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A note on the implied volatility of floating strike Asian options

Elisa Alòs[1] · Jorge A. León[2]

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Abstract
In this paper, we study the short-time behavior of the implied volatility for short-time floating strike Asian options. Our method is based on Malliavin calculus techniques and allows us to construct an approximation formula for the corresponding option prices. Numerical examples are given.

Keywords Floating strike Asian options · Kirk's formula · Malliavin calculus · Derivative operator in the Malliavin calculus sense · Skorohod integral

Mathematical Subject Classification 91B28 · 91B70 · 60H07

JEL Classification G12 · G13

1 Introduction

This paper is devoted to the study of floating strike Asian options, that is, European options whose payoff is of the form

$$(S_T - A_T)_+,$$

where S denotes the asset price process and $A_T := k\frac{1}{T}\int_0^T S_u \mathrm{d}u$, for some positive constant k. These options can be seen as *random strike* options (RSO), where the strike

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✉ Jorge A. León
jleon@ctrl.cinvestav.mx

1 Dpt. d'Economia i Empresa and Barcelona GSE, Universitat Pompeu Fabra, c/Ramon Trias Fargas, 25-27, 08005 Barcelona, Spain

2 Control Automático, CINVESTAV-IPN, Apartado Postal 14-740, 07000 Mexico City, Mexico

is allowed to be random. Other classical examples of RSOs include exchange options, spread options or forward starting options. It is well known that, under the Black–Scholes model, some RSOs have an explicit expression for their price. This is the case of exchange options (see Margrabe 1978) or forward start options (see Rubinstein 1991; Wilmott 2002). In other cases, this explicit expression is not available, and only some approximations can be used. For example, spread option prices can be approximated by Kirk's formula (see Kirk 1995). In the case of floating strike Asian options, there are not explicit pricing formulas and we need to apply numerical methods (see, for example, Rogers and Shi 1995; Vecer 2001; Dubois and Lelièvre 2004, among others). We remark that Asian options have a relevant role in energy markets (see Benth and Detering 2015).

In Alòs and León (2016), we provided a systematic procedure to construct short-time option pricing approximation formulas for RSOs. Toward this end, we defined the implied volatility of our RSO as the inverse of the Black–Scholes function, but where the strike was taken to be equal to the expectation of the corresponding random strike. Notice that, due to the randomness of the strike, the behavior of this implied volatility is not the same as in the vanilla case. In particular, we recall that this implied volatility is not a constant, even in the context of the Black–Scholes model. The methodology of these paper consists in computing the short-time level and skew of this implied volatility by means of a change of numéraire and Malliavin calculus techniques. This approach was proved to be an efficient tool in the study of spread options (see again Alòs and León 2016). Nevertheless, the proofs in this work are not valid when the strike is adapted to the same Brownian motion as the asset price. Then, these results cannot be directly applied to the study of floating strike options.

Our aim in this paper is to adapt the ideas in Alòs and León (2016) to develop an approximation formula for floating strike Asian options. Toward this end, we will apply Malliavin calculus techniques to compute the short-time level and skew of the corresponding implied volatility. This will allow us to construct an approximation for this implied volatility and then, an approximation for the Asian option price.

The paper is organized as follows. Section 2 is devoted to present the framework and the notation that we use in this paper. In Sect. 3, we prove a decomposition formula for the option price that will allow us to deduce a first-order approximation result. In Sect. 4, we compute the derivative of the implied volatility with respect to the parameter k. Section 5 is devoted to study the short-time limit of this derivative. This will allow us to construct a second-order approximation formula. The numerical examples in Sect. 6 show that this formula is highly accurate and improves the first-order approximation presented in Sect. 3. Finally, our conclusions are presented in Sect. 7.

2 Statement of the model and notation

In this paper, we consider a Black–Scholes model, where we consider $r = 0$ for the sake of simplicity. More precisely, we assume the following model for the log price of a stock under a risk-neutral probability measure Q:

$$dX_t = -\frac{\sigma^2}{2}dt + \sigma dW_t. \tag{1}$$

Here W is a Brownian motion and σ is a positive constant. We denote by $\mathcal{F}^W$ the augmentation under the underlying probability measure of the filtration generated by W.

This work is devoted to study floating strike Asian options with payoff $h(X_T) := (S_T - A_T)_+$, where $S_t = e^{X_t}$, $t \in [0, T]$, and $A_T = k\frac{1}{T}\int_0^T S_u du$. Notice that the corresponding option price at $t < T$ is given by

$$V_t = E\left[\left(e^{X_T} - A_T\right)_+ | \mathcal{F}_t^W\right]. \tag{2}$$

In the sequel, we will make use of the following notation:

- $M_t^T := E\left[A_T | \mathcal{F}_t^W\right]$. Notice that

$$\begin{aligned} M_t^T &= k\frac{1}{T}\left(S_t(T-t) + \int_0^t S_u du\right) \\ &= kS_t F(T,t) + k\frac{1}{T}\int_0^t S_u du, \end{aligned} \tag{3}$$

where $F(T, t) := \frac{T-t}{T}$. Moreover, the integration by parts formula implies

$$dM_t^T = k\sigma S_t F(T,t) dW_t. \tag{4}$$

- $v_t := \left(\frac{Y_t}{T-t}\right)^{\frac{1}{2}}$, with $Y_t := \int_t^T a_u^2 du$, where

$$a_u^2 du := \sigma^2 du - 2\frac{d\left\langle M^T, X\right\rangle_u}{M_u^T} + \frac{d\left\langle M^T, M^T\right\rangle_u}{\left(M_u^T\right)^2}.$$

