



Groundwater extraction among overlapping generations: a differential game approach

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Received: 2 December 2019 / Accepted: 20 June 2020 / Published online: 7 July 2020
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Abstract

Groundwater is a common resource that has been wasted for years. Today, we pay the consequences of such inappropriate exploitation and we are aware that it is necessary to realize policies in order to guarantee the use of this resource for future generations. In fact, the irrational exploitation of water by agents, nevertheless it is a renewable resource, may cause its exhaustion. In our paper, we develop a differential game to determine the efficient extraction of groundwater resource among overlapping generations. We consider intragenerational as well as intergenerational competition between extractors that exploit the resource in different time intervals, and so the horizons of the players in the game are asynchronous. Feedback equilibria have been computed in order to determine the optimal extraction rate of “young” and “old” agents that coexist in the economy. The effects of the withdrawal by several generations are numerically and graphically analyzed in order to obtain results on the efficiency of the groundwater resource.

Keywords Differential games · Resource extraction · Overlapping generation · Feedback Nash equilibrium

JEL Classification C73 · Q25 · J11 · C72

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1 Introduction

The intensive exploitation of groundwater resources represents an issue that must be addressed not only for the current generations but also for future ones. In fact, the social, economic and environmental consequences due to the intensive use of the groundwater require greater attention and a more parsimonious use policy to preserve these resources for future generations. Some negative consequences can be, for instance, land subsidence, increment in the vulnerability of agriculture, increases in pumping costs and so on. Groundwater resources are often exploited under a common property regime; that is, the access is restricted to land owners situated over the aquifer. Gisser and Sanchez (1980) propose their seminal work about aquifer management regime, and they estimate that welfare gains from policy intervention are inefficient with respect to competitive outcomes. Gisser and Sanchez's theoretical prediction is that if the storage capacity of the aquifer is relatively large, then the two approaches would be very close. These results have produced a large literature about groundwater management, see Koundouri (2004). Using the game theory approach, some authors assume that farmers are myopic and take decisions over a short period of time, without considering the impact of the other users or future generations on the available water stock. Instead, other authors propose the exploitation of the water resource in the long period. Negri (1989) determines open-loop and feedback Nash solutions showing that the open-loop equilibrium captures only the pumping cost externality, instead in the feedback approach, also the strategic externalities emerge. However, if the objective function of the problem is concave, Provencher and Burt (1993) show that the feedback solution increases in inefficiency in comparison with the socially optimal outcome. Starting from Gisser and Sanchez' model, Rubio and Casino (2001, 2003) propose a differential game studying open-loop and feedback scenarios over an infinite planning horizon and determine analytical solutions of the social optimum. Their results are in line with those of Negri (1989) and Gisser and Sanchez (1980). In fact, they show that strategic behavior increases the exploitation of the aquifer compared with the open-loop solution, but if the groundwater storage capacity is large, then the difference between the social optimum and private extraction is negligible. Esteban and Albiac (2011) introduce ecosystem damages showing that these environmental externalities can change results substantially. In Biancardi and Maddalena (2018), Biancardi and Maddalena (2019), authors consider heterogeneous farmers in terms of behavior. In particular, cooperators and outsiders are considered in the exploitation of the water resource.

The problem of groundwater exploitation is strictly connected with intergenerational equity. In fact, extraction activity of present generations influences the choices of future generations. In the past, authors that have analyzed the intertemporal exploitation of renewable resources have been, for instance, Chiarella et al. (1984), Kaitala (1993), Clemhout et al. (1985), Jørgensen and Yeung (1996) and so on. The quoted literature assumes that the players have identical infinite time horizons. The sustainability of resource extraction among generations has been considered for the first time by Burton (1993) that analyzes intertemporal preferences and intergenerational equity in a renewable resource harvesting. Mourmouras (1993) examines government policies in an overlapping generations model with renewable resources. Carrera and

Moran (1995) assume that agents make decisions only in the first period and analyze the dynamics obtained by the equilibrium solution in an overlapping generations model. However, these papers do not address the problem from the perspective of the game theory analysis.

Following the approach given by Jørgensen and Yeung (1999) and Grilli (2008), we propose a differential game to determine the efficient extraction of groundwater resources among overlapping generations considering intragenerational as well as intergenerational competition. Our approach is based on the asynchronous horizons of the players resulting from overlapping generations. We assume that players that use the groundwater resource are active during their entire economic life span. We obtain feedback Nash equilibria which describe strategies of extractors. Finally, we emphasize our results in terms of efficiency (see Banker 1980; Banker et al. 1989) of the groundwater pumping system proposing some numerical applications. The paper is organized in the following way. Section 2 presents the model, and in Sect. 3, we investigate feedback Nash equilibrium. In Sect. 4, we propose some numerical applications, and finally, in Sect. 5, we conclude.

2 The model

The model proposes a dynamic equation for the water table and a set of net benefits from groundwater use. The differential equation which describes the dynamic of the water table is obtained as the difference between natural recharge and net extractions

$$\dot{H} = \frac{1}{\Delta S} [R + (\gamma - 1)W]; \quad H(0) = H_0 \quad (1)$$

where H is the height of the aquifer, R denotes the natural recharge, ΔS is the area of the aquifer, $0 < \gamma < 1$ is the constant return flow coefficient of irrigation water, W is the amount of the groundwater pumped, and H_0 is the initial value of groundwater table level in the first period $t_0 = 0$.

