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Oligopoly models with different learning and production time scales

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Abstract
We propose a modelling approach to study Cournotian oligopolies of boundedly rational firms which continuously update production decisions on the basis of information collected periodically. The model consists of a system of differential equations with piecewise constant arguments, which can be recast into a system of difference equations. Considering different economic settings, we study the local stability of equilibrium, proving the destabilizing role of the time lag between two consecutive learning activities. We investigate some particular families of oligopolies showing the occurrence of both flip and Neimark–Sacker bifurcations, as well as the evidence of multistability with the coexistence between different attractors, occurring when oligopolies consisting of both technologically different and identical firms are studied.

Keywords Cournot oligopolies · Learning and production decisions · Differential equations with piecewise constant argument · Stability · Bifurcations · Multistability

JEL Classification L13 · C62

1 Introduction

In Cournotian oligopolies of boundedly rational firms, production decisions are updated over time on the basis of collected information about the competitors' choices

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and the environment. Several heuristics have been proposed to model the lack of complete knowledge of the economy by the firms, in order to study the effects on the resulting dynamics of fundamental market characteristics, such as the shape of demand and cost functions, the size of the oligopoly, the heterogeneity among firms and the possibility for them to evolutionary switch among different decision rules.

If the modelling approach is based on dynamical systems, the literature can be divided into two families, namely a former one adopting differential equations, and hence assuming that economic variables are continuously updated, and a latter one based on difference equations, so that the adjustment mechanisms periodically take place, at a discrete time level. In addition, a different framework has been proposed to overcome these approaches. In particular, it has been assumed that economic choices (which take place continuously) are based on the past information given by the economic performance to a specific data (discrete delay differential equations) or a weighted average of past economic performances (distributed delay differential equations). Such modelling approaches date back to the contributions in Howroyd and Russell (1984) and Howroyd et al. (1986) and give rise to a large research strand developed through the years (see, e.g. Matsumoto and Szidarovszky 2011).

In order to update output levels, firms actually have to face two kinds of tasks, as they must collect information about the economic setting, performing a *learning activity*, and they must decide the future production levels, relying on information at their disposal, taking *production decisions*. Both adopting continuous and discrete dynamical systems, the two tasks are assumed to be performed at the same time. However, the learning activity requires economic resources and time, so in general it is actually carried out just periodically, differently from production, that continuously takes place and whose level is adapted more frequently. This suggests that a proper modelling should not neglect the existence of two different time scales for the learning activity and the production decisions that can be properly represented by a continuous time level for the quantity adjustment mechanism and a discrete one for the process of collecting information.

In the present contribution, we pursue this approach extending to oligopolies the research we started for a monopoly in Cavalli and Naimzada (2016). We consider oligopolies consisting of firms that, to choose their strategy, adopt a gradient mechanism, i.e. an heuristic based on a rule of thumb in which decisions depend on the profitability signal coming from a local estimate of marginal profits. The production choices are updated at a continuous time level depending on the marginal profits, an estimation of which is conversely assumed to be obtained only at discrete times. The model results in a system of differential equations with piecewise constant arguments (DEPCA). We stress that even if time-multiscale modelling and hybrid systems are quite common in several scientific fields, their application to economics is still limited (see Lamantia and Radi 2015; Cavalli and Naimzada 2016). Moreover, from the mathematical point of view, the difference of the present setting with that in the literature about delayed differential equations is that in latter one time delays are constant, while in the present one time delays are not constant.

Studying a general market setting, we show that as in the monopolistic setting (Cavalli and Naimzada (2016)), the presence of a discrete time level for the updating of information is a potential source of instabilities in the system, arising as the frequency of estimation of the local marginal profits decreases. Differently from Cavalli and Naimzada (2016), the strategic interaction between players makes it possible that production decisions follow both periodic, chaotic and quasi-periodic trajectories. Moreover, path dependency arises, with the occurrence of multistability.

The remainder of the paper is organized as follows: in Sect. 2, we introduce the model and we study it. In Sect. 3, we present several examples and numerical simulations. Section 4 collects conclusions and future research aims.

2 Model and analysis

We consider a Cournotian oligopoly, consisting of $i = 1, \ldots, N$ firms that, supplying homogeneous goods, strategically compete in quantities. The ith firm produces quantity $q_i \geq 0$ and faces production costs described by function $c_i : [0, +\infty) \to [0, +\infty)$, $q_i \mapsto c_i(q_i)$, which is increasing and twice differentiable.[1] The economy is characterized by a generic twice differentiable, strictly decreasing inverse demand function $p : [0, +\infty) \to [0, +\infty)$, $Q \mapsto p(Q)$ where Q is the aggregated quantity $Q = \sum_{i=1}^{N} q_i$.

As a consequence, the profit functions $\pi_i : [0, +\infty) \to \mathbb{R}$ of each firm result

$$\pi_i(q_i, q_{-i}) = p(Q)q_i - c_i(q_i),$$

where q_{-i} is the vector collecting all strategies but the ith one. We assume that demand and cost functions are such that there exists a unique equilibrium $\mathbf{q}^* = (q_i^*)_{i=1,\ldots,N}$, with strictly positive components $q_i^* > 0$ that satisfy

$$q_i^* = \underset{q_i \geq 0}{\operatorname{argmax}}\, \pi_i\left(q_i, \mathbf{q}_{-i}^*\right), \; i = 1, \ldots, N, \tag{1}$$

where $\mathbf{q}_{-i}^*$ is the vector collecting all q_j^* for $j \neq i$. We stress $\mathbf{q}^*$ corresponds to the Nash equilibrium of a game where players are the firms, strategies are chosen in the set of feasible quantities $q_i \geq 0$ and payoffs are given by the profit functions. As shown in Bischi et al. (2010), a sufficient condition to guarantee the existence and the uniqueness of $\mathbf{q}^*$ is that the profit function of each player is strictly concave with respect to variable q_i, i.e.