Note that

$$\begin{aligned} a_u^2 &= \sigma^2 - 2k\frac{\sigma^2 S_u F(T,u)}{M_u^T} + k^2\frac{\sigma^2 S_u^2 F^2(T,u)}{(M_u^T)^2} \\ &= \sigma^2\left(1 - k\frac{S_u F(T,u)}{M_u^T}\right)^2 \\ &= \frac{k^2}{T^2}\sigma^2\left(\frac{\int_0^u S_\theta d\theta}{M_u^T}\right)^2 \\ &= \sigma^2\left(\frac{\int_0^u S_\theta d\theta}{S_u(T-u) + \int_0^u S_\theta d\theta}\right)^2, \end{aligned}$$

which does not depend on k. Moreover, although the right-hand side of the last equality depends on T, we denote it by a_u^2 in order to simplify the notation. Also, notice that

$$\begin{aligned}\lim_{T\to 0}\frac{1}{T}\int_0^T a_u^2\mathrm{d}u &= \lim_{T\to 0}\frac{k^2}{T^3}\int_0^T \sigma^2\left(\frac{\int_0^u S_\theta\mathrm{d}\theta}{M_u^T}\right)^2\mathrm{d}u\\ &= \lim_{T\to 0}\frac{\sigma^2}{T^3}\int_0^T u^2\mathrm{d}u\\ &= \frac{\sigma^2}{3}. \end{aligned} \tag{5}$$

- $\mathrm{BS}(t,x,k,\sigma)$ denotes the price of an European call option under the classical Black–Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price k and interest rate 0. Remember that in this case:

$$\mathrm{BS}(t,x,k,\sigma)=e^x N(d_+)-kN(d_-),$$

  where N denotes the cumulative probability function of the standard normal law and

$$d_\pm := \frac{x-x_t^*}{\sigma\sqrt{T-t}}\pm\frac{\sigma}{2}\sqrt{T-t},$$

  with $x_t^* := \ln k$.
- $\mathcal{L}_{\mathrm{BS}}\left(\sigma^2\right)$ stands for the Black–Scholes differential operator, in the log variable, with volatility σ:

$$\mathcal{L}_{\mathrm{BS}}\left(\sigma^2\right)=\partial_t+\frac{1}{2}\sigma^2\partial_{xx}^2-\frac{1}{2}\sigma^2\partial_x.$$

  It is well known that $\mathcal{L}_{\mathrm{BS}}\left(\sigma^2\right)\mathrm{BS}(\cdot,\cdot,\sigma)=0$.

Now we describe some basic notation that is used in this article. For this, we assume that the reader is familiar with the elementary results of the Malliavin calculus, as given, for instance, in Nualart (2006).

We denote by $\mathbb{D}_W^{1,2}$ the domain of the derivative operator D^W in the Malliavin calculus sense. $\mathbb{D}_W^{1,2}$ is a dense subset of $L^2(\Omega)$, and D^W is a closed and unbounded operator from $L^2(\Omega)$ into $L^2([0,T]\times\Omega)$. We also consider the iterated derivatives $D^{W,n}$, for $n>1$, whose domains are denoted by $\mathbb{D}_W^{n,2}$.

The adjoint of the derivative operator D^W, denoted by δ^W, is an extension of the Itô integral in the sense that the set $L_a^2([0,T]\times\Omega)$ of square integrable and adapted processes (with respect to the filtration generated by W) is included in $\mathrm{Dom}\delta^W$ and the operator δ^W restricted to $L_a^2([0,T]\times\Omega)$ coincides with the Itô integral. We make use of the notation $\delta^W(u)=\int_0^T u_t\mathrm{d}W_t$ and $\delta^W(u\mathbf{1}_{[0,t]})=\int_0^t u_s\mathrm{d}W_s$. We recall that $\mathbb{L}_W^{n,2}:=L^2([0,T];\mathbb{D}_W^{n,2})$ is included in the domain of δ^W for all $n\geq 1$.

Remark 1 Assume model (1), where σ is a positive constant. Then, for all $r < u$,

$$D_r^W a_u^2 = \frac{2k^2}{T^2}\sigma^2 \left(\frac{\int_0^u S_\theta \mathrm{d}\theta}{M_u^T} \right) \frac{M_u^T \int_r^u D_r S_\theta \mathrm{d}\theta - \int_0^u S_\theta \mathrm{d}\theta (D_r M_u^T)}{(M_u^T)^2}. \tag{6}$$

Now, taking into account (3), we get

$$\begin{aligned} D_r^W M_u^T &= k(D_r S_u) F(T,u) + k\frac{1}{T}\int_r^u D_r S_\theta \mathrm{d}\theta \\ &= k\sigma S_u F(T,u) + \frac{k\sigma}{T}\int_r^u S_\theta \mathrm{d}\theta. \end{aligned} \tag{7}$$