As in Gisser and Sanchez (1980), we assume that the demand of water is a negatively sloped linear function

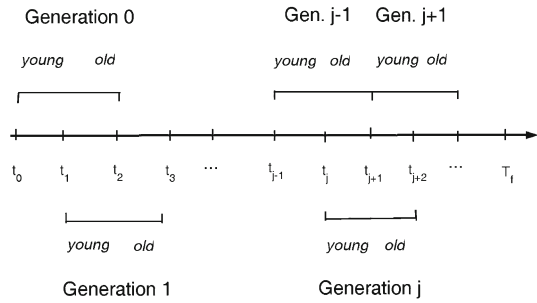
$$W = g + kP \quad (2)$$

where $k < 0$ and $g > 0$ are the price coefficient and the intercept, respectively. Under free market, the current value of the marginal physical product of the water is

$$P = \frac{W}{k} - \frac{g}{k} \quad (3)$$

We suppose that there are $J + 1$ overlapping generations that access the resource for their needs and that in each generation there are $n \geq 2$ identical resource extractors. Let denote by $w_i^j(s)$ the rate of water pumping of the representative extractor i in generation j at time s , for $i \in N = \{1, 2, \dots, n\}$ and $j \in \{0, 1, 2, \dots, J\}$. Following Jørgensen and Yeung (1999) and Grilli (2008), the economy life span of each generation is T , except for the first one, namely generation 0, which is $T/2$. Any generation

Fig. 1 Overlapping generation structure



co-exists with an “old” (preceding) generation in the first half of its life span and a “young” (subsequent) generation in the second half. Generation $j \in \{1, 2, \dots, J\}$ starts its pumping activities at time t_j and ends these activities at time $t_j + T$. Thus, extraction activities of this generation are restricted to the time interval $[t_j, t_j + T]$; in the interval $[t_j, t_j + T/2]$, there are n “old” extractors from generation $j - 1$ and n “new” extractors from generation j . The “old” extractors are in the second half of their life span and will exit the game at the instant $[t_j + T/2]$. The “young” extractors compete with the “old” ones until $[t_j + T/2]$. Then, they will become “old” and face the competition of the new generation $j + 1$.

In the last period, after generation $J - 1$ has left the game, generation J remains alone in the last sub-interval. The following relationships underline the dynamic of overlapping generations: $[t_j, t_j + T/2] = [t_{j-1} + T/2, t_{j-1} + T]$ and $[t_j + T/2, t_j + T] = [t_{j+1}, t_{j+1} + T/2]$.

Figure 1 shows the structure of overlapping generations.

The amount of groundwater pumped in every time interval is:

$$W^j(s) = \begin{cases} \sum_{h=1}^n w_h^{j-1}(s) + \sum_{h=1}^n w_h^j(s) & \text{for } s \in [t_j, t_j + T/2] \\ \sum_{h=1}^n w_h^j(s) + \sum_{h=1}^n w_h^{j+1}(s) & \text{for } s \in [t_j + T/2, t_j + T] \end{cases}$$

and each farmers’ revenue is:

$$R^j(s) = \begin{cases} \left[\frac{\sum_{h=1}^n w_h^{j-1}(s) + \sum_{h=1}^n w_h^j(s)}{k} - \frac{g}{k} \right] w_i^j(s) & \text{for } s \in [t_j, t_j + T/2] \\ \left[\frac{\sum_{h=1}^n w_h^j(s) + \sum_{h=1}^n w_h^{j+1}(s)}{k} - \frac{g}{k} \right] w_i^{j+1}(s) & \text{for } s \in [t_j + T/2, t_j + T] \end{cases}$$

The total payoff of player $i \in N$ in generation $j \in \{1, 2, \dots, J - 1\}$ is:

$$\begin{aligned} \Pi_i^j &= \int_{t_j}^{t_j+T/2} e^{-r(s-t_j)} \\ &\times \left\{ \frac{\left[\sum_{h=1}^n w_h^{j-1}(s) + \sum_{h=1}^n w_h^j(s) \right] w_i^j(s)}{k} - \left(\frac{g}{k} \right) w_i^j(s) - [c_0 - c_1 H(s)] w_i^j(s) \right\} ds \\ &+ \int_{t_j+T/2}^{t_j+T} e^{-r(s-t_j-T/2)} \\ &\times \left\{ \frac{\left[\sum_{h=1}^n w_h^j(s) + \sum_{h=1}^n w_h^{j+1}(s) \right] w_i^j(s)}{k} - \left(\frac{g}{k} \right) w_i^j(s) - [c_0 - c_1 H(s)] w_i^j(s) \right\} ds \end{aligned} \tag{4}$$

where $r > 0$ is a discount rate which is common to all extractors in all generations and

$$C_i^j [w_i^j(s), H(s)] = [c_0 - c_1 H(s)] w_i^j(s) \tag{5}$$

are the costs of water pumping that depend on the quantity of water extracted $w_i^j(s)$ and on the height of the groundwater table $H(s)$. In particular, $c_0 > 0$ represents the fixed cost, with respect to the aquifer height, linked with the hydrologic cone and $c_1 > 0$ is the marginal pumping cost of water extracted. This assumption implies that the cost per unit of water pumped increases when the height of water table goes down and it is proportional to extraction rate $w_i^j(s)$.

The payoff of the first generation (generation 1) is given only of the second expression on the right-hand side of Eq. (4). Otherwise, the payoff of the last generation (generation J) is given by the two expressions of the right-hand side of Eq. (4), with $w_i^{J+1} = 0$ for all $i \in N$.

The natural growth of the groundwater resource stock can be written in the following way:

$$\dot{H}(s) = \frac{1}{\Delta S} \left[R + (\gamma - 1) \sum_{h=1}^n w_h^{j-1}(s) + (\gamma - 1) \sum_{h=1}^n w_h^j(s) \right] \text{ for } s \in [t_j, t_j + T/2]$$

and

$$\dot{H}(s) = \frac{1}{\Delta S} \left[R + (\gamma - 1) \sum_{h=1}^n w_h^j(s) + (\gamma - 1) \sum_{h=1}^n w_h^{j+1}(s) \right] \text{ for } s \in [t_j + T/2, t_j + T] \tag{6}$$

3 Feedback Nash equilibrium

We study the following differential game

$$\max_{w_i^j} \Pi_i^j = \max_{w_i^j} \int_{t_j}^{T_f} e^{-r(s-t_j)} \times \left\{ \frac{\left[\sum_{h=1}^n w_h^{j-1}(s) + \sum_{h=1}^n w_h^j(s) + \sum_{h=1}^n w_h^{j+1}(s) \right] w_i^j(s)}{k} - \left(\frac{g}{k} \right) w_i^j(s) - [c_0 - c_1 H(s)] w_i^j(s) \right\} ds \tag{7}$$

subject to the dynamic of the groundwater given by Eq. (6).

