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Optimality and duality in nonsmooth semi-infinite optimization, using a weak constraint qualification

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Abstract

Variational analysis, a subject that has been vigorously developing for the past 40 years, has proven itself to be extremely effective at describing nonsmooth phenomenon. The Clarke subdifferential (or generalized gradient) and the limiting subdifferential of a function are the earliest and most widely used constructions of the subject. A key distinction between these two notions is that, in contrast to the limiting subdifferential, the Clarke subdifferential is always convex. From a computational point of view, convexity of the Clarke subdifferential is a great virtue. We consider a nonsmooth multiobjective semi-infinite programming problem with a feasible set defined by inequality constraints. First, we introduce the weak Slater constraint qualification and derive the Karush–Kuhn–Tucker types necessary and sufficient conditions for (weakly, properly) efficient solution of the considered problem. Then, we introduce two duals of Mond–Weir type for the problem and present (weak and strong) duality results for them. All results are given in terms of Clarke subdifferential.

Keywords Semi-infinite programming · Multiobjective optimization · Constraint qualification · Optimality conditions

JEL Classification Optimization Techniques · Programming Models · Dynamic Analysis C61

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1 Introduction

A multiobjective semi-infinite programming (MOSIP in brief) is an optimization problem where two or more objectives are to be minimized on a set of feasible solutions described by infinitely many inequality constraint functions. Optimality and duality conditions of MOSIP have been studied by many authors: (Guerra-Vazquez and Todorov 2016; Kanzi et al. 2018) in linear case, (Goberna and Kanzi 2017; Goberna et al. 2013) in convex case, (Caristi et al. 2010) in smooth case, and (Kanzi 2015; Kanzi and Nobakhtian 2013) in locally Lipschitz case. Also, Gao presented some sufficient and duality results for MOSIPs under the various generalized convexity assumptions in (Gao 2012, 2013). In almost all of the articles in MOSIP theory, the Fritz–John-type (Karush–Kuhn–Tucker-type) necessary optimality conditions are justified for continuous problems (under Slater constraint qualification); continuous MOSIPs and Slater constraint qualification will be defined in Sect. 3. The first aim of this paper is to replace these conditions by two weaker conditions, named PLV property and weak Slater constraint qualification. Also, in all the articles mentioned above (with the exception of Caristi et al. (2010), sufficient optimality conditions and duality results are presented under convexity, quasiconvexity\pseudoconvexity, and invexity assumptions for objective and restriction functions. Caristi et al. (2010) took the (Φ, ρ)-invexity to state of optimality and duality theorems for MOSIPs with differentiable data. Another aim of this paper is to extension of (Φ, ρ)-invexity for nondifferentiable functions and also is to apply this concept for nonsmooth MOSIPs. We should mention that the (Φ, ρ)-invexity was generalized for nonsmooth functions in some references (e.g., Antczak 2015, 2012), but our definition and results are different from theirs. Some of the issues that soared in the optimization theory, in the past four decades and suddenly declined, are the topics related to the invex functions, their extensions, and applications. The concept of invexity was originated by Hanson and Mond (1981) but so named by Craven (1981) for differentiable functions. They used this concept to extend the concept of convexity in presenting sufficient optimality conditions and duality results. After Hanson, this concept was considered by many researchers. Invexity was generalized to nonsmooth functions by Phoung et al. (1995), by Reiland (1990), and by Craven (1986). Also, the invexity was used in optimality conditions and for alternative theorems by (Hanson 1999; Brandao et al. 2000), respectively. Some researchers such as Antczak have selected the extension of the concepts of invexity and convexity as their main research areas (see, e.g., Antczak 2002, 2009a, b). These studies extended the previous concepts in a way that sufficient and duality results could remain valid. These concepts are researched in two separate forms, smooth and nonsmooth. The procedure to extend convexity was not limited to invexity. But other concepts such as abstract convex functions by Rubinov (2000), DC functions by Horst and Thoai (1999), and star-shaped functions by Cambini and Martein (2009) were also introduced. Of course, the quasiconvex analysis also plays a crucial role in weakening the convexity condition (Penot 1998, 2000). Since these concepts are not related to this study, they are not delved into. But *why was invexity taken out of focus*? The authors present some reasons for that: First, it seems that the extensions of invex functions become so abstract in some levels that one can hardly find a model for them in the real world and they can only be used as some artificial samples. Of course, we do not consider them as very serious because the "theory of optimization" is one of the crossing points of pure and applied mathematics and it is natural for it to inherit the abstractness from pure mathematics (specifically from mathematical analysis), while in some issues, it fully functions within the framework of the applied mathematics (e.g., modeling). Another reason seems to be the similarity (and sometimes sameness) of proving the theorems. In fact, in many theorems which are proved under various extensions of invexity, only the underlying assumptions will change and the foundations will remain unchanged and these repetitions will drastically reduce the attraction and novelty of the subject. Repetition of techniques in some articles cannot be ignored, and we think nobody would approve them. Unfortunately, this is a plague which many scientific papers suffer from (including papers related to optimization) and invexity is not the only subject affected by it. But the main reason for lack of interest in researchers on invexity is the publication of article by Zalinescu (2014). In fact, it may be claimed that the heaviest strike on invexity has been caused by this beautiful, exact, and clever paper. Zalinescu (2014) in his article, which is a criticism of several other papers, mentions some basic points.

- One is that, in the definition of invex functions (according to some references), a logical instrument exists and its consequence is that all functions should be inevitably invex. It is clear that this problem can be solved by a replacement in the assumption of existence of $\eta(x, y)$. Fortunately, this replacement for $\eta(x, y)$ in the definition of invex function does not invalidate the proofs of previous theorems and the verdicts are still valid (as long as the authors have studied).
- Another problem that Zalinescu highlights is the falsity of some of the theorems in the invexity-related articles to which many have referred. It is clear that it is not the fault by invexity and negligence in inference and argument we have caused these problems.