$$\frac{\partial^2 \pi_i(q_i, \mathbf{q}_{-i})}{\partial q_i^2} = p''(Q)q_i + 2p'(Q)q_i - c_i''(q_i) < 0, \tag{2}$$

In what follows, we always assume that (2) holds true. Indeed, assumption (2) is guaranteed if we impose

$$g_1(q_i, Q) = p''(Q)q_i + p'(Q) \leq 0, \tag{3a}$$

[1] For the sake of simplicity, in this section we assume that the involved functions are all defined on $[0, +\infty)$, but this can be indeed relaxed and all results still hold with minor adjustments.

and

$$g_2(q_i, Q) = p'(Q) - c_i''(q_i) < 0, \tag{3b}$$

for each $i = 1, \ldots, N$. Beyond the usual condition $g_2(q_i, Q) < 0$, assumption $g_1(q_i, Q) \leq 0$ is guaranteed by either concave or "not too convex" demand functions. We stress that such setting is only sufficient to provide the existence and uniqueness of the equilibrium. Economically significant inverse demand functions fail to fulfil it, as, for example, the isoelastic function $p(Q) = 1/Q$.

We assume that the firms populating the oligopoly are boundedly rational. More specifically, they have neither perfect foresight of their competitors' future production decisions nor a global knowledge of the demand function, so they try to update their current output level by means of some kind of rule of thumb. From the modelling point of view, this essentially requires the introduction of time in order to describe how the production decisions $q_{i,t}$ evolve over time t. During such adjustment process, firms actually have to face two kinds of tasks, namely the collection of information about the economic setting (*learning activity*), and the choice of the future production level, relying on information at their disposal in order to improve realized profits (*production decisions*). Since such activities are not performed with the same frequency, we adopt two different time scales for the learning activity and the production decisions, assuming a continuous time level $t \in \mathbb{R}^+$ for the quantity adjustment mechanism and a discrete one t_n, $n \in \mathbb{N}$ for the updating information process, where t_n is a strictly increasing sequence of positive real numbers. In what follows, we study the setting in which learning activities are realized by each firm at the same time and occur at periodic intervals, i.e. we set $t_n = n\sigma$, where $\sigma > 0$ and $1/\sigma$ represents the frequency of learning activities. This means that at each time t_n firms collect information about the environment and use them to adapt the output levels for $t \in [t_n, t_{n+1})$, before the next learning activity performed at time t_{n+1}. The motivation for a synchronous updating of market information is the following one. The literature about market research (see, e.g. Mooradian et al. 2014) suggests that it should be carried on both periodically, to learn about the market the agents are working with, and when significant changes occur (e.g. when a new product is introduced in the market, or when the market structure and/or its composition changes). Since the goal of the present paper is not to study what happens when the studied market changes, the main motivation for realization of market research relies on the periodic collection of information about the market. Especially, when the market is controlled by a few, large firms, institutional regulations require to periodically provide a record of the activities in terms of a financial report, in which the future development of the business and plans to achieve predetermined goals are presented. This requires the collection of information about the market situations, and since institutional constraints require to present financial reports at periodical intervals and close to specific periods of the year (e.g. quarterly or yearly), it is natural to consider a synchronous timing for learning activities (i.e. taking place near the end of each year quarter or close to the end of the year). Moreover, firms often make reference to the same agency providing results to each of them on the basis of a unique market research.

To describe the heuristic adopted by the firms, we assume that by means of market experiments realized at t_n, each oligopolist is able to know the exact value of their marginal profits corresponding to the current output level so that we can write

$$\begin{aligned}\pi_i'(q_i(t_n), \mathbf{q}_{-i}(t_n)) &= \frac{\partial \pi_i}{\partial q_i}\left(q_i\left(t_n\right), \mathbf{q}_{-i}\left(t_n\right)\right)\\ &= \frac{\partial \pi_i}{\partial q_i}\left(q_i\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right), \mathbf{q}_{-i}\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right)\right),\ i = 1, \ldots, N,\end{aligned} \tag{4}$$

for $t \in [t_n, t_{n+1})$, where $\lfloor x \rfloor$ is the floor function, providing the greatest integer that is less or equal to x. The size and the sign of (4) provide the profitability signal proportionally to which the production level is adapted,[2] so that the production on $t \in [t_n, t_{n+1})$ increases (resp. decreases or does not change) if $\pi_i'(q_i(t_n), \mathbf{q}_{-i}(t_n)) > 0$ (resp. $\pi_i'(q_i(t_n), \mathbf{q}_{-i}(t_n)) < 0$ or $\pi_i'(q_i(t_n), \mathbf{q}_{-i}(t_n)) = 0$). Proportionality is regulated by the agents' reactivity to the profitability signal, modelled by a function $v_i : [0, +\infty) \to [0, +\infty),\ q \mapsto v_i(q)$ which is strictly increasing with $v_i(0) = 0$.