The game begins at the instant $t_1 = t_0 + T/2$ when the initial generation (generation 0) starts its “old” interval and the first generation enters into the market of the water extraction as “young” generation. The terminal instant of the game is $T_f = t_j + T/2$ if $s \in [t_j, t_j + T/2]$ or $T_f = t_j + T$ if $s \in [t_j + T/2, t_j + T]$.

We determine feedback Nash equilibria of the game so that every agent takes decisions about their behavior in dependence of the groundwater level H , taking as given the decision rules of the other players.

Let denote the value function of extractor $i \in \{1, 2, \dots, n\}$ in generation $j \in \{1, 2, \dots, J\}$ by

$$V_i^j(H, t) = \begin{cases} V_i^{j*}(H, t) & \text{for } t \in [t_j, t_j + T/2], \\ V_i^{j**}(H, t) & \text{for } t \in [t_j + T/2, t_j + T] \end{cases} \tag{8}$$

The value functions $V_i^j(H, t)$ must satisfy the continuity conditions that are equivalent to $V_i^{j*}(H, t_j + T/2) = V_i^{j**}(H, t_j + T/2)$ and the final condition $V_i^{j**}(H, t_j + T) = 0$ for the i th player for generation j . This last condition is used as terminal condition in order to solve the problem when an agent is “old” and exits the game. The value function $V_i^{j*}(H, t_j + T/2)$ is used as terminal condition by “young” agents. Obviously, the first generation, (generation 0), lives alone in the interval $[t_0, t_0 + T/2]$ and determines its optimal strategy when it is “old.” Its terminal condition becomes $V_i^{0**}(H, t_0 + T/2) = 0$.

In the subinterval $[t_j, t_j + T/2]$, the game played by the i th agent from the generation j and by the i th agent from the generation $j - 1$ is the following:

$$\max_{w_i^j} \Pi_i^j = \max_{w_i^j} \int_{t_j}^{t_j+T/2} e^{-r(s-t_j)}$$

$$\times \left\{ \frac{\left[\sum_{h=1}^n w_h^{j-1}(s) + \sum_{h=1}^n w_h^j(s) \right]}{k} w_i^j(s) - \left(\frac{g}{k} \right) w_i^j(s) - [c_0 - c_1 H(s)] w_i^j(s) \right\} ds$$

$$+ V_i^{j**}(H, t_j + T/2) \tag{9}$$

$$\max_{w_i^{j-1}} \Pi_i^{j-1} = \max_{w_i^{j-1}} \int_{t_{j-1}+T/2}^{t_{j-1}+T} e^{-r(s-t_j-T/2)}$$

$$\times \left\{ \frac{\left[\sum_{h=1}^n w_h^{j-1}(s) + \sum_{h=1}^n w_h^j(s) \right]}{k} w_i^{j-1}(s) \right.$$

$$\left. - \left(\frac{g}{k} \right) w_i^{j-1}(s) - [c_0 - c_1 H(s)] w_i^{j-1}(s) \right\} ds \tag{10}$$

subject to the groundwater dynamic:

$$\dot{H}(s) = \frac{1}{\Delta S} \left[R + (\gamma - 1) \sum_{h=1}^n w_h^{j-1}(s) + (\gamma - 1) \sum_{h=1}^n w_h^j(s) \right]$$

Following Basar and Olsder (1995), a set of strategies $w_i^{j*}(t) = \phi_i^{j*}(H, t)$ and $w_i^{j-1**}(t) = \phi_i^{j-1**}(H, t)$ is a feedback Nash equilibrium if there exist value functions $V_i^{j*}(H, t)$ and $V_i^{j-1**}(H, t)$ that satisfy the Hamilton–Jacobi–Bellman equations:

$$-\frac{\partial V_i^{j*}(H, t)}{\partial t} = \max_{w_i^j} \left\{ \left[\frac{\left(\sum_{h=1}^n \phi_h^{j-1**}(H, t) + \sum_{h=1, h \neq i}^n \phi_h^{j*}(H, t) + w_i^j(t) \right) w_i^j(t)}{k} \right. \right.$$

$$- \left(\frac{g}{k} \right) w_i^j(t) - [c_0 - c_1 H(t)] w_i^j(t) \Big] e^{-r(t-t_j)}$$

$$+ \frac{\partial V_i^{j*}(H, t)}{\partial H} \left[\frac{1}{\Delta S} \left(R + (\gamma - 1) \sum_{h=1}^n \phi_h^{j-1**}(H, t) \right. \right.$$

$$\left. \left. + (\gamma - 1) \sum_{h=1, h \neq i}^n \phi_h^{j*}(H, t) + (\gamma - 1) w_i^j(t) \right) \right] \Big\} \tag{11}$$

with the boundary condition:

$$V_i^{j*}(H, t_j + T/2) = V_i^{j**}(H, t_j + T/2) \quad \forall H \geq 0$$

and

$$\begin{aligned}
 -\frac{\partial V_i^{j-1**}(H, t)}{\partial t} = \max_{w_i^{j-1}} & \left\{ \left[\frac{\left(\sum_{h=1, h \neq i}^n \phi_h^{j-1**}(H, t) + w_i^{j-1}(t) + \sum_{h=1}^n \phi_h^{j*}(H, t) \right) w_i^{j-1}(t)}{k} \right. \right. \\
 & - \left(\frac{g}{k} \right) w_i^{j-1}(t) - [c_0 - c_1 H(t)] w_i^{j-1}(t) \Big] e^{-r(t-t_j-T/2)} \\
 & + \frac{\partial V_i^{j-1**}(H, t)}{\partial H} \left[\frac{1}{\Delta S} \left(R + (\gamma - 1) \sum_{h=1, h \neq i}^n \phi_h^{j-1**}(H, t) \right. \right. \\
 & \left. \left. + (\gamma - 1) w_i^{j-1}(t) + (\gamma - 1) \sum_{h=1}^n \phi_h^{j*}(H, t) \right) \right] \Big\} \tag{12}
 \end{aligned}$$

with the boundary condition:

$$V_i^{j-1**}(H, t_{j-1} + T) = 0 \quad \forall H \geq 0$$

In the subinterval $[t_j + T/2, t_j + T]$, the game played by the i th agent from the generation j and by the i th agent from the generation $j + 1$ is the following:

$$\begin{aligned}
 \max_{w_i^j} \Pi_i^j = \max_{w_i^j} & \int_{t_j+T/2}^{t_j+T} e^{-r(s-t_j-T/2)} \\
 & \times \left\{ \frac{\left[\sum_{h=1}^n w_h^j(s) + \sum_{h=1}^n w_h^{j+1}(s) \right] w_i^j(s)}{k} \right. \\
 & \left. - \left(\frac{g}{k} \right) w_i^j(s) - [c_0 - c_1 H(s)] w_i^j(s) \right\} ds + V_i^{j**}(H, t_j + T) \tag{13}
 \end{aligned}$$

$$\begin{aligned}
 \max_{w_i^{j+1}} \Pi_i^{j+1} = \max_{w_i^{j+1}} & \int_{t_{j+1}}^{t_{j+1}+T/2} e^{-r(s-t_{j+1})} \\
 & \times \left\{ \frac{\left[\sum_{h=1}^n w_h^j(s) + \sum_{h=1}^n w_h^{j+1}(s) \right] w_i^{j+1}(s)}{k} \right. \\
 & \left. - \left(\frac{g}{k} \right) w_i^{j+1}(s) - [c_0 - c_1 H(s)] w_i^{j+1}(s) \right\} ds \tag{14}
 \end{aligned}$$

subject to the groundwater dynamic:

$$\dot{H}(s) = \frac{1}{\Delta S} \left[R + (\gamma - 1) \sum_{h=1}^n w_h^j(s) + (\gamma - 1) \sum_{h=1}^n w_h^{j+1}(s) \right]$$

A set of strategies $w_i^{j**}(t) = \phi_i^{j**}(H, t)$ and $w_i^{j+1*}(t) = \phi_i^{j+1*}(H, t)$ is a feedback Nash equilibrium if there exist value functions $V_i^{j**}(H, t)$ and $V_i^{j+1*}(H, t)$ that satisfy the Hamilton–Jacobi–Bellman equations:

$$\begin{aligned}
 -\frac{\partial V_i^{j**}(H, t)}{\partial t} = \max_{w_i^j} & \left\{ \left[\frac{\left(\sum_{h=1}^n \phi_h^{j+1*}(H, t) + \sum_{h=1, h \neq i}^n \phi_h^{j**}(H, t) + w_i^j(t) \right) w_i^j(t)}{k} \right. \right. \\
 & - \left(\frac{g}{k} \right) w_i^j(t) - [c_0 - c_1 H(t)] w_i^j(t) \left. \right] e^{-r(t-t_j-T/2)} \\
 & + \frac{\partial V_i^{j**}(H, t)}{\partial H} \left[\frac{1}{\Delta S} \left(R + (\gamma - 1) \sum_{h=1}^n \phi_h^{j+1*}(H, t) \right. \right. \\
 & \left. \left. + (\gamma - 1) \sum_{h=1, h \neq i}^n \phi_h^{j**}(H, t) + (\gamma - 1) w_i^j(t) \right) \right] \left. \right\} \tag{15}
 \end{aligned}$$

with the boundary condition:

$$V_i^{j**}(H, t_j + T) = 0 \quad \forall H \geq 0$$

and

$$\begin{aligned}
 -\frac{\partial V_i^{j+1*}(H, t)}{\partial t} = \max_{w_i^{j+1}} & \left\{ \left[\frac{\left(\sum_{h=1, h \neq i}^n \phi_h^{j+1*}(H, t) + w_i^{j+1}(t) + \sum_{h=1}^n \phi_h^{j**}(H, t) \right) w_i^{j+1}(t)}{k} \right. \right. \\
 & - \left(\frac{g}{k} \right) w_i^{j+1}(t) - [c_0 - c_1 H(t)] w_i^{j+1}(t) \left. \right] e^{-r(t-t_{j+1})} \\
 & + \frac{\partial V_i^{j+1**}(H, t)}{\partial H} \left[\frac{1}{\Delta S} \left(R + (\gamma - 1) \sum_{h=1, h \neq i}^n \phi_h^{j+1*}(H, t) \right. \right. \\
 & \left. \left. + (\gamma - 1) w_i^{j+1}(t) + (\gamma - 1) \sum_{h=1}^n \phi_h^{j**}(H, t) \right) \right] \left. \right\} \tag{16}
 \end{aligned}$$

with the boundary condition:

$$V_i^{j+1*}(H, t_{j+1} + T/2) = V_i^{j+1**}(H, t_{j+1} + T/2) \quad \forall H \geq 0$$

See ‘‘Appendix’’ for the solution of the game.

It is important to underline that, solving the game for generations j and $j - 1$ in the interval $[t_j, t_j + T/2]$, we obtain the result of the game in the next interval $[t_j + T/2, t_j + T]$. In fact, in this last period generation j acts as ‘‘old’’ generation and generation $j + 1$ acts as ‘‘young’’ generation.

In order to compute the solutions of Eqs. (33) and (34) in ‘‘Appendix,’’ given the linear quadratic structure of the model, we guess that the optimal value functions are quadratic, and consequently, the equilibrium strategies are linear with respect to the state variable. Precisely, we postulate that:

$$V_i^{j*}(H, t) = [A_i^{j*}(t)H^2 + B_i^{j*}(t)H + C_i^{j*}(t)]e^{-r(t-t_j)} \tag{17}$$

for $(H, t) \in R^+ \times [t_j, t_j + T/2]$ and

$$V_i^{j**}(H, t) = [A_i^{j**}(t)H^2 + B_i^{j**}(t)H + C_i^{j**}(t)]e^{-r(t-t_j-T/2)} \tag{18}$$

for $(H, t) \in R^+ \times [t_j + T/2, t_j + T]$.

