Ramsey rule with forward/backward utility for long-term yield curves modeling

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Abstract

This paper draws a parallel between the economic and financial points of view in the modeling of long-term yield curves and provides new results on asymptotic long rates. The Ramsey rule, which is the reference equation in the economic literature to compute long-term discount rates, links endogenous discount rate and marginal utility of aggregate optimal consumption at equilibrium. This paper proposes a unified framework and a financial interpretation of the economic discount rate given by the Ramsey rule, using marginal utility indifference prices for non-replicable zero-coupon bonds. Optimal discounted pricing kernel is at the core of this unifying approach and is determined through an optimization program that can be posed backward or forward. The dynamics and the long-term behavior of the marginal utility yield curve is studied in both settings. Special attention is paid to its dependency on the initial wealth of the economy, as well as on the time-horizon in the backward setting, extending previous results in the literature.

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Introduction

Modeling accurately long-term interest rates is a crucial challenge in many financial topics, such as the financing of ecological projects, or the pricing of longevity-linked securities or any other investment with long-term impact. The standard valuation methodology to evaluate such investment projects relies on a cost–benefit analysis. Once this cost–benefit analysis has been conveyed, the main question arising is how to compare valuations of projects' impacts with different temporally distributed cash-flows. This is especially crucial when they have different and long maturities. To answer this question, one key ingredient is the discount rate, used to compute the present value of each cash flow. As in Gollier (2012), the discount rate is defined here as the minimum rate of return to implement a non-risky investment project. For the evaluation of risky projects, this discount rate should be adapted to take into account the degree of uncertainty of the project: different methodologies exist, for example relying on CAPM (capital asset pricing model), or transforming each future cash flow into its certainty equivalent, which is related to the marginal indifference price (see Sect. 4.2.2). Interest rates are at the core of three main sectors: government bond and public policies, fixed income market and rate derivatives, and private long-term investment project and pensions funds. Each sector addresses its own issues, and consequently has its own point of view and its own modeling. Besides, as mentioned by Piazzesi (2010), "in most industrialized countries, the central bank seems to be able to move the short term of the yield curve. What matters for aggregate demand, however, are long-term yields". In the meantime, the debate on ecological issues and global warning has replaced to the front of the stage the difficulty to reach a census on the notion of rates which are standard both in economy and finance. Therefore, it is important to bring coherence to those points of view, and even more since the Covid-19 crisis that prompts us to think "long term" and "global". Our paper aims first to provide a unified framework to highlight similarities and differences between those approaches. Thanks to its formalism, the mathematics help in clarifying the notions in a unified way, while being as neutral as possible. Second, we propose a new model to evaluate rates, based on dynamic utilities and indifference pricing. Our model, which is linked to financial markets, still offers interesting economic interpretations, as well a mathematical robustness. Particular attention is paid on the dependency of rates on the initial wealth of the economy, and on the time-horizon, which is often downplayed in the literature.

Based on the equilibrium theory, an extensive literature has been developed to propose an endogenous definition of the economic discount rate. The Ramsey rule, introduced in 1928 by Ramsey in his seminal work (Ramsey 1928), is the reference equation to compute the discount rate. It has been further discussed by numerous economists such as Gollier (2010, 2012) and Weitzman (1998, 2007). The issue is addressed at a macroeconomic level, where long-run interest rates have not necessarily the same meaning as in financial markets. We call them "economic" interest rates because they are affected mainly by structural characteristics of the economy. The Ramsey rule links the discount rate with the marginal utility of aggregate consumption at the economic equilibrium. Besides, the financial framework is based on a no-arbitrage condition and links yield curves and zero-coupon bonds prices. Since the zero-coupon bond market is highly illiquid for long maturities, we use utility indif-

ference pricing for the evaluation of these non-replicable contingent claims. For a small amount of transaction, this pricing method leads to a linear pricing rule (see Davis 1998) called the *Davis price* or the *marginal utility price*. Then, according to the Ramsey rule, we show that equilibrium interest rates and marginal utility interest rates coincide. The economic and financial frameworks are actually very close: both rely on a similar optimization problem that determines the optimal discounted pricing kernel used to evaluate claims under the historical (also called physical) probability measure. The discounted pricing kernels are the key processes for yield curve modeling and provide a unifying approach for the economic and financial viewpoints. One main difference is that in the economic framework, it is the spot interest rate r (which is the drift term of the optimal discounted pricing kernel) that is determined endogenously by the market clearing condition at the equilibrium, while in the financial framework, r is exogenous and it is the orthogonal diffusion coefficient of the optimal discounted pricing kernel that is determined at the optimum. As utility functions are at the cornerstone of the Ramsey rule and its financial interpretation using marginal utility indifference price, this paper also provides an in-depth comparison analysis of the standard backward setting (in which the utility function at a time horizon T_H is given) and the forward setting (in which the initial utility is the one that is given). To satisfy time-consistency, the preference criterion should satisfy a dynamic programming principle, that is also called market consistency. Musiela and Zariphopoulou (2007, 2010) were the first to suggest to use instead of the classic criterion the concept of progressive dynamic utilities that have been further studied by El Karoui and Mrad (2013) and El Karoui et al. (2018) in a consumption framework. Progressive utilities give an adaptive way to model possible changes over the time of preferences of an agent, which is particularly important in this context of long-term decision making. They also provide a flexible tool to aggregate preferences of heterogeneous economic actors (see El Karoui et al. 2017). Contrary to the standard approach in which the optimal processes are computed through a backward analysis and emphasizing their dependency on the time-horizon of the optimization problem, the problem here is posed forward, leading to time-coherent optimal processes and putting emphasis on their monotonicity with respect to their initial values.

The paper is organized as follows. Section 1 introduces the Ramsey rule and highlights some features of the standard economic framework. Section 2 is dedicated to basic concepts of the economic equilibrium and financial no-arbitrage frameworks, highlighting their similarities and their differences. The related optimization problem that determines the discounted pricing kernel is presented in both the backward and forward settings. Section 3 develops those concepts in an Itô model. Section 4 provides a pathwise version of the Ramsey rule, written in terms of the optimal discounted pricing kernel and proposes a financial interpretation of the Ramsey rule and of the economic discount rates, using marginal utility indifference pricing. Utility indifference price with logarithmic utility corresponds to the benchmark approach of Platen and Heath (2006), but this special case does not allow us to capture the dependency on initial conditions such as the initial wealth. The yield curve dynamics is studied in Sect. 5, and using general marginal indifference price, special attention is paid on the dependency of the interest rates on the global wealth of the economy. Section 6 is devoted to the long-term behavior of the instantaneous forward rate and zero-coupon

rates, as well as to aggregated rates. In particular, in the case of backward power utilities, we provide a new relation between the orthogonal diffusion coefficient of the optimal discounted pricing kernel and the zero-coupon bond price. As a consequence, for non-replicable zero-coupon bonds, the time-horizon dependency of the discounted pricing kernel process and its orthogonal diffusion coefficient implies long-term yield curves that have a diffusion component and thus that are not necessarily monotonous in time. This extends previous results of Dybvig et al. (1996) and El Karoui et al. (1997) that did not take into account this time-horizon dependency (that only occurs in incomplete market). We illustrate our results with the important example of mixture of power utilities that corresponds to the aggregation of investors having different Constant Relative Risk Aversion (CRRA). We prove that when the maturity tends to infinity, the asymptotic long aggregate rate is the lowest individual asymptotic rate. The asymptotic limit with respect to the wealth of the economy is also studied: when the wealth tends to infinity the aggregate zero-coupon price converges to the one priced by the less risk averse agent, whereas when the wealth tends to zero, it converges to the one priced by the more risk averse agent. Finally, technical details and proofs on utility indifference pricing are postponed in the "Appendix".

1 The Ramsey rule

For the financing of ecological projects reducing global warming and any other investment with a long-term impact, it is necessary to model accurately long-run interest rates. In general, these issues are addressed at macroeconomic level, where long-run interest rates have not necessarily the same interpretation as in financial market. To avoid confusion, we refer to it as *socially efficient or economic* interest rates, because they would be mainly affected by structural characteristics of the economy and be low-sensitive to monetary policy. Correct estimates of these rates are useful for long-term decisions, and understanding their determinants is important.

General macroeconomic models often assume that at equilibrium, the sum of agents' choices is mathematically equivalent to the optimal decision of one individual, called the representative agent. More precisely, the economy is represented by the strategy of a risk-averse representative agent, whose utility function from consumption rate at date t is denoted $v(t, c)$. The macroeconomics literature typically relates the *economic equilibrium* rate to the time preference rate and to the average rate of productivity growth. Indeed, if one considers a small perturbation around the equilibrium that consists in investing (a small amount) in a project which is financed by a reduction of aggregate consumption, then, using a first degree Taylor approximation, it implies that the discount rate is related to the marginal rate of substitution between current and future consumption. In 1928, Ramsey in his seminal paper (Ramsey 1928) was the first to establish an economic model used to construct a scientific basis for the discount rate, which leads to the following definition.

Definition 1.1 We call Ramsey rule the link between the discount rate and the marginal utility of the optimal aggregate consumption (written below between time $t = 0$ and T)

$$R_0^e(T) = -\frac{1}{T} \ln \mathbb{E} \left[\frac{v_c(T, c_T^*)}{v_c(0, c_0^*)} \right] \qquad (1.1)$$

where c^* is the optimal consumption trajectory.

The Ramsey rule emphasizes the key role played by the marginal utility of consumption in the evaluation of the discount rate. This marginal utility will be interpreted hereafter as a discounted pricing kernel and creates a bridge between the economic and financial points of view.

1.1 Ramsey rule in a standard economic framework

In modern dynamic macroeconomics, it is standard to represent intertemporal behavior by a time separable intertemporal utility function with constant relative risk aversion θ ($0 < \theta < 1$) and time preference parameter (also called rate of impatience) λ: typically, the utility is proportional to $e^{-\lambda t} \frac{c^{1-\theta}}{1-\theta}$. In the seminal paper of Ramsey (1928), the optimal consumption is a deterministic function $c_t^* = c_0^* \exp(gt)$ (with g being the growth rate of the economy) and the Ramsey rule (1.1) becomes $R_0^e(T) = \lambda + \theta g$, in which the parameters should be calibrated in accordance with the time-horizon. Although this equation is very simple, there is no consensus on the parameter values. In the Stern review on the climate change (Stern and Stern 2007) and which addresses time horizons covering two centuries, $\theta = 0.1$, $g = 1.3\%$ $\lambda = 0.1\%$, which leads to a discount rate of 1.4%, whereas the UK-treasury uses a discount rate of 2.5% for maturity of 100 years. Thus, 1 million of dollars in 100 years is equivalent today either to 250,000 dollars or 82,000 dollars, depending on which rate is taken. In order to add some randomness in the future optimal consumption, the consumption process is frequently modeled by a geometric Brownian motion $c_t^* = c_0^* \exp(gt + \varphi W_t)$, still leading to a flat curve $R_0^e(T) = \lambda + \theta g - \frac{1}{2}\theta^2 \varphi^2$. The Ramsey rule is still the reference equation in macroeconomics and it was revisited by numerous economists, such as Gollier (2010, 2012) and Weitzman (1998, 2007).

1.2 Discussion on the robustness

Despite the tractability of simple models, they nevertheless lead to shortcuts that often hide certain dependencies and determinants. For example, the use of a power utility function is not an innocuous assumption and implies that the rate does not depend on the initial level of consumption (or equivalently on the initial wealth of the economy). Besides, economic rates are very sensitive to the rate of preference for the present, which can be viewed as the intensity of an independent exponential random horizon (see Remark 2.1). In this paper, we particularly focus on those dependencies by highlighting them in the notation, when needed.

Robustness with respect to the time preference parameter In an infinite time horizon, the time component of the utility is often taken as a discount rate of the form $e^{-\lambda t}$ where λ is interpreted as the time preference rate. In his seminal paper (Ramsey 1928), Ramsey prefers not to discount later enjoyments in comparison with earlier ones, "a

practice which is ethically indefensible and arises merely from the weakness of the imagination". Then, to overcome the problem of ill-posedness of the underlying optimization problem, he introduces a "maximum obtainable rate of enjoyment or utility" called "Bliss". In their paper (Hourcade and Lecocq 2004) motivated by ecological issue, Lecocq and Hourcade also emphasize the difficulty to calibrate this parameter λ. They interpret it as a preference of no sacrifice for the present. If we expect to consume more in the future, this parameter gives a lower bound for the Ramsey rule: indeed relation (1.1) applied to a time separable utility function with exponential decay at rate λ and assuming increasing expected consumption implies that $R_0^e(T) \geq \lambda$.

Robustness with respect to form of the utility In the Ramsey rule (1.1), apart from the time preference parameter λ, another key component is the marginal utility v_c. At equilibrium, the marginal rates of substitution $v_c(T, c_T)/v_c(0, c_0)$ between consumption at date 0 and at date T are equalized across agents and equal to the marginal rate of substitution of a representative agent whose consumption is equal to the aggregate consumption in the economy. The utility function of this representative agent is characterized by a risk tolerance (which is just the inverse of the absolute risk aversion) which is the mean of the absolute risk tolerance of all agents evaluated at their actual level of consumption, see (Wilson 1968). This means that at equilibrium, the utility of the representative agent is supposed to aggregates preferences of the heterogenous economic actors. This aggregation is very complex, and the aggregate utility is unlikely to have a simple expression, unless all agents are identical. In particular, it is shown in El Karoui et al. (2017) that assuming a consistent power utility for the representative agent actually implies that all agents have a power utility with the same risk aversion. Besides, in the presence of generalized long-term uncertainty, the decision scheme must evolve: economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions according to the evolution of the knowledge and to direct experiences, see (Hourcade and Lecocq 2004). Market-consistent progressive (also called forward) utilities (see Definition 2.1) provide a flexible framework to tackle those issues. They allow to take into account accurately the aggregation of preferences and to overcome the dependency in the time preference parameter, while leading to time-coherent strategies.

The next section is dedicated to basic concepts of the economic equilibrium and financial no-arbitrage frameworks. The purpose is to briefly present both the economic and financial points of view, and to point out the differences, that may be quite subtle. Although the results in this section are not completely new, we aim at providing a mathematical unifying framework, to shed a new light on concepts that are sometimes posed as evidence. The concept of preference criterion is central in this mathematical framework. We therefore recall briefly the definition of a utility function and its conjugate.

A UTILITY FUNCTION u is a strictly concave, increasing, and nonnegative function on $\mathbb{R}^+$, with continuous marginal utility u_z, satisfying the Inada conditions, $\lim_{z \mapsto \infty} u_z(z) = 0$ and $\lim_{z \mapsto 0} u_z(z) = +\infty$ to prevent 0 consumption at optimum. The risk aversion is measured by the ratio $R_A(u)(z) = -u_{zz}(z)/u_z(z)$ and the relative risk aversion by $R_A^r(u)(z) = z\, R_A(u)(z)$.