We can now write the model in terms of the following differential system with piecewise constant arguments (DEPCA)

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = v_i(q_i(t))\frac{\partial \pi_i}{\partial q_i}\left(q_1\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right), q_2\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right), \ldots, q_n\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right)\right) \tag{5}$$

with $q_i(0) > 0,\ i = 1, \ldots, N$, namely each initial output level is strictly positive.[3] Continuous System (5) can be recast as an equivalent system of difference equations.[4] We stress that thanks to the regularity assumptions on involved functions, for each choice of the initial conditions, the solution of (5) is unique (see, e.g.Wiener 1993). Since $q_i(t) = 0$ is solution to (5), this guarantees that if we consider strictly positive initial output level, each production choice $q_i(t)$ is strictly positive for any $t > 0$. Integrating each equation of (5) on $t \in [n\sigma, (n+1)\sigma)$, we easily obtain

$$\begin{aligned}q_i(t) = F_i^{-1}\Bigg(&F_i(q_i(n\sigma))\\ &+ \frac{\partial \pi_i}{\partial q_i}\left(q_1\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right), q_2\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right), \ldots, q_n\left(\left\lfloor \frac{t}{\sigma}\right\rfloor \sigma\right)\right)(t - n\sigma)\Bigg),\end{aligned} \tag{6}$$

where F_i is an antiderivative of $1/v_i(q)$. We stress that thanks to the assumptions on $v_i(q)$, F_i is strictly increasing and invertible.

From (6), we can obtain a difference equation linking production decisions at times $t_{n+1} = (n+1)\sigma$ to those at times $t_n = n\sigma$. Since solutions $q_i(t)$ to System (5) are

[2] For more details about the economic description and interpretation of the gradient adjustment mechanism, we refer to Bischi et al. (2010) and Cavalli and Naimzada (2015).

[3] We note that if $q_i(0) = 0$, from (5) we have $q_i(t) = 0$, so the ith firm actually does not take part in the market.

[4] We do not enter into mathematical details about conditions under which (5) is well defined and can be solved for any $t > 0$. For a detailed mathematical description on the whole process of transformation of a DEPCA into a discrete time difference equation, we refer to Wiener (1993) and Cavalli and Naimzada (2016).

continuous functions, we have

$$q_i((n+1)\sigma) = \lim_{t\to(n+1)\sigma} q_i(t) = F_i^{-1}\left(F_i(q_i(n\sigma)) + \sigma\pi_i'(q_i(n\sigma), \mathbf{q}_{-i}(n\sigma))\right)$$

so, setting $q_{i,n} = q_i(n\sigma)$ for each i and $n \in N$, we obtain a system of difference equations $G = (G_1, G_2, \ldots, G_N) : [0, +\infty)^N \to [0, +\infty)^N$, $\mathbf{q} \mapsto G(\mathbf{q})$ defined by

$$q_{i,n+1} = G_i(\mathbf{q}_n) = F_i^{-1}\left(F_i(q_{i,n}) + \sigma\pi_i'(q_{i,n}, \mathbf{q}_{-i,n})\right), \tag{7}$$

where $\mathbf{q}_n \in \mathbb{R}^N$ collects output choice of firms at time period n. As we remarked, since assuming $q_i(0) > 0$ for any $i = 1, \ldots, N$ we have $q_i(t) > 0$ for any $t > 0$, we also have $q_{i,n} > 0$ for any $n > 0$. In the remaining part of this section, we focus on the possible steady states of (7) and on their stability considering different assumptions on the economic setting.

Proposition 1 *Under assumption (2), if $q_i(0) > 0$ for $i = 1, \ldots, N$, System (7) has exactly a unique steady-state coincident with equilibrium $\mathbf{q}^*$.*

Proof Setting $q_{i,n+1} = q_{i,n} = q_i$ for $i = 1, \ldots, N$, we obtain

$$q_i = F_i^{-1}\left(F_i(q_i) + \sigma\pi_i'(q_i, \mathbf{q}_{-i})\right),\ 1, \ldots, N.$$

The strict monotonicity of F_i immediately provides $\pi_i'(q_i, \mathbf{q}_{-i}) = 0$ for $i = 1, \ldots, N$, which is fulfilled at $\mathbf{q}^*$, which is then a steady state for (7). Thanks to the concavity of π_i, it is also unique.

We note that assumption (2) is only sufficient to guarantee that the solution of (1) is the unique steady state of (7), but if it does not hold true, model (7) may have multiple steady states.

Concerning stability, we firstly consider a very general setting, in order to cast a first glance at the role of σ.

Proposition 2 *Under assumption (2), for each oligopoly there exists a suitably large value σ_b such that $\mathbf{q}^*$ is unstable for $\sigma \in (\sigma_b, +\infty)$.*
Moreover, either under condition (3) or if $Np''(Q)q_i + (N+1)p'(Q) - c''(q_i) < 0$ when condition (3a) is violated, then there exists a suitably small value σ_a such that $\mathbf{q}^*$ is locally asymptotically stable for $\sigma \in (0, \sigma_a)$.

Proof The Jacobian matrix of G is $J = (j_{ik})_{i,k\in\{1,\ldots,N\}}$, where

$$j_{ii} = \frac{F_i'(q_i) + \sigma\frac{\partial^2\pi_i}{\partial q_i^2}(q_i, \mathbf{q}_{-i})}{F_i'\left(F_i^{-1}\left(F_i(q_i) + \sigma\pi_i'(q_i, q_{-i})\right)\right)}$$

and

$$j_{ik} = \frac{\sigma\frac{\partial^2\pi_i}{\partial q_i q_k}(q_i, \mathbf{q}_{-i})}{F_i'\left(F_i^{-1}\left(F_i(q_i) + \sigma\pi_i'(q_i, \mathbf{q}_{-i})\right)\right)}.$$

From

$$\frac{\partial \pi_i}{\partial q_i}(q_i, \mathbf{q}_{-i}) = p'(Q)q_i + p(Q) - c_i'(q_i)$$

and

$$\frac{\partial \pi_i}{\partial q_k}(q_i, \mathbf{q}_{-i}) = p'(Q)q_i,$$

we can straightforwardly obtain the expressions of the second-order derivatives, respectively, given by (2) and

$$\frac{\partial^2 \pi_i}{\partial q_i q_k}(q_i, \mathbf{q}_{-i}) = p''(Q)q_i + p'(Q).$$

