \subset f_r^\tau} \widehat{\theta}_1^A(\tau+1,\, f_r^\tau)$, where $\widehat{\theta}_1^A(\tau+1,\, f_r^\tau)$ is the generic realization of the random variable $\widehat{\theta}_1^A(\tau+1)$, and exploit the definition of V in (10) to obtain

$$\psi(f_k^t)V(f_k^t) = \sum_{f_h^{t+1} \subset f_k^t} \psi(f_h^{t+1}) \cdot \left\{ \left[S_1(f_h^{t+1}) + d_1(f_h^{t+1})\right] \sum_{f_k^t \subset f_r^\tau} \widehat{\theta}_1^A(\tau+1,\, f_r^\tau) \right\}, \qquad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1.$$

Observe then that the quantity $\left[S_1(f_h^{t+1}) + d_1(f_h^{t+1})\right] \sum_{f_k^t \subset f_r^\tau} \widehat{\theta}_1^A(\tau+1,\, f_r^\tau)$ is a realization of the random variable $[S_1(t+1) + d_1(t+1)] \sum_{\tau=1}^{t+1} \widehat{\theta}_1^A(\tau)$. To establish (12), therefore, one only needs to show that

$$[S_1(t+1) + d_1(t+1)] \sum_{\tau=1}^{t+1} \widehat{\theta}_1^A(\tau) = V(t+1) \quad \text{for } t = 0, 1, \ldots, T-1.$$

To do so, one exploits the facts that $\widehat{\theta}$ is self-financing, that is, $x_{\widehat{\theta}}(t) = 0$ for $t = 1, \ldots, T-1$, that $V(T) = x_{\widehat{\theta}}(T) = d_1(T) \sum_{t=1}^{T} \widehat{\theta}_1^A(t)$, and that $S_1(T) = 0$. Indeed, $[S_1(T) + d_1(T)] \sum_{\tau=1}^{T} \widehat{\theta}_1^A(\tau) = V(T)$ is an immediate consequence of $S_1(T) = 0$. For $t < T - 1$, instead, from the self-financing condition and the definition of generated cashflow process in (1) we have $0 = x_{\widehat{\theta}}(t) = d_1(t+1)\sum_{\tau=1}^{t+1}\widehat{\theta}_1^A(\tau) - \widehat{\theta}_1^A(t+2)S_1(t+1)$, which implies that $d_1(t+1)\sum_{\tau=1}^{t+1}\widehat{\theta}_1^A(\tau) = \widehat{\theta}_1^A(t+2)S_1(t+1)$, so that, again using the definition of V in (10), we have

$$\begin{aligned}
[S_1(t+1)+\ d_1(t+1)]&\sum_{\tau=1}^{t+1}\widehat{\theta}_1^A(\tau)\\
&= S_1(t+1)\sum_{\tau=1}^{t+1}\widehat{\theta}_1^A(\tau) + \widehat{\theta}_1^A(t+2)S_1(t+1)\\
&= S_1(t+1)\sum_{\tau=1}^{t+2}\widehat{\theta}_1^A(\tau)\\
&= V(t+1)
\end{aligned}$$

which shows that (12) holds for all $\psi \in \Psi$, and establishes our claim. □

Now, given any probability Q equivalent to P,[17] we use it together with the value process V to define the $\mathbb{F}$-adapted and $\Re^J$-valued process $S^{(Q,V)}$ as follows:

$$S^{(Q,V)}(t) = \begin{cases} V(t)E_Q\left[\sum_{\tau=t+1}^{T}\frac{d(\tau)}{V(\tau)}\;\middle|\; F_t\right] & t = 0,\ldots,T-1 \\ 0 & t = T. \end{cases} \tag{14}$$

Once again, we interpret the components $S_j^{(Q,V)}(t)$ of $S^{(Q,V)}(t)$ as the ex-dividend prices for the J assets in the case of zero bid-ask spreads, and denote by $(S^{(Q,V)}, d)$ the resulting price-dividend system. In this case, it is readily seen that the gains process of $(S^{(Q,V)}, d)$ denominated in units of V, $\left\{\frac{S^{(Q,V)}(t)}{V(t)} + \sum_{\tau=0}^{t}\frac{d(\tau)}{V(\tau)}\right\}$, is by construction a Q-martingale. Therefore, V satisfies the definition of *numéraire* for the price-dividend system $(S^{(Q,V)}, d)$ with zero bid-ask spreads, and Q of equivalent martingale measure associated with the numéraire V (see Girotto and Ortu (1997, 2000)). We then denote by $\mathcal{Q}(V)$ the set of probabilities Q equivalent to P such that $S^{(Q,V)}$ lies inside the bid-ask spread of $\left(S^A, S^B, d\right)$, that is,

$$\mathcal{Q}(V) = \left\{Q \sim P \;\middle|\; S^B \le S^{(Q,V)} \le S^A\right\},$$

and establish our final result.