Considering what has been mentioned in above, we still believe that invexity and its extensions could be studied more, and for this purpose, we investigate the concept of (Φ, ρ)-invexity which has been introduced by Caristi et al. (2010) and try to present its nonsmooth version. Of course, it should be mentioned that, in this study, if we replace "(Φ, ρ)-invex" by "invex," the results will still be original which are the extensions of the existing theorems in other articles and we have added the concept of (Φ, ρ)-invexity to these extensions so that our results could be more general. We organize the paper as follows. In the next section, we provide the preliminary results to be used in the rest of the paper. In Sect. 3 (resp. 4), we present some necessary and sufficient optimality conditions and duality results for weak (resp. proper) efficient solutions of nonsmooth MOSIPs.

2 Preliminaries

In this section, we briefly overview some notions of nonsmooth analysis widely used in formulations and proofs of main results of the paper. Nonsmooth analysis refers to differential analysis in the absence of differentiability. It can be regarded as a subfield of that vast subject known as nonlinear analysis. While nonsmooth analysis has classical roots, it is only in the last decades that the subject has grown rapidly. To the point, in fact, that further development has sometimes appeared in danger of being stymied, due to the plethora of definitions and unclearly related theories. For more details, discussion, and applications, see (Clarke 1983; Hiriart-Urruty and Lemarechal 1991). Our notation and terminology are basically standard. As usual, $||x||$ stands for the Euclidean norm of $x \in \mathbb{R}^n$, and $\mathbb{B}_n$ denotes the closed unit ball in $\mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, we write $x \leqq y$ (resp. $x < y$) when $x_i \leq y_i$ (resp. $x_i < y_i$) for all $i \in \{1, \ldots, n\}$. Moreover, we write $x \leq y$ when $x \leqq y$ and $x \neq y$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n. Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote by $\overline{A}$, $conv(A)$, and $cone(A)$, the closure of A, the convex hull and convex cone (containing the origin) generated by A, respectively. Also, we denote the Clarke tangent cone of A at $\hat{x} \in \overline{A}$ by $\Gamma(A, \hat{x})$, i.e.,

$$\Gamma(A, \hat{x}) := \Big\{ v \in \mathbb{R}^n \mid \forall \{x_r\} \subseteq A,\ x_r \to \hat{x},\ \forall t_r \downarrow 0, \\ \exists v_r \to v \text{ such that } x_r + t_r v_r \in A \ \ \forall r \in \mathbb{N} \Big\}.$$

Let $\hat{x} \in \mathbb{R}^n$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of φ at $\hat{x}$ in the direction $v \in \mathbb{R}^n$ and the Clarke subdifferential of φ at $\hat{x}$ are, respectively, given by

$$\varphi^0(\hat{x}; v) := \limsup_{y \to \hat{x},\ t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}$$

and

$$\partial_c \varphi(\hat{x}) := \Big\{ \xi \in \mathbb{R}^n \mid \langle \xi, v \rangle \leq \varphi^0(\hat{x}; v) \quad \text{for all } v \in \mathbb{R}^n \Big\}.$$

The Clarke subdifferential is a natural generalization of the classical derivative since it is known that when function φ is continuously differentiable at $\hat{x}$, $\partial_c \varphi(\hat{x}) = \{\nabla \varphi(\hat{x})\}$. Moreover, when a function φ is convex, the Clarke subdifferential coincides with $\partial \varphi(\hat{x})$, the subdifferential in the sense of convex analysis, i.e.,

$$\partial \varphi(\hat{x}) := \Big\{ \xi \in \mathbb{R}^n \mid \varphi(x) \geq \varphi(\hat{x}) + \big\langle \xi, x - \hat{x} \big\rangle \quad \forall\, x \in \mathbb{R}^n \Big\}.$$

It is worth to observe that if $\hat{x}$ is a minimizer of locally Lipschitz function ϕ on a set C, then

$$0 \in \partial_c \phi(\hat{x}) + N(C, \hat{x}),$$

where $N(C, \hat{x})$ denotes the Clarke normal cone of C at $\hat{x}$, i.e.,

$$N(C, \hat{x}) := \Big\{ x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0, \quad \forall a \in \Gamma(C, \hat{x}) \Big\}.$$

In the following theorem, we summarize some important properties of the Clarke subdifferential from Clarke (1983) which are widely used in what follows.

Theorem 1 *Let φ and ϕ be functions from $\mathbb{R}^n$ to $\mathbb{R}$ which are Lipschitz near $\hat{x}$. Then,*

(i) $\partial_c(\varphi+\phi)(\hat{x}) \subseteq \partial_c\varphi(\hat{x}) + \partial_c\phi(\hat{x})$.

(ii) $\partial_c\big(\lambda\varphi\big)(\hat{x}) = \lambda\partial_c\varphi(\hat{x}), \quad \forall\lambda \in \mathbb{R}$.

(iii) $\partial_c\Big(\max\{\varphi,\phi\}\Big)(\hat{x}) \subseteq conv\Big(\partial_c\varphi(\hat{x}) \cup \partial_c\phi(\hat{x})\Big)$.

(iv) $\partial_c\varphi(\hat{x})$ *is a nonempty, convex, and compact subset of* $\mathbb{R}^n$.