THE CONJUGATE OR DUAL UTILITY $\tilde{u}$ is the Fenchel–Legendre convex conjugate transformation of the utility function u, given by $\tilde{u}(\zeta) = \sup_{z>0}\big(u(z) - \zeta z\big)$. In particular, $\tilde{u}(\zeta) \geq u(z) - \zeta z$ and the maximum is attained at $u_z(z) = \zeta$. Under Inada conditions, $\tilde{u}$ is twice continuously differentiable, strictly convex, strictly decreasing, with $\tilde{u}(0^+) = u(+\infty)$, $\tilde{u}(+\infty) = u(0^+)$. Moreover, the marginal utility u_z is the inverse of the opposite of the marginal conjugate utility $\tilde{u}_\zeta$; that is $u_z^{-1}(\zeta) = -\tilde{u}_\zeta(\zeta)$; $\tilde{u}(\zeta) = u\big(-\tilde{u}_\zeta(\zeta)\big) + \tilde{u}_\zeta(\zeta)\,\zeta$, and $u(z) = \tilde{u}\big(u_z(z)\big) + z\,u_z(z)$. These strategic relations are also applied with stochastic utilities U (throughout the paper, we adopt the convention of capital letter for stochastic utility and small letter for deterministic utility).

2 The discounted pricing kernel: an unifying approach

We draw hereafter some parallels and comparisons between the economic and the financial frameworks for the modeling of interest rates and we present formally some strategic tools that are common to both frameworks. In particular we are concerned with the computation of the optimal aggregate consumption (c_t^*) that appears in the Ramsey rule (1.1). Overall, it is related to an optimization problem of the representative agent. His choice variables are how much to consume or save at each point in time, how much to invest in each security, under the constraint that no bankruptcy is permitted. His optimization problem is to maximize the expected utility over the class of admissible wealth-consumption processes subject to a continuous time budget constraint to be written down.

2.1 An economic and financial model setup

The economic and financial setups have a lot of similarities. For now, we just present the global picture and we emphasize points that are strategic for the paper: namely the time-horizon T_H, the initial conditions, the existence of a representative agent and his preference criterion. We refer to Björk (2020) for a detailed economic framework, and Sect. 3 for a general financial model.

The universe consists in long-lived securities (also called technology in economics) and a riskless security (a bank account) with short rate (r_t). The dynamic strategy of an investor is characterized by the portfolio investment π and a (nonnegative) consumption plan c that should be chosen in an admissible set denoted $\mathcal{A}$: in particular the corresponding wealth $X^{\pi,c}$ should remain positive (no bankruptcy). The set $\mathcal{X}^c$ of admissible wealth may have different forms, depending on the framework and the optimization problem that is considered. Usually it is a positive convex cone. The trades are assumed to occur continuously in time without any friction: no transaction costs and no taxes, and securities are infinitely divisible. The following optimization program has to be solved in both financial and economic frames; in the usual setting it is formulated on a given horizon T_H, and is written at time $t = 0$ as follows (given $X_0 = x$):

$$\mathcal{U}(0, x) := \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\Big(u(T_H, X_{T_H}^{\pi,c}) + \int_0^{T_H} v(t, c_t)\mathrm{d}t\Big). \qquad (2.1)$$

In the backward financial formulation, the utilities u and v of terminal wealth (at T_H) and of consumption rate are given. To ensure time-consistency, it is important to identify which "terminal" criterion $\mathcal{U}(T, .)$ should be considered at any intermediate date $T \leq T_H$, while still leading to the same optimal strategy and the same value $\mathcal{U}(0, x)$ that is satisfying

$$\text{for any } T \leq T_H, \ \mathcal{U}(0, x) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\Big(\mathcal{U}(T, X_T^{\pi,c}) + \int_0^T v(t, c_t)\mathrm{d}t\Big).$$

Under regularity assumptions, this criterion is given by the "value function" $\mathcal{U}(T, z)$ given the wealth $X_T = z$ at time T (not to be confused with the initial wealth $X_0 = x$)

$$\mathcal{U}(T, z) = \text{"}\sup_{(\pi,c)\in\mathcal{A}}\text{"} \mathbb{E}\Big(u(T_H, X_{T_H}^{\pi,c}(T, z)) + \int_T^{T_H} v(s, c_s)\mathrm{d}s | X_T = z\Big), \ a.s.. \quad (2.2)$$

This time-consistency translates into a martingale property of the preference process $\mathcal{U}(t, X_t^*) + \int_0^t v(s, c_s^*)\mathrm{d}s$ along the optimal strategy. This property, known as the dynamic programming principle, is the key common feature of the different points of view considered in this paper. In all of them the utility of consumption $(v(t, .), t > 0)$ is given, and the question arising is to find the utility $(\mathcal{U}(t, .), t > 0)$ of wealth, but they mainly differ by their boundary conditions.

IN THE BACKWARD SETTING, $\mathcal{U}(T_H, .) = u(T_H, .)$ is given, and the unknown is the optimal strategy (X^*, c^*) as well as $\mathcal{U}(t, .)$, also called "indirect" utility, possibly stochastic. Nevertheless, it is not trivial to prove that $\mathcal{U}$ defined by (2.2) is indeed concave.

IN THE FORWARD SETTING, there is no intrinsic time-horizon T_H and it is the initial utility $\mathcal{U}(0, .)$ which is given. Then the unknown is the utility process $(\mathcal{U}(t, .), t > 0)$ associated to an optimal strategy (X^*, c^*).

AT THE ECONOMIC EQUILIBRIUM, the formulation of the problem is close to the forward formulation. At equilibrium, the optimal portfolio is given by the market clearing condition $\pi^e = 1$. The unknown is still the utility $(\mathcal{U}(t, .), t > 0)$ and a consumption rate c^e, such that the pair (X^{π^e, c^e}, c^e) is optimal.

In this paper the preference criteria of agents are modeled by a pair of progressive utilities $(\mathbf{U}, \mathbf{V})$, that is a family of stochastic utility processes such that for any t, $(U(t, z), V(t, c))$ are some utility functions. As discussed above, it is natural to impose that the progressive utility system satisfies a dynamic programming principle, also called market consistency given the investment universe $\mathscr{X}^c$.

Definition 2.1 (*Consistent progressive utility system*). A progressive utility system $(\mathbf{U}, \mathbf{V})$ is said to be $\mathscr{X}^c$-consistent if

(i) for any admissible wealth $X^{\pi,c} \in \mathscr{X}^c$ with consumption rate c, the preference process $\mathcal{G}_t^{\pi,c} = U(t, X_t^{\pi,c}) + \int_0^t V(s, c_s)\mathrm{d}s$ is a positive supermartingale.

(ii) there exists an optimal strategy such that the preference process

$$\mathcal{G}_t^* = U(t, X_t^{\pi^*, c^*}) + \int_0^t V(s, c_s^*) ds \text{ is a martingale.}$$

The value function system $(\mathcal{U}(t, .), v(t, .))$ of the classic consumption optimization problem is an example of a $\mathscr{X}^c$-consistent system defined from its terminal condition $\mathcal{U}(T_H, z) = u(T_H, z)$. Conversely, a $\mathscr{X}^c$-consistent system $(\mathbf{U}, \mathbf{V})$ is the value function system of some investment-consumption problem, with stochastic terminal condition $U(T_H, .)$ for any time horizon T_H. The forward and backward settings differ by their boundary conditions, the terminal utility is given in the standard case and the initial one in the forward case. This point induces major differences in the interpretation and in the mathematical treatment of the utility's characterization. In particular, progressive utilities put emphasis on the initial conditions, such as the initial wealth of the economy, which is often downplayed with standard utilities.

2.2 The discounted pricing kernels and the dual problem

As for any concave optimization problem, it is useful to associate the dual convex problem based on the orthogonal cone $\mathscr{Y}$ of the convex cone $\mathscr{X}^c$, and whose elements Y are called discounted pricing kernels. They are also called stochastic discount factor in the economic literature, or state price density process in the financial literature. The discounted pricing kernels $Y \in \mathscr{Y}$ are characterized by the property that for any admissible strategy (π, c), the current wealth plus the cumulative consumption, both discounted by Y, is a positive local martingale (and thus supermartingale), namely $(Y_t X_t^{\pi,c} + \int_0^t Y_s c_s ds)$ is supermartingale. This implies that for any $T \geq 0$, $\mathbb{E}\left(Y_T X_T + \int_0^T Y_s c_s ds\right) \leq x$. This inequality, also known as the *budget constraint*, provides a necessary condition of admissibility, directly written in terms of the terminal wealth X_T and the consumption process $(c_s)_{s \in [0,T]}$. Discounted pricing kernels are related to the following dual convex problem written at time $t \leq T_H$ in the backward setting, with $\tilde{u}$ (resp.$\tilde{v}$) is the conjugate utility of u (resp. v) given in (2.1)

$$\tilde{\mathcal{U}}(t, \zeta) = \text{"}\inf_{Y \in \mathscr{Y}}\text{"} \mathbb{E}\left(\tilde{u}(T_H, Y_{T_H}(t, \zeta)) + \int_t^{T_H} \tilde{v}(s, Y_s) ds | Y_t = \zeta\right), \text{ a.s.} \quad (2.3)$$

Note that the dynamic programming principle for the primal and dual preference processes, together with the martingale property of $(Y_t^* X_t^* + \int_0^t Y_s^* c_s^* ds)$, implies that $\tilde{\mathcal{U}}(t, .)$ is indeed the Fenchel–Legendre convex conjugate of $\mathcal{U}(t, .)$. In the general forward setting, the $\mathscr{X}^c$-market consistency property on the primal progressive utility system $(\mathbf{U}, \mathbf{V})$ translates naturally into a market consistency property on the dual progressive utilities $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$, given the dual set $\mathscr{Y}$.

Definition 2.2 (*Consistent dual progressive utility system*). A dual progressive utility system $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ is said to be $\mathscr{Y}$-consistent if

(i) for any admissible dual process $Y \in \mathscr{Y}$, $\tilde{J}_t = \tilde{U}(t, Y_t) + \int_0^t \tilde{V}(s, Y_s) ds$ is a submartingale.

(**ii**) there exists an optimal process Y^* in $\mathscr{Y}$ such that

the optimal preference process $\tilde{J}_t^* = \widetilde{U}(t, Y_t^*) + \int_0^t \widetilde{V}(s, Y_s^*) \mathrm{d}s$ is a martingale.

Besides, thanks to the time-consistency, the dual relation at terminal date $Y_{T_H}^* = u_x'(T_H, X_{T_H}^*)$ translates at any date $t \leq T_H$ (with the corresponding relation for the consumption):

Proposition 2.1 (El Karoui et al. 2018, Corollary 4.9). *Let* $(\mathbf{U}, \mathbf{V})$ *be a* $\mathscr{X}^c$*-consistent progressive utility system[1] satisfying regularities conditions. Then the optimal processes are linked by the first-order relation: for any t, $Y_t^*(y) = U_z(t, X_t^*(x)) = V_c(t, c_t^*(c_0))$ with $y = u_z(x) = v_c(c_0)$. Equivalently in terms of the dual utilities $X_t^*(x) = -\widetilde{U}_\zeta(t, Y_t^*(y))$ and $c_t^*(c_0) = -\widetilde{V}_\zeta(t, Y_t^*(y))$.*

This shows that the marginal utility of consumption that appears in the Ramsey rule can be interpreted as an optimal discounted pricing kernel.

Remark 2.1 In the backward approach, the optimal processes are denoted $(c^{*,H}, X^{*,H})$ and $Y^{*,H}$, the additional symbol H underlining the dependency of the optimal processes on the optimization horizon T_H. This dependency is analog to the sensitivity of economic discount rate in the pure time preference parameter λ raised in Sect. 1.2. Indeed, if one take in (2.1) a random time horizon T_H as an independent exponential random variable with mean[2] $1/\lambda$ (and a vanishing wealth at T_H), then the criterion becomes $\mathbb{E}(\int_0^\infty e^{-\lambda t} v(c_t) \lambda \mathrm{d}t)$.

After this overview of main concepts involved in the modeling of discount rates, we develop them in a classic Itô framework.

3 Discounted pricing kernel in an Itô framework

3.1 The financial no-arbitrage framework in incomplete markets

The financial investment universe is assumed to be an incomplete Itô market, defined on a standard filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ that supports a n-standard Brownian motion W [see for example (Karatzas et al. 1987; Karatzas and Shreve 2001) or (Skiadas 2007)]. The market is characterized by the short rate (r_t), the n-dimensional risk premium vector (η_t), and by the $d \times n$ volatility matrix (σ_t) of the risky assets ($d \leq n$). In finance, the processes r, η and σ are usually taken exogenous. We assume that $\int_0^T (|r_t| + \|\eta_t\|^2) \mathrm{d}t < \infty$, for any $T > 0$, a.s. We specify here the class of admissible strategies in terms of (κ_t, ρ_t) where[3] $\kappa_t = \sigma_t^{tr} \pi_t$, $c_t = \rho_t X_t$: π_t is $\mathbb{R}^d$-valued and corresponds to the proportion of wealth invested in the risky assets, while ρ_t

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[1] The deterministic initial utilities $U(0, .)$ and $V(0, .)$ are denoted $u(.)$ and $v(.)$.

[2] For example, $\lambda = 0.1$ corresponds to a random horizon T_H with mean of 10 years.

[3] The upperscript ^{tr} denotes the matrix transpose.

is the (nonnegative) wealth-proportional consumption rate. Using this parametrization in (κ, ρ), the self-financing dynamics of a positive wealth process has a multiplicative form[4]

$$dX_t^{\kappa,\rho} = X_t^{\kappa,\rho}[(r_t - \rho_t)\,dt + \kappa_t.(dW_t + \eta_t dt)] \quad X_0^{\kappa,\rho} = x. \tag{3.1}$$

The existence of a multivariate risk premium η formulates the absence of arbitrage opportunity. A self-financing strategy (κ, ρ) is admissible if the portfolio κ_t lives in a given progressive family of vector spaces $\mathcal{R}_t$ $a.s.$, which expresses the incompleteness of the market. The set $\mathscr{X}^c$ of admissible wealth processes with admissible (κ, ρ) is a convex cone. Since from (3.1), the impact of the risk premium on the wealth dynamics only appears through the term $\kappa_t.\eta_t$ for $\kappa_t \in \mathcal{R}_t$, there is a "minimal" risk premium $\eta_t^{\mathcal{R}}$, the projection of η_t on the space $\mathcal{R}_t$, to which we refer in the sequel. For any $x \in \mathbb{R}^n$, $x^{\mathcal{R}}$ denotes the orthogonal projection of x onto $\mathcal{R}$ and $x^\perp$ the orthogonal projection onto $\mathcal{R}^\perp$. To avoid technicalities, we assume throughout the paper that all processes satisfy the necessary measurability and integrability conditions such that the following formal manipulations and statements are meaningful.

In this Itô setting, the class $\mathscr{Y}$ of the discounted pricing kernels is characterized as follows.