Hence, we obtain that, for all $r < u$,

$$\begin{aligned} &M_u^T \int_r^u D_r^W S_\theta \mathrm{d}\theta - \int_0^u S_\theta \mathrm{d}\theta (D_r^W M_u^T) \\ &= k\sigma S_u F(T,u)\int_r^u S_\theta \mathrm{d}\theta + \frac{k\sigma}{T}\left(\int_0^u S_\tau \mathrm{d}\tau\right)\left(\int_r^u S_\theta \mathrm{d}\theta\right) \\ &\quad - k\sigma S_u F(T,u)\left(\int_0^u S_\theta \mathrm{d}\theta\right) - \frac{k\sigma}{T}\left(\int_r^u S_\theta \mathrm{d}\theta\right)\left(\int_0^u S_\theta \mathrm{d}\theta\right) \\ &= -k\sigma F(T,u) S_u \int_0^r S_\theta \mathrm{d}\theta. \end{aligned} \tag{8}$$

Therefore, (6) and (8) give us that

$$D_r^W a_u^2 = -\frac{2\sigma^3 k^3 S_u F(T,u)}{T^2}\left(\frac{\int_0^u S_\theta \mathrm{d}\theta}{M_u^T}\right)\frac{\int_0^r S_\theta \mathrm{d}\theta}{(M_u^T)^2}. \tag{9}$$

In a similar way, for $\tau < r < u$,

$$\begin{aligned} D_\tau^W D_r^W a_u^2 = &-\frac{2\sigma^3 k^3 (D_\tau S_u) F(T,u)}{T^2}\left(\frac{\int_0^u S_\theta \mathrm{d}\theta}{M_u^T}\right)\frac{\int_0^r S_\theta \mathrm{d}\theta}{(M_u^T)^2} + \frac{2\sigma^3 S_u k^3 F(T,u)}{T^2} \\ &\times\left[\frac{k\sigma F(T,u) S_u \left(\int_0^r S_\theta \mathrm{d}\theta\right)\int_0^\tau S_\theta \mathrm{d}\theta}{(M_u^T)^4}\right. \\ &\left. + \left(\frac{\int_0^u S_\theta \mathrm{d}\theta}{M_u^T}\right)\left(\frac{2M_u^T (D_\tau^W M_u^T)\left(\int_0^r S_\theta \mathrm{d}\theta\right)}{(M_u^T)^4} - \frac{(M_u^T)^2 D_\tau^W \left(\int_0^r S_\theta \mathrm{d}\theta\right)}{(M_u^T)^4}\right)\right]. \end{aligned} \tag{10}$$

3 A decomposition result and a first-order approximation formula

Before proving an extension of the Hull and White formula, we state the following result, which is needed in the remaining of the paper.

Lemma 2 *Let* $0 \le t < T$ *and* $n \ge 0$. *Then, there exists* $C > 0$ *such that*

$$\left|\left((\partial_x^{n+2} - \partial_x^{n+1})\mathrm{BS}(t, X_t, M_t^T, v_t)\right)\right| \le C M_t^T \left(\int_t^T a_\theta^2 \mathrm{d}\theta\right)^{-\frac{1}{2}(n+1)}.$$

Proof This result follows from a direct computation of the derivatives of the function BS and the fact that the function f defined by $f(x) := xe^{-x}$ is bounded. □

Lemma 3 *Let* $p \ge 1$ *and* $t \in [0, T)$. *Then,*

$$E\left((M_t^T)^p \left(\int_t^T a_\theta^2 \mathrm{d}\theta\right)^{-\frac{p}{2}(n+1)}\right) < \infty.$$

Proof Since σ is a constant, we clearly have that $E((M_t)^p) < \infty$, for any $p \ge 1$. Hence, we only need to show that $E\left(\left(\int_t^T a_\theta^2 \mathrm{d}\theta\right)^{-p}\right) < \infty$, for $p > 0$. To do so, we study the integral $\int_t^T \left(1 - k\frac{S_u F(T,u)}{M_u^T}\right)^2 \mathrm{d}u$ now.

We have, by (3),

$$\begin{aligned}
\int_t^T \left(1 - k\frac{S_u F(T,u)}{M_u^T}\right)^2 \mathrm{d}u &= \int_t^T \frac{1}{(M_u^T)^2}\left(M_u^T - kS_u F(T,u)\right)^2 \mathrm{d}u \\
&= \int_t^T \frac{1}{(M_u^T)^2}\left(\frac{k}{T}\int_0^u S_r \mathrm{d}r\right)^2 \mathrm{d}u \\
&= \left(\frac{k}{T}\right)^2 \int_t^T \left(\frac{\int_0^u S_r \mathrm{d}r}{kS_u F(T,u) + \frac{k}{T}\int_0^u S_r \mathrm{d}r}\right)^2 \mathrm{d}u \\
&= \int_t^T \left(\frac{\int_0^u S_r \mathrm{d}r}{S_u(T-u) + \int_0^u S_r \mathrm{d}r}\right)^2 \mathrm{d}u.
\end{aligned}$$