So, recalling that $F_i'(q_i) = 1/v_i(q_i)$ and that $p'(Q^*)q_i^* + p(Q^*) - c_i'(q_i^*) = 0$, we have that the Jacobian matrix evaluated at the equilibrium is $J^* = \left(j_{ik}^*\right)_{i,k\in\{1,\ldots,N\}}$ where

$$j_{ii}^* = 1 + \sigma v_i(q_i^*) \left(p''(Q^*)q_i^* + 2p'(Q^*) - c_i''(q_i^*)\right)$$

and

$$j_{ik}^* = \sigma v_i(q_i^*) \left(p''(Q^*)q_i^* + p'(Q^*)\right).$$

Matrix J^* has then the form

$$J^* = \begin{pmatrix} a_1 & b_1 & \cdots & \cdots & b_1 \\ b_2 & a_2 & b_2 & \cdots & b_2 \\ \vdots & & \vdots & & \vdots \\ b_N & b_N & \cdots & b_N & a_N \end{pmatrix} \tag{8}$$

where $a_i = j_{ii}^*$ and $b_i = b_{ik}^*$. Local asymptotic stability of $\mathbf{q}^*$ for the discrete problem (7) holds provided that $\rho(J^*) < 1$, where $\rho(J^*)$ is the spectral radius of J^*. A lower bound for the spectral radius is indeed provided by $|\,\mathrm{tr}(J^*)|/N$, which in the present case gives

$$\rho(J^*) \geq \left| 1 + \frac{\sum_{i=1}^N \sigma v_i(q_i^*) \left(p''(Q^*)q_i^* + 2p'(Q^*) - c_i''(q_i^*)\right)}{N} \right|. \tag{9}$$

Since (2) holds at $\mathbf{q}^*$, for any

$$\sigma > \sigma_b = -\frac{2}{\min_i v_i(q_i^*) \left(p''(Q^*)q_i^* + 2p'(Q^*) - c_i''(q_i^*)\right)}$$

the term inside the absolute value in the right-hand side of (9) is less than -1, so $\rho(J^*) > 1$ and $\mathbf{q}^*$ is unstable.

For the upper bound of $\rho(J^*)$, we distinguish two cases. If (3) holds true, and hence $j_{ik} < 0$ thanks to (3), we can obtain an estimation of the eigenvalues of J^* mimicking the proof in Quandt (1967), where a matrix with a similar structure is analyzed for a differential problem. For the reader's sake, we report the steps of the proof.

To compute the characteristic polynomial, we subtract the first column of $J_\lambda = J^* - \lambda I$ to each other column, obtaining

$$\begin{pmatrix} a_1 - \lambda & b_1 - (a_1 - \lambda) & \cdots & \cdots & b_1 - (a_1 - \lambda) \\ b_2 & a_2 - \lambda - b_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ b_N & 0 & 0 & \cdots & a_N - \lambda - b_N \end{pmatrix}.$$

The determinant, computed using the first column, is

$$p(\lambda) = (a_1 - \lambda)\prod_{i=2}^{N}(a_i - \lambda - b_i) + \sum_{j=2}^{N} b_j \prod_{i=1, i\neq j}^{N}(a_i - \lambda - b_i). \tag{10}$$

Without loss of generality, we can assume

$$a_i - b_i < a_{i+1} - b_{i+1} \tag{11}$$

for $i = 1, \ldots, N-1$, so, recalling the second condition in (3), $a_N - b_N = 1 + \sigma v_N(q_N^*)(p'(Q^*) - c_N''(q_N^*)) < 1$.

Evaluating the characteristic polynomial at $a_1 - b_1$, all terms but the first addend vanish, obtaining

$$p(a_1 - b_1) = b_1 \prod_{i=2}^{N}(a_i - b_i - (a_1 - b_1))$$

which is negative since $b_1 < 0$ and each of the remaining factors is positive thanks to (11). Evaluating the characteristic polynomial at $a_j - b_j$ for $j > 1$, the first addendum vanishes, as well as all the remaining terms but the ith one, namely

$$p(a_j - b_j) = b_j \prod_{i=1, i\neq j}^{N}\big(a_i - b_i - (a_j - b_j)\big)$$

whose sign, recalling that $b_j < 0$ and (11), is $-(-1)^{j-1} = (-1)^j$. This means that the characteristic polynomial has N alternating signs at $a_i - b_i$ for $i = 1, \ldots, N$, which allows identifying $N-1$ solutions $a_i - b_i < \lambda_{i+1} < a_{i+1} - b_{i+1} < 1$. The leading order term in the characteristic polynomial is $(-1)^N\lambda^N$, which means that $\lim_{\lambda\to-\infty} p(\lambda) = +\infty$, so since $p(a_1 - b_1) < 0$, we have $\lambda_1 < a_1 - b_1$. This allows concluding that all the eigenvalues are real and less than 1, so a simple application of Gerschgorin's Theorem to J^* shows that there exists a suitably small σ_a such that $\lambda_1 > -1$.