Definition 3.1 (*Discounted pricing kernel*). A nonnegative Itô semimartingale Y^ν is an admissible discounted pricing kernel if for any admissible consumption plan (κ, ρ), the process $\left(Y_t^\nu X_t^{\kappa,\rho} + \int_0^t Y_s^\nu X_s^{\kappa,\rho} \rho_s ds\right)$ is a local martingale. The dynamics of Y^ν is then given by

$$dY_t^\nu = Y_t^\nu\left[-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}).dW_t\right], \quad \nu_t \in \mathcal{R}_t^\perp, \quad Y_0^\nu = y. \tag{3.2}$$

The *minimal* discounted pricing kernel Y^0 corresponds to $\nu \equiv 0$

$$yY_t^0 = y\exp\left(-\int_0^t r_s ds - \int_0^t \eta_s^{\mathcal{R}}.dW_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 ds\right). \tag{3.3}$$

Note that $\mathscr{Y}$ does not depend on the presence of the consumption process and is uniquely characterized by the financial market. The volatility process $\sigma^Y = (\nu - \eta^{\mathcal{R}})$ of Y^ν consists of two components: the minimal risk premium $\eta^{\mathcal{R}}$ that lies in $\mathcal{R}$ and an orthogonal component ν that lies in $\mathcal{R}^\perp$. Observe that any discounted pricing kernel $Y_t^\nu(y)$, starting from y at time 0, is the product of Y_t^0 by the exponential local martingale $L_t^{\perp,\nu} = \exp\left(\int_0^t \nu_s.dW_s - \frac{1}{2}\int_0^t ||\nu_s||^2 ds\right)$, since $\eta_s^{\mathcal{R}}.\nu_s \equiv 0$. The inverse of the minimal discounted pricing kernel, $\frac{1}{Y^0}$, is the admissible market numeraire, also called GOP (Growth Optimal Portfolio), see (El Karoui et al. 1995; Platen and Heath 2006), or Filipovic and Platen 2009).

A discounted pricing kernel involves both a discounted factor $\exp(-\int_0^t r_s ds)$ (with the process r that may be stochastic) and a martingale density process corresponding

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[4] In this paper, the scalar product of two vectors $\mathfrak{X}$ and $\mathfrak{Z}$ (of the same dimension) will be denoted by $\mathfrak{X}.\mathfrak{Z}$ or sometimes by $< \mathfrak{X}, \mathfrak{Z} >$.

to a change of probability measure. In a complete market in which all risks could be hedged, the orthogonal set $\mathcal{R}^\perp$ is trivial and reduced to $\nu = 0$. This is the standard economic framework. In the economic framework, it is the short rate r and thus the drift term of the optimal discounted pricing kernel that is determined at the optimum (at the equilibrium), whereas in the financial framework, r is exogenous and it is the orthogonal component ν that is determined at the optimum.

3.2 Itô dynamics of the utility process

In this Itô framework it is natural to take progressive utility as "regular"[5] Itô random field with differential decomposition

$$dU(t, z) = \beta(t, z)dt + \gamma(t, z).dW_t.$$

In the standard backward framework, the initial value of the value function $\mathcal{U}$ is usually not explicit and is computed through a backward analysis, starting from its given terminal utility (possibly random) $U(T_H, .)$ at time T_H. For consistent progressive utilities, the initial value is given and the problem is solved forward, and the emphasis is placed on the monotonicity of optimal processes with respect to the initial condition. We refer to El Karoui et al. (2018) for explicit regularity conditions and characterization of the consistent pairs of consistent utilities of investment and consumption and the optimal policies. The optimal portfolio is given by

$$z\kappa_t^*(z) = z\sigma_t^{tr}\pi^*(t, z) = -\frac{U_z(t, z)}{U_{zz}(t, z)}\left(\eta_t^{\mathcal{R}} + \frac{\gamma_z^{\mathcal{R}}(t, z)}{U_z(t, z)}\right) \tag{3.4}$$

with the additional (compared to the deterministic case) risk premium term $\frac{\gamma_z^{\mathcal{R}}(t,z)}{U_z(t,z)}$ coming from the diffusion term of the progressive utility **U**. The market consistency implies the following HJB constraint

$$\beta(t, z) = -U_z(t, z)zr_t + \frac{1}{2}U_{zz}(t, z)\|z\kappa_t^*(z)\|^2 - \tilde{V}\big(t, U_z(t, z)\big). \tag{3.5}$$

CONSISTENT PROGRESSIVE POWER UTILITIES A consistent progressive utility system with constant relative risk aversion (also called power utility) is necessarily a pair of power utilities that are time-separable, with the same risk aversion coefficient θ $(0 < \theta < 1)$

$$U^{(\theta)}(t, z) = Z_t^u \frac{z^{1-\theta}}{1-\theta} \quad \text{and} \quad V^{(\theta)}(t, c) = Z_t^v \frac{c^{1-\theta}}{1-\theta}. \tag{3.6}$$

The positive processes Z^u and Z^v are linked by a SDE satisfied by Z^u and that is given by the HJB drift constraint (3.5) [see El Karoui et al. (2018, Sect. 4.2) for the study of progressive power utilities with consumption]. One important feature is that the

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[5] Explicit regularity conditions are given in El Karoui et al. (2018, Sect. 4).

optimal processes for power utilities are linear with respect of their initial condition. Power utility is the usual framework of the Ramsey rule.

Before interpreting and generalizing the Ramsey rule in this financial forward setting, Sect. 3.3 points out that this forward approach is in fact very natural when considering an economic equilibrium.

3.3 Determining the equilibrium spot rate

For evaluating public policies, the economy is usually assumed to be at equilibrium. Nevertheless, it must be kept in mind that this assumption puts strong constraints on the economic framework that could be considered (see He and Leland 1993 and El Karoui and Mrad 2021). A power utility function, together with a geometric Brownian motion for the discounted pricing kernel Y^*, provides a classic example of such an equilibrium, which is usually stated in a Markovian setting. Let us first recall the definition of an equilibrium [see Dumas and Luciano (2017, Chapter 11)]. For sake of simplicity, we state it in the simple case of a one-dimensional market, with no purely financial/inside security and a productive/outside security, whose dynamics is given exogenously (with drift coefficient μ_t and volatility σ_t).

Definition 3.2 At time t, an equilibrium is an allocation π_t^*, a consumption level c_t^*, a rate of interest r_t^*, such that the representative agent is at the optimum and the market (for the productive/outside security as well as for the riskless security) clears. Market-clearing conditions are as follows:

- The supply-equals-demand condition for productive/outside security: $\pi^* = 1$.
- The zero-net supply condition for the riskless security.

The equilibrium is expressed in terms of the representative agent's value function $\mathcal{U}(t, z)$ (Eq. (2.2)), with deterministic utilities $u(T_H, .)$ and $v(t, .)$. By identifying the optimal investment to 1 (cf. (3.4) with the diffusion term of $\mathcal{U}$ equal to zero in a backward Markovian setting), the market clearing condition (on the risky securities) determines the risk premium as a function of the relative risk aversion of the utility process $\mathcal{U}$:

$$\eta(t, x) = \eta(t, X_t^*(x)) \quad \text{with } \eta(t, z) = -\sigma_t \frac{z\,\mathcal{U}_{zz}(t, z)}{\mathcal{U}_z(t, z)} = \sigma_t \mathrm{R}_A^r(\mathcal{U})(t, z). \quad (3.7)$$

This determines endogenously the equilibrium rate

$$r_t^*(x) = r(t, X_t^*(x)) \quad \text{with } r(t, z) = \mu_t + \sigma_t^2 z \frac{\mathcal{U}_{zz}(t, z)}{\mathcal{U}_z(t, z)}. \quad (3.8)$$

Deterministic power (CRRA) utilities $u(z) = \frac{z^{1-\theta}}{1-\theta}$ and deterministic coefficients σ, μ is the standard model used in economy; it is an important case in which computations simplify and the existence of an equilibrium can be stated. It notably implies, using (3.8), that the equilibrium rate does not depend on the wealth process X^*. Nevertheless,

this case hides some important features on the dependency of the optimal processes and rates on initial conditions, as we will see in Sects. 5.2 or 6.4.2.

For a general utility function u that is not necessarily of power type, the existence of an equilibrium is not guaranteed and the relations given here are conditioned to its existence.

This simple equilibrium model has numerous extensions, as the famous one proposed by Cox–Ingersoll–Ross (1985). One sought feature of this model was that it yields positive rate (but nowadays the desire of having model with positive rates is not current anymore). Taking into account the presence of a financial market, Cox et al. (1985) adopted an equilibrium approach to endogenously determine the term structure of interest rates. In their model, the dynamics of the production process and the utility function depend on an exogenous stochastic factor which in some way influences the economy. At equilibrium, all purely financial assets are in zero net supply. The risk-free rate and the financial assets prices are determined endogenously such that the representative agent is not better off by trading in the money market, i.e. he is indifferent between an investment in the production opportunity and the risk-free instrument. This is related to the theory of indifference pricing that will be used in the sequel (see Sect. 4.2). Then assuming a CIR dynamic for the exogenous stochastic factor implies also a CIR dynamics for the equilibrium short rate.

To summarize, in the equilibrium approach, the short rate is determined endogenously and does not appear in the equilibrium optimal wealth process dynamics $\mathrm{d}X_t^* = (\mu_t X_t^* - c_t^*)\mathrm{d}t + X_t^* \sigma_t \mathrm{d}W_t$ (the terms in the short rate r cancel due to the market clearing conditions) nor in the HJB equation: replacing the expression of the equilibrium rate (3.8) into the HJB equation (3.5) yields[6],

$$\mathcal{U}_t(t, z) + v(t, c_t^*) + (\mu_t z - c_t^*)\mathcal{U}_z(t, z) + \frac{1}{2}\sigma_t^2 z^2 \mathcal{U}_{zz}(t, z) = 0,$$

which is linear in $\mathcal{U}_z$ and $\mathcal{U}_{zz}$. In fact, the utility function at time T_H is not given and is part of the processes that should be determined at equilibrium. Besides, the expression for the short rate (3.8), together with the dynamics of the wealth process X^* shows that the problem is naturally posed forward in the equilibrium setting. Remark that in the no-arbitrage financial framework, the bank account and utility functions are given exogenously. In turn, when the market is incomplete, the excesses of return of some less basic assets, such as some bonds, are the one that are endogenously determined in the arbitrage approach.

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[6] When the time horizon T_H is an exponential variable, the terminal condition disappears and is replaced by a linear term of order 0 in the HJB equation

$$\mathcal{U}_t(t, z) + v(t, c_t^*) + (\mu_t z - c_t^*)\mathcal{U}_z(t, z) + \frac{1}{2}\sigma_t^2 z^2 \mathcal{U}_{zz}(t, z) - \lambda \mathcal{U}(t, z) = 0.$$

4 Pathwise Ramsey rule and its financial interpretation

In light of Sect. 2, we provide a pathwise extension of the Ramsey rule and its financial interpretation, based on marginal utility indifference pricing.

4.1 A pathwise Ramsey rule

In the sequel, the upper-script .* denotes interchangeably optimal process of the forward and backward formulation, keeping in mind that, for the backward formulation, the statements are valid up to time T_H, with optimal processes that may depend on T_H. We focus on the optimality relations given by Proposition 2.1

$$\begin{cases} c_t^*(c_0) = -\tilde{V}_\zeta(t, Y_t^*(y)) & \text{i.e.} \quad V_c(t, c_t^*(c_0)) = Y_t^*(y),\ t \geq 0 \\ \quad c_0 = -\tilde{v}_\zeta(y) & \text{i.e.} \qquad v_c(c_0) = y. \end{cases} \tag{4.1}$$

Remark that a parametrization in y is equivalent to a parametrization in the initial wealth x or in the initial consumption rate c_0, based on the one to one correspondence $v_c(c_0) = u_z(x) = y$. The forward point of view emphasizes the key role played by the monotonicity of Y with respect to the initial condition y (under regularity conditions of the progressive utilities). Then as function of y, c_0 is decreasing, and $c_t^*(c_0)$ is an increasing function of c_0. This question of monotonicity is frequently avoided, maybe because with power utility functions $Y_t^*(y)$ is linear in y.

Equation (4.1) may be interpreted as a pathwise Ramsey rule, between the marginal utility of the optimal consumption and the optimal discounted pricing kernel:

$$\frac{V_c(t, c_t^*(c_0))}{v_c(c_0)} = \frac{Y_t^*(y)}{y}, \quad t \geq 0 \quad \text{with} \quad v_c(c_0) = y. \tag{4.2}$$

This one to one correspondence between the optimal consumption and the optimal discounted pricing kernel holds at any date t, that is why we call it a "pathwise Ramsey rule". Remark that formulating this pathwise relation (4.2) in terms of the optimal consumption leads to an expression that only involves the utility process **V** of the consumption, which contrary to **U**, is a given process. Formulating the pathwise relation (4.2) in terms of the wealth would have involved the utility **U** which is complex to compute, **U** being the value function of the optimization problem.

The Ramsey rule leads to a description of the equilibrium yield curve as a function of the optimal discounted pricing kernel Y^*, $R_0^e(T)(y) = -\frac{1}{T} \ln \mathbb{E}[Y_T^*(y)/y]$ which allows us to give a financial interpretation in terms of zero-coupon bonds. More dynamically in time, we define for $t < T$ and denoting by $\delta := (T - t)$ the time to maturity

$$R_t^e(\delta)(y) = R_t^e(T - t)(y) := \frac{-1}{T - t} \ln \mathbb{E}\left[\frac{V_c(T, c_T^*(c_0))}{V_c(t, c_t^*(c_0))} \Big| \mathcal{F}_t\right]$$

$$= \frac{-1}{T - t} \ln \mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)} \Big| \mathcal{F}_t\right]. \tag{4.3}$$

The Ramsey rule brings us to study the quantity $\mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\right]$. In the context of a financial complete market, it is well-known that this quantity corresponds to the price at date t of zero-coupon bonds (maturing at time T). Nevertheless, its interpretation for incomplete market is less trivial and will be investigated in Sect. 4.2. Before going on with the financial interpretation of this equilibrium yield curve given in terms of the discounted pricing kernel, we recall that in the equilibrium framework the short-term interest rate r_t is endogenous and fixed at equilibrium to satisfy the market clearing condition of the aggregate demands. On the contrary, in the financial no-arbitrage framework, the short rate is exogenous and the discounted pricing kernel is optimized not through its drift r_t but through its orthogonal diffusion coefficient v_t. In the financial no-arbitrage context, the optimization procedure impacts only the form on the yield curve (through the risk premium), and not the beginning of the curve. This helps to understand how yield curve movements of the short end (monitored by a central bank) translate into long-term yield. For this financial interpretation purpose, it is natural to link zero-coupon bonds and the equilibrium yield curve.