Hence, the Hölder inequality, together with the convention $Z^* = \exp\left(\max_{s\in[0,T]} |\sigma W_s - \frac{\sigma 2}{2}s|\right)$, implies

$$\begin{aligned}
&\int_t^T \left(1 - k\frac{S_u F(T,u)}{M_u^T}\right)^2 \mathrm{d}u \\
&\ge \frac{1}{T-t}\left(\int_t^T \frac{\int_0^u S_r \mathrm{d}r}{S_u(T-u) + \int_0^u S_r \mathrm{d}r}\mathrm{d}u\right)^2 \\
&\ge \frac{1}{(T-t)(Z^*)^2}\left(\int_t^T \frac{\int_0^u S_r \mathrm{d}r}{(T-u) + \int_0^u 1 \mathrm{d}r}\mathrm{d}u\right)^2 \\
&= \frac{1}{(T-t)(TZ^*)^2}\left(\int_t^T \int_0^u S_r \mathrm{d}r \mathrm{d}u\right)^2
\end{aligned}$$

$$\geq \frac{\exp\left(-4\max_{s\in[0,T]}\left|\sigma W_s - \frac{\sigma 2}{2}s\right|\right)}{(T-t)T^2}\left(\int_t^T\int_0^u \mathrm{d}r\mathrm{d}u\right)^2$$
$$= \frac{\exp\left(-4\max_{s\in[0,T]}\left|\sigma W_s - \frac{\sigma 2}{2}s\right|\right)}{4(T-t)T^2}\left(T^2-t^2\right)^2$$
$$= \frac{\exp\left(-4\max_{s\in[0,T]}\left|\sigma W_s - \frac{\sigma 2}{2}s\right|\right)}{4(T-t)T^2}((T+t)(T-t))^2$$
$$\geq \frac{(T-t)}{4}\exp\left(-4\max_{s\in[0,T]}\left|\sigma W_s - \frac{\sigma 2}{2}s\right|\right).$$

Thus, we have shown that the inequality

$$\left(\int_t^T\left(1-k\frac{S_u F(T,u)}{M_u^T}\right)^2\mathrm{d}u\right)^{-1}$$
$$\leq \frac{4}{(T-t)}\exp\left(4\max_{s\in[0,T]}\left|\sigma W_s - \frac{\sigma 2}{2}s\right|\right)$$
$$\leq \frac{4e^{2T\sigma^2}}{(T-t)}\exp\left(4\max_{s\in[0,T]}|\sigma W_s|\right).$$

On the other hand, by de la Peña and Eisenbaum (1997), we have

$$E\left(\exp\left(p\max_{s\in[0,T]}|W_s|\right)\right) \leq 80\exp\left(18^2\frac{p^2}{2}T\right) < \infty,$$

for any $p \geq 1$. Thus, the proof is complete. □

Theorem 4 *Consider model* (1)*. Then, it follows that*

$$V_0 = E\left(\mathrm{BS}(0, X_0, M_0^T, v_0)\right) + \frac{\sigma}{2}E\int_0^T\left(\partial_{xxx}^3 - \partial_{xx}^2\right)\mathrm{BS}(u, X_u, M_u^T, v_u)\Lambda_u^W\Gamma_u\mathrm{d}u, \quad (11)$$

where $\Lambda_u^W := D_u^W\int_u^T a_r^2\mathrm{d}r$ *and* $\Gamma_u := \frac{\int_0^u S_\theta\mathrm{d}\theta}{S_u(T-u)+\int_0^u S_\theta\mathrm{d}\theta}$.

Proof This proof is similar to the one of the main theorems in Alòs et al. (2007) (Theorem 4.2), so we only sketch it. Notice that $\mathrm{BS}(T, X_T, M_T^T, v_T) = V_T$. Then, from (2), we have

$$V_t = E(\mathrm{BS}(T, X_T, k_T, v_T)|\mathcal{F}_t).$$

Now, using the Itô's formula to the process

$$t \to \mathrm{BS}(t, X_t, M_t^T, v_t)$$

and proceeding as in Alòs et al. (2007) (see also Nualart 2006), we can write

$$
\begin{aligned}
\mathrm{BS}(T, X_T, M_T^T, v_T) &= \mathrm{BS}(0, X_0, M_0^T, v_0) \\
&\quad + \int_0^T \mathcal{L}_{\mathrm{BS}}(v_u^2)\mathrm{BS}(u, X_u, M_u^T, v_u)\mathrm{d}u \\
&\quad + \int_0^T \partial_x \mathrm{BS}(u, X_u, M_u^T, v_u)\sigma_u \mathrm{d}W_u \\
&\quad + \int_0^T \partial_k \mathrm{BS}(u, X_u, M_u^T, v_u)\mathrm{d}M_u^T \\
&\quad + \int_0^T \partial_{xk}^2 \mathrm{BS}(u, X_u, M_u^T, v_u) d\left\langle M^T, X\right\rangle_u \\
&\quad + \frac{1}{2}\int_0^T \partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{v_u^2 - a_u^2}{v_u(T-u)}\mathrm{d}u \\
&\quad + \int_0^T \partial_{x\sigma}^2 \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\sigma\Lambda_u^W}{2v_u(T-u)}\mathrm{d}u \\
&\quad + k\int_0^T \partial_{k\sigma}^2 \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\Lambda_u^W \sigma S_u F(T,u)}{2v_u(T-u)}\mathrm{d}u \\
&\quad + \frac{1}{2}\int_0^T \left(\partial_{xx}^2 - \partial_x\right)\mathrm{BS}(u, X_u, M_u^T, v_u)\left(\sigma^2 - v_u^2\right)\mathrm{d}u \\
&\quad + \frac{1}{2}\int_0^T \partial_{kk}^2 \mathrm{BS}(u, X_u, M_u^T, v_u) d\left\langle M^T, M^T\right\rangle_u .
\end{aligned}
$$