If (3) is not fulfilled, an upper bound of $\rho(J^*)$ is given by $||J^*||_\infty$, for which, provided that σ is sufficiently small to have $a_i > 0$, we have

$$||J^*||_\infty = 1 + \sigma v_j(q_j^*)\left(p''(Q^*)q_j^* + 2p'(Q^*) - c_j''(q_j^*)\right) + (N-1)b_j \qquad (12)$$

for some j. Recalling (2), the existence of σ_a is then only guaranteed under the supplementary assumption provided in the theorem. We stress that if $b > 0$, it is possible that $\rho(J^*)$ actually coincides with the right-hand side of (12). If we consider the homogeneous case of identical cost functions $c_i \equiv c$, for which the equilibrium consists of identical components $q_i^* \equiv \hat{q}^*$, and identical reaction functions $v_i \equiv v$, matrix J^* becomes

$$\hat{J}^* = (a-b)I + bE, \qquad (13)$$

where I is the identity matrix and E is the matrix with all entries equal to one, while $a = 1 + \sigma v(\hat{q}^*)(p''(Q^*)\hat{q}^* + 2p'(Q^*) - c''(\hat{q}^*))$ and $b = \sigma v(\hat{q}^*)(p''(Q^*)\hat{q}^* + p'(Q^*))$. Matrix (13), in which diagonal elements are equal to a and off-diagonal ones are equal to b, is circulant and has $N-1$ eigenvalues equal to $a-b$ and an eigenvalue equal to $a+(N-1)b$ (see, for example, Cavalli et al. 2015), which then coincides with the right-hand side in (12).

In agreement with what found in Cavalli and Naimzada (2016), if the time interval between two learning activities is sufficiently large, instabilities can occur in the economic system due to the superimposition of the two distinct time scales. The economic rationale is evident: as an example, let us assume that at time t_a the observed profitability signal suggests to increase the production level, and this will be done until $t = t_b$, at which a new learning activity is performed. If such signal is very strong or if the production decisions are updated using it for an excessively long time period, at time t_b output levels can have been increased by a too large extent so that the new profitability signal at time t_b would result opposite with respect to that at t_a. This leads to a sequence of periods in which production levels alternatively increase and decrease. If such endogenous fluctuations reduce, production trajectories converge to the optimal output choice, otherwise non-convergent dynamics occur. On the other hand, if, with respect to size N of the market, the inverse demand function is not "too convex", we then have that if the profitability signal is observed frequently, a milder adaption of production decisions occurs and allows the convergence to the equilibrium output level.[5]

To provide more details about stability and to understand the kind of unstable dynamics arising when it is lost, we need to consider more particular situations. Assuming (3) allows studying stability of $\mathbf{q}^*$ for a quite general family of economic settings, as shown in the next proposition.

[5] As evident from the previous considerations and from the mathematical structure of (7), the role of σ and v_i is very similar, so Proposition 2 and the subsequent comments can be rephrased in terms of the agents' reactivity, too.

Proposition 3 *Under assumption (3), equilibrium* $\mathbf{q}^*$ *is locally asymptotically stable on* $\sigma \in [0, \sigma_a)$*, where* σ_a *is the smallest value at which*

$$1 + \sum_{i=1}^{N} \frac{\sigma v_i(q_i^*) \left(p''(Q^*)q_i^* + p'(Q^*)\right)}{\left(2 + \sigma v_i(q_i^*) \left(p'(Q^*) - c_i''(q_i^*)\right)\right)} > 0 \tag{14}$$

is violated. At $\sigma = \sigma_a$*, a flip bifurcation occurs. When firms are identical with respect to technology* $c(q) = c_i(q)$ *and reaction functions* $v(q) = v_i(q)$*,* $i = 1, \ldots, N$*, condition (14) simplifies as*

$$2 + \sigma v(\mathbf{q}^*) \left((N+1)p'(Q^*) - c''(\mathbf{q}^*)\right) > 0, \tag{15}$$

where $\mathbf{q}^*$ *are the identical components of the equilibrium.*

Proof We conclude the analysis of eigenvalues started in the proof of Proposition (2). Local asymptotic stability of $\mathbf{q}^*$ for the discrete problem (7) holds provided that $|\lambda_i| < 1$, $i = 1, \ldots, N$, where λ_i are the eigenvalues of J^*. We already showed that $\lambda_i < 1$, so a sufficient condition for local stability is $p(-1) > 0$, where p is defined in (10) and provides

$$\begin{aligned}
&(a_1 + 1)\prod_{i=2}^{N}(a_i + 1 - b_i) + \sum_{j=2}^{N} b_j \prod_{i=1, i\neq j}^{N} (a_i + 1 - b_i) \\
&= \prod_{i=2}^{N}(a_i + 1 - b_i)\left(a_1 + 1 + (a_1 + 1 - b_1)\sum_{j=2}^{N}\frac{b_j}{a_j + 1 - b_j}\right) \\
&= \prod_{i=2}^{N}(a_i + 1 - b_i)\left(a_1 + 1 - b_1 + (a_1 + 1 - b_1)\sum_{j=1}^{N}\frac{b_j}{a_j + 1 - b_j}\right) \\
&= \prod_{i=1}^{N}(a_i + 1 - b_i)\left(1 + \sum_{j=1}^{N}\frac{b_j}{a_j + 1 - b_j}\right) > 0
\end{aligned}$$

Recalling (11), from $-1 < \lambda_1 < a_1 - b_1 < a_i - b_i$ we have that the previous inequality is satisfied if and only if

$$1 + \sum_{j=1}^{N}\frac{b_j}{a_j + 1 - b_j} > 0.$$

Using the definition of a_i and b_i allows obtaining (14). Noting that in the homogeneous setting we indeed have $q_i^* = q_j^*$, $1 \le i, j \le N$, from (14) we easily find (15).