4.2 Marginal indifference pricing interpretation of the Ramsey rule

In this section, we investigate the financial interpretation of the Ramsey rule. The financial point of view focuses more on the financial products than the rates, namely in this context on the zero-coupon bonds, which is a contract that pays 1 at a given date T. We thus want to interpret, in terms of price of zero-coupon bonds, the quantities $\mathbb{E}\left[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\right]$ for all $t < T$. This question is related to a more general issue in finance that consists in the pricing of a bounded contingent claim ξ_T paid at date T ($\xi_T = 1$ in the case of zero-coupon bond). We thus address this pricing issue for replicable and non-replicable claims, with both backward (in this case $T \leq T_H$) and forward approaches. When all risks are replicable, then the price is uniquely determined as the value of the replicating portfolio (by no-arbitrage arguments). When some risks remain not replicable, several valuation methodologies exist (such as super-replicating prices or indifference prices), leading to different prices or bid-ask prices; we refer the interested reader to the "Appendix" for further discussion. To evaluate small amounts of non-replicable claims, we will consider the marginal utility indifference pricing. This pricing procedure consists in choosing an optimal discounted pricing kernel Y^* among the set $\mathscr{Y}$ of all admissible discounted pricing kernels.

4.2.1 Valuation of replicable payoffs

The valuation of a (bounded) contingent claim ξ_T (paid at date T) is done through the choice of a discounted pricing kernel Y^v, the price at time t being then given by the expectation $\mathbb{E}\left[\frac{Y_T^v(y)}{Y_t^v(y)}\xi_T\big|\mathcal{F}_t\right]$. The question that arises is the choice of this discounted pricing kernel Y^v. As mentioned in Definition 3.1, any discounted pricing kernel Y_t^v is written as the product of the so-called minimal discounted pricing kernel $Y_t^0(y) = y Y_t^0 = y \exp\left(-\int_0^t r_s \mathrm{d}s - \int_0^t \eta_s^{\mathcal{R}}.\mathrm{d}W_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 \mathrm{d}s\right)$ and an orthogonal local martingale $L_t^{\perp,v}(y) = \exp\left(\int_0^t v_s(y).\mathrm{d}W_s - \frac{1}{2}\int_0^t ||v_s(y)||^2 \mathrm{d}s\right)$. In finance r_t and $\eta_t^{\mathcal{R}}$

are exogenous, while $\nu_t \in \mathcal{R}_t^\perp$ is endogenous and may depend on y. The minimal discounted pricing kernel Y^0 plays a "universal" rule and any Y^ν differs only in the orthogonal part $L^{\perp,\nu}(y)$. Y^0 includes both the short-term interest rate r and the risk premium $\eta^{\mathcal{R}}$, it can be decomposed as $Y_t^0 = e^{-\int_0^T r_s ds} L_T^{\mathcal{R}}$ with $L_T^{\mathcal{R}} = \exp\big(-\int_0^t \eta_s^{\mathcal{R}} . dW_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 ds\big)$ an exponential martingale which corresponds to the density process of a change of probability.

If the bounded contingent claim ξ_T is *replicable* by an admissible self-financing portfolio, its market price $p^m(\xi_T)$ (p^m when it is not ambiguous) is the value of the replicating portfolio (by no-arbitrage). Thus, p_t^m is a bounded process such that $Y_t^\nu(y) p_t^m$ is a martingale for any discounted pricing kernel $Y^\nu(y)$, and in particular for $y Y_t^0$. This leads to the classic pricing formula of a replicable contingent claim $p_t^m(\xi_T) = \mathbb{E}\big[\frac{Y_T^0}{Y_t^0}\xi_T | \mathcal{F}_t\big] = \mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T | \mathcal{F}_t\big]$.

Therefore, for replicable payoff, the price is uniquely given by $\mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T | \mathcal{F}_t\big]$, whatever the discounted pricing kernel Y^ν. In finance, it is interpreted as the risk neutral conditional expectation of the discounted claim between t and T,

$$p_t^m(\xi_T) = \mathbb{E}\left[\frac{Y_T^0}{Y_t^0}\xi_T | \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds}\xi_T | \mathcal{F}_t\big] \qquad (4.4)$$

where $\mathbb{Q}$ is the minimal risk-neutral probability with density $L_T^{\mathcal{R}}$ with respect to $\mathbb{P}$ (on $\mathcal{F}_T$). Under the risk neutral probability $\mathbb{Q}$, all assets and admissible self-financing portfolios have the same return r_t. Remark also that in a complete market (which is the natural framework of equilibrium modeling), any contingent claim is replicable, and Y^0 is the only discounted pricing kernel. *In conclusion, for replicable zero-coupon bonds, equilibrium yield curve (4.3) and market yield curve have the same expression in terms of the discounted pricing kernel.*

However, for long maturities, this replicable assumption is very strong (even if the payoff of the zero coupon is constant, the short-term interest rate and the risk premium are stochastic). If the contingent claim is not replicable, the price is not uniquely determined and different discounted pricing kernel Y^ν may lead to different prices $\mathbb{E}\big[\frac{Y_T^\nu(y)}{Y_t^\nu(y)}\xi_T | \mathcal{F}_t\big]$. What is the financial interpretation of the Ramsey rule in this context? It is important to point out that the Ramsey rule is a marginal linear pricing rule that is computed for relative small amounts. The following section relates it with the marginal utility indifference pricing. Indeed, similarly to the heuristic of the Ramsey rule recalled in (1.1), the marginal utility indifference price is also a linear price that corresponds to a small perturbation of first order around an equilibrium.

4.2.2 Marginal indifference pricing

When hedging strategies cannot be implemented, the nominal amount of the transaction becomes an important risk factor. One way to evaluate non-replicable claims is the utility based indifference pricing, which is a nonlinear pricing rule. The utility *indifference price* is the price at which the investor is indifferent from investing or

not in the contingent claim. We consider the two following maximization problems stated at time $t = 0$ to simplify the notations (this can be easily extended to any time $t \leq T$). The first one without the claim ξ_T has already been introduced previously

$$\mathcal{U}^T(x) := \sup_{(\kappa, \rho) \in \mathcal{A}^c} \mathbb{E}\left[U\left(T, X_T^{\kappa, \rho}(x)\right) + \int_0^T V(s, c_s) \mathrm{d}s \right]. \tag{4.5}$$

The terminal utility $U(T, .)$ is then perturbed by the random payment $q\xi_T$, leading to the second maximization problem

$$\mathcal{U}^{\xi, T}(x, q) := \sup_{(\kappa, \rho) \in \mathcal{A}^c} \mathbb{E}\left[U\left(T, X_T^{\kappa, \rho}(x)\right) - q\,\xi_T + \int_0^T V(s, c_s) \mathrm{d}s \right]. \tag{4.6}$$

The *utility indifference price*[7] is the cash amount $\widehat{p}_{0,T}^q(x, \xi_T, q)$ determined by the relationship

$$\mathcal{U}^{\xi, T}\left(x + \widehat{p}_{0,T}^q(x, \xi_T, q), q\right) = \mathcal{U}^T(x). \tag{4.7}$$

As in (1.1), (4.7) provides the additional initial wealth $\widehat{p}^q$ that offsets the loss of providing a q-quantity of the claim ξ_T at time T.

When the investors are aware of their sensitivity to the non-replicable risk, they can try to transact for only a little amount in the risky contract, which corresponds to the zero marginal rate of substitution p_T^u (*u* for utility), also called *Davis price* (1998) or *marginal indifference price*. This is a classic pricing approach in economics, less frequently used in option pricing. The *marginal utility indifference price* is determined by the relationship

$$p_{0,T}^u(x, \xi_T) := \lim_{q \to 0} \frac{\partial \widehat{p}_{0,T}^q}{\partial q}(x, \xi_T, q). \tag{4.8}$$

The marginal utility price is characterized by the optimal discounted pricing kernel of the consumption optimization problem (4.5).

Proposition 4.1 *Let $Y^*(y)$ be the optimal discounted pricing kernel associated with the (forward or backward) consumption optimization problem. For any nonnegative contingent claim ξ_T delivered at time T, the marginal utility price is given at any time $t \leq T$ by*

$$p_{t,T}^u(x, \xi_T) = \mathbb{E}\left[\xi_T \frac{Y_T^*(y)}{Y_t^*(y)} | \mathcal{F}_t \right], \quad y = u_z(x). \tag{4.9}$$

Proof We refer to the "Appendix" for the proof, as well as a discussion on the time-coherence of this pricing rule, in the backward and forward settings (see Proposition 7.1). □

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[7] If $q > 0$ (resp. $q < 0$) it is a selling (resp. a buying) indifference price. For $q > 0$ one should assume that $q\xi_T$ is super replicable at price x.

Using the marginal utility indifference pricing, the price of the contingent claim is computed as the expectation under a pricing measure. $1/Y^*(y)$ can be interpreted as the optimal market numeraire. In the case of a logarithmic utility criterion, $Y^*(y)$ is the minimal discounted pricing kernel Y^0 (that does not depend on y) and we recover the pricing rule given by the benchmark approach of Platen and Heath (2006), as $1/Y^0$ coincides with the Growth Optimal Portfolio. This special case does not allow us to capture the dependency on initial conditions such that the initial wealth. This pricing rule is also related to the "local expectations hypothesis" of Piazzesi (2010), in which the transition from the data-generating measure $\mathbb{P}$ to the pricing measure $(Y_T^*.\mathbb{P})$ is tied to preference parameters.

We point out that the marginal utility price is a linear pricing rule; this means that there exists a consensus on this price for a small amount, but investors are not sure to have liquidity at this price. Nevertheless, this linear pricing rule may not be well adapted for larger nominal amount of transaction and highly illiquid market. From a financial viewpoint, this linear pricing rule given by the discounted pricing kernel Y^* allows to enrich the financial market with the zero-coupon bonds whose prices become coherent assets under $(Y^*.\mathbb{P})$. In this extended market, the minimal discounted pricing kernel Y^0 is then replaced by Y^*.

From an economic viewpoint, utility indifference pricing relies on the disturbance of a partial equilibrium by adding a new contingent claim/asset that should be financed. A complete market cannot be disturbed by a new asset because any contingent claim/asset can be hedged. But in incomplete markets the equilibrium is not perfect and the new claims to be financed have an impact on it. In the case of new claims whose size are small, the disturbance is marginal, leading to a marginal utility indifference price. This indicates similarities between the marginal utility indifference price and the Ramsey rule. We now interpret the previous results on the marginal utility pricing of zero-coupon bonds in terms of the yield curve.

4.3 Marginal utility yield curve

As usual, we use the generic notation $\big(B(t, T),\ t \le T\big)$ for the price at time t of a zero-coupon bond paying one unit of cash at maturity T. In finance, the market yield curve $(\delta \to R_t(\delta))$ is expressed in term of the time to maturity $\delta = T - t$ and is defined through the price of a zero-coupon bonds by $B(t, T) = \exp(-(T - t)R_t(T - t))$. We use the previous results of Sects. 4.2.1 and 4.2.2 concerning the pricing of contingent claims: the case of a zero-coupon bond corresponds to a contract that delivers 1 at maturity T, i.e. $\xi_T = 1$.

(i) If the zero-coupon bonds are replicable, then there is no ambiguity about their prices, as any discounted pricing kernel Y leads to the same price (see (4.4))

$$B^0(t, T) = \mathbb{E}\left[\frac{Y_T}{Y_t}|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{Y_T^0}{Y_t^0}|\mathcal{F}_t\right].$$

In practice, this pricing rule $B^0(t, T) = \mathbb{E}\big(\frac{Y_T^0}{Y_t^0}\big|\mathcal{F}_t\big)$, using the minimal pricing kernel Y^0, is often used as a benchmark, even if the bonds are not replicable. It corresponds to the benchmark approach of Platen and Heath (2006). In these case, the price does not depend on y (for exogenous r, $\eta^{\mathcal{R}}$).

(ii) For non-hedgeable zero-coupon bond, we can apply the marginal indifference pricing rule (with consumption) based on the u-optimal pricing kernel $Y_t^*(y)$. Although it is important to emphasize the dependence of the optimal pricing kernel $Y^*(y)$ on the utility, we avoid this dependence to simplify the notations. Similarly, the marginal utility price at time t of a zero-coupon bond depends on the utility only through the optimal discounted pricing kernel $Y^*(y)$, we denote it by $B^*(t, T)(y)$ (note that it depends on y): $B^*(t, T)(y) = \mathbb{E}\Big[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\Big]$. Based on the link between optimal discounted pricing kernel and optimal consumption,

$$B^*(t, T)(y) = \mathbb{E}\Big[\frac{Y_T^*(y)}{Y_t^*(y)}\big|\mathcal{F}_t\Big] = \mathbb{E}\Big[\frac{V_c\big(T, c_T^*(c_0)\big)}{V_c\big(t, c_t^*(c_0)\big)}\big|\mathcal{F}_t\Big], \quad y = v_c(c_0) = u_z(x)$$

(4.10)

where V_c is given by the first-order relation (4.2). According to the Ramsey rule (4.3), equilibrium interest rates and marginal utility interest rates are the same, in terms of the discounted pricing kernel. One should keep in mind that in the equilibrium framework the discounted pricing kernel is determined at equilibrium through the spot rate r_t endogenously, while it is optimized through its orthogonal diffusion coefficient v_t in the financial setting. Besides, it is worth emphasizing that the marginal utility prices are only valid for small trades. Indeed for non-replicable claims, the size of the transactions is an important source of risk; for larger trades, the first-order approximation given by the marginal utility price is no more accurate, and we should add a correcting second-order term or use indifference pricing (see Appendix, Theorem 7.2).

5 Yield curves dynamics and their volatilities

The increase of the fixed income market in size and number of products has transformed the way of considering the links between rates of different maturities, leading to leave the economic theory of rational expectation for the principle of no-arbitrage between bonds of different terms. Initiated by Vasicek in 1977, this evolution has matured with Heath–Jarrow–Morton theory (192) and the theory of bond as a numeraire in El Karoui et al. (1995). Note that this point of view that follows from the no-arbitrage principle is relevant for a day by day management of the rate fluctuations, but does not replace the analysis of the economic fundamentals that explain the broad patterns of the fluctuations. This section revisits the previous results on the yield curve, using Heath–Jarrow–Morton (HJM) theory in incomplete market, for both the economic and financial viewpoints, and both the forward and backward frameworks (in the backward

approach, the maturity T of the zero-coupon should be taken smaller than the horizon T_H).

The notion of forward contracts will be used, such as the forward zero-coupon bonds, whose price $B_t(T_0, T)$ is the price at time t of a bond starting at time T_0 and paying one unit of cash at time $T > T_0$. By non-arbitrage, $B_t(T_0, T) = B(t, T)/B(t, T_0)$. The family of forward instantaneous rates $\left(f(t, T_0) = -\partial_T \ln \left(B_t(T_0, T) \right) |_{T=T_0} \right)$ takes also a large place in the HJM theory.