Hence, the fact that $\mathcal{L}_{\mathrm{BS}}(v_u^2)\mathrm{BS}(u, X_u, M_u^T, v_u) = 0$, and taking expectations, we can establish

$$
\begin{aligned}
&E\left(\mathrm{BS}(T, X_T, M_T^T, v_T)\right) \\
&\quad = E\left\{\mathrm{BS}(0, X_0, M_0^T, v_0) + \int_0^T \partial_{xk}^2 \mathrm{BS}(u, X_u, M_u^T, v_u) d\left\langle M^T, X\right\rangle_u\right. \\
&\qquad + \frac{1}{2}\int_0^T \partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{v_u^2 - a_u^2}{v_u(T-u)}\mathrm{d}u \\
&\qquad + \frac{1}{2}\int_0^T \partial_{x\sigma}^2 \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\sigma\Lambda_u^W}{v_u(T-u)}\mathrm{d}u \\
&\qquad + k\int_0^T \partial_{k\sigma}^2 \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\sigma S_u F(T,u)\Lambda_u^W}{2v_u(T-u)}\mathrm{d}u \\
&\qquad + \frac{1}{2}\int_0^T \left(\partial_{xx}^2 - \partial_x\right)\mathrm{BS}(u, X_u, M_u^T, v_u)\left(\sigma^2 - v_u^2\right)\mathrm{d}u \\
&\qquad \left. + \frac{1}{2}\int_0^T \partial_{kk}^2 \mathrm{BS}(u, X_u, M_u^T, v_u) d\left\langle M^T, M^T\right\rangle_u\right\} .
\end{aligned}
$$

Consequently, the classical relationships between the Greeks

$$\partial^2_{xx}\mathrm{BS} - \partial_x \mathrm{BS} = \partial_\sigma \mathrm{BS}\frac{1}{\sigma(T-u)}$$
$$\partial^2_{xk}\mathrm{BS} = -\partial_\sigma \mathrm{BS}\frac{1}{k\sigma(T-u)}$$
$$\partial^2_{kk}\mathrm{BS} = \partial_\sigma \mathrm{BS}\frac{1}{k^2\sigma(T-u)}$$

give

$$\begin{aligned}
&E\left(\mathrm{BS}(T, X_T, M_T^T, v_T)\right)\\
&\quad = E\left\{\mathrm{BS}(0, X_0, M_0^T, v_0) - \int_0^T \partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{1}{M_u^T v_u(T-u)} d\left\langle M^T, X\right\rangle_u\right.\\
&\qquad + \frac{1}{2}\int_0^T \partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{v_u^2 - a_u^2}{v_u(T-u)}\mathrm{d}u\\
&\qquad + \frac{1}{2}\int_0^T \partial^2_{x\sigma} \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\sigma\Lambda_u^W}{v_u(T-u)}\mathrm{d}u\\
&\qquad + k\int_0^T \partial^2_{k\sigma} \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\Lambda_u^W \sigma S_u F(T,u)}{2v_u(T-u)}\mathrm{d}u\\
&\qquad + \frac{1}{2}\int_0^T \partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)\left(\sigma^2 - v_u^2\right)\frac{1}{v_u(T-u)}\mathrm{d}u\\
&\qquad \left. + \frac{1}{2}\int_0^T \partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{1}{M_u^2 v_s(T-u)} d\left\langle M^T, M^T\right\rangle_u\right\}.
\end{aligned}$$

That is,

$$\begin{aligned}
&E\left(\mathrm{BS}(T, X_T, M_T^T, v_T)\right)\\
&\quad = E\left\{\mathrm{BS}(0, X_0, M_0^T, v_0) + \int_0^T \frac{\partial_\sigma \mathrm{BS}(u, X_u, M_u^T, v_u)}{v_u(T-u)}\right.\\
&\qquad \times\left[-\frac{d\left\langle M^T, X\right\rangle_u}{M_u^T} + \frac{1}{2}\left(v_u^2 - a_u^2\right)\mathrm{d}u + \frac{1}{2}\left(\sigma^2 - v_u^2\right)\mathrm{d}u + \frac{1}{2}\frac{d\left\langle M^T, M^T\right\rangle_u}{\left(M_u^T\right)^2}\right]\\
&\qquad + \int_0^T \partial^2_{x\sigma} \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\sigma\Lambda_u^W}{2v_u(T-u)}\mathrm{d}u\\
&\qquad \left. + k\int_0^T \partial^2_{k\sigma} \mathrm{BS}(u, X_u, M_u^T, v_u)\frac{\Lambda_u^W \sigma S_u F(T,u)}{2v_u(T-u)}\mathrm{d}u\right\}.
\end{aligned}$$

Since $a_u^2 \mathrm{d}u := \sigma^2 \mathrm{d}u - 2\frac{d\langle M^T, X\rangle_u}{M_u^T} + \frac{d\langle M^T, M^T\rangle_u}{(M_u^T)^2}$, we obtain

$$
\begin{aligned}
& E\left(\mathrm{BS}(T, X_T, M_T^T, v_T)\right) \\
& \quad = E\left\{\mathrm{BS}(0, X_0, M_0^T, v_0) + \int_0^T \partial_{x\sigma}^2 \mathrm{BS}(u, X_u, M_u^T, v_u) \frac{\sigma \Lambda_u^W}{2v_u(T-u)} \mathrm{d}u \right. \\
& \qquad \left. + k \int_0^T \partial_{k\sigma}^2 \mathrm{BS}(u, X_u, M_u^T, v_u) \frac{\Lambda_u^W \sigma S_u F(T,u)}{2v_u(T-u)} \mathrm{d}u \right\}.
\end{aligned}
$$