[17] Since by assumption, Ω is finite and P is strictly positive on $2^\Omega/\{\emptyset\}$, in this framework Q is equivalent to P as long as it is itself strictly positive on $2^\Omega/\{\emptyset\}$.

Theorem 4. *If $\left(S^A, S^B, d\right)$ satisfies Condition 2, the following facts hold.*

1. *The set Ψ of UF state prices is in one-to-one correspondence with the set $\mathcal{Q}(V)$ via*

$$\psi(f_k^t) = \frac{V(0)}{V(f_k^t)} Q(f_k^t), \qquad \begin{array}{l} k = 1, \ldots, s_t \\ t = 0, \ldots, T \end{array} \tag{15}$$

$$Q(\{\omega\}) = \frac{\psi(\{\omega\})V(\{\omega\})}{V(0)}, \qquad \forall \omega \in \Omega. \tag{16}$$

2. *Statements 1 to 3 in Theorem 3 are equivalent to the existence of a probability Q equivalent to P and such that the process $S^{(Q,V)}$ in (14) satisfies $S^B \le S^{(Q,V)} \le S^A$.*
3. *If $\left(S^A, S^B, d\right)$ is arbitrage-free, then*

$$[-\pi(-m), \pi(m)] = cl\left\{ V(0)E_Q\left[\sum_{t=1}^{T} \frac{m(t)}{V(t)}\right] \middle| \; Q \in \mathcal{Q}(V) \right\}, \qquad \forall m \in \Re^{L-1}. \tag{17}$$

Proof. To establish the first fact, for $Q \in \mathcal{Q}(V)$ observe that the vector ψ with coordinates given by (15) is manifestly a strictly positive vector in $\Re^L$ with first coordinate equal to 1. By the argument in the proof of Theorem 3, to show that such ψ is a *UF* state-price vector we only need to show that it satisfies the following set of inequalities:

$$\psi(f_k^t)S_j^B(f_k^t) \le \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \psi(f_h^\tau)d_j(f_h^\tau) \le \psi(f_k^t)S_j^A(f_k^t),$$
$$j = 1, \ldots, J, \quad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1.$$

To see this, we substitute $\dfrac{V(0)}{V(f_k^t)} Q(f_k^t)$ to $\psi(f_k^t)$ in the above relation and rearrange to obtain the following equivalent system:

$$S_j^B(f_k^t) \le V(f_k^t) \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \frac{Q(f_h^\tau)}{Q(f_k^t)} \frac{d_j(f_h^\tau)}{V(f_h^\tau)} \le S_j^A(f_k^t), \qquad j = 1, \ldots, J, \quad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1. \tag{18}$$

Observe now that the quantities that lie between $S_j^B(f_k^t)$ and $S_j^A(f_k^t)$ in (18) are the realizations of the random variable $V(t)E_Q\left[\sum_{\tau=t+1}^{T} \frac{d_j(\tau)}{V(\tau)} \middle| \; F_t\right]$. By reference to (14) we see that the vector ψ with coordinates given by (15) being a *UF* state-price vector is equivalent to requiring that

$$S_j^B(t) \le S_j^{(Q,V)}(t) \le S_j^A(t), \quad \begin{array}{l} j = 1, \ldots, J \\ t = 0, \ldots, T-1, \end{array} \tag{19}$$