Instead of starting with a given dynamic for the short rate r and deducing the zero-coupon bonds and their volatilities (as it is the case for example for the Vasicek model), the Heath–Jarrow–Morton framework adopts a reverse approach based on the prices of zero-coupon bonds and their volatility. It is worth emphasizing that in the HJM approach the spot rate is not given and is deduced from the volatility process, and of the initial conditions of the forward rates ($f(0, T)$). Thus, in what follows, we focus on the volatility family of the zero-coupon bonds that characterizes the dynamics of the yield curve. It is important to highlight that this characteristic is determined directly by the martingale property of the process $(Y_t^*(y) B^*(t, T)(y))_{t \in [0,T]}$, in both the economic and the financial viewpoints. The subtle difference consists of the endogeneity for the economic viewpoint (resp. exogeneity for the financial viewpoint) of the spot rate r that may depend (or not) on y. This dependency of the rates on the initial wealth of the economy x (through the one to one relation $y = u_z(x)$) is investigated in Sect. 5.2 .

5.1 Heath Jarrow Morton framework for forward rates

Recall that any discounted pricing kernel $Y^*(y)$ is characterized by its volatility process $\sigma^{Y^*}(y) := \nu^*(y) - \eta^{\mathcal{R}}(y)$ (resp. $-\eta^{\mathcal{R}}(y)$ for Y^0), where $\eta^{\mathcal{R}}(y)$ is the minimal risk premium (that lies in $\mathcal{R}$) and $\nu^*(y)$ is the orthogonal component that lies in $\mathcal{R}^\perp$. In the economic framework $\eta^{\mathcal{R}}(y)$ is endogenous, while in the financial setting it is exogenous and usually taken independent of y. $\sigma^{Y^*}(y)$ does not depend on the maturity T, but may depend on the horizon T_H in the backward framework, through the orthogonal component $\nu^*(y)$. The dynamics of the associated bonds $B^*(t, T)(y)$ differ by their volatility vectors, denoted by $\Gamma^*(t, T)(y)$ that are assumed to be progressive processes with the convention $\Gamma^*(t, T)(y) = 0$ a.s. for $t \geq T$. In the sequel, we use the usual short notation for exponential martingale[8], $\mathcal{E}_t(\phi) := \exp\left(\int_0^t \phi_s . dW_s - \frac{1}{2} \int_0^t \|\phi_s\|^2 ds \right)$. The study is based on the martingale property of the process $Y_t^*(y) B^*(t, T)(y)$ (resp. $Y_t^0 B^0(t, T)$), whose volatility $\left(\sigma_t^{Y^*}(y) + \Gamma^*(t, T)(y) \right)$ is the sum of the volatilities of each term, and whose terminal value is $Y_T^*(y)$. Thus, the exponential martingale $Y_t^*(y) B^*(t, T)(y)$ has the following representation:

$$Y_t^*(y) B^*(t, T)(y) = y B^*(0, T)(y) \mathcal{E}_t \left(\sigma^{Y^*}(y) + \Gamma^*(., T)(y) \right). \tag{5.1}$$

The same formula holds for $Y_t^0 B^0(t, T)$ ($\nu^* \equiv 0$). As a byproduct, (5.1) written for $t = T$ provides another formula for the random variable $Y_T^*(y) =$

---

[8] Additional assumptions on ϕ, of Novikov type, are necessary to ensure that this local martingale is a true martingale, see e.g. (Novikov 1973) or (Krylov 2019).

$y \exp(-\int_0^T r_s ds)\mathcal{E}_T\left(\sigma^{Y^*}(y)\right)$: observing that $B^*(T, T)(y) = 1$,

$$Y_T^*(y) = Y_T^*(y)B^*(T, T)(y) = yB^*(0, T)(y)\mathcal{E}_T\left(\sigma^{Y^*}(y) + \Gamma^*(., T)(y)\right). \quad (5.2)$$

Identifying the two formulas for the random variable $Y_T^*(y)$ yields, where Cst(y) is a deterministic term

$$\int_0^T r_s(y)ds = \text{Cst}(y) - \int_0^T \Gamma^*(s, T)(y).dW_s + \frac{1}{2}\int_0^T ||\Gamma^*(s, T)(y) + \sigma_s^{Y^*}(y)||^2 - ||\sigma_s^{Y^*}(y)||^2 ds$$

$$= \text{Cst}(y) - \int_0^T \Gamma^*(s, T)(y).(dW_s + \sigma_s^{Y^*}(y)ds) + \frac{1}{2}\int_0^T ||\Gamma^*(s, T)(y)||^2 ds. \quad (5.3)$$

The instantaneous forward rates are defined by $f^*(t, T)(y) = -\partial_T \ln B^*(t, T)(y)$. They represent the instantaneous rate of the forward zero-coupon bond defined at time t with starting date T. The limit of the instantaneous forward rate, when the maturity T tends to the current date t, is the spot rate r of no-arbitrage:

$$\lim_{T \to t} f^*(t, T)(y) = r_t(y). \quad (5.4)$$

The instantaneous forward rates are easier to compute than the rates $R_t^*(T - t)(y)$ themselves: indeed they are computed directly from (5.1) by taking the logarithmic derivative of the product $Y_t^*(y)B^*(t, T)(y)$ with respect to the maturity T.

Proposition 5.1 *We recall that* $\sigma^{Y^*}(y) = v^*(y) - \eta^{\mathcal{R}}(y)$ *is the volatility process of* $Y^*(y)$. *We assume that the volatility vectors* $\Gamma^*(t, T)(y)$ *are differentiable with respect to* T *with locally bounded derivative* $\gamma^*(t, T)(y) := \partial_T \Gamma^*(t, T)(y)$. *Then the instantaneous forward rates satisfy*

$$\begin{cases} f^*(t, T)(y) = f^*(0, T)(y) - \int_0^t \gamma^*(s, T)(y).\left(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s, T)(y))ds\right), \\ \hspace{9cm} (5.5) \\ df^*(t, T)(y) = -\gamma^*(t, T)(y).\left(dW_t - (\sigma_t^{Y^*}(y) + \Gamma^*(t, T)(y))dt\right). \hspace{1cm} (5.6) \end{cases}$$

The yield curve $\delta \mapsto R_t^*(\delta)(y)$ *is obtained as the primitive of the forward rate curve:*

$$R_t^*(\delta)(y) = \frac{1}{\delta}\int_0^\delta f^*(t, t + s)(y)ds. \quad (5.7)$$

The market practice that uses the minimal pricing kernel Y^0 (for which $v = 0$) as benchmark induces a instantaneous forward rate $f^0(t, T)(y)$ instead of $f^*(t, T)(y)$. We compute below the dynamics of the difference between the instantaneous forward rates.

Dynamics of the error $f^*(t, T) - f^0(t, T)$

The difference $\Delta f(t, T)(y) := f^*(t, T)(y) - f^0(t, T)(y)$ $(\Delta f(T, T) = 0)$ between the instantaneous forward rates has the following dynamics (with similar notations for $\Delta \gamma$ and $\Delta \Gamma$)

$$
\begin{aligned}
d_t(\Delta f(t, T))(y) \\
= d_t f^*(t, T)(y) - d_t f^0(t, T)(y) \\
= -\Delta \gamma(t, T)(y).(dW_t + \eta_t^{\mathcal{R}}(y)dt) + <\gamma^*(t, T)(y), \nu_t^*(y)> dt \\
+ <\gamma^*(t, T)(y), \Gamma^*(t, T)(y)> dt - <\gamma^0(t, T)(y), \Gamma^0(t, T)(y)> dt \\
= -\Delta \gamma(t, T)(y).(dW_t - \sigma_t^{Y^*}(y)dt) + <\Delta \gamma(t, T)(y), \Gamma^*(t, T)(y)> dt \\
+ <\gamma^0(t, T)(y), \Delta \Gamma(t, T)(y) + \nu_t^*(y)> dt.
\end{aligned}
$$

The dynamics of the "error" of using the minimal discounted pricing kernel Y^0 (benchmark approach) instead of $Y^*(y)$ is similar to the dynamics (5.5) of a forward rate, plus the additional source term $<\gamma^0(t, T)(y), \Delta \Gamma(t, T)(y) + \nu_t^*(y)> dt$.

5.2 Exogenous spot rate and wealth dependency

To investigate the wealth dependency of the rates, one should keep in mind that the parameter y is directly linked to the wealth of the economy, through the one to one relation $y = u_z(x)$. Note that writing (5.5) in a backward formulation and since $f^*(T, T)(y) = r_T(y)$ from equation (5.4),

$$
f^*(t, T)(y) = r_T(y) + \int_t^T \gamma^*(s, T)(y).(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s, T)(y))ds). \quad (5.8)
$$

It appears that the spot rate seems to be depending on y (and thus on the initial wealth x), even for an exogenous spot rate. This dependency is conveyed by the orthogonal component $\nu^*(y)$ of the diffusion coefficient $\sigma^{Y^*}(y)$. Nevertheless, it is usual in the financial modeling to take the spot rate r_t and the minimal risk premium $\eta^{\mathcal{R}}$ independent of the initial parameter y, on the contrary to the economic framework in which they are endogenous and thus naturally depend on y. But this assumption implies a constraint on the initial slope of the instantaneous forward rates. The dynamics of the spot rate and the condition under which r does not depend on y are given in the following proposition.

Proposition 5.2 (Properties of the spot rate). *The spot rate is given by*

$$
r_t(y) = f^*(0, t)(y) - \int_0^t \gamma^*(s, t)(y).(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s, t)(y))ds), \quad (5.9)
$$

and its dynamics is given by

$$
dr_t(y) = \partial_\delta f^*(t, t)(y)dt - \gamma^*(t, t)(y).(dW_t - \sigma_t^{Y^*}(y)dt). \quad (5.10)
$$

This implies that for exogenous spot rate r that does not depend on y, $\gamma^*(t, t)$ and $\partial_\delta f^*(t, t)(y) + \gamma^*(t, t).\sigma_t^{Y^*}(y)$ do not depend on y.

Remark Since the yield curve $R_t^*(\delta)$ is a more natural market data than the instantaneous forward rate $f^*(t, t + \delta)$, it is interesting to write its initial slope in terms of the initial slope of the yield curve, namely $\partial_\delta f^*(t, t)(y) = 2\partial_\delta R_t^*(0)(y)$.

Proof Note that Eq. (5.9) is a backward formulation of (5.5). Contrary to the differential form (5.6), T is not fixed anymore, instead $T = t + \delta$ with $\delta \to 0$. Therefore, as in Musiela and Rutkowski (2005), we denote $r(t, \delta)(y) := f^*(t, t + \delta)(y)$. To get its dynamics, we apply Itô's formula to equation (5.5):

$$f^*(t, T)(y) = f^*(0, T)(y) - \int_0^t \gamma^*(s, T)(y).\big(dW_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s, T)(y))ds\big),$$

with $T = t + \delta$ which is of finite variation, and thus, we get

$$dr(t, \delta)(y) = \partial_\delta r(t, \delta)(y)dt + \gamma^*(t, t + \delta)(y).\Gamma^*(t, t + \delta)(y)dt$$
$$-\gamma^*(t, t + \delta)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big). \tag{5.11}$$

When the time to maturity δ goes to zero, using the relation $r_t(y) = f^*(t, t)(y)$ and the fact that $\Gamma^*(t, t)(y) = 0$, the dynamics of the spot rate is given by

$$dr_t(y) = \partial_\delta f^*(t, t)(y)dt - \gamma^*(t, t)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big).$$

This implies that for exogenous spot rate r, γ^* does not depend on y on the diagonal and $\partial_\delta f^*(t, t)(y) + \gamma^*(t, t)(y).\sigma_t^{Y^*}(y)$ does not depend on y. Besides, the initial slope of the instantaneous forward rate can be interpreted in terms of the initial slope of the yield curve. Indeed, differentiating (5.7) w.r.t. δ, one gets $\partial_\delta R_t^*(\delta) = -\frac{R_t^*(\delta) - r_t}{\delta} + \frac{f^*(t, t + \delta) - r_t}{\delta}$. Since $r_t = R_t^*(0) = f^*(t, t)$, passing to the limits when $\delta \to 0$, yields $\partial_\delta f^*(t, t)(y) = 2\partial_\delta R_t^*(0)(y)$. Thus, the dynamics of the spot rate can also be written as

$$dr_t(y) = 2\partial_\delta R_t^*(0)(y)dt - \gamma^*(t, t)(y).\big(dW_t - \sigma_t^{Y^*}(y)dt\big).$$

□

We now illustrate these constraints of exogenous spot rates in an affine framework with deterministic volatilities.

A Gaussian affine framework The Vasicek model (1977) was the first model for interest rate coming from a financial point of view. It is stated in a complete market and its starting point is the dynamics of the spot rate (r_t) which is assumed to be an Ornstein–Uhlenbeck process. As a consequence, all the rates in the Vasicek model are affine and Gaussian. We provide here a similar affine framework in an incomplete

market. We only assume that the instantaneous forward rates $f^*(t, T)(y)$ are affine function of $r_t(y)$

$$f^*(t, T)(y) = \Lambda(t, T)(y)r_t(y) + \Upsilon(t, T)(y), \ \Lambda(t, T)(y) \text{ and } \Upsilon(t, T)(y) \text{ deterministic,}$$
(5.12)

together with the hypothesis of a deterministic diffusion coefficient for the spot rate. Then differentiating this identity with respect to T, and replacing into (5.10) implies an Ornstein–Uhlenbeck dynamics for the spot rate, with $a_t(y) := -\partial_\delta \Lambda(t, t)(y)$ and $b_t(y) := \partial_\delta \Upsilon(t, t)(y)$

$$dr_t(y) = (b_t(y) - a_t(y)r_t(y)) \, dt - \gamma^*(t, t)(y).\left(dW_t - \sigma_t^{Y^*}(y)dt\right).$$

Furthermore, identifying the diffusion coefficient in (5.12) and (5.6) implies that $\gamma^*(t, T)(y) = \Lambda(t, T)(y)\gamma^*(t, t)(y)$. Besides, differentiating $r(t, \delta) = f(t, t + \delta)$ using relation (5.12) and identifying with (5.11) the term in $r_t(y)dt$ implies $\partial_t \Lambda(t, T)(y) - a_t(y)\Lambda(t, T)(y) = 0$, hence $\Lambda(t, T)(y) = e^{-\int_t^T a_u(y)du}$ since $\Lambda(T, T)(y) = 1$. Therefore, we have proved that the affine structure (5.12) induces a time-dependent version of the standard Vasicek model with $\gamma^*(t, T)(y) = e^{-\int_t^T a_u(y)du}\gamma^*(t, t)(y)$. If in addition the volatility $\sigma^{Y^*}(y)$ is deterministic then this affine model is also Gaussian.[9]

Illustration of Proposition 5.2: If the spot rate r does not depend on y, then the diffusion coefficient $\gamma^*(s, s)$ is independent of y. Remark also that if r_t does not depend on y, then $\mathbb{E}(r_t(y))$ does not either, and this implies that a is also independent of y. To summarize, in this affine model, if the spot rate r does not depend on the initial condition y then $\gamma^*(t, t)$, a_t and the drift $b_t(y) - a_t r_t + \gamma^*(t, t).\sigma_t^{Y^*}(y)$ do not depend on y. We recover the result of Proposition 5.2, since in this affine framework, $\partial_\delta f^*(t, t)(y) = b_t(y) - a_t r_t$ as a direct consequence of (5.12). This approach can be generalized into a multidimensional affine model, as in Duffie et al. (2003).