Now, taking into account that

$$
1 - k\frac{S_u F(T,u)}{M_u^T} = \frac{\int_0^u S_\theta \mathrm{d}\theta}{S_u(T-u) + \int_0^u S_\theta \mathrm{d}\theta}
$$

the proof is complete. Notice that, due to Remark 1, and Lemmas 2 and 3, all the integrals in this proof are well defined. □

Remark 5 Notice that, from the above theorem, $E\left(\mathrm{BS}(0, X_0, M_0^T, v_0)\right)$ can be seen as a first-order approximation for the option price. Moreover, (5) gives us that a short-time approximation for this term will be given by

$$
E\left(\mathrm{BS}(0, X_0, M_0^T, \frac{\sigma}{\sqrt{3}})\right). \tag{12}
$$

4 Derivative of the implied volatility

Let $I_t(k)$ denote the implied volatility process, which satisfies by definition $V_t = \mathrm{BS}(t, X_t, M_t^T, I_t(k))$.

In this section, we prove a formula for its at-the-money derivative that we use in Sect. 5 to study the short-time behavior of the implied volatility as a function of k

Proposition 6 *Assume model* (1). *Then, it follows that*

$$
\frac{\partial I_0}{\partial k}(1) = \left.\frac{E\left(\int_0^T (M_u^T \partial_k F(u, X_u, M_u^T, v_u) - \frac{1}{2} F(u, X_u, M_u^T, v_u))\mathrm{d}u\right)}{\partial_\sigma \mathrm{BS}(0, X_0, M_0^T, I_0(1))}\right|_{k=1}, \quad a.s.
$$

where

$$
F(u, X_u, M_u^T, v_u) = \frac{1}{2}\left(\partial_{xxx}^3 - \partial_{xx}^2\right) \mathrm{BS}(u, X_u, M_u^T, v_u) \sigma \Lambda_u^W \Gamma_u.
$$

Proof Using Theorem 4 and the expression $V_t = \mathrm{BS}(t, X_t, M_t^T, I_t(k))$, we obtain

$$
\frac{\partial V_0}{\partial k} = \partial_k \mathrm{BS}(0, X_0, M_0^T, I_0(k)) \frac{M_0^T}{k} + \partial_\sigma \mathrm{BS}(0, X_0, M_0^T, I_0(k)) \frac{\partial I_0(k)}{\partial k} \tag{13}
$$

and

$$\frac{\partial V_0}{\partial k} = E(\partial_k \mathrm{BS}(0, X_0, M_0^T, v_0))\frac{M_0^T}{k} + \frac{1}{k}E\left(\int_0^T M_u^T \partial_k F(u, X_u, M_u^T, v_u)\mathrm{d}u\right). \tag{14}$$

We can check that the expectation $E\left(\int_t^T M_u^T \partial_k F(u, X_u, M_u^T, v_u)\mathrm{d}u\right)$ is well defined and finite a.s. due to Lemmas 2 and 3. Thus, (13) and (14) imply

$$\begin{aligned}\frac{\partial I_0}{\partial k}(1) = &\frac{1}{\partial_\sigma \mathrm{BS}(0, X_0, M_0^T, I_0(1))}\\ &\times\Big[M_0^T(E(\partial_k \mathrm{BS}(0, X_0, M_0^T, v_0)) - M_0^T \partial_k \mathrm{BS}(0, X_0, M_0^T, I_0(1)))\\ &\quad + E\left(\int_0^T M_u^T \partial_k F(u, X_u, M_u^T, v_u)\mathrm{d}u\right)\Big]\Big|_{k=1}.\end{aligned} \tag{15}$$

Now using the fact that

$$M_0^T \partial_k \mathrm{BS}(0, X_0, M_0^T, \sigma)\Big|_{k=1} = -e^{X_0} N\left(-\frac{1}{2}\sigma\sqrt{T-t}\right)\Big|_{k=1}.$$

Straightforward calculations and Theorem 4 lead us to

$$\begin{aligned}&M_0^T E(\partial_k \mathrm{BS}(0, X_0, M_0^T, v_0) - M_0^T \partial_k \mathrm{BS}(0, X_0, M_0^T, I_0(1))\Big|_{k=1}\\ &= \frac{1}{2}E(\mathrm{BS}(0, X_0, M_0^T, v_0) - V_0)\Big|_{k=1}\\ &= -\frac{1}{2}E\left(\int_0^T F(u, X_u, M_u^T, v_u)\mathrm{d}u\right)\Big|_{k=1}.\end{aligned}$$

This, together with (15), implies that the result holds. □

5 Short-time behavior and second-order approximation formulas

In this section, we study the short-time behavior of the implied volatility in order to describe its dependence on the asset price. More precisely, this section is devoted to study the limit of $\frac{\partial I_0}{\partial k}(1)$ as $T \downarrow 0$.