3 On the weak efficiency

In this paper, we consider the following multiobjective semi-infinite programming problem:

$$\begin{array}{ll} \text{(P)} & \inf\ \big(f_1(x), f_2(x), \ldots, f_p(x)\big) \\ \text{s.t.} & g_t(x) \leq 0 \quad t \in T, \\ & x \in \mathbb{R}^n, \end{array}$$

where f_i, $i \in I := \{1, 2, \ldots, p\}$ and g_t, $t \in T$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$, and the index set T is arbitrary, not necessarily finite (but nonempty). An important feature of problem (P) is that the index set T is arbitrary, i.e., may be infinite and also noncompact. When T is finite, (P) is a multiobjective optimization problem, and when $p = 1$ and T is infinite, (P) is a semi-infinite optimization problem. The feasible set of (P) is denoted by M, i.e.,

$$M := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \quad \forall t \in T\}.$$

For each $\hat{x} \in M$, set

$$F_{\hat{x}} := \bigcup_{i\in I} \partial_c f_i(\hat{x}) \quad \text{and} \quad G_{\hat{x}} := \bigcup_{t\in T(\hat{x})} \partial_c g_t(\hat{x}),$$

where $T(\hat{x}) := \{t \in T \mid g_t(\hat{x}) = 0\}$. A feasible point $\hat{x}$ is said to be efficient solution [resp. weakly efficient solution] for (P) if and only if there is no $x \in M$ satisfying $f(x) \leq f(\hat{x})$ [resp. $f(x) < f(\hat{x})$]. Recall that the problem (P) is said to be continuous when T is a compact metric space, $g_t(x)$ is a continuous function of (t, x) in $T \times \mathbb{R}^n$, and $t \mapsto \partial_c g_t(x)$ is an upper semicontinuous (set-valued) mapping for each $x \in \mathbb{R}^n$. At almost all articles in (multiobjective) semi-infinite programming, the continuity of problem is assumed, even in differentiable case Caristi et al. (2010). The continuity of (P) implies the compactness of $F_{\hat{x}} \cup G_{\hat{x}}$ by Kanzi (2015); then, the strict separation theorem implies that $0 \in conv(F_{\hat{x}} \cup G_{\hat{x}})$ when $\hat{x}$ is a weakly efficient solution of the considered problem, and hence, the Fritz–John (FJ)-type necessary condition is satisfied for continuous (P) at a weakly efficient solution. For extension of this well-known result to noncontinuous (P), we recall the following definition from (Kanzi 2011, 2014).

Definition 1 We say that (P) has the Pshenichnyi–Levin–Valadier (PLV in short) property at $\hat{x} \in M$, if $\Psi(\cdot)$ is finite-valued Lipschitz around $\hat{x}$, and

$$\partial_c \Psi(\hat{x}) \subseteq conv\Big(\bigcup_{t \in T(\hat{x})} \partial_c g_t(\hat{x}) \Big) = conv\big(G_{\hat{x}}\big),$$

where $\Psi(\cdot)$ is defined as

$$\Psi(x) := \sup_{t \in T} g_t(x), \qquad \forall x \in M.$$

It should be observed from (Goberna and Kanzi 2017; Kanzi 2014) that the PLV property is strictly weaker than continuity for (P). Thus, the following simple theorem is better than its continuous versions Caristi et al. (2010).

Theorem 2 (FJ necessary condition) *Let $\hat{x}$ be a weakly efficient solution of (P). If the PLV property holds at $\hat{x}$, then there exist $\alpha_i \geq 0$ (for $i \in I$), and $\beta_t \geq 0$, (for $t \in T(\hat{x})$), with $\beta_t \neq 0$ for finitely many indexes, such that*

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \beta_t \partial_c g_t(\hat{x}), \quad \textit{and} \quad \sum_{i=1}^{p} \alpha_i + \sum_{t \in T(\hat{x})} \beta_t = 1.$$

Proof It is easy to see that $\hat{x}$ is a global minimizer for the function

$$\vartheta(x) := \max\{\Theta(x), \Psi(x)\},$$

where $\Theta(x) := \max_{i \in I}\{f_i(x) - f_i(\hat{x})\}$ and $\Psi(x)$ is defined as Definition 1. Thus, by PLV property we deduce that

$$0_n \in \partial_c \vartheta(\hat{x}) \subseteq conv\big(\partial_c \Theta(\hat{x}) \cup \partial_c \Psi(\hat{x})\big) \subseteq conv\Big(\big[conv(F_{\hat{x}})\big] \cup \big[conv(G_{\hat{x}})\big]\Big).$$

This means $0_n \in conv(F_{\hat{x}} \cup G_{\hat{x}})$, as required. □

As well as in the classical case, the optimality implies the Karush–Kuhn–Tucker (KKT) condition, provided some constraint qualifications are satisfied (see Caristi et al. 2010; Gao 2012, 2013; Goberna and Kanzi 2017; Guerra-Vazquez and Todorov 2016; Goberna et al. 2013; Kanzi 2015; López and Vercher 1983). Slater's condition (SC) is said to be satisfied for problem (P) if there exists a Slater point, $(x^*) < 0$ for all $t \in T$. It is easy to see that the SC cannot play the role of such constraint qualification without additional convexity assumption on the restriction functions. Also, the SC works only for continuous (multiobjective) semi-infinite problems (SIP). In fact, the

Slater constraint qualification (SCQ) for (multiobjective) SIP consists of the following three conditions (see López and Vercher 1983):

$$\begin{cases} \text{(I): Continuity of } (P). \\ \text{(II): SC.} \\ \text{(III): Convexity of } g_t \text{ functions for } t \in T. \end{cases}$$

In the line of extension of KKT necessary condition for (P), we are going to change: (I) the continuity of (P) by PLV property; (II) the SC by weak Slater's condition, introduced in Definition 2; and (III) the convexity of g_ts by (Φ, ρ)-invexity of them, defined in Definition 3. We will use the following weaker form of SC for SIP appeared in Hettich and Kortanek (1993).

Definition 2 We say that the weak Slater's condition (WSC in brief) is satisfied for (P) at $x_0 \in M$ if for each finite index set $T^* \subseteq T(x_0)$, there exists a point $x_{T^*} \in \mathbb{R}^n$ such that $g_t(x_{T^*}) < 0$ for all $t \in T^*$. Also, (P) is satisfied in global WSC (GWSC, briefly) if for each finite index set $T^* \subseteq T$, there exists a point $x_{T^*} \in \mathbb{R}^n$ such that $g_t(x_{T^*}) < 0$ for all $t \in T^*$.

Clearly, WSC at a point is strictly weaker than GWSC. The following example shows that the GWSC is strictly weaker than SC.