which holds true since $Q \in \mathcal{Q}(V)$. This shows that (15) defines a mapping of $\mathcal{Q}(V)$ into Ψ, a mapping which is injective since, for any $Q_1, Q_2 \in \mathcal{Q}(V)$ such that $Q_1(\{\omega\}) \ne Q_2(\{\omega\})$ for some $\omega \in \Omega$, then $\psi_1(\{\omega\}) \ne \psi_2(\{\omega\})$ whenever ψ_1, ψ_2 are associated with Q_1, Q_2 via (15). To complete the proof, we need to show that the mapping introduced by (15) is onto Ψ. To this end, given any $\psi \in \Psi$ we define Q via (16). Since V and ψ are strictly positive, clearly $Q(\{\omega\}) > 0\ \forall\, \omega \in \Omega$. By (11) in Lemma 1 and the fact that the first coordinate of ψ is 1, we have $\sum_{\omega\in\Omega} Q(\{\omega\}) = \frac{1}{V(0)} \sum_{\omega\in\Omega} \psi(\{\omega\})V(\{\omega\}) = \frac{1}{V(0)} V(0) = 1$, so that Q is indeed a strictly positive probability on $2^\Omega \backslash \{\emptyset\}$. We show now that $Q \in \mathcal{Q}(V)$, that is, that the process $S^{(Q,V)}$ obtained from Q via (14) satisfies $S^B \le S^{(Q,V)} \le S^A$. To this end, since $S_j^B(T) = S_j^A(T) = 0\ \forall\, j$ by assumption, and $S_j^{(Q,V)}(T) = 0\ \forall\, j$ by construction, it is enough to show that the components of $S^{(Q,V)}$ satisfy (19) or, equivalently, (18). To see this, we exploit (11) in Lemma 1 to obtain

$$Q(f_k^t) = \sum_{\omega\in f_k^t} Q(\{\omega\}) = \sum_{\omega\in f_k^t} \frac{\psi(\{\omega\})V(\{\omega\})}{V(0)} = \frac{\psi(f_k^t)V(f_k^t)}{V(0)}, \qquad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1. \tag{20}$$

Substituting (20) into (18), we see that Q belongs to $\mathcal{Q}(V)$ as long as ψ satisfies

$$S_j^B(f_k^t) \le V(f_k^t) \sum_{\tau=t+1}^{T} \sum_{f_h^\tau \subset f_k^t} \frac{\psi(f_h^\tau)}{\psi(f_k^t)} d_j(f_h^\tau) \le S_j^A(f_k^t),$$
$$j = 1, \ldots, J, \quad k = 1, \ldots, s_t, \quad t = 0, \ldots, T-1,$$

that is, as long as the ψ used to define Q in (16) is indeed a *UF* state-price vector. The proof is then completed upon observing that, by (20),

$$\psi(f_k^t) = \frac{V(0)}{V(f_k^t)} Q(f_k^t),$$

that is, the *UF* state-price vector associated with Q via (15) is the ψ used to define Q in (16).

Fact 2 is now an immediate consequence of Fact 1, of the observation that Condition 2 implies the internality condition, and of Theorem 3. Finally, Fact 3 is an immediate consequence of Fact 1, of (4) in Theorem 2, and of the set $\mathcal{Q}(V)$ being convex. □

We conclude by observing that, since any $Q \in \mathcal{Q}(V)$ is a martingale measure associated with the numéraire V in a securities market with zero bid-ask spread and such that $S^B \leq S \leq S^A$, we can interpret the quantity $V(0)E_Q\left[\sum_{t=1}^{T} \frac{m(t)}{V(t)}\right]$ in (17) as the arbitrage value of the future cashflow m in any such market. Therefore, (17) supplies the martingale counterpart of the characterization of the value functional $\pi(m)$ as the least upper bound of the arbitrage values assigned to m in the securities market with zero bid-ask spreads underlying (S^A, S^B, d) supplied by (4) in Theorem 2.

4.3. Comparison with the literature

It is now useful to compare Theorems 3 and 4 with the martingale-based characterizations of absence of arbitrage opportunities in securities markets with bid-ask spreads of Naik (1995, Theorem 2), and of Jouini and Kallal (1995, Theorem 3.2).

Naik, in particular, considers an event-tree securities market model with bid-ask spreads in which the assets pay no intermediate dividends. One of the assets, moreover, is a pure-discount bond with zero bid-ask spread, with time t price denoted by $S_0(t)$. Naik shows that absence of arbitrage opportunities is equivalent to the existence of a probability Q equivalent to P such that $S^B(t) \leq S_0(t)E_Q[d(T)\,|F_t] \leq S^A(t)$ for $t < T$. Comparing with our results, we see that our Theorem 4 extends Naik's Theorem 2 to the case of intermediate dividends, and of an asset with zero bid-ask spreads required only to pay non-negative intermediate dividends and strictly positive terminal ones. Our Theorem 3, moreover, also relaxes the requirement of the existence of an asset with zero bid-ask spreads.