6 Asymptotic long-run rates

We are interested in the dynamics behavior of the yield curve, when the maturity goes to infinity

$$R_t^*(\infty)(y) := \lim_{T \to +\infty} R_t^*(T)(y).$$
(6.1)

Recalling the relation $R_t^*(T)(y) = \frac{1}{T}\int_0^T f^*(t, t + s)(y)ds$, we study hereafter the asymptotic limit of the forward rate $f^*(t, \infty)(y) := \lim_{T \to +\infty} f^*(t, T)(y)$ and by Cesaro's Lemma[10] we deduce the limit of the yield curve from the one of the instan-

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[9] In fact, up to a change of probability that depends on $\sigma^{Y^*}(y)$, this affine model is always Gaussian.

[10] See (Korevaar 2013). Note that this is only a sufficient condition: the two limits of $f^*(t, T)(y)$ and $R_t^*(T)(y)$ are not equivalent from a strict mathematical point of view, but are equal when both of them

taneous forward rate. Recalling (5.5)

$$f^*(t, T)(y) = f^*(0, T)(y) - \int_0^t \gamma^*(s, T)(y).\big(\mathrm{d}W_s - (\sigma_s^{Y^*}(y) + \Gamma^*(s, T)(y))\mathrm{d}s\big),$$

we have to study together the behavior of the stochastic integral $\int_0^t \gamma^*(s, T)(y).\mathrm{d}W_s$ and of the finite variation process $\int_0^t \gamma^*(s, T)(y).(\sigma_s^{Y^*}(y) + \Gamma^*(s, T)(y))\mathrm{d}s$, for a fixed t and when T is large.[11] A particular attention is paid on the parameters : the initial value y, or the time horizon T_H. Notably the backward and forward frameworks induce different asymptotic behaviors, as detailed hereafter. This extends previous results of Dybvig et al. (1996) and El Karoui et al. (1997).

6.1 Asymptotic long-run rates with backward utilities

We study the yield curve dynamics for infinite maturity, first in the framework of backward utility, for which the orthogonal component $\nu^{*,H}(y)$ of $\sigma^{Y^*,H}(y)$, as well as the volatility $\Gamma^{*,H}(., T)(y)$, depend on the time-horizon T_H, and consequently impacts the long-term behavior of the yield curve. Remark that in previous papers on long-term rates such as Dybvig et al. (1996) and El Karoui et al. (1997), this dependency on T_H that only happens in incomplete market (otherwise the orthogonal component ν is zero) is not taken into account. This explains why we have more various long-term behaviors for rates in the backward setting in incomplete markets. We thus highlight this dependency by using the index H, and to fix the idea, as T tends to infinity, we take $T_H = T$.

Proposition 6.1 *In the backward case, when the maturity T_H tends to infinity, the instantaneous forward rate $f^{*,H}(t, T_H)(y)$ converges uniformly in L^2 (toward a finite limit) if the following limits exist a.s. in $\mathbb{R}$*

$$\begin{cases} k_s(y) := \lim_{T_H \to +\infty} \gamma^{*,H}(s, T_H)(y), & (6.2) \\ \mathfrak{g}_s(y) := \lim_{T_H \to +\infty} \gamma^{*,H}(s, T_H)(y).\big(\nu_s^{*,H}(y) + \Gamma^{*,H}(s, T_H)(y)\big). & (6.3) \end{cases}$$

The dynamics of the asymptotic long instantaneous forward rate $f^*(t, \infty)(y)$ is

$$f^*(t, \infty)(y) = f^*(0, \infty)(y) - \int_0^t k_s(y).\mathrm{d}W_s + \int_0^t \big(\mathfrak{g}_s(y) - <k_s^{\mathcal{R}}(y), \eta_s^{\mathcal{R}}>\big)\mathrm{d}s. \tag{6.4}$$

---

exist. For the converse result one need for example a monotonicity condition of $u \to f^*(t, u)$ to deduce the infinite limit of f^* from the one of R^*.

[11] We assume sufficient regularity conditions on the coefficients of the SDE satisfied by the process $f^*(t, T)$ (typically $\gamma^*(s, T)(y)$ uniformly bounded in T by an L^2-integrable process, as in El Karoui et al. (1997)) to use convergence results of SDE.

(i) *Having simultaneously the limit $k_.(y)$ not equal to zero $ds \otimes d\mathbb{P}$ a.s. with a finite limit $\mathfrak{g}_.(y)$ is possible only if $\frac{1}{T_H} < k_s^\perp(y), v_s^{*,H}(y) >= -||k_s(y)||^2 + \mathcal{O}(\frac{1}{T_H})$ $ds \otimes d\mathbb{P}$. Then the instantaneous forward rates for infinite maturity are finite and their dynamics (6.4) have a diffusion component.*

(ii) *If $\frac{1}{T_H} < k_s^\perp(y), v_s^{*,H}(y) > \neq -||k_s(y)||^2 + \mathcal{O}(\frac{1}{T_H})$ $ds \otimes d\mathbb{P}$, the limit $\mathfrak{g}_s(y)$ is finite only if $k_s(y) \equiv 0$ $ds \otimes d\mathbb{P}$ a.s., and the usual form holds for the asymptotic instantaneous forward rates*

$$f^*(t, \infty)(y) = f_0(y) + \int_0^t \mathfrak{g}_s(y) ds.$$

Proof We have to study the limit of the terms in (5.5), where in the backward case the orthogonal component of $\sigma^{Y^*}(y)$, namely $v^{*,H}(y)$, may depend on T_H and has to be taken into account to compute the limit.

First, remark that if $\gamma^{*,H}(s, T_H)(y)$ converges (which is equivalent to $\gamma^{*,H,\mathcal{R}}(s, T_H)(y)$ and $\gamma^{*,H,\perp}(s, T_H)(y)$ converge), then the stochastic integral in (5.5) converges. Besides,

$$\gamma^{*,H}(s, T_H)(y).\left(\sigma_s^{Y^*}(y) + \Gamma^{*,H}(s, T_H)(y)\right)$$
$$= - < \gamma^{*,H,\mathcal{R}}(s, T_H)(y), \eta_s^{\mathcal{R}} > + < \gamma^{*,H}(s, T_H)(y), v_s^{*,H}(y)$$
$$+ \Gamma^{*,H}(s, T_H)(y) > .$$

Since $\eta_s^{\mathcal{R}}$ does not depend on T_H, (6.2) and (6.3) imply that the right hand side converges a.s. and the dynamics is given by (6.4).

We recall that $\gamma^*(t, T)(y) = \partial_T \Gamma^*(t, T)(y)$. Therefore, by Cesaro's Lemma, when $T_H \rightarrow +\infty$, $< \gamma^{*,H}(s, T_H)(y), \Gamma^{*,H}(s, T_H)(y) >$ is asymptotically equivalent to $T_H ||\gamma^{*,H}(s, T_H)(y)||^2$. Thus, if $k_s(y) = \lim_{T_H \rightarrow +\infty} \gamma^{*,H}(s, T_H)(y)$ is not equal to zero a.s., then $< \gamma^{*,H}(s, T_H)(y), v_s^{*,H}(y) + \Gamma^{*,H}(s, T_H)(y) >$ converges if and only if $\frac{1}{T_H} < k_s^\perp(y), v_s^{*,H}(y) >= -||k_s(y)||^2 + \mathcal{O}(\frac{1}{T_H})$.

Otherwise, to ensure the limit $\mathfrak{g}_s(y)$ to be finite, one should have $\lim_{T_H \rightarrow +\infty} \gamma^{*,H}(s, T_H)(y) = 0$ $ds \otimes d\mathbb{P}$ a.s., which implies that there is no stochastic integral in the dynamics (6.4), which is then given by $f_t(y) = f_0(y) + \int_0^t \mathfrak{g}_s(y) ds$. $\square$

By applying again Cesaro's Lemma, this time on the rates $f^{*,H}(t, T_H)(y)$ and $R_t^{*,H}(T_H)(y) = \frac{1}{T_H} \int_0^{T_H} f^{*,H}(t, t + s)(y) ds$, we conclude that $R^*(t, \infty)(y) = f^*(t, \infty)(y)$.

The diffusion component in the dynamics (6.4) of asymptotic long rates is a consequence of the dependency on T_H of the orthogonal $v^{*,H}$ of the optimal discounted pricing kernel $Y^{*,H}$. To specify the dynamics (6.4), one need to determine the links between the orthogonal diffusion coefficients $v^{*,H}$ and $\Gamma^{*,H,\perp}(., T_H)$, which is not an easy task in full generality. Nevertheless, the computations are tractable for power utilities, which is the natural setting for the Ramsey rule (cf. Sects. 3.3 and 1.1).

6.2 Yield curve properties with backward power utilities

The following theorem provides a new and non-asymptotic relation between the orthogonal diffusion coefficient of the optimal discounted pricing kernel and the zero-coupon bond price.

Theorem 6.2 *For backward power utilities, the orthogonal diffusion coefficient $v^{*,H}$ of the optimal discounted pricing kernel $Y^{*,H}$ and $\Gamma^{*,H,\perp}(., T_H)$ of the zero-coupon bond price are linked by the relation*

$$v_t^{*,H} = -\Gamma^{*,H,\perp}(t, T_H), \quad 0 \le t \le T_H. \tag{6.5}$$

Proof $v^{*,H}$ is the orthogonal diffusion coefficient of the optimal discounted pricing kernel $Y^{*,H}$, solution of the dual optimization problem. According to Definition 2.2, the dual problem relies on the submartingale/martingale property of the preference process $\left(\tilde{\mathcal{U}}(t, Y_t^v) + \int_0^t \tilde{v}(s, Y_s^v) ds\right)$, which is sometimes better to write in a multiplicative form. It is then equivalent to study the submartingale/martingale property of $\left(\exp\left(\int_0^t \frac{\tilde{v}(s, Y_s^v)}{\tilde{\mathcal{U}}(s, Y_s^v)} ds\right) \tilde{\mathcal{U}}(t, Y_t^v)\right)$.

In the backward power framework, the terminal dual utility from wealth $\tilde{\mathcal{U}}(T_H, .)$ and the dual utilities from consumption $\tilde{v}(s, .)$ are given: they are dual power utilities, with the same risk aversion parameter θ, $\quad \tilde{\mathcal{U}}(T_H, y) = Z_{T_H}^{\tilde{u}} y^{\frac{\theta-1}{\theta}}$, and for $s \in [0, T_H]$, $\quad \tilde{v}(s, y) = Z_s^{\tilde{v}} y^{\frac{\theta-1}{\theta}}$, where $Z_s^{\tilde{v}}$ is a given process and $Z_{T_H}^{\tilde{u}}$ is a given $\mathcal{F}_{T_H}$-random variable.

Then, as recalled in (3.6), $\tilde{\mathcal{U}}$ is also time-separable with risk aversion parameter θ. This implies that for $s \in [0, T_H]$, $\frac{\tilde{v}(s, Y_s^v)}{\tilde{\mathcal{U}}(s, Y_s^v)} = \tilde{Z}_s$ where $\tilde{Z}$ is a progressive process that does not depend[12] on v. The backward dual optimization problem (2.3) turns out to find $v \in \mathcal{R}^\perp$ that minimizes the drift of $\tilde{\mathcal{U}}(T_H, Y_s^v)$ that is the drift of $(Y_{T_H}^v)^{\frac{\theta-1}{\theta}}$

$$(Y_{T_H}^v)^{\frac{\theta-1}{\theta}} = \exp\left(-\frac{\theta-1}{\theta}\int_0^{T_H} r_s ds + \frac{\theta-1}{\theta}\int_0^{T_H} \sigma_s^{Y^v}.dW_s - \int_0^{T_H}\frac{\theta-1}{2\theta}||\sigma_s^{Y^v}||^2 ds\right).$$

Using relation (5.3) with the discounted pricing kernel $Y^{v,H}$ instead of Y^*, leading to zero-coupon prices $B^{v,H}(t, T)$ $\left(B^{v,H}(0, T) = B(0, T) \text{ and } B^{v,H}(T, T) = 1\right)$ with volatility $\Gamma^{v,H}$, we have (where Cst denotes a deterministic constant that does not depend on v)

$$(Y_{T_H}^v)^{\frac{\theta-1}{\theta}} = \text{Cst} \exp\left(\frac{\theta-1}{\theta}\int_0^{T_H}(\Gamma^{v,H}(s, T_H) + \sigma_s^{Y^v}).dW_s - \frac{\theta-1}{2\theta}\int_0^{T_H}||\Gamma^{v,H}(s, T_H) + \sigma_s^{Y^v}||^2 ds\right)$$

$$= \text{Cst} \, \mathcal{E}_{T_H}\left(\frac{\theta-1}{\theta}(\Gamma^{v,H}(., T_H) + \sigma^{Y^v})\right) \exp\left(-\frac{\theta-1}{2\theta}\int_0^{T_H}||\Gamma^{v,H}(s, T_H) + \sigma_s^{Y^v}||^2 ds\right)$$

$$\exp\left(\int_0^{T_H}\frac{1}{2}(\frac{\theta-1}{\theta})^2||\Gamma^{v,H}(s, T_H) + \sigma_s^{Y^v}||^2 ds\right)$$

---

[12] This has been proved in El Karoui et al. (2018, Sect. 4.2).

$$= \text{Cst}\, \mathcal{E}_{T_H}\left(\frac{\theta-1}{\theta}(\Gamma^{\nu,H}(.,T_H)+\sigma^{Y^\nu})\right)\exp\left(\frac{1-\theta}{2\theta^2}\int_0^{T_H}||\Gamma^{\nu,H}(s,T_H)+\sigma_s^{Y^\nu}||^2 ds\right).$$

This implies that the minimization problem is equivalent to minimize (in ν) the quadratic form

$$\frac{1-\theta}{2\theta^2}||\sigma_t^{Y^\nu}+\Gamma^{\nu,H}(t,T_H)||^2 = \frac{1-\theta}{2\theta^2}||\nu_t^H - \eta_t^{\mathcal{R}}+\Gamma^{\nu,H}(t,T_H)||^2$$

which achieves its minimum at $\nu_t^{*,H}=-\Gamma^{*,H,\perp}(t,T_H)$.   □

Even in this simple framework of backward power utilities, the backward approach and relation (6.5) imply a diffusion component in the dynamics of asymptotic long rates. Recall that for power utilities, the optimal discounted pricing kernel is linear with respect to its initial condition y, which implies that the interest rates do not depend on y.