Example 1 Let $T := \mathbb{N} \cup \{0\}$, and

$$g_t(x) := \begin{cases} x - \dfrac{1}{t} & \text{if} \quad t \in \mathbb{N}, \\ -x & \text{if} \quad t = 0. \end{cases}$$

It is easy to check that $M = \{0\}$, and SC does not hold. Now, assume that T^* is a finite subset of T. Take $\max(T^*) := q$ and $x_{T^*} := \frac{1}{q+1}$. Then,

$$g_t(x_{T^*}) = \begin{cases} x_{T^*} - \dfrac{1}{t} = \dfrac{1}{q+1} - \dfrac{1}{t} < 0 & \text{if} \quad t \in \mathbb{N} \cap T^*, \\ \\ -x_{T^*} = -\dfrac{1}{q+1} < 0 & \text{if} \quad t = 0. \end{cases}$$

Thus, the GWSC holds.

Definition 3 Suppose that the functions $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and the nonempty set $X \subseteq \mathbb{R}^n$ are given. A locally Lipschitz function $\hbar : \mathbb{R}^n \to \mathbb{R}$ is said to be (Φ, ρ)-invex at $x^* \in X$ with respect to X, if for each $x \in X$ one has:

$$\Phi\big(x, x^*, 0_n, r\big) \geq 0 \quad \text{for all } r \geq 0, \tag{1}$$

$$\Phi(x, x^*, ., .) \text{ is convex on } \mathbb{R}^n \times \mathbb{R}, \tag{2}$$

$$\Phi\Big(x, x^*, \xi, \rho(x, x^*)\Big) \leq \hbar(x) - \hbar(x^*), \quad \forall \xi \in \partial_c \hbar(x^*). \tag{3}$$

Remark 1 Since in Antczak (2015), Antczak and Stasiak (2011), Antczak (2012) the considered ρ is real number, Definition 2 is more general than them.

As mentioned by Antczak (2015), the definition of (Φ, ρ)-invexity generalizes the almost all concepts of invexity and convexity. Also, some (Φ, ρ)-invex functions are presented by Antczak (2015) which are not convex. The following example shows that the (Φ, ρ)-invexity is strictly weaker than invexity.

Example 2 Consider a function $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x, y, u, w) := \begin{cases} w - \dfrac{u}{3y^2}|x^3 - y^3| & \text{if} \quad y \neq 0, \\ w|x^3| & \text{if} \quad y = 0. \end{cases}$$

Let x and $\hat{x}$ be arbitrary elements of $\mathbb{R}$. Since $\Phi(x, \hat{x}, ., .)$ is a linear function and

$$\Phi(x, y, 0, r) = \begin{cases} r & \text{if} \quad y \neq 0, \\ r|x^3| & \text{if} \quad y = 0, \end{cases}$$

conditions (1) and (2) hold. Take $\rho(x, y) := -1$ for all $x, y \in \mathbb{R}$, and $\hbar(x) := x^3$. Since $\hbar(.)$ is continuously differentiable on $\mathbb{R}$, $\partial_c \hbar(\hat{x}) = \{3\hat{x}^2\}$. Thus, the following relations show that $\hbar(.)$ is a (Φ, ρ)-invex function at each $\hat{x} \in \mathbb{R}$ with respect to $\mathbb{R}$,

$$\Phi(x, \hat{x}, 3\hat{x}^2, -1) = \begin{cases} -1 - |x^3 - \hat{x}^3| & \text{if} \quad \hat{x} \neq 0, \\ -|x^3| & \text{if} \quad \hat{x} = 0, \end{cases}$$

$$\leq x^3 - \hat{x}^3 = \hbar(x) - \hbar(\hat{x}).$$

Furthermore, as it follows by Ben-Israel and Mond (1986), $\hbar(.)$ is not an invex function on $\mathbb{R}$ with respect to any $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

It is noteworthy that the strength of the Example 2 is that $\hbar(.)$ is (Φ, ρ)-invex for any $\hat{x} \in \mathbb{R}^n$ but its weakness is that $\hbar(.)$ is continuously differentiable and ρ is a fixed number. The following example presents a function $\hbar : \mathbb{R} \to \mathbb{R}$ that is nonsmooth at $x^* = 0$ and (Φ, ρ)-invex at $\hat{x}$ for a ρ that depends on x and x^*.

Example 3 Consider the functions $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, respectively, defined as

$$\Phi(x, y, u, w) := \begin{cases} w - \dfrac{u}{3y^2}|x^3 - y^3| & \text{if} \quad y \neq 0, \\ (w + u^2)|x^3| & \text{if} \quad y = 0. \end{cases}$$

$$\rho(x, y) := \begin{cases} (1 - \frac{1}{x^2}) + xy & \text{if} \quad x > 0, \\ -(1 + \frac{1}{x^2}) + y & \text{if} \quad x < 0, \\ 1 + y & \text{if} \quad x = 0. \end{cases}$$

Let $x^* = 0$ and $\hbar : \mathbb{R} \to \mathbb{R}$ as $\hbar(x) := x^3 - |x|$, obviously the conditions (1) and (2) hold. For checking condition (3), we observe that for each $\xi \in \partial_c \hbar(x^*) = [-1, 1]$ and $x \in \mathbb{R}$ we have:

$$\Phi(x, x^*, \xi, \rho(x, x^*)) = (\rho(x, 0) + \xi^2)|x^3| \leq \rho(x, 0)|x^3|$$

$$= \begin{cases} (1 - \frac{1}{x^2})|x^3| & \text{if} \quad x > 0, \\ -(1 + \frac{1}{x^2})|x^3| & \text{if} \quad x < 0, \\ |x^3| & \text{if} \quad x = 0. \end{cases}$$

$$= \begin{cases} (x^3 - x) & \text{if} \quad x > 0, \\ (x^3 + x) & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

$$= (x^3 - |x|) = \hbar(x) = \hbar(x) - \hbar(x^*).$$

Everywhere in the following, we will assume X equals to feasible solution of (P), i.e., $X = M$, but for the sake of simplicity we will omit to mention X. The following definition is motivated by above comments.