Looking at (14), we can see that the role of parameter σ on stability is the same as in Proposition 2. In this case, we can be more precise, since as $\sigma \to 0$, the right-hand side of (14) approaches 1, and hence, the equilibrium is stable for suitably small values of σ. Since both $p''(Q^*)q_i^* + p'(Q^*)$ and $p'(Q^*) - c_i''(q_i^*)$ are negative, for small values of σ the fraction in the summation is negative and increases in absolute value as σ increases. In particular, there exists a threshold value of σ for which (14) holds with the equality, and $\mathbf{q}^*$ becomes unstable by means of a flip bifurcation crossing the unique stability threshold. From (14), we have information also about the role of the inverse demand function on the stability. As it becomes steeper (i.e. ceteris paribus, the price elasticity of demand decreases) or more concave at the equilibrium, more likely instabilities can arise in the output choices, as (14) is violated for smaller values of σ and for smaller agents' reactivity.

If we remove assumption (3a), stability condition is no more true and when stability is lost, different kinds of unstable dynamics may arise. If we restrict to case of homogeneous firms (i.e. identical cost functions $c_i \equiv c$ and reaction functions $v_i \equiv v$), we again have a period-doubling bifurcation when stability is lost, as shown in the next proposition.

Proposition 4 *In an oligopoly of N identical firms, if* (3a) *is violated and* $Np''(Q^*)\hat{q}^* + (N+1)p'((Q^*)) - c''(\hat{q}^*) < 0$*, then the equilibrium is locally asymptotically stable provided that*

$$\sigma < -\frac{2}{v(\hat{q}^*)(p'(Q^*) - c''(\hat{q}^*))}, \tag{16}$$

where we set $q_i^* = \hat{q}^*, i = 1, \ldots, N$*. If* (16) *is violated, a flip bifurcation occurs.*

Proof As shown at the end of the proof of Proposition 2, the eigenvalues of the Jacobian matrix $\hat{J}^*$ in the homogeneous case are real and correspond to $a - b$ with multiplicity $N-1$ and $a + (N-1)b$. Thanks to the assumption, we have $b > 0$ so we need $-1 < a - b < a + (N-1)b < 1$. The rightmost inequality is guaranteed by the assumption in the theorem, and the leftmost follows from (16).

The previous two propositions rule out the possibility to have a Neimark–Sacker bifurcation under assumption (3a) or in an homogeneous setting. To look for a possible Neimark–Sacker bifurcation, we need to focus on a suitably convex demand function and consider a non-homogeneous setting. In this case, general conditions are hard to obtain and are not very readable even in the case of a duopoly. We will study this scenario through an example in the next section.

Before presenting numerical simulations, we draw the attention on the role of the frequency of learning activities on firms' welfare. In order to provide production decisions that converge towards the Nash equilibrium strategy, frequently carrying out learning activity allows to suitably correct production decisions towards $\mathbf{q}^*$. On the other hand, each learning process requires a fixed cost, which can significantly affect profits when its frequency is elevated. From these points of view, the intuition is that the "best" values for σ should be those smaller than the stability threshold and close to it, in order to guarantee at the same time optimal production choices and to minimize market researches' costs. We will numerically address this aspect in the next section.

3 Numerical simulations

In this section, we focus on some examples to investigate through simulations the possible complex dynamics arising when stability is lost. In what follows, we assume that firms can be heterogeneous only with respect to their technology, and we consider identical linear reactivity functions $v_i(q) = \alpha q$ for all the agents $i = 1, \ldots, N$, where $\alpha > 0$ represents the reaction speed. From such expression of v_i, we obtain $F_i(q_i) = \ln(q_i)/\alpha$, so model (7) becomes

$$q_{i,n+1} = q_{i,n} e^{\sigma \alpha \pi_i'(q_{i,n}, \mathbf{q}_{-i,n})}. \tag{17}$$

Once more, we underline that economic feasibility of output level trajectories is guaranteed, as the right-hand side of System (17) maps set $(0, +\infty)^N$ into itself. Finally, without loss of generality, we can set $\alpha = 1$.

In the first family of simulations, we consider an inverse linear demand function $p(Q) = a - bQ$, with positive parameters b and a, and we assume homogeneous quadratic cost functions $c_i(q_i) = d_i q_i^2 + C,\ i = 1, \ldots, N$, where $d_i > 0$, so that marginal profits are $\pi_i'(q_i, \mathbf{q}_{-i}) = a - bQ - bq_i - 2d_i q_i,\ i = 1, \ldots, N.$ Term C in cost function accounts for the cost of each learning activity, and in what follows, it is set equal to $C = 0.1$. We stress that such setting satisfies assumptions (3), so stability is regulated by Proposition 3. Firstly, we consider a duopoly ($N = 2$) of heterogeneous firms, for which a simple computation provides the unique positive Nash equilibrium

$$q_i^* = \frac{a(b + 2d_{-i})}{4bd_i + 4bd_{-i} + 4d_i d_{-i} + 3b^2}, i = 1, 2. \tag{18}$$

Using (18) in (14), we obtain

$$\frac{(2 - a\sigma)\left(4bd_1 + 4bd_2 + 4d_1 d_2 + 3b^2\right)}{8bd_1 + 8bd_2 + 8d_1 d_2 + 6b^2 - a\sigma(b + 2d_1)(b + 2d_2)} > 0,$$

from which the stability condition is $\sigma < 2/a$. In fact, if it holds true, the numerator is indeed positive as well as the denominator, which is linearly decreasing in σ and, at $\sigma = 2/a$, equals $4b(b + d_1 + d_2) > 0$. The first simulation we present is obtained setting $a = 0.05,\ b = 1,\ d_1 = 0.1$ and $d_2 = 0.2$, so we have $(q_1^*, q_2^*) \approx (3.051, 1.6949)$ and stability threshold is $\sigma = 2$. In the top left plot of Fig. 1, we report a couple of bifurcation diagrams for q_1 on increasing σ, obtained setting the initial datum, respectively, equal to $(q_1(0), q_2(0)) = (2, 1)$ (red diagram) and $(q_1(0), q_2(0)) = (1, 1)$ (black diagram), from which it is evident the coexistence between different attractors. This means that depending on the current productions choices, when dynamics do not converge towards the equilibrium, future output trajectories can become significantly different. In the top right plot of Fig. 1, we report the basins of attraction of the period-4 (black) and the period-8 (red) cycles towards which we have convergence when $\sigma = 2.66$. As σ increases, an attractor consisting of four closed invariant curves arises from a Neimark–Sacker bifurcation of the period-4 cycle, firstly coexisting with a period-16 cycle and then with a complex attractor (bottom left plot of Fig. 1) whose