So we have that:

$$\frac{dV_i^{j*}}{dt} = \left[\frac{dA_i^{j*}(t)}{dt} H^2 + \frac{dB_i^{j*}(t)}{dt} H + \frac{dC_i^{j*}(t)}{dt} \right] e^{-r(t-t_j)} - r e^{-r(t-t_j)} [A_i^{j*}(t) H^2 + B_i^{j*}(t) H + C_i^{j*}(t)] \tag{19}$$

$$\frac{dV_i^{j*}}{dH} = [2HA_i^{j*}(t) + B_i^{j*}(t)] e^{-r(t-t_j)} \tag{20}$$

and

$$\frac{dV_i^{j**}}{dt} = \left[\frac{dA_i^{j**}(t)}{dt} H^2 + \frac{dB_i^{j**}(t)}{dt} H + \frac{dC_i^{j**}(t)}{dt} \right] e^{-r(t-t_j)} - r e^{-r(t-t_j)} [A_i^{j**}(t) H^2 + B_i^{j**}(t) H + C_i^{j**}(t)] \tag{21}$$

$$\frac{dV_i^{j**}}{dH} = [2HA_i^{j**}(t) + B_i^{j**}(t)] e^{-r(t-t_j)} \tag{22}$$

Let denote by $A_i^{j**}(t) = A^{**}(t - t_j - T/2)$; $A_i^{j*}(t) = A^*(t - t_j)$; $B_i^{j**}(t) = B^{**}(t - t_j - T/2)$; $B_i^{j*}(t) = B^*(t - t_j)$; $C_i^{j**}(t) = C^{**}(t - t_j - T/2)$; $C_i^{j*}(t) = C^*(t - t_j)$. Substituting the previous notations in Eqs. (33) and (34), we obtain the following system:

$$\begin{aligned} \frac{dA^*(t-t_j)}{dt} &= -4\alpha_6[A^{**}(t-t_{j-1}-T/2) + A^*(t-t_j)]^2 \\ &\quad + (r-2\alpha_5)A^*(t-t_j) - \alpha_3 - 2\alpha_8A^{**}(t-t_{j-1}-T/2) \\ \frac{dB^*(t-t_j)}{dt} &= -4\alpha_6[B^{**}(t-t_{j-1}-T/2) + B^*(t-t_j)] \\ &\quad \times [A^{**}(t-t_{j-1}-T/2) + A^*(t-t_j)] \\ &\quad + (r-\alpha_5)B^*(t-t_j) - \alpha_8B^{**}(t-t_{j-1}-T/2) \\ &\quad - 2\alpha_7A^{**}(t-t_{j-1}-T/2) - \alpha_2 - 2\alpha_4A^*(t-t_j) \\ \frac{dC^*(t-t_j)}{dt} &= -rC^*(t-t_j) - \alpha_1 - \alpha_4B^*(t-t_j) - \alpha_7B^{**}(t-t_{j-1}-T/2) \\ \frac{dA^{**}(t-t_j-T/2)}{dt} &= -4\alpha_6[A^*(t-t_{j+1}) + A^{**}(t-t_j-T/2)]^2 \\ &\quad + (r-2\alpha_5)A^{**}(t-t_j-T/2) - \alpha_3 - 2\alpha_8A^*(t-t_{j+1}) \\ \frac{dB^{**}(t-t_j-T/2)}{dt} &= -4\alpha_6[B^*(t-t_{j+1}) + B^{**}(t-t_j-T/2)] \\ &\quad \times [A^*(t-t_{j+1}) + A^{**}(t-t_j-T/2)] \\ &\quad + (r-\alpha_5)B^{**}(t-t_j-T/2) - \alpha_8B^*(t-t_{j+1}) \\ &\quad - 2\alpha_7A^*(t-t_{j+1}) - \alpha_2 - 2\alpha_4A^{**}(t-t_j-T/2) \\ \frac{dC^{**}(t-t_j-T/2)}{dt} &= -rC^{**}(t-t_j-T/2) - \alpha_1 - \alpha_4B^{**}(t-t_j-T/2) \\ &\quad - \alpha_7B^*(t-t_{j+1}) \end{aligned}$$

The boundary conditions on the equations system are:

$$\begin{aligned} A^*(T/2) &= A^*(0); & B^*(T/2) &= B^*(0); \\ C^*(T/2) &= C^*(0) & \text{for } t &\in [t_j, t_j + T/2] \end{aligned}$$

and

$$A^{**}(T/2) = B^{**}(T/2) = C^{**}(T/2) = 0 \quad \text{for } t \in [t_j + T/2, t_j + T]$$

Substituting the value functions into the expressions (29) and (30), we obtain the level of pumping rates:

$$\begin{aligned} \phi_i^{j*}(H, t) &= \frac{g + k[c_0 - c_1 H(t)] - k(n + 1)[2H(t)A^*(t - t_j) + B^*(t - t_j)] \frac{(\gamma - 1)}{\Delta S}}{(1 + 2n)} \\ &+ \frac{nk[2H(t)A^{**}(t - t_{j-1} - T/2) + B^{**}(t - t_{j-1} - T/2)] \frac{(\gamma - 1)}{\Delta S}}{(1 + 2n)} \end{aligned} \quad (23)$$

for $t \in [t_j, t_j + T/2]$ and

$$\begin{aligned} \phi_i^{j**}(H, t) &= \frac{g + k[c_0 - c_1 H(t)] + nk[2H(t)A^*(t - t_{j+1}) + B^*(t - t_{j+1})] \frac{(\gamma - 1)}{\Delta S}}{(1 + 2n)} \\ &- \frac{k(n + 1)[2H(t)A^{**}(t - t_j - T/2) + B^{**}(t - t_j - T/2)] \frac{(\gamma - 1)}{\Delta S}}{(1 + 2n)} \end{aligned} \quad (24)$$

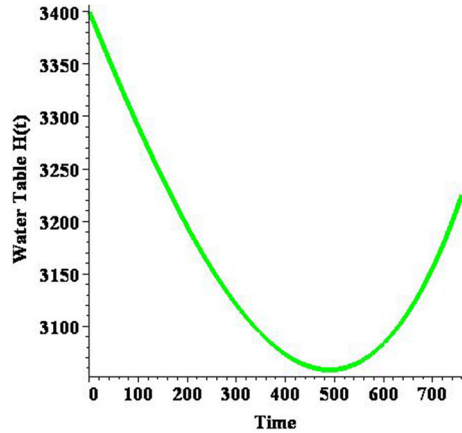
for $t \in [t_j + T/2, t_j + T]$.

Equations (23) and (24) are linear in the state variable H , and the pumping rates are linear with respect to the height of the water table. The following numerical analysis allows us to obtain economical results and better interpretations about the influence of the model parameters on the Feedback equilibria.