Proposition 7 *Assume that model (1) holds. Then,*

$$\begin{aligned}
&\partial_\sigma \mathrm{BS}(0, X_0, M_0^T, I_0(1)) \frac{\partial I_0}{\partial k}(1) \\
&= \frac{1}{2} E \left(\left(M_0^T \partial_k - \frac{1}{2} \right) \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) \mathrm{BS}(0, X_0, M_0^T, v_0) \int_0^T \sigma \Lambda_u^W \Gamma_u \mathrm{d}u \right) \Big|_{k=1} \\
&+ h(T),
\end{aligned}$$

where $E\left(\frac{h(T)}{\sqrt{T}}\right) \to 0$ *as* $T \to 0$.

Proof Proposition 6 yields

$$\begin{aligned}
&\partial_\sigma \mathrm{BS}(0, X_0, M_0^T, I_0(1)) \frac{\partial I_0}{\partial k}(1) \\
&= \frac{1}{2} E \left(\int_0^T \left(M_u^T \partial_k - \frac{1}{2} \right) \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) \mathrm{BS}(u, X_u, M_u^T, v_u) \sigma \Lambda_u^W \Gamma_u \mathrm{d}u \right).
\end{aligned}$$

Now we prove that the right-hand side in the above equality is equal to

$$\frac{1}{2} E \left(L(0, X_0, M_0^T, v_0) \int_0^T \sigma \Lambda_u^W \Gamma_u \mathrm{d}u \right) + h\,(T)\,, \tag{16}$$

with $L(u, X_u, M_u^T, v_u) = \left(M_u^T \partial_k - \frac{1}{2} \right) \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) \mathrm{BS}(u, X_u, M_u^T, v_u)$ and $E\left(\frac{h(T)}{\sqrt{T}}\right) \to 0$ as $T \to 0$. In fact, we can write

$$\begin{aligned}
&\frac{1}{2} E \int_0^T (M_u^T \partial_k - \frac{1}{2}) \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) \mathrm{BS}(u, X_u, M_u^T, v_u) \sigma \Lambda_u^W \Gamma_u \mathrm{d}u \\
&= \frac{1}{2} E \int_0^T L(u, X_u, M_u^T, v_u) \sigma \Lambda_u^W \Gamma_u \mathrm{d}u \\
&\quad + \frac{1}{2} E \int_0^T (M_u^T - M_0^T) \partial_k \left(\partial_{xxx}^3 - \partial_{xx}^2 \right) \mathrm{BS}(u, X_u, M_u^T, v_u) \sigma \Lambda_u^W \Gamma_u \mathrm{d}u \\
&=: T_1 + T_2.
\end{aligned} \tag{17}$$

Now the proof will be decomposed into two steps.

Step 1 Applying Itô formula to

$$L(u, X_u, M_u^T, v_u) \left(\int_u^T \sigma \Lambda_r^W \Gamma_r \mathrm{d}r \right)$$

as in the proof of Theorem 4 and taking expectations, we obtain that

$$
\begin{aligned}
&\frac{1}{2}E\left(\int_0^T L(u,X_u,M_u^T,v_u)\sigma\Lambda_u^W\Gamma_u\mathrm{d}u\right)\\
&=\frac{1}{2}E\left(L(0,X_0,M_0^T,v_0)\left(\int_0^T\sigma\Lambda_u^W\Gamma_u\mathrm{d}u\right)\right)\\
&\quad+\frac{1}{4}E\left(\int_0^T(\partial_{xxx}^3-\partial_{xx}^2)L(u,X_u,M_u^T,v_u)\sigma\Lambda_u^W\right.\\
&\qquad\left.\times\left(\int_u^T\sigma\Lambda_r^W\Gamma_r\mathrm{d}r\right)\mathrm{d}u\right)\\
&\quad+\frac{1}{4}E\left(\int_0^T\partial_k(\partial_{xx}^2-\partial_x)L(u,X_u,M_u^T,v_u)\sigma S_uF(T,u)\Lambda_u^W\right.\\
&\qquad\left.\times\left(\int_u^T\sigma\Lambda_r^W\Gamma_r\mathrm{d}r\right)\mathrm{d}u\right)\\
&\quad+\frac{1}{2}E\left(\int_0^T\partial_xL(u,X_u,M_u^T,v_u)\sigma\left(\int_u^T(D_u^W(\Lambda_r^W\Gamma_r))\sigma\mathrm{d}r\right)\mathrm{d}u\right)\\
&\quad+\frac{1}{2}E\left(\int_0^T\partial_kL(u,X_u,M_u^T,v_u)\,\sigma S_uF(T,u)\left(\int_u^T(D_u^W(\Lambda_r^W\Gamma_r))\sigma\mathrm{d}r\right)\mathrm{d}u\right)\\
&=:\frac{1}{2}E\left(L(0,X_0,M_0^T,v_0)\left(\int_0^T\sigma\Lambda_u^W\Gamma_u\mathrm{d}u\right)\right)+\tilde{T}_1+\tilde{T}_2+\tilde{T}_3+\tilde{T}_4.
\end{aligned}
$$

Remark 1 and Lemmas 2 and 3 give us that $\tilde{T}_1+\tilde{T}_2+\tilde{T}_3+\tilde{T}_4=O(T)$.