Corollary 6.3 *For backward power utilities, the asymptotic long instantaneous forward rate $f^*(t,\infty)$ (that may be infinite) is given by*

$$f^*(t,\infty) = f^*(0,\infty) - \int_0^t k_s . dW_s + \int_0^t \mathfrak{g}_s ds, \tag{6.6}$$

$$with \quad \begin{cases} k_s := \lim_{T_H\to+\infty} \gamma^{*,H}(s,T_H), \\[2mm] \mathfrak{g}_s := \lim_{T_H\to+\infty}\frac{1}{T_H}||\Gamma^{*,H,\mathcal{R}}(s,T_H)||^2. \end{cases}$$

(i) *If $k_s^{\mathcal{R}}$ is not equal to zero $dt \otimes d\mathbb{P}$ a.s., then $f^*(t,\infty)$ is infinite.*
(ii) *Otherwise, $f^*(t,\infty) = f^*(0,\infty) - \int_0^t k_s^\perp . dW_s + \int_0^t \mathfrak{g}_s ds.$*

Proof Applying Proposition 6.1 with $\nu_t^{*,H}=-\Gamma^{*,H,\perp}(t,T_H)$ and using Cesaro's Lemma

$$\lim_{T_H\to+\infty}\gamma^{*,H}(s,T_H).(\nu_s^{*,H}+\Gamma^{*,H}(s,T_H)) = \lim_{T_H\to+\infty}\frac{1}{T_H}||\Gamma^{*,H,\mathcal{R}}(s,T_H)(y)||^2.$$

If $k_s^{\mathcal{R}}(y) = \lim_{T_H\to+\infty}\gamma^{*,H,\mathcal{R}}(s,T_H) \neq 0$ $dt \otimes d\mathbb{P}$ a.s., then $\lim_{T_H\to+\infty}\frac{1}{T_H}$ $||\Gamma^{*,H,\mathcal{R}}(s,T_H)(y)||^2$ and $f^*(t,\infty)$ are infinite. Otherwise $f^*(t,\infty)(y) = f^*(0,\infty)(y) - \int_0^t k_s^\perp . dW_s + \int_0^t \mathfrak{g}_s ds.$ □

Even in this simple framework of backward power utilities, the long-run yield curves (if they are not infinite) have a diffusion component and thus are not monotonous in time. This differs from the framework of forward utility for which they are non-decreasing processes, as detailed below.

6.3 Asymptotic long-run rates with forward utility

We study the yield curve dynamics for infinite maturity, in the framework of forward utility, for which the orthogonal diffusion coefficient $\nu^*(y)$ does not depend on the time-horizon. As a consequence the limit behavior is more straightforward compared to the backward case and has no diffusion component. In particular, in this forward setting, we recover the results of Dybvig et al. (1996) and El Karoui et al. (1997).

Proposition 6.4 *In the forward case, the asymptotic long instantaneous forward rate* $f^*(t, \infty)(y)$ *is*

 (i) *infinite if* $\lim\limits_{T \to +\infty} \gamma^*(t, T)(y)$ *exists and is not equal to zero* $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ *a.s.*
(ii) *Otherwise,* $f^*(t, \infty)(y) = f^*(0, \infty)(y) + \int_0^t g_s(y)\mathrm{d}s$ *with* $g_s(y) = \lim_{T \to +\infty} \frac{1}{T}||\Gamma^*(s, T)(y)||^2$.
    So, the asymptotic long forward rate $f^*(t, \infty)(y)$ *is a non-decreasing process in time starting from* $f^*(0, \infty)(y)$, *constant if* $g_s(y) \equiv 0$ $\mathrm{d}s \otimes \mathrm{d}\mathbb{P}$ *a.s.*

As a corollary, by Cesaro's Lemma, $R^*(t, \infty)(y) = f^*(t, \infty)(y)$.

Proof The proof is based on the following observation (using Cesaro's Lemma)

$$\lim_{T \to +\infty} \frac{1}{T}\Gamma^*(t, T)(y) = \lim_{T \to +\infty} \frac{1}{T}\int_0^T \gamma^*(t, u)(y)\mathrm{d}u = \lim_{T \to +\infty} \gamma^*(t, T)(y).$$

(i) If $\lim_{T \to +\infty} \gamma^*(t, T)(y)$ exists and is not equal to zero $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ a.s. then

$$\lim_{T \to +\infty} \Gamma^*(t, T)(y) = \infty \ \ a.s \text{ and } \ l_t(y) \text{ is infinite.}$$

(ii) Otherwise, $\int_0^T \gamma^*(s, T)(y).\mathrm{d}W_s$ and $\int_0^T \gamma^*(s, T)(y).\sigma_s^{Y^*}(y)\mathrm{d}s$ converge to zero and

$$l_t(y) = l_0(y) + \int_0^t g_s(y)\mathrm{d}s, \quad \text{where } g_t(y) \text{is the nonnegative process}$$

$$g_t(y) = \lim_{T \to +\infty} (\gamma^*(t, T)(y).\Gamma^*(t, T)(y)) = \lim_{T \to +\infty} T||\gamma^*(t, T)(y)||^2$$

$$= \lim_{T \to +\infty} \frac{1}{T}||\Gamma^*(s, T)(y)||^2.$$

$\square$

Throughout this paper, we have pointed out the key role of the discounted pricing kernel Y^* in the computation of the Ramsey rule and the yield curve, such Y^* being optimal relatively to a given preference criterion. A natural question arising is how to handle the heterogeneity of economic actors that may have different preferences and thus different discounted pricing kernel Y^*. To do this, considering N investors

characterized by their utility U^{θ_i}, we aggregate the discounted pricing kernels as follows:

$$Y^*(y) := \sum_{i=1}^{N} Y^{*,\theta_i}(y^{\theta_i}(y)), \quad y = \sum_{i=1}^{N} y^{\theta_i}(y).$$

We propose to study the impact of aggregation on the yield curve, in particular for infinite maturity, or when the wealth of the economy tends to 0 or ∞.

6.4 Aggregation of utilities

As pointed out in El Karoui et al. (2017), aggregating discounted pricing kernels corresponds to the aggregation of utilities. We concentrate of aggregating power utilities, since as explained in Sect. 3.3, power utility functions is an important case of utility functions, in which computations are tractable and the existence of an equilibrium can be stated. Besides, El Karoui and Mrad (2021) proved that the utility functions that are compatible with an equilibrium can be written as mixtures of power utilities.

Let us consider an economy composed of N investors, with consistent power utilities characterized by (constant) relative risk aversion parameters $\theta_1 < \cdots < \theta_N$. Then, their optimal discounted pricing kernels $Y_t^{*,\theta_i}(y)$ are linear in y with coefficient $\bar{Y}_t^{*,\theta_i}$ and the individual price of zero-coupon bonds with maturity T does not depend on y and is given by $B^{*,\theta_i}(t,T) = \mathbb{E}\left(\frac{\bar{Y}_T^{*,\theta_i}}{\bar{Y}_t^{*,\theta_i}}|\mathcal{F}_t\right)$. The aggregate indifference zero-coupon bond price $B^*(0,T)(y)$, computed at time 0 for simplicity, is given by

$$B^*(0,T)(y) = \frac{1}{y}\sum_{i=1}^{N} y^{\theta_i}(y) B^{*,\theta_i}(0,T), \quad \text{with} \quad y = \sum_{i=1}^{N} y^{\theta_i}(y). \quad (6.7)$$

6.4.1 Asymptotic limit for infinite maturity

For any agent, we define his asymptotic long rate

$$R_0^{*,\theta_i}(\infty) := \lim_{T \to \infty} R_0^{*,\theta_i}(T) = \lim_{T \to \infty}\left(-\frac{1}{T}\ln(B^{*,\theta_i}(0,T))\right).$$

The following proposition shows that when the maturity tends to infinity, the asymptotic long aggregate rate is the one with the lowest asymptotic long rate. This is a similar result to that in Cvitanic et al. (2011, Sect. 7).

Proposition 6.5 *We consider the aggregation of N heterogeneous agents having CRRA utility functions, and we denote by $R_0^*(T)(y)$ the corresponding aggregate indifference rate. Then the asymptotic long aggregate rate*

$$R_0^*(\infty) := \lim_{T \to \infty} R_0^*(T)(y) = \min_{i \in [\![1;N]\!]} R_0^{*,\theta_i}(\infty)(possibly\ infinite).$$

Proof First remark that if for any $i \in [\![1; N]\!]$, $R_0^{*,\theta_i}(T)(y)$ have the same limit (infinite or not) then it is straightforward to see that the aggregate yield curve $R_0^*(T)(y)$ converges to this limit. We define $\mathfrak{I} := \operatorname*{argmin}_{i \in [\![1;N]\!]} R_0^{*,\theta_i}(\infty)$, and we choose $i_o \in \mathfrak{I}$. Then

$$
\begin{aligned}
R_0^*(T)(y) &= -\frac{1}{T} \ln(B^*(0, T)(y)) \\
&= -\frac{1}{T} \ln\left(\frac{y^{\theta_{i_o}}(y)}{y} B^{*,\theta_{i_o}}(0, T) \right) - \frac{1}{T} \ln\left(1 + \sum_{i \neq i_o} \frac{y^{\theta_i}(y) B^{*,\theta_i}(0, T)}{y^{\theta_{i_o}}(y) B^{*,\theta_{i_o}}(0, T)} \right) \\
&= R_0^{*,\theta_{i_o}}(T) - \frac{1}{T} \ln\left(\frac{y^{\theta_{i_o}}(y)}{y} \right) \\
&\quad - \frac{1}{T} \ln\left(1 + \sum_{i \neq i_o} \frac{y^{\theta_i}(y)}{y^{\theta_{i_o}}(y)} e^{-T\left(R^{*,\theta_i}(0,T) - R^{*,\theta_{i_o}}(0,T)\right)} \right).
\end{aligned}
\tag{6.8}
$$

If $i \notin \mathfrak{I}$, $e^{-T\left(R^{*,\theta_i}(0,T) - R^{*,\theta_{i_o}}(0,T)\right)} \to 0$ since $\lim_{T \to \infty}(R_0^{*,\theta_i}(T) - R_0^{*,\theta_{i_o}}(T)) > 0$. Thus, the factor inside the logarithm is greater than one and for, large T, is smaller than $(N+1)e^{T \max_{i \in \mathfrak{I}} |R_0^{*,\theta_i}(T) - R_0^{*,\theta_{i_o}}(T)|}$. Therefore, the last term (6.8) converges to zero since for all $i \in \mathfrak{I}$, $\lim_{T \to \infty}(R_0^{*,\theta_i}(T) - R_0^{*,\theta_{i_o}}(T)) = 0$. We conclude that

$$
R_0^*(\infty) = \lim_{T \to \infty} R_0^*(T)(y) = \lim_{T \to \infty} R_0^{*,\theta_{i_o}}(T) = \min_{i \in [\![1;N]\!]} R_0^{*,\theta_i}(\infty).
$$

$\square$

6.4.2 Asymptotic limit with respect to the initial wealth

Power utility functions imply equilibrium rates that do not depend on the wealth process of the economy (see Sect. 3.3) and thus does not allow to capture some important features concerning the impact of the wealth of the economy on the rates. This can be circumvented with aggregation of power utilities, which provides a more flexible preference criterion. Thus, we study hereafter the asymptotic behavior of the aggregate zero-coupon bond price $B^*(0, T)(y)$ for small and large wealth $x = u_z^{-1}(y)$, and for any maturity T.

If any investor is endowed at time 0 with a proportion α_i of the initial global wealth x ($\sum_{i=1}^N \alpha_i = 1$), then $y^{\theta_i}(y) = u_z^{\theta_i}(\alpha_i x) = (\alpha^i x)^{-\theta_i}$, $y = \sum_{i=1}^N y^{\theta_i}(y) = u_z(x)$ and

$$
B^*(0, T)(y) = \frac{1}{y} \sum_{i=1}^N y^{\theta_i}(y) B^{*,\theta_i}(0, T) = \frac{\sum_{i=1}^N (\alpha_i x)^{-\theta_i} B^{*,\theta_i}(0, T)}{\sum_{i=1}^N (\alpha_i x)^{-\theta_i}}.
\tag{6.9}
$$

Proposition 6.6 *We consider the aggregation of N heterogeneous agents having CRRA utility functions. When the wealth tends to infinity the aggregate zero-coupon price*

converges to the one priced by the less risk averse agent, whereas when the wealth tends to zero, it converges to the one priced by the more risk averse agent.

Proof We use (6.9), and the fact that for power utility u^{θ_i}, $y^{\theta_i}(y) = u_z^{\theta_i}(\alpha^i x) = (\alpha^i x)^{-\theta_i}$. When the wealth tends to infinity (corresponding to $y = u_z(x)$ tends to zero) the discrete random measure $\frac{\sum_{i=1}^{N} y^{\theta_i}(y)\, \delta_{\theta_i}(\theta)}{y}$ converges toward a Dirac measure that charges the agent with the smallest risk aversion θ_i and, respectively, toward the largest risk aversion θ_i when the wealth tends to zero (corresponding to y tends to infinity):

$$\lim_{y \to 0} B^*(0, T)(y) = B_0^{\theta_1}(T) \text{ and } \lim_{y \to +\infty} B^*(0, T)(y) = B_0^{\theta_N}(T).$$

This is coherent with the result of Cvitanic et al. (2011, Corollary 4.6).   □

This can be generalized into a continuum of heterogeneous investors indexed by θ, with any utility function (not necessarily power) and having different weights in the economy [see (El Karoui et al. 2017, Sect. 3)].

Conclusion

This paper draws a parallel between financial and economic discount rates and provides a financial interpretation of the Ramsey rule, using consistent pair of progressive utilities of investment and consumption and using marginal utility indifference price (Davis price) for the pricing of non-replicable zero-coupon bonds. We have highlighted that forward utilities provide a more flexible framework than standard backward utilities, which induce time dependency on the time horizon ; this difference between forward and backward approaches is particularly relevant in the computation of the infinite maturity yield curve. The case of power utilities is also developed, in order to provide tractable computations and to remain deliberately close to the economic equilibrium setting. Nevertheless, power utilities imply that the optimal processes are linear with respect to their initial conditions, and due to this simplification, power utilities are not able to catch the impact of the wealth of the economy on the discount rates. Considering aggregation of power utilities, which is equivalent to an aggregation of discounted pricing kernels, overcomes this issue while keeping tractable formulas. This arises naturally in a context of heterogeneous investors, while being compatible with the existence of an equilibrium. Our approach can also be related to multi-curve modeling that attracts significant attention since the crisis, see (Grbac and Runggaldier 2015).

In this paper, we have chosen a framework close to the one of the economic equilibrium framework, with a linear pricing rule (given by the marginal utility price), and for illustrative purpose, we have provided explicit examples in Gaussian markets. We would like to point out the limitations of such framework and to suggest some extensions. Indeed, models that are linear with respect to the noise could result to an underestimation of extreme risks, especially for the long term, and one would like to give more importance to the randomness of the economy. Alternative models to

Gaussian markets for interest rate are affine models and quadratic Gaussian models, for which calculations can be carried out. A short-rate model is affine if it is a linear combination of an affine state space process, whose conditional characteristic function is exponential affine with respect to the initial value. Affine models lead to tractable pricing formula, using Riccati's equations, see for example (El Karoui et al. 2014) in the context of the Ramsey rule. Quadratic Gaussian models are factor models where interest rates are quadratic functions of underlying Gaussian factors, see (Beaglehole and Tenney 1991; Karoui and Durand 1998), or (Jamshidian 1996), among others. Quadratic Gaussian models allow an extra quadratic term of the state variable in the expression for the short rate. For these quadratic short-rate models similar properties hold as for the affine models—as well as analytical and computational tractability— in which the zero-coupon price changes to an expression with an extra quadratic term. Besides, marginal utility price is a linear pricing rule which means that investors agree on this price for a small amount, but they are not sure to have liquidity at this price. For larger nominal amount of transaction and highly illiquid market, the size of the transaction impacts the price. One may use utility indifference pricing, which induces a bid ask spread. Nevertheless, computing utility indifference prices is often a difficult task. An alternative is to use a second-order expansion of the Davis price, which is more tractable. This is developed in the Appendix.