Definition 4 Let $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a given function, and $\hat{x} \in M$. We say that (P) satisfies the Φ-weak SCQ (Φ-WSCQ, briefly) at $\hat{x}$, if

- the PLV property holds at $\hat{x}$,
- WSC is satisfied at $\hat{x}$,
- for each $t \in T(\hat{x})$, the g_t function is (Φ, ρ_t)-invex at $\hat{x}$ for some given function $\rho_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Normally, we are interested to show Karush–Kuhn–Tucker necessary condition for (P) under Φ-WSCQ assumption. In fact, the following theorem guaranties that Φ-WSCQ is a constraint qualification.

Theorem 3 (KKT necessary condition) *Let $\hat{x}$ be a weak efficient solution of (P). Suppose that the Φ-WSCQ is satisfied at $\hat{x}$ with $\rho_t(x, \hat{x}) \geq 0$ for every $(x, t) \in \mathbb{R}^n \times T$. Then, there exist $\alpha_i \geq 0$ (for $i \in I$) with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, (for $t \in T(\hat{x})$), with $\beta_t \neq 0$ for finitely many indexes, such that*

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \beta_t \partial_c g_t(\hat{x}). \tag{4}$$

Proof Applying Theorem 2, we find some $\alpha_i \geq 0$ and $\xi_i \in \partial_c f_i(\hat{x})$ for $i \in I$, $\beta_t \geq 0$ and $\zeta_t \in \partial_c g_t(\hat{x})$ for $t \in T^* \subseteq T(\hat{x})$ with $|T^*| < \infty$, such that

$$\sum_{i \in I} \alpha_i \xi_i + \sum_{t \in T^*} \beta_t \zeta_t = 0_n, \qquad \sum_{i \in I} \alpha_i + \sum_{t \in T^*} \beta_t = 1.$$

All we need to prove is that at least one α_i should be positive. If it is not this case, then

$$\sum_{t\in T^*} \beta_t \zeta_t = 0_n, \qquad \sum_{t\in T^*} \beta_t = 1.$$

Thus, owing to $\sum_{t\in T^*} \beta_t \rho_t(x_{T^*}, \hat{x}) \geq 0$ and Definition 2, we get

$$\begin{aligned} 0 \leq \Phi\Big(x_{T^*}, \hat{x}, \sum_{t\in T^*} \beta_t \zeta_t, \sum_{t\in T^*} \beta_t \rho_t(x_{T^*}, \hat{x})\Big) \\ \leq \sum_{t\in T^*} \beta_t \Phi\big(x_{T^*}, \hat{x}, \zeta_t, \rho(x_{T^*}, \hat{x})\big) \\ \leq \sum_{t\in T^*} \beta_t \big(g_t(x_{T^*}) - g_t(\hat{x})\big) = \sum_{t\in T^*} \beta_t g_t(x_{T^*}) < 0. \end{aligned}$$

This contradiction justifies the result. □

In order to recall the role of (Φ, ρ)-invexity in KKT sufficient condition, we recall the following result from Kanzi (2017). It is noteworthy that the following theorem is proved in Caristi et al. (2010) for MPVCs with differentiable data, and for the case $p = 1,\ |T| < \infty$ in Antczak and Stasiak (2011).

Theorem 4 (KKT sufficient condition) *Suppose that there exist a feasible solution $\hat{x} \in M$, and scalars $\alpha_i \geq 0$ (for $i \in I$) with $\sum_{i=1}^{p} \alpha_i = 1$, and a finite set $T^* := \{t_1, t_2, \ldots, t_m\} \subseteq T(\hat{x})$, and scalars $\beta_{j_s} \geq 0$ (for $s \in \{1, 2, \ldots, m\}$) such that*

$$0 \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(\hat{x}) + \sum_{s=1}^{m} \beta_{t_s} \partial_c g_{t_s}(\hat{x}). \tag{5}$$

Moreover, if the f_i functions are generalized (Φ, ρ_i)-invex at $\hat{x}$ and the g_t functions are generalized (Φ, ρ_t)-invex at $\hat{x}$ (for $(i, t) \in I \times T(\hat{x})$), and $\sum_{i=1}^{p} \alpha_i \rho_i(x, \hat{x}) + \sum_{s=1}^{m} \beta_{t_s} \rho_{t_s}(x, \hat{x}) \geq 0$ for all $x \in M$, then $\hat{x}$ is a weak efficient solution for (P).

Now, in accordance with Caristi et al. (2006) and similar to Antczak (2009a, b), we introduce a dual of Mond and Weir (1981) type for (P) that is in connection with weak efficiency. For $y \in \mathbb{R}^n$, $T^* \subseteq T$ with $|T^*| < \infty$, and $\beta := (\beta_t)_{t\in T^*} \geq 0_{|T^*|}$, put

$$f^\sharp(y, \beta, T^*) := \Big(f_1(y) + \sum_{t\in T^*} \beta_t g_t(y), \ldots, f_p(y) + \sum_{t\in T^*} \beta_t g_t(y)\Big).$$

Consider the following dual problem:

$$(D_1^\sharp): \qquad \max \big\{f^\sharp(y, \beta, T^*) \mid \exists \alpha := (\alpha_i)_{i\in I}, \quad (\alpha, y, \beta, T^*) \in M_1^\sharp\big\},$$

where the feasible set $M_1^\sharp$ is defined by

$$M_1^\sharp := \Big\{(\alpha, y, \beta, T^*) \mid y \in \mathbb{R}^n,\ T^* \subseteq T,\ |T^*| < \infty,\ (\beta_t)_{t\in T^*} \geq 0_{|T^*|},$$
$$(\alpha_i)_{i\in I} \geq 0_p, \sum_{i=1}^{p} \alpha_i = 1,\ \ 0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(y) + \sum_{t\in T^*} \beta_t \partial_c g_t(y)\Big\}.$$

Since the proofs of following two theorems are similar to Antczak (2015) and Caristi et al. (2010), we omit them. Notice that we can rewrite the proofs of Theorems 9 and 10 (which will be expressed in the next section) with a little change for the following theorems.