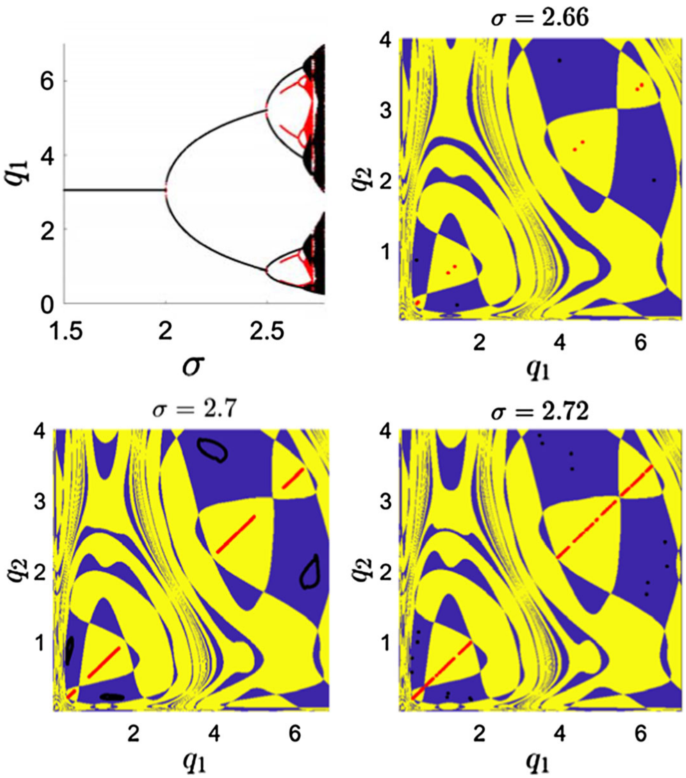


Fig. 1 Duopoly of heterogeneous firms with linear demand function. Top left plot: bifurcation diagrams on increasing σ, showing coexistence. Top right and bottom plots: basins of attractions and coexisting attractors (color figure online)

points are "synchronized" on a line of equation $q_2 = (b + 2d_1)/(b + 2d_2)q_1$. If σ is further increased, such attractor grows (bottom right plot of Fig. 1) and finally collides with the boundary of its basin and disappears. The articulated structure of the basins of attraction makes quite difficult to predict production trajectories (e.g. some points that are close to the red (resp. black) attractor lie in blue (resp. yellow) regions, which means that they converge towards the black (resp. red) attractor). Moreover, even in the case of a linear demand function, the flip bifurcation analytically found in the previous section can then evolve in both, periodic, chaotic and quasi-periodic trajectories.

A similar level of complexity persists even if we consider oligopolies consisting of more than 2 firms or in the case of homogeneous players. In the next example, we assume identical cost functions (i.e. $d_i = d, i = 1, \ldots, N$), so the unique Nash equilibrium $\mathbf{q}^*$ has identical components $q_i^* = a/((N+1)b + 2d)$. Moreover, thanks to the homogeneity, the equilibrium is stable under condition (15), which in the present case reduces to $2 - a\sigma > 0$.

If consider $N = 4$ firms and we set $a = 0.05$, $b = 1$ and $d = 0.2$, we obtain $q_i^* \approx 1.5385$ which put into (16) immediately gives $\sigma < 2$. In Fig. 2, we report a couple of bifurcation diagrams for q_1 on increasing σ, obtained setting the initial

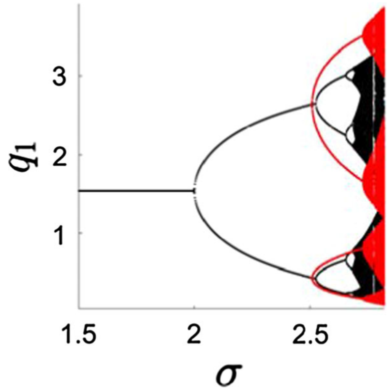


Fig. 2 Oligopoly of $N = 4$ homogeneous firms with linear demand function. Bifurcation diagrams on increasing σ, showing coexistence (color figure online)

datum, respectively, equal to $(q_1(0), q_2(0), q_3(0), q_4(0)) = (0.1, 0.2, 0.3, 0.4)$ (red diagram) and $q_i(0) = 0.5$, $i = 1, \ldots, 4$, (black diagram), which confirm coexistence of qualitatively different attractors, even in a homogeneous setting. Again, unstable dynamics possibly consist of either periodic, quasi-periodic or complex trajectories.