Jouini and Kallal consider instead a model with an infinite-dimensional state-space although they maintain the no intermediate dividends assumption and they require the pure-discount bond with zero bid-ask spread to have constant unit price. Also, they restrict dynamic trading to self-financing strategies. In this framework, Jouini and Kallal first characterize absence of multiperiod free lunches, the infinite state-space counterpart of absence of arbitrage opportunities, in terms of existence of *underlying frictionless linear pricing rules* (see their Theorem 2.1, Proposition 3.1 and Lemma 1). In Theorem 3.2, then, they establish the following three results. First, they show that absence of multiperiod free lunches is equivalent to the existence of a process S that lies inside the bid-ask spread and is a martingale with respect to some probability Q equivalent to P. Second, they establish a one-to-one correspondence between the set of such probabilities Q and the set of underlying frictionless linear pricing rules. Third, they characterize the minimum cost for super-replicating a future payoff m as the supremum over the set of probabilities Q of the expected values of m. Since absence of multiperiod free lunches is equivalent to absence of arbitrage opportunities if the state-space is finite, and the *UF* state-prices introduced in this paper are the finite-dimensional counterparts of Jouini and Kallal's underlying frictionless linear pricing rules, our results provide finite-dimensional extensions of the results of Jouini and Kallal. In particular, our Theorem 3 extends the first result in their Theorem 3.2 to the case in which, although the number of states is finite, intermediate dividend payments are accounted for, non-self-financing trading is permitted, and a positive bid-ask spread is allowed on all assets. Our Theorem 3 extends instead the other two results of their Theorem 3.2 to the case of intermediate dividend payments, non-self-financing trading, and an asset with zero bid-ask spreads required only to pay non-negative intermediate dividends and strictly positive terminal ones.

5. Conclusion

In this paper, we have supplied several characterizations of absence of arbitrage opportunities in a securities market model with bid-ask spreads. First, we have defined the linear programming problem $\mathcal{P}[m]$ that computes the minimum cost to super-replicate a future cashflow m. In Theorem 1, we have characterized absence of arbitrage in terms of properties of $\pi(m)$, the value functional of $\mathcal{P}[m]$, and of existence of optimal solutions to $\mathcal{P}[m]$. Next, we have defined the *UF state-prices* for $\left(S^A, S^B, d\right)$ and, in Theorem 2, we have employed the dual of $\mathcal{P}[m]$ to show that absence of arbitrage is also equivalent to the existence of *UF* state-prices. We have then supplied a martingale-based characterization of absence of arbitrage. Specifically, we have transformed the *UF* state-prices into state-price deflators and, in Theorem 3, we have employed these state-price deflators to show that $\left(S^A, S^B, d\right)$ is arbitrage-free if and only if there exists a price-dividend system (S, d) which has zero bid-ask spreads, is itself arbitrage-free, and satisfies $S^B \leq S \leq S^A$. Finally, we have considered the special case in which one of the traded assets has zero bid-ask spread, and distributes a strictly positive dividend at liquidation and non-negative dividends otherwise. We have denoted by V the value of the self-financing strategy that is required to buy one share of this asset at the initial date, reinvesting the dividends received over time. In Theorem 4, we have obtained two further results for this special case. First, we have established a one-to-one correspondence between the set of *UF* state-prices and the set of strictly positive probabilities Q such that, if all prices and dividends are denominated in units of V, the conditional expected value computed under Q of the cumulative future dividends lies inside the bid-ask spread. Moreover, we have shown that the minimum cost $\pi(m)$ for super-replicating a future cashflow m is the supremum over all such probabilities Q of the expected value of m denominated in units of V.

To conclude, we observe that our results can be employed to provide arbitrage-based bounds on the prices of derivative securities in the presence of bid-ask spreads. Given a derivative security with future cashflow n, for instance, it can be shown that the minimum cost $\pi(n)$ to super-replicate n must constitute an upper bound to the derivative's bid price, if second-type arbitrage opportunities are to be prevented. Likewise, $-\pi(-n)$ must constitute a lower bound to the derivative's ask price to prevent second-type arbitrage opportunities. A detailed analysis of this and other related results is presented in Baccara and Ortu (2001).

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