Theorem 5 (weak duality for $(D_1^\sharp)$) *Let $x \in M$ and $(\alpha, y, \beta, T^*) \in M_1^\sharp$. If the f_i functions for $i \in I$ are (Φ, ρ_i)-invex at y, the g_t functions for $t \in T^*$ are (Φ, ρ_t)-invex at y, and $\sum_{i=1}^{p} \alpha_i \rho_i(x, y) + \sum_{t\in T^*} \beta_t \rho_t(x, y) \geq 0$, then $f(x) \nless f^\sharp(y, \beta, T^*)$.*

Theorem 6 (strong duality for $(D_1^\sharp)$) *Suppose that $\hat{x}$ is a weak efficient of (P) and that Φ-WSCQ is satisfied at $\hat{x}$. If each f_i functions (for $i \in I$) is (Φ, ρ_i)-invex at $\hat{x}$ with $\rho_i(x, \hat{x}) \geq 0$ for every $x \in \mathbb{R}^n$, then there exist $\hat{\alpha}_i \geq 0$ (for $i \in I$) with $\sum_{i=1}^{p} \hat{\alpha}_i = 1$, and $T_1^* \subseteq T(\hat{x})$ with $|T_1^*| < \infty$, and $\hat{\beta}_t \geq 0$ (for $t \in T_1^*$), such that $(\hat{\alpha}, \hat{x}, \hat{\beta}, T_1^*) \in M_1^\sharp$ and $f(\hat{x}) = f^\sharp(\hat{x}, \hat{\beta}, T_1^*)$. Furthermore, $(\hat{\alpha}, \hat{x}, \hat{\beta}, T_1^*)$ is a weak efficient solution for dual problem $(D_1^\sharp)$.*

4 On the proper efficiency

Proper efficiency is a very important notion used in studying multiobjective optimization problems. There are many definitions of proper efficiency in literature, as those introduced by Geoffrion, Benson, Borwein, and Henig; see Guerraggio et al. (1994) for a comparison among the main definitions of this notion. We recall the following definition from Gopfert et al. (2003).

Definition 5 A point $\hat{x} \in M$ is called a properly efficient solution of (P) when there exists a $\lambda > 0_p$ such that

$$\langle \lambda, f(\hat{x})\rangle \leq \langle \lambda, f(x)\rangle, \qquad \forall x \in M.$$

As proved in Ehrgott (2005), the above definition of proper efficiency is weaker than its other definitions (under some assumed conditions). Thus, the following theorem can be extended to other sense of proper efficiency under further assumptions.

Theorem 7 (KKT strong necessary condition) *Let $\hat{x}$ be a proper efficient solution of (P). Suppose that the Φ-WSCQ is satisfied at $\hat{x}$ with $\rho_t(x, \hat{x}) \geq 0$ for every $x \in \mathbb{R}^n$. Then, there exist $\alpha_i > 0$ (for $i \in I$) with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, (for $t \in T(\hat{x})$), with $\beta_t \neq 0$ for finitely many indexes, such that (4) fulfilled.*

Proof By the definition of proper efficiency, there exist some scalars $\lambda_i > 0$ (for $i \in I$) such that $\hat{x}$ is a minimizer of the following scalar semi-infinite problem:

$$\min_{x \in M} \sum_{i=1}^{p} \lambda_i f_i(x).$$

Applying Theorem 3, we get

$$0_n \in \partial\Big(\sum_{i=1}^{p} \lambda_i f_i(\cdot)\Big)(\hat{x}) + \sum_{t \in T(\hat{x})} \mu_t \partial_c g_t(\hat{x}) \subseteq \sum_{i=1}^{p} \lambda_i \partial_c f_i(\hat{x}) + \sum_{t \in T(\hat{x})} \mu_t \partial_c g_t(\hat{x}),$$

for some $\mu_t \geq 0$, $(t \in T(\hat{x}))$, with $\mu_t \neq 0$ for finitely many indexes. For each $i \in I$ take $\alpha_i := \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i}$, and for each $t \in T(\hat{x})$ put $\beta_t := \frac{\mu_t}{\sum_{i=1}^{p} \lambda_i}$. □

Now, we state the sufficient condition for proper efficiency of (P) as follows.

Theorem 8 (KKT strong sufficient condition) *Let $\hat{x} \in M$. Suppose that there exist some scalars $\alpha_i > 0$ (for $i \in I$) with $\sum_{i=1}^{p} \alpha_i = 1$, a finite set $T^* := \{t_1, t_2, \ldots, t_m\} \subseteq T(\hat{x})$, and some scalars $\beta_{t_s} \geq 0$ (for $s \in \{1, 2, \ldots, m\}$) such that (4) holds. Moreover, if the f_i functions for $i \in I$ are (Φ, ρ)-invex at $\hat{x}$ (this means that all ρ_is are equal to ρ for $i \in I$), and the g_{t_s} functions for $s \in \{1, \ldots, m\}$ are (Φ, ρ_{t_s})-invex at $\hat{x}$, and $\sum_{i=1}^{p} \alpha_i \rho_i(x, \hat{x}) + \sum_{s=1}^{m} \beta_{t_s} \rho_{t_s}(x, \hat{x}) \geq 0$ for all $x \in M$, then $\hat{x}$ is a proper efficient solution for (P).*

Proof Owing to (4), we can find some $\xi_i \in \partial_c f_i(\hat{x})$ and $\zeta_{t_s} \in \partial_c g_{t_s}(\hat{x})$ for $i \in I$ and $s \in \{1, \ldots, m\}$) such that

$$\sum_{i=1}^{p} \alpha_i \xi_i + \sum_{s=1}^{m} \beta_{t_s} \zeta_{t_s} = 0_n, \qquad \sum_{i=1}^{p} \alpha_i + \sum_{s=1}^{m} \beta_{t_s} > 0.$$