4 Numerical applications

The numerical simulation is performed on the basis of parameters given in Gisser and Sanchez (1980). We estimate that the return flow coefficient $\gamma = 0.27$, the natural recharge $R = 173,000$ ac ft/year, the interest rate $r = 0.10$, the area of aquifer $\Delta S = 135,000$ ac ft/year, the fixed cost, with respect to the aquifer height, linked with the hydrologic cone $c_0 = 125$ dollars/ac ft, the marginal pumping cost per acre foot of water pumped $c_1 = 0.035$ dollars/ac ft per foot of lift and the initial water table elevation $H_0 = 3400$ feet above sea level. Moreover, the intercept of the demand for water function stated in Gisser and Sanchez (1980) is $g = 470,365$ ac ft/year, and the decrease in demand for water per 1\$ increase in price is $k = -3259$ ac ft/year. We assume that the deadline of our game is $T_f = 760$ years and, about the longevity

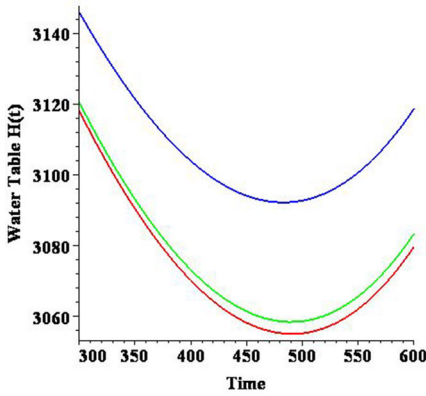
Fig. 2 Optimal groundwater table with $T = 60$ and $n = 100$



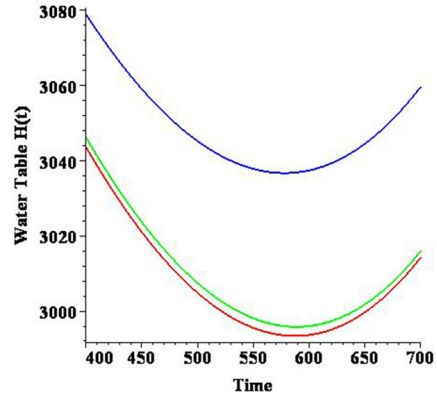
of each generation, we consider three scenarios: $T = 60, 80$ and 100 . For instance, if $T = 60$ years, this means that two generations coexist every 30 years. Assuming that $t_0 = 0$, the game starts at time $t_1 = t_0 + \frac{T}{2} = 30$ years and the number of generations until the deadline T_f is $J = 26$. Moreover, concerning the number of economic agents who operate in the area ΔS , we consider three cases: a low density and therefore $n = 10$, a medium density with $n = 100$ and a high density, and so $n = 500$.

Solving numerically the system (23), we determine the value functions $A^*(t - t_j)$, $B^*(t - t_j)$, $C^*(t - t_j)$ and $A^{**}(t - t_j - T/2)$, $B^{**}(t - t_j - T/2)$, $C^{**}(t - t_j - T/2)$. Substituting these values in Eqs. (23) and (24), we determine the pumping level $\phi_i^{j*}(H, t)$ and $\phi_i^{j**}(H, t)$ for the i th player when is young and old, respectively, considering that he belongs to generation j . Obviously, all players in the j th generation are symmetric. Finally, substituting these values in Eq. (6), we obtain the dynamic of water table and we realize the optimal path of groundwater table $H(t)$.

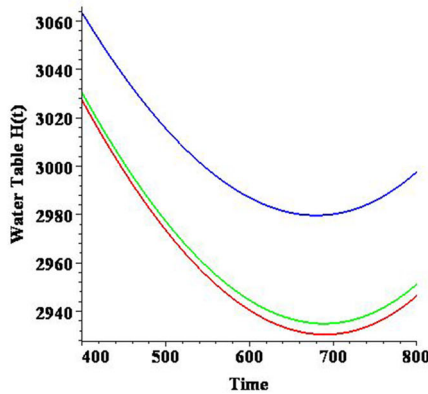
Some interesting interpretations in terms of efficiency can be deduced from the analysis of Figs. 2 and 3a–c. First of all, it is clear that the trend of the height of the water, determined with the optimum pumping levels, is convex. Obviously, this means that the initial generations make a more intensive use of the aquifer. After reaching a minimum threshold, the future generations commit themselves to guaranteeing protection and bringing back the height of the aquifer to its initial level. This behavior can be determined by an increase in pumping costs caused by a lowering of the aquifer which leads to lower withdrawals. For instance, when $T = 60$ and $n = 100$, as illustrated in Fig. 2, the minimum water table level is reached at time $t = 490$ years during the sixteenth generation. After that, a more conservative exploitation will begin. A comparison among the different scenarios can be made analyzing Fig. 3. It is clear that an increase in population size n entails a greater exploitation of the aquifer. Furthermore, the exploitation rate is more sensitive when the population size is low. This means that, when the community expands, it puts in place measures to avoid an overexploitation of aquifer. However, an increase in the longevity of generations causes a negative effect



(a) Comparison with $T = 60$



(b) Comparison with $T = 80$



(c) Comparison with $T = 100$

Fig. 3 Evolution of optimal path of water table elevation $H(t)$ in the three scenarios. The green line denotes the case with $n = 10$. The blue line the case $n = 100$, while the red line $n = 500$ (color figure online)

on the groundwater level and also postpones the instant of time in which the minimum level is obtained.

Another interesting application is the evolution of $H(t)$ when the natural recharge R changes. As shown in Table 1 and Fig. 4, we observe that a scarcity of the rainfall rate leads to a greater minimum level of the aquifer.