7 Appendix

This Appendix provides theoretical details and proofs on utility indifference pricing, on the time-coherence of the marginal utility price in both the forward and backward settings, as well as the derivation of the second-order development of the utility indifference price with respect to the amount of claim.

7.1 Utility indifference pricing

When the payoff ξ_T of the claim is not replicable, there are different ways to evaluate the risk coming from the non-replicable part, while taking into account the size of the transaction. A way is the pricing by indifference that leads to a bid-ask spread. The utility *indifference price* $\widehat{p}_{0,T}^q(x, \xi_T, q)$ is the price at which the investor is indifferent from investing or not in the contingent claim ; it is given by the nonlinear relationship

$$\mathcal{U}^{\xi,T}(x + \widehat{p}_{0,T}^q(x, \xi_T, q), q) = \mathcal{U}^T(x). \qquad (7.1)$$

where the two maximization problems[13] (with and without the claim ξ_T) have been introduced in Sect. 4.2

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[13] To ease the notations, we will often write $\mathcal{U}^\xi$ and $\mathcal{U}$ rather than $\mathcal{U}^{\xi,T}$ and $\mathcal{U}^T$.

$$
\begin{cases}
\mathcal{U}^T(x) = \sup_{(\kappa,\rho)\in\mathcal{A}^c} \mathbb{E}[U(T, X_T^{\kappa,\rho}) + \int_0^T V(s, c_s)\mathrm{d}s]. & (7.2) \\[2em]
\mathcal{U}^{\xi,T}(x, q) := \sup_{(\kappa,\rho)\in\mathcal{A}^c} \mathbb{E}[U(T, X_T^{\kappa,\rho} - q\,\xi_T) + \int_0^T V(s, c_s)\mathrm{d}s]. & (7.3)
\end{cases}
$$

REMARK: The formulation of the utility indifference pricing problem is the same for forward and backward utilities, with the appropriate utility process **U** that should be considered in the definitions (7.2) and (7.3). In both cases, the utility indifference pricing problem is posed backward, with the natural maturity T which is the date of payment of the claims, and the associated optimal processes depend on T. The literature usually considers the utility indifference pricing problem in the backward framework (that is with $U(T_H, .)$ a given deterministic function, and $T \leq T_H$), see for example (Davis 1998), the survey of Hobson (Henderson and Hobson 2009) or (Carmona and Nualart 1990). If $T < T_H$, thanks to the dynamic programming principle, the stochastic utility $U(T, .)$ that should be considered in (7.2) and (7.3) is the value function at time T of the backward optimization problem with utility $U(T_H, .)$ at time T_H. In the forward framework, $U(T, .)$ is the forward utility itself at time T (and T is not restricted to be less than T_H). In what follows, we consider both the forward and backward settings and we comment the differences when needed. We use the index H (such as $Y_{\cdot}^{*,H}$) to emphasize the time horizon dependency in the backward optimization problem.

7.2 Marginal indifference pricing and time-coherence

For a small amount of the claim, one can use marginal indifference price, which corresponds to the zero marginal rate of substitution $p_{0,T}^u(x, \xi_T) := \lim_{q\to 0} \frac{\partial \hat{p}_{0,T}^q}{\partial q}(x, \xi_T, q)$ as defined in (4.8). In this section, we prove Proposition 4.1 that characterizes the marginal indifference price in terms of the optimal discounted pricing kernel Y^*, and we investigate the time-coherence of this linear pricing rule.

Marginal indifference price is defined for any maturity $T \in [0, +\infty[$ in the forward case and for any $T \leq T_H$ in the backward case. In the backward case, the value function $U(T, .)$ depends on the horizon T_H. In particular, if the contingent claim ξ_T is delivered at time $T \leq T_H$, then ξ_T can be invested between time T and T_H into any admissible portfolio $X_{\cdot}(T, \xi_T)$ (martingale under $Y^{*,H}$) and computing the marginal utility price with terminal payoff $\xi_{T_H} = X_{T_H}(T, \xi_T)$ leads to the same price, as explained below.

Proposition 7.1 *Let $(Y_t^*(y))$ be the optimal discounted pricing kernel associated with a (forward or backward) consumption optimization problem. For any nonnegative contingent claim ξ_T delivered at time T, the marginal utility price is given at any time $t \leq T$ by*

$$
p_{t,T}^u(x, \xi_T) = \mathbb{E}\big[\xi_T \frac{Y_T^*(y)}{Y_t^*(y)} | \mathcal{F}_t\big], \quad y = \mathcal{U}_z(0, x). \tag{7.4}
$$

*(i) **In the forward case**, the pricing rule is time-coherent:*
     for all T and T', with $T \leq T'$

$$p_{t,T'}^u(x, \xi_{T'}) = p_{t,T}^u(x, \xi_T(t, x)) \text{ with } \xi_T(t, x) = p_{T,T'}^u(X_T^*(t, x), \xi_{T'}). \quad (7.5)$$

*(ii) **In the backward case**, the time-coherence property (7.5) is satisfied*

- *for $T \leq T' \leq T_H$ with $\xi_T(t, x) = \xi_T^H(t, x) = p_{T,T'}^{u,H}(X_T^{*,H}(t, x), \xi_{T'})$.*
- *for $T \leq T'$ with $T' > T_H$ if the utility function $U(T', .)$ at the horizon T' is the consistent progressive utility starting from $u(T_H, .)$ at time T_H.*

Proof To simplify the notations, the proof is given for $t = 0$ (the dynamic version can be proved in the same way) and the indifference price is denoted $\widehat{p}^q := \widehat{p}_{0,T}^q(x, \xi_T, q)$.

Following (Davis 1998), we compute the marginal indifference price of any contingent claim as follows. Denote by $(X^{*,q}(x), c^{*,q}(x))$ the optimal strategy of the optimization program (7.3) (q-quantity of the claim ξ_T), such that

$$\mathbb{E}\Big[U(T, X_T^{*,q}(x) - q\xi_T) + \int_0^T V(s, c_s^{*,q}(x)) \mathrm{d}s\Big] = \mathcal{U}^\xi(0, x, q).$$

Thanks to the envelope theorem we can invert optimization and differentiation along the optimal paths (see Milgrom and Segal 2002); in our setting, the q-derivative concerns the random variables $U\big(T, X_T^{\kappa,c}(x) - q\xi_T\big) + \int_t^T V(s, c_s(x))\mathrm{d}s$. Then

$$\partial_q \mathcal{U}^\xi(0, x, q) = -\mathbb{E}\big(U_z(T, X_T^{*,q}(x) - q\xi_T)\xi_T\big).$$

On the other hand, since by definition $\mathcal{U}^\xi(0, x, q) = \mathcal{U}(0, x - \hat{p}^q)$

$$\partial_q \mathcal{U}^\xi(0, x, q) = \partial_q \mathcal{U}(0, x - \hat{p}^q) = -\frac{\partial \hat{p}^q}{\partial q} \mathcal{U}_z(0, x - \hat{p}^q),$$

we obtain the q-sensitivity of the indifference price

$$\frac{\partial \hat{p}^q}{\partial q} = \frac{\mathbb{E}\big(U_z(T, X_T^{*,q}(x) - q\xi_T)\xi_T\big)}{\mathcal{U}_z(0, x - \hat{p}^q)}. \quad (7.6)$$

This quantity depends on the optimal process $X_T^{*,q}(x)$ which is not easy to compute, but at the limit in $q = 0$, it becomes, since $\lim_{q \to 0} X_T^{*,q} = X_T^*$

$$\begin{aligned}
p_{0,T}^u(x, \xi_T) &= \lim_{q \to 0} \frac{\partial \hat{p}^q}{\partial q}(x, \xi_T) \\
&= \frac{\mathbb{E}\big[\xi_T U_z(T, X_T^*(x))\big]}{\mathcal{U}_z(0, x)}.
\end{aligned}$$

The marginal pricing rule is linear and associated with the pricing kernel

$$\frac{U_z(T, X_T^*(x))}{\mathcal{U}_z(0, x)} = \frac{Y_T^*(\mathcal{U}_z(0, x))}{\mathcal{U}_z(0, x)}.$$

(i) In the forward case, for any maturity T', we have

$$p_{0,T'}^u(x, \xi_{T'}) = \frac{1}{u_z(x)} \mathbb{E}\big[U_z(T', X_{T'}^*(x))\, \xi_{T'}\big]$$

$$= \mathbb{E}\big[\xi_{T'} \frac{Y_{T'}^*(y)}{y}\big].$$

In particular, for any $T \leq T'$, one can easily prove (7.5):

$$p_{0,T'}^u(x, \xi_{T'}) = \frac{1}{u_z(x)} \mathbb{E}\big[U_z(T', X_{T'}^*(x))\, \xi_{T'}\big]$$

$$= \frac{1}{u_z(x)} \mathbb{E}\big[\frac{1}{U_z(T, X_T^*(x))} \mathbb{E}\big[U_z(T', X_{T'}^*(x))\, \xi_{T'}|\mathcal{F}_T\big] U_z(T, X_T^*(x))\big]$$

$$= \frac{1}{u_z(x)} \mathbb{E}\big[p_{T,T'}^u(X_T^*(x), \xi_{T'}) U_z(T, X_T^*(x))\big]$$

$$= p_{0,T}^u(x, p_{T,T'}^u(X_T^*(t, x), \xi_{T'})).$$

(ii) In the backward case, if the maturity of the claim is $T \leq T_H$, then the amount ξ_T may be invested in any admissible portfolio $X_.(T, \xi_T)$ such that $(X_t(T, \xi_T)\, Y_t^{*,H}(y))_{T \leq t \leq T_H}$ is a martingale and taking $\xi_{T'} = X_{T'}(T, \xi_T)$, $T' \in [T, T_H]$. Then the proof of (7.5) in the backward case is identical to the one of the forward case as soon as $T \leq T' \leq T_H$:

$$p_{0,T'}^{u,H}(x, \xi_{T'}) = \mathbb{E}\left[\mathbb{E}\left(X_{T'}(T, \xi_T) \frac{Y_{T'}^{*,H}(y)}{y} \right) |\mathcal{F}_T \right]$$

$$= \mathbb{E}\big[\xi_T \frac{Y_T^{*,H}(y)}{y}\big] = p_{0,T}^{u,H}(x, \xi_T), \quad y = u_z(x).$$

The backward marginal utility pricing is a well-posed pricing rule only for $T \leq T_H$. Nevertheless, for $T' > T_H$, in order to still have (7.5), the utility function should be extended between T_H and T' in a time-coherent way in order to get the optimal Y^* until T'.   □

As mentioned before, the marginal utility indifference pricing rule is not well adapted for larger nominal amount of transaction and highly illiquid market. A correcting term of Davis' price consists in providing a second-order development of the utility indifference price, with respect to the number of claim q. In the backward case, this

has first been studied by Henderson (2002) in the Black and Scholes model for power and exponential utilities, and it has been generalized in a semimartingale financial model and backward utility function by Kramkov and Sirbu (2006, Theorem A.1). Theorem 7.2 provides a more direct proof for forward utility.

7.3 Second-order extension of the marginal utility price

The following result provides a second-order expansion of the utility indifference price, for small quantity q of the claim ξ_T.

Theorem 7.2 *Suppose the optimal strategy $X^{*,q}(x)$ of the optimization program (7.3) to be continuously differentiable [14] with respect to q. The utility indifference price at time t of a q-quantity of the claim ξ_T delivered at time T admits the following second-order expansion in the neighborhood of $q = 0$*

$$\widehat{p}^q_{t,T}(x, \xi_T) = q p^u_{t,T}(x, \xi_T)\left(1 + q \frac{U_{zz}(t, X^*_t(x))}{U_z(t, X^*_t(x))} p^u_{t,T}(x, \xi_T)\right)$$
$$+ q^2 \frac{\mathbb{E}\big(U_{zz}(T, X^*_T(x))(\partial_q X^{*,q}_T(x)|_{q=0} - \xi_T)\xi_T\big)}{U_z(t, x)} + o(q^2) \quad (7.7)$$

recalling the Davis price $p^u_{t,T}(x, \xi_T) = \mathbb{E}\big[\xi_T \frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t\big], \quad y = u_z(x) = U_z(0, x).$

Remark that the term $R_A(u) = -\frac{U_{zz}(t,z)}{U_z(t,z)}$ is the absolute risk aversion coefficient. Besides, the term $\partial_q X^{*,q}_T(x)|_{q=0}$ makes it difficult to compute explicitly this second-order term.

Proof We prove the result at time $t = 0$, the dynamic version is obtained in the same way. From (7.6),

$$U_z(0, x - \hat{p}^q)(\partial_q \hat{p}^q) = \mathbb{E}\big(U_z(T, X^{*,q}_T(x) - q\xi_T)\xi_T\big).$$

Differentiating again with respect to q, it follows under regularity assumptions

$$U_z(0, x - \hat{p}^q)(\partial^2_q \hat{p}^q) - U_{zz}(0, x - \hat{p}^q)(\partial_q \hat{p}^q)^2$$
$$= \mathbb{E}\big(U_{zz}(T, X^{*,q}_T(x) - q\xi_T)(\partial_q X^{*,q}_T(x) - \xi_T)\xi_T\big).$$

Then, since $\hat{p}^q \to 0$ and $(\partial_q \hat{p}^q) \to p^u$ when $q \to 0$

$$\partial^2_q \hat{p}^q|_{q=0} = \frac{\mathbb{E}\big(U_{zz}(T, X^*_T(x))(\partial_q X^{*,q}_T(x)|_{q=0} - \xi_T)\xi_T\big) + U_{zz}(0, x)(p^u)^2}{U_z(0, x)}.$$

---

[14]   In the semimartingale framework, this regularity is obtained from that of the SDE coefficients with respect to the parameter q. See (Kunita 1997) for regularity of semimartingales with parameters.

Therefore, the second-order expansion of $\hat{p}^q$ in the neighborhood of $q = 0$ is

$$\hat{p}^q = qp^u(1 + qp^u\frac{U_{zz}(0, x)}{U_z(0, x)}) + q^2\frac{\mathbb{E}\big(U_{zz}(T, X_T^*(x))(\partial_q X_T^{*,q}(x)|_{q=0} - \xi_T)\xi_T\big)}{U_z(0, z)}$$
$$+o(q^2).$$

$\square$

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