Put $\widehat{\alpha}_i := \frac{\alpha_i}{\sum_{i=1}^{p} \alpha_i + \sum_{s=1}^{m} \beta_{t_s}}$ and $\widehat{\beta}_{t_s} := \frac{\beta_{t_s}}{\sum_{i=1}^{p} \alpha_i + \sum_{s=1}^{m} \beta_{t_s}}$. Due to assumptions of theorem, the following inequalities hold:

$$\begin{aligned} 0 &\leq \Phi\Big(x, \hat{x}, \sum_{i=1}^{p} \widehat{\alpha}_i \xi_i + \sum_{s=1}^{m} \widehat{\beta}_{t_s} \zeta_{t_s}, \sum_{i=1}^{p} \widehat{\alpha}_i \rho_i(x, \hat{x}) + \sum_{s=1}^{m} \widehat{\beta}_{t_s} \rho_{t_s}(x, \hat{x})\Big) \\ &\leq \sum_{i=1}^{p} \widehat{\alpha}_i \Phi\big(x, \hat{x}, \xi_i, \rho_i(x, \hat{x})\big) + \sum_{s=1}^{m} \widehat{\beta}_{t_s} \Phi\big(x, \hat{x}, \zeta_{t_s}, \rho_{t_s}(x, \hat{x})\big). \end{aligned} \tag{6}$$

On the other hand, the (Φ, ρ_{t_s})-invexity of g_{t_s} functions implies that for each $s \in \{1, \ldots, m\}$ we get

$$\Phi\big(x, \hat{x}, \zeta_{t_s}, \rho_{t_s}(x, \hat{x})\big) \leq g_{t_s}(x) - g_{t_s}(\hat{x}) = g_{t_s}(x) \leq 0. \tag{7}$$

From (6) and (7), we conclude that

$$\sum_{i=1}^{p} \widehat{\alpha}_i \Phi\big(x, \hat{x}, \xi_i, \rho_i(x, \hat{x})\big) \geq 0 \Rightarrow \sum_{i=1}^{p} \alpha_i \Phi\big(x, \hat{x}, \xi_i, \rho_i(x, \hat{x})\big) \geq 0. \tag{8}$$

Employing [Antczak and Stasiak (2011), Proposition 11] and $\sum_{i=1}^{p} \alpha_i = 1$, we deduce that $\sum_{i=1}^{p} \alpha_i f_i$ is a (Φ, ρ)-invex function. Therefore, (8) yields that

$$\sum_{i=1}^{p} \alpha_i f_i(\hat{x}) \leq \sum_{i=1}^{p} \alpha_i f_i(x), \qquad \forall x \in M.$$

Since $\alpha_i > 0$ for all $i \in I$, we can take $\lambda_i = \alpha_i$ in Definition 5, and the proof is complete. □

For given $y \in \mathbb{R}^n$, $T^* \subseteq T$ with $|T^*| < \infty$, and $\beta := (\beta_t)_{t \in T^*} \geq 0_{|T^*|}$, set the $f^\sharp(y, \beta, T^*)$ function as previous section. In connection with the problem (P), we consider the following dual of Mond–Weir-type problem (which is different with $(D_1^\sharp)$):

$$(D_2^\sharp): \qquad \max \left\{ f^\sharp(y, \beta, T^*) \mid \exists \alpha := (\alpha_i)_{i \in I}, \quad (\alpha, y, \beta, T^*) \in M_2^\sharp \right\},$$

in which $M_2^\sharp$ is defined as

$$M_2^\sharp := \Big\{ (\alpha, y, \beta, T^*) \mid y \in \mathbb{R}^n, \ T^* \subseteq T, \ |T^*| < \infty, \ (\beta_t)_{t \in T^*} \geq 0_{|T^*|},$$
$$(\alpha_i)_{i \in I} > 0_p, \sum_{i=1}^{p} \alpha_i = 1, \ 0_n \in \sum_{i=1}^{p} \alpha_i \partial_c f_i(y) + \sum_{t \in T^*} \beta_t \partial_c g_t(y) \Big\},$$

Remark 2 It should be noted that the dual problems presented in some articles are constructed with infinite index sets (unlike the problems $(D_1^\sharp)$ and $(D_2^\sharp)$); to study such problems, we can refer to surveys (Hettich and Kortanek 1993; López and Still 2007) and their references. Clearly, dual problems defined by a finite index set (including the problems $(D_1^\sharp)$ and $(D_2^\sharp)$) are more efficient and usable in terms of application.

The next two theorems describe (weak and strong) duality relations between the prime problem (P) and the dual problem $(D_2^\sharp)$.

Theorem 9 (weak duality for $(D_2^\sharp)$) *Let $x \in M$ and $(\alpha, y, \beta, T^*) \in M_2^\sharp$. If the f_i functions for $i \in I$ are (Φ, ρ_i)-invex at y, the g_t functions for $t \in T^*$ are (Φ, ρ_t)-invex at y, and $\sum_{i=1}^{p} \alpha_i \rho_i(x, y) + \sum_{t \in T^*} \beta_t \rho_t(x, y) \geq 0$, then $f(x) \nleq f^\sharp(y, \beta, T^*)$.*

Proof Assume on the contrary that $f(x) \leq f^{\sharp}(y, \beta, T^*)$. Thus, there exists an index $k \in I$ such that

$$\begin{cases} f_i(x) \leq f_i(y) + \sum_{t \in T^*} \beta_t g_t(y), & \forall i \in I \setminus \{k\}, \\ f_k(x) < f_k(y) + \sum_{t \in T^*} \beta_t g_t(y). \end{cases} \tag{9}$$

For each $i \in I$ and $t \in T^*$, set

$$\widetilde{\alpha}_i := \frac{\alpha_i}{1 + \sum_{t \in T^*} \beta_t}, \quad \text{and} \quad \widetilde{\beta}_t := \frac{\beta_t}{1 + \sum_{t \in T^*} \beta_t}.$$