Finally, another important parameter that influences the optimal evolution of the groundwater height $H(t)$ is the return flow coefficient γ . In fact, this parameter is low if the community has a scarce technical availability about the irrigation and vice versa, it increases if there is adequate technology that preserves the height of the water. Figure 5 illustrates three different cases assuming $\gamma = 0.10, 0.30$ and 0.60 . We remark how, when γ is low, the groundwater level reaches a minimum value of $H_{\min} = 2904.24$ after $t = 486$ years and it increases to $H_{\min} = 3087.94$ when $t = 487$ years. On the

Table 1 Minimum level of the optimal water height H_{min} and instant of time t_{min} in which the minimum is reached when R changes

R	H_{min}	t_{min}
50,000	2679.50	684
75,000	2690.00	645
100,000	2793.58	606
125,000	2890.52	566
150,000	2980.95	528

Fig. 4 Optimal water table evolution $H(t)$ when R changes. The gray line denotes $R = 150,000$, while the turquoise denotes $R = 125,000$; purple, brown and gold lines represent $R = 100,000$, $R = 75,000$ and $R = 50,000$, respectively (color figure online)

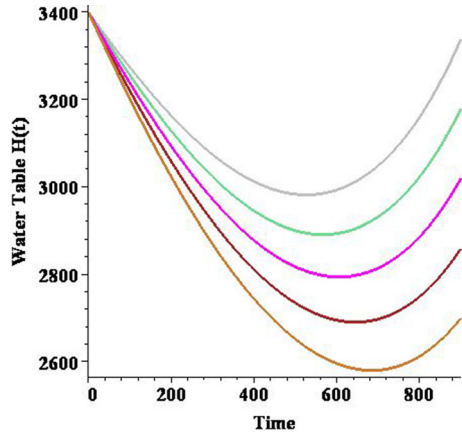
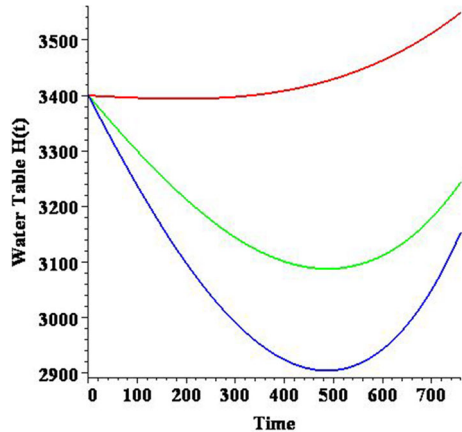


Fig. 5 Optimal water table when γ changes. The blue line denotes $\gamma = 0.10$, and the green one represents $\gamma = 0.30$, while the red line denotes $\gamma = 0.60$ (color figure online)



other hand, when the parameter γ is high, the aquifer level is protected reaching a minimum level $H_{min} = 3394.24$ in $t = 190$ years.

5 Conclusions

In our paper, we have proposed a differential game in order to study the efficient extraction of groundwater resource among agents belong to overlapping generations. We

have considered intragenerational and intergenerational competition between extractors that exploit the resource in different time intervals, and so the horizons of the players in the game are asynchronous. Feedback equilibria have been computed in order to determine the optimal extraction rate of “young” and “old” agents that coexist in the economy. The use of Feedback solution captures the strategic externalities and so it considers that each player adopts a state-dependent decision rule. We have shown that the optimal pumping rates are linear in the state variable H . By a numerical analysis, we have emphasized how the technology, the size and the longevity of agents that make up the generations and environmental parameters, such as the rainfall rate, influence the efficiency of the water pumping system.

Appendix A

Performing the maximization problem in Eqs. (11) and (12) with respect to w_i^j and w_i^{j-1} , for $t \in [t_j, t_j + T/2]$, we obtain:

$$\frac{2\phi_i^{j*}(H, t) + \sum_{h=1}^n \phi_h^{j-1**}(H, t) + \sum_{h=1, h \neq i}^n \phi_h^{j*}(H, t) - g}{k} = [c_0 - c_1 H(t)] - \frac{\partial V_i^{j*}(H, t)}{\partial H} \cdot \frac{(\gamma - 1)}{\Delta S} e^{r(t-t_j)} \tag{25}$$

and

$$\frac{2\phi_i^{j-1**}(H, t) + \sum_{h=1, h \neq i}^n \phi_h^{j-1**}(H, t) + \sum_{h=1}^n \phi_h^{j*}(H, t) - g}{k} = [c_0 - c_1 H(t)] - \frac{\partial V_i^{j-1**}(H, t)}{\partial H} \cdot \frac{(\gamma - 1)}{\Delta S} e^{r(t-t_j)} \tag{26}$$

Adding in Eqs. (25) and (26) over all extractors living in the time interval $[t_j, t_j + T/2]$, that is, all the n “young” agents for the generation j and the n “old” agents for the generation $j - 1$, we obtain:

$$\frac{2n\phi_i^{j*}(H, t) + n^2\phi_h^{j-1**}(H, t) + n(n - 1)\phi_h^{j*}(H, t)}{k} = \frac{ng}{k} + n[c_0 - c_1 H(t)] - \sum_{i=1}^n \frac{\partial V_i^{j*}(H, t)}{\partial H} \cdot \frac{(\gamma - 1)}{\Delta S} e^{r(t-t_j)} \tag{27}$$

for all i th extractor in the j th generation and

$$\frac{2n\phi_i^{j-1**}(H, t) + n^2\phi_h^{j*}(H, t) + n(n - 1)\phi_h^{j-1**}(H, t)}{k}$$

$$= \frac{ng}{k} + n[c_0 - c_1H(t)] - \sum_{i=1}^n \frac{\partial V_i^{j-1**}(H, t)}{\partial H} \cdot \frac{(\gamma - 1)}{\Delta S} e^{r(t-t_j)} \tag{28}$$

for all i th extractor in the $j - 1$ th generation. Equations (27) and (28) represent a system of linear equations in $\phi_i^{j*}(H, t)$ and $\phi_i^{j-1**}(H, t)$. The system admits a unique solution:

$$\begin{aligned} \phi_i^{j-1**}(H, t) = & \frac{g + k[c_0 - c_1H(t)] - k(n + 1) \frac{\partial V_i^{j-1**}(H, t)}{\partial H} \cdot \frac{(\gamma-1)}{\Delta S} e^{r(t-t_j)}}{(1 + 2n)} \\ & + \frac{nk \frac{\partial V_i^{j*}(H, t)}{\partial H} \cdot \frac{(\gamma-1)}{\Delta S} e^{r(t-t_j)}}{(1 + 2n)} \end{aligned} \tag{29}$$

$$\begin{aligned} \phi_i^{j*}(H, t) = & \frac{g + k[c_0 - c_1H(t)] - k(n + 1) \frac{\partial V_i^{j*}(H, t)}{\partial H} \cdot \frac{(\gamma-1)}{\Delta S} e^{r(t-t_j)}}{(1 + 2n)} \\ & + \frac{nk \frac{\partial V_i^{j-1**}(H, t)}{\partial H} \cdot \frac{(\gamma-1)}{\Delta S} e^{r(t-t_j)}}{(1 + 2n)} \end{aligned} \tag{30}$$