Step 2 Here we consider the term T_2 given in (17). From Remark 1 and Lemma 2, we have that there is a constant $C>0$ such that

$$
|T_2|\leq C\int_0^T\left|M_u^T-M_0^T\right|M_u^T\left(\int_u^T a_\theta^2\mathrm{d}\theta\right)^{-3/2}(T-u)\mathrm{d}u.
$$

Therefore, the proof of Lemma 3 yields that there exists $G\in L^p(\Omega)$, for $p\geq 1$, such that

$$
\begin{aligned}
|T_2|&\leq E\left(G\left(\sup_{s\in[0,T]}M_s^T\left|M_s^T-M_0^T\right|\right)\right)\int_0^t(T-u)^{-3/2}(T-u)\mathrm{d}u\\
&=2\sqrt{T}E\left(G\left(\sup_{s\in[0,T]}M_s^T\left|M_s^T-M_0^T\right|\right)\right),
\end{aligned}
$$

which, together with Step 1, implies that the result is true. Thus, the proof is complete. □

Now we can state the main result of this paper.

Theorem 8 *Consider model* (1). *Then,*

$$\lim_{T\to 0}\frac{\partial I_0}{\partial k}(1) = -\frac{\sqrt{3}}{40}. \tag{18}$$

Proof We can write

$$\partial_\sigma \mathrm{BS}(0, X_0, M_0^T, I_0(1))\Big|_{k=1} = \frac{\exp(X_0)e^{\frac{-I_0(1)^2 T}{8}}\sqrt{T}}{\sqrt{2\pi}}$$

and

$$\left(M_0^T\partial_k - \frac{1}{2}\right)\left(\partial_{xxx}^3 - \partial_{xx}^2\right)\mathrm{BS}(0, X_0, M_0^T, v_0)\Big|_{k=1}$$
$$= \frac{\exp(X_0)}{\sqrt{2\pi}}e^{-\frac{v_t^2 T}{8}}\left(v_0^{-3}T^{-\frac{3}{2}} + \frac{1}{4}v_0^{-2}T^{-1}\right).$$

Then, Lemma 3 and Proposition 7 imply that

$$\begin{aligned}
\lim_{T\to 0}\frac{\partial I_0}{\partial k}(1) &= \frac{\sigma}{2}\lim_{T\to 0} v_0^{-3}T^{-2}\int_0^T \Lambda_u^W \Gamma_u \mathrm{d}u \\
&= -\sigma^3 \lim_{T\to 0} v_0^{-3}T^{-6}\int_0^T u^2(T-u)\left(\int_u^T r\mathrm{d}r\right)\mathrm{d}u \\
&= -\frac{\sigma^3}{2}\lim_{T\to 0} v_0^{-3}T^{-6}\int_0^T u^2(T-u)^3\mathrm{d}u \\
&= -\frac{\sigma^3}{120}\lim_{T\to 0} v_0^{-3} \\
&= -\frac{\sqrt{3}}{40}.
\end{aligned} \tag{19}$$

Now the proof is complete. □

The previous result gives us, using Taylor expansions, the following short-time approximation for the implied volatility

$$\hat{I}_0(k) = \frac{\sigma_0}{\sqrt{3}} - \frac{\sqrt{3}}{40}(k-1).$$

Then, the corresponding approximation for the option price will be given by

$$\hat{V}_0(k) = \mathrm{BS}(0, X_0, M_0^T, \hat{I}_0(k)). \tag{20}$$

Table 1 Approximated prices ($T = 0.1$)

k	MC	1st order	2nd order	Error (1st) (%)	Error (2nd) (%)
0.9	10.5474	10.5330	10.5598	– 0.14	0.12
0.95	6.6054	6.5951	6.6178	– 0.17	0.37
1	3.6334	3.6406	3.6406	0.20	0.20
1.05	1.7301	1.7517	1.7275	1.20	– 0.15
1.1	0.7072	0.7325	0.6996	3.57	– 1.10

Table 2 Approximated prices ($T = 0.5$)

k	MC	1st order	2nd order	Error (1st) (%)	Error (2nd) (%)
0.9	13.7743	13.7247	13.8258	– 0.36	0.37
0.95	10.6914	10.6740	10.7314	– 0.16	0.44
1	8.1084	8.1293	8.1293	0.26	0.26
1.05	6.0106	6.0685	6.0080	0.96	– 0.04
1.1	4.3538	4.4456	4.3315	2.11	– 0.51

Table 3 Approximated prices ($T = 1$)

k	MC	1st order	2nd order	Error (1st) (%)	Error (2nd) (%)
0.9	16.6697	16.6152	16.7671	– 0.33	0.56
0.95	13.8843	13.8646	13.9466	– 0.14	0.45
1	11.4313	11.4766	11.4766	0.37	0.37
1.05	9.3278	9.4299	9.3436	1.10	0.17
1.1	7.5439	7.6962	7.5265	2.02	– 0.23

6 Numerical examples

This section is devoted to check numerically the goodness of the approximations proposed in the previous sections. We consider $\sigma = 0.5$ and $S_0 = 100$. The benchmark values for the option prices have been obtained using a 10^7 simulations Monte Carlo scheme with antithetic variates. In the following tables, we compare (for $T = 0.1, 0.5, 1$ and for $k = 0.90, 0.95, 1, 1.05, 1$) the values of the MC approximation with the results given by first-order approximation formula (12) and second-order approximation formula (20). The corresponding errors are given in % of the MC value. We can see the maximum error is less for the second-order formula. Moreover, this second approximation improves clearly the first one in the case $k > 1$ (Tables 1, 2, 3).

7 Conclusions

By means of Malliavin calculus, we have developed closed-form approximation formulas for short-time floating strike Asian options. The obtained approximations are simple closed-form explicit expressions, and the numerical analysis show they are extremely accurate.

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