Since $(\alpha, y, \beta, T^*) \in M_2^{\sharp}$, we have $\sum_{i=1}^{p} \widetilde{\alpha}_i + \sum_{t \in T^*} \widetilde{\beta}_t = 1$, and

$$\sum_{i=1}^{p} \widetilde{\alpha}_i \xi_i + \sum_{t \in T^*} \widetilde{\beta}_t \zeta_t = 0_n, \tag{10}$$

for some $\xi_i \in \partial_c f_i(y),\ i \in I$, and $\zeta_t \in \partial_c g_t(y),\ t \in T^*$. It follows from (9) and $(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_p) > 0_p$ that

$$\begin{aligned} \sum_{i=1}^{p} \widetilde{\alpha}_i \big(f_i(x) - f_i(y)\big) &< \sum_{i=1}^{p} \Big[\widetilde{\alpha}_i \sum_{t \in T^*} \beta_t g_t(y)\Big] \\ &= \Big(\sum_{t \in T^*} \beta_t g_t(y)\Big)\Big(\sum_{i=1}^{p} \widetilde{\alpha}_i\Big) \leq \sum_{t \in T^*} \beta_t g_t(y). \end{aligned} \tag{11}$$

At the same time, Eq. (10) implies that

$$\begin{aligned} 0 &\leq \Phi\Big(x, y, \sum_{i=1}^{p} \widetilde{\alpha}_i \xi_i + \sum_{t \in T^*} \widetilde{\beta}_t \zeta_t, \sum_{i=1}^{p} \widetilde{\alpha}_i \rho_i(x, y) + \sum_{t \in T^*} \widetilde{\beta}_t \rho_t(x, y)\Big) \\ &\leq \sum_{i=1}^{p} \widetilde{\alpha}_i \Phi\big(x, y, \xi_i, \rho_i(x, y)\big) + \sum_{t \in T^*} \widetilde{\beta}_t \Phi\big(x, y, \zeta_t, \rho_t(x, y)\big) \\ &\leq \sum_{i=1}^{p} \widetilde{\alpha}_i \big(f_i(x) - f_i(y)\big) + \sum_{t \in T^*} \beta_t \big(g_t(x) - g_t(y)\big) \\ &\leq \sum_{i=1}^{p} \widetilde{\alpha}_i \big(f_i(x) - f_i(y)\big) - \sum_{t \in T^*} \beta_t g_t(y), \end{aligned}$$

where the last inequality holds by $x \in M$. The above inequality contradicts (11), and the proof is complete. □

Theorem 10 (strong duality for $(D_2^{\sharp})$) *Suppose that $\hat{x}$ is a properly efficient solution of (P) and that Φ-WSCQ is satisfied at $\hat{x}$. If each f_i functions (for $i \in I$) is (Φ, ρ_i)-invex at $\hat{x}$ with $\rho_i(x, \hat{x}) \geq 0$ for every $x \in \mathbb{R}^n$, then there exist $\hat{\alpha}_i > 0$ (for $i \in I$)*

with $\sum_{i=1}^{p} \hat{\alpha}_i = 1$, and $T_1^* \subseteq T(\hat{x})$ with $|T_1^*| < \infty$, and $\hat{\beta}_t \geq 0$ (for $t \in T_1^*$), such that $(\hat{\alpha}, \hat{x}, \hat{\beta}, T_1^*) \in M_2^\sharp$ and $f(\hat{x}) = f^\sharp(\hat{x}, \hat{\beta}, T_1^*)$. Furthermore, $(\hat{\alpha}, \hat{x}, \hat{\beta}, T_1^*)$ is an efficient solution for dual problem $(D_2^\sharp)$.

Proof According to Theorem 7, there exist $\hat{\alpha} := (\hat{\alpha}_i)_{i \in I} > 0_p$, with $\sum_{i=1}^{p} \hat{\alpha}_i = 1$, and $T_1^* \subseteq T(\hat{x})$ with $|T_1^*| < \infty$, and $\hat{\beta}_t \geq 0,\ t \in T_1^*$ such that

$$0_n \in \sum_{i=1}^{p} \hat{\alpha}_i \partial_c f_i(\hat{x}) + \sum_{t \in T_1^*} \hat{\beta}_t \partial_c g_t(\hat{x}).$$

This means $(\hat{\alpha}, \hat{x}, \hat{\beta}, T_1^*) \in M_2^\sharp$. Obviously, $t \in T_1^* \subseteq T(\hat{x})$ implies

$$f^\sharp(\hat{x}, \hat{\beta}, T_1^*) := \Big(f_1(\hat{x}) + \sum_{t \in T_1^*} \beta_t g_t(\hat{x}), \ldots, f_p(\hat{x}) + \sum_{t \in T_1^*} \hat{\beta}_t g_t(\hat{x})\Big) = f(\hat{x}). \tag{12}$$

Now, invoking the weak duality result in Theorem 9, and take (12) into account, we get $f^\sharp(\hat{x}, \hat{\beta}, T_1^*) \nleq f^\sharp(y, \beta, T^*)$ for any $(\alpha, y, \beta, T^*) \in M_2^\sharp$. This gives us that $(\hat{x}, \hat{\beta}, T_1^*)$ is an efficient solution for the dual problem $(D_2^\sharp)$. □

Remark 3 Note that our strong duality result appeared in Theorem 10 is not in an ordinary way; that is, the solution of the dual problem is not guaranteed properly efficient, only efficient, although the solution to the prime one is properly efficient. It is noteworthy that in [Antczak (2012), Theorem 19] a relationship has been established between the properly efficient solutions of primary and dual problems; and this is made possible by choosing the concept of properly efficient in Geoffrion type in Antczak (2012), which is weaker than properly efficient in type of Definition 5 [see Ehrgott (2005), Theorem 3.11].

Remark 4 Similar to Caristi et al. (2010), we can define quasi (Φ, ρ)-quasiinvex functions and (Φ, ρ)-pseudoinvex functions, and then, we can prove some weaker optimality and duality results for (P). Since the proof of these extensions are similar to previous theorems, we omit them.

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Conflict of interest The authors declare that they have no conflict of interest.

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