Substituting Eqs. (29) and (30) into HJB given by Eqs. (11) and (12), we obtain the following system of partial differential equations (dropping for notational convenience the (H, t) arguments of the value functions):

$$\begin{aligned} - \frac{\partial V_i^{j*}}{\partial t} = & \left[\frac{-(\gamma - 1)^2 n^2 e^{r(t-t_j)} k}{(2n + 1)^2 (\Delta S)^2} \right] \left(\frac{\partial V_i^{j*}}{\partial t} \right)^2 \\ & + \left\{ \frac{4[(R + g(\gamma - 1))n^2 + n(\frac{1}{2}g(\gamma - 1) + R) + \frac{1}{4}(R + g(\gamma - 1))]}{\Delta S(2n + 1)^2} \right. \\ & \left. + \frac{[4(\gamma - 1)\Delta S(c_0 - c_1H)(n^2 + \frac{1}{2}n + \frac{1}{4}) - 2(\gamma - 1)^2 n^2 e^{r(t-t_j)} \frac{\partial V_i^{j-1**}}{\partial H}]}{(\Delta S)^2(2n + 1)^2} \right\} \\ & \times \left(\frac{\partial V_i^{j*}}{\partial t} \right) - \left\{ \frac{[(c_0 - c_1H)\Delta S + (\gamma - 1)e^{r(t-t_j)} n \frac{\partial V_i^{j-1**}}{\partial H}] k + \Delta Sg}{(\Delta S)^2(2n + 1)^2 k e^{r(t-t_j)}} \right\}^2 \tag{31} \\ - \frac{\partial V_i^{j-1**}}{\partial t} = & \left[\frac{-(\gamma - 1)^2 n^2 e^{r(t-t_j)} k}{(2n + 1)^2 (\Delta S)^2} \right] \left(\frac{\partial V_i^{j-1**}}{\partial t} \right)^2 \\ & + \left\{ \frac{4[(R + g(\gamma - 1))n^2 + n(\frac{1}{2}g(\gamma - 1) + R) + \frac{1}{4}(R + g(\gamma - 1))]}{\Delta S(2n + 1)^2} \right. \\ & \left. + \frac{[4(\gamma - 1)\Delta S(c_0 - c_1H)(n^2 + \frac{1}{2}n + \frac{1}{4}) - 2(\gamma - 1)^2 n^2 e^{r(t-t_j)} \frac{\partial V_i^{j-1**}}{\partial H}]}{(\Delta S)^2(2n + 1)^2} \right\} \\ & \times \left(\frac{\partial V_i^{j-1**}}{\partial t} \right) - \left\{ \frac{[(c_0 - c_1H)\Delta S + (\gamma - 1)e^{r(t-t_j)} n \frac{\partial V_i^{j*}}{\partial H}] k + \Delta Sg}{(\Delta S)^2(2n + 1)^2 k e^{r(t-t_j)}} \right\}^2 \tag{32} \end{aligned}$$

Rearranging Eqs. (31) and (32), we obtain that:

$$\begin{aligned}
 -\frac{\partial V_i^{j*}}{\partial t} &= e^{-r(t-t_j)} \left(\alpha_1 + \alpha_2 H + \alpha_3 H^2 \right) + \alpha_4 \left(\frac{\partial V_i^{j*}}{\partial H} \right) + \alpha_5 H \left(\frac{\partial V_i^{j*}}{\partial H} \right) \\
 &+ \alpha_7 \left(\frac{\partial V_i^{j-1**}}{\partial H} \right) + \alpha_8 H \left(\frac{\partial V_i^{j-1**}}{\partial H} \right) \\
 &+ \alpha_6 e^{r(t-t_j)} \left[\left(\frac{\partial V_i^{j*}}{\partial H} \right) + \left(\frac{\partial V_i^{j-1**}}{\partial H} \right) \right]^2
 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
 -\frac{\partial V_i^{j-1**}}{\partial t} &= e^{-r(t-t_j)} \left(\alpha_1 + \alpha_2 H + \alpha_3 H^2 \right) + \alpha_4 \left(\frac{\partial V_i^{j-1**}}{\partial H} \right) + \alpha_5 H \left(\frac{\partial V_i^{j-1**}}{\partial H} \right) \\
 &+ \alpha_7 \left(\frac{\partial V_i^{j*}}{\partial H} \right) + \alpha_8 H \left(\frac{\partial V_i^{j*}}{\partial H} \right) \\
 &+ \alpha_6 e^{r(t-t_j)} \left[\left(\frac{\partial V_i^{j-1**}}{\partial H} \right) + \left(\frac{\partial V_i^{j*}}{\partial H} \right) \right]^2
 \end{aligned} \tag{34}$$

where

$$\begin{aligned}
 \alpha_1 &= -\frac{(kc_0 + g)^2}{k(2n + 1)^2}; \quad \alpha_2 = \frac{2c_1(c_0k + g)}{(2n + 1)^2}; \quad \alpha_3 = \frac{-c_1^2k}{(2n + 1)^2}; \\
 \alpha_4 &= \frac{[(\gamma - 1)(kc_0 + g) + R](4n^2 + 1) + 2n[(\gamma - 1)(kc_0 + g) + 2R]}{\Delta S(2n + 1)^2}; \\
 \alpha_5 &= \frac{-(\gamma - 1)(4n^2 + 2n + 1)kc_1}{\Delta S(2n + 1)^2}; \quad \alpha_6 = \frac{-(\gamma - 1)^2n^2k}{(2n + 1)^2(\Delta S)^2}; \\
 \alpha_7 &= -\frac{2(\gamma - 1)gn}{\Delta S(2n + 1)^2}; \quad \alpha_8 = \frac{2(\gamma - 1)c_1nk}{\Delta S(2n + 1)^2}.
 \end{aligned}$$

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