In the last simulation, we consider a nonlinear isoelastic inverse demand function $p(Q) = 1/Q$ and we assume heterogeneous linear cost functions $c_i(q_i) = d_i q_i + C/\sigma$, $d_i \neq d_j$. We stress that the analytical results of the previous section do not apply to this case, since (3) is violated by $p(Q) = 1/Q$ and we deal with heterogeneous firms. The equilibrium in the duopolistic case is $q_i^* = d_{-i}/(d_i^2 + 2d_i d_{-i} + d_{-i}^2)$, $i = 1, 2$, while the Jacobian matrix of the two-dimensional system evaluated at $\mathbf{q}^*$ is given by

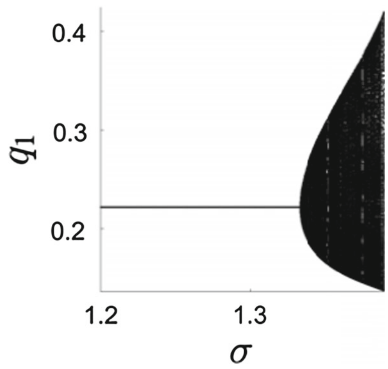
$$J^* = \begin{pmatrix} \dfrac{d_1 + d_2 - 2d_1 d_2 \sigma}{d_1 + d_2} & -\dfrac{d_2 \sigma\,(d_1 - d_2)}{d_1 + d_2} \\ \dfrac{d_1 \sigma\,(d_1 - d_2)}{d_1 + d_2} & \dfrac{d_1 + d_2 - 2d_1 d_2 \sigma}{d_1 + d_2} \end{pmatrix},$$

so stability is guaranteed by the usual conditions

$$\begin{cases} 1 + \operatorname{tr}(J^*) + \det(J^*) = \dfrac{{d_1}^2 d_2 \sigma^2 + d_1 {d_2}^2 \sigma^2 - 8d_1 d_2 \sigma + 4d_1 + 4d_2}{d_1 + d_2} > 0, \\ 1 - \operatorname{tr}(J^*) + \det(J^*) = d_1 d_2 \sigma^2 > 0, \\ 1 - \det(J^*) = -\dfrac{d_1 d_2 \sigma\,(d_1 \sigma + d_2 \sigma - 4)}{d_1 + d_2} > 0. \end{cases}$$

Solving the previous conditions with respect to σ, we find that the first one is always satisfied as the numerator is a second degree convex polynomial with negative determinant $-d_1 d_2$. Since the second condition always holds true, stability is indeed guaranteed under the last condition, namely if $\sigma < 4/(d_1 + d_2)$, and when it is violated a Neimark–Sacker bifurcation occurs. In Fig. 3, we show the bifurcation diagram for variable q_1 when $d_1 = 1$ and $d_2 = 2$, for which the stability threshold is $\sigma < 4/3$. In this case, when the equilibrium becomes unstable, output levels follow quasi-periodic trajectories.

Fig. 3 Duopoly of heterogeneous firms with isoelastic demand function. Bifurcation diagram on increasing σ, showing the emergence of quasi-periodic dynamics



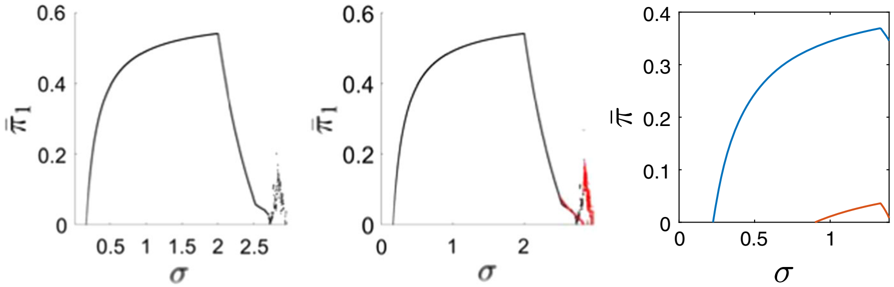


Fig. 4 Profits corresponding to parameter settings of the simulations reported in Figs. 1, 2 and 3

If, at the same time T, we compare average realized profits for each parameter setting of the simulations reported in Figs. 1, 2 and 3, we, respectively, obtain the diagrams reported in Fig. 4. As we can see, if σ is very small, fixed costs of the frequent learning activities cause negative profits. Profits increases with the lag between two consecutive market research until σ reaches the stability threshold value. If the profitability signal is too infrequently updated so that firms are not able to approach the optimal production strategies, erratic output levels make profits decrease. This confirms that the best learning frequencies that firms can choose are those smaller than and close to the stability threshold $\bar{\sigma}$. It is worth noticing that increasing the learning frequency from that optimal (i.e. decreasing σ from $\bar{\sigma}$) has (at least for suitably moderate changes) a milder effect on profits that decreasing it.

4 Conclusions

We proposed and studied an approach based on differential equations with piecewise constant arguments to take into account that production and learning activities can be carried on by firms at different time scales (respectively, continuous and discrete). A central role for the stability of the equilibrium is played by the frequency of the learning activities. If the lag between two subsequent market researches is too large, output choices can follow complex trajectories even if the reactivity of the agents to profitability signal is moderate. Such result is in agreement with what was found in Cavalli and Naimzada (2016) about a monopoly, but in the present setting dynamics can result in both periodic, quasi-periodic and chaotic trajectories and different attractors can coexist. The occurrence of such phenomena is then linked to the presence of strategic interaction, indeed lacking in the monopolistic setting, and is possible even if identical players are considered, i.e. in a homogeneous framework.

A central assumption on which we have built and studied the model is that the learning activities are carried on by all the firms at the same, exogenously determined time. In future research, we want to take into account the possibility for the each firm to choose the (possibly different) times at which information are collected, on the basis of some measure of the performance achieved with the last information set at disposal. This is challenging from the mathematical point of view and interesting from the economical one, as it would allow investigating the effect of introducing such a kind of heterogeneity among the firms.

Preliminary numerical investigations seem to suggest that stability, when each firm updates information with a different frequency, is more affected by the less frequent updating than by the more frequent one.

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