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Poverty trap, boom and bust periods and growth. A nonlinear model for non-developed and developing countries

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Abstract
This work investigates the qualitative and quantitative dynamics of a Solow–Swan growth model with differential savings as proposed by Böhm and Kaas (J Econ Dyn Control 24:965–980, 2000) assuming the shifted Cobb–Douglas (SCD) production function (see Capasso et al. in Nonlinear Anal. 11:3858–3876, 2010) which makes it possible to consider the long-run dynamics of non-developed and developing countries as well as that of developed economies. The resulting model is described by a nonlinear discontinuous map generating both a poverty trap and complex dynamics. Furthermore, multistability phenomena may emerge: besides the "vicious circle of poverty", long-run behaviours may include boom and bust periods. Complex basins can emerge, hence, economic policies trying to raise the capital per capita may fail and economies may be captured by the poverty trap.

Keywords Solow model · Poverty trap · Growth dynamics · Multistability · Discontinuous map

JEL Classification C61 · C62 · E2 · O1 · O4

1 Introduction

Recall the neoclassical Solow–Swan growth model [see Solow (1956) and Swan (1956)] describing the dynamics of the growth process and the long-run evolution of the economic system. The related discrete time model is given by

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$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + sf(k_t)]$$

where $y_t = f(k_t)$ is the production function in intensive form (mapping capital per worker $k_t \geq 0$ into output per worker $y_t \geq 0$), $n \geq 0$ is the labour force growth rate, $\delta \in (0, 1]$ is the depreciation rate of capital, $s \in (0, 1)$ is the saving rate and $t \in \mathbb{N}$. When the Cobb–Douglas (CD) production function is considered, i.e. $f(k_t) = Ak_t^{\alpha}$, almost all trajectories converge to the positive steady state. In recent works, different technologies have been taken into account, proving that fluctuations or more complex features can be produced. For instance, with constant elasticity of substitution (CES) or variable elasticity of substitution (VES) [for CES see Brianzoni et al. (2007, 2009), Masanjala and Papageorgiou (2004) and Papageorgiou and Saam (2008), while for VES see Brianzoni et al. (2012b), Cheban et al. (2013) and Karagiannis et al. (2005)] it has been found that non-simple dynamics may arise if the elasticity of substitution is sufficiently low. Evidently, the elasticity of substitution between production factors plays a crucial role in the theory of economic growth. Moreover, it represents one of the determinants of the long-run equilibrium level [for the correlation between elasticity of substitution and capital per capita levels see Klump and de La Grandville (2000) and Miyagiwa and Papageorgiou (2003)].

As Azariadis and Stachurski (2005) showed the above-mentioned production functions do not take into account the differences in production technology between rich and poor countries. Whereas, non-concave growth models may be able to generate poverty trap, thus describing what occurs in non-developed economies. (The condition for which a country needs a critical level of physical capital before a growth dynamic could be observed.), Brianzoni et al. (2012a) considered a non-concave production function and showed that also for non-developed or developing countries complicated dynamics emerge if the elasticity of substitution is sufficiently low, confirming that the elasticity of substitution is responsible for the creation and propagation of complexity and that the origin too can be an attractor.

Starting from the works by Kaldor (1957, 1956) and Pasinetti (1962), attention has been given to the influence of different saving propensities of workers and shareholders over the dynamics of economic growth: if the aggregate saving propensity is dependent on income distribution, then multiple equilibria may exist. Böhm and Kaas (2000) showed that complexity can be generated in neoclassical growth models with differential savings.

In this paper, we study the discrete time one-sector Solow–Swan growth model with differential savings as given by Böhm and Kaas (2000) while assuming that the technology is described by the shifted Cobb–Douglas (SCD) production function as proposed by Capasso et al. (2010). As in Brianzoni et al. (2012a, 2015), the use of a non-concave production function states the existence of a poverty trap. Notice that, although CES and VES production functions properly describe developed economies, they are not able to explain dynamics related to non-developed countries. On the other hand, the SCD production function implies a minimum level of physical capital essential for production, a requirement of capital needed in order to observe increasing returns. This kind of production functions is often considered in the literature to describe the growth dynamics of developing countries. Indeed, as Azariadis and Stachurski (2005) thoroughly explain, poor economies are often characterised by market failure, inefficient practices, "institution failure" and also social norms and conventions which cause the well-know "vicious circle of poverty". These considerations make the model economically significant for the analysis of the growth dynamics of developing countries: can a poor economy escape the poverty trap? Which is the required initial investment? If a developing country is out of the poverty trap, could its economy fall back into it? A first step towards an answer to these questions will be given.

From a mathematical point of view, when the SCD production function is considered, the resulting model is described by a nonlinear discontinuous map, a type of framework recently considered in several economic models [see, among all, Böhm and Kaas (2000) and Tramontana et al. (2011, 2014, 2015)] since recent mathematical tools make it possible to investigate economic phenomena defined by discontinuous systems. Our main goals are to describe the qualitative and quantitative long-run dynamics of the growth model and evaluate the relation between elasticity of substitution and capital per capita equilibrium levels in not-yet-developed countries. The results of our analysis show that complex dynamics, multistability phenomena and non-connected basins of attraction may emerge. In such cases, the initial condition becomes crucial, since the long-run behaviours of the economy may be predictable or unpredictable depending on the structure of the basins of attraction, determining the possibility to forecast if a country might go towards a poverty trap or towards a developed economy. Moreover, as in Klump and de La Grandville (2000), a positive correlation between elasticity of substitution and long-term dynamics is exhibited.

The rest of the paper is organised as follows. In Sect. 2, we present the model and we discuss its properties. In Sect. 3, we analyse the existence and the stability of the steady states while in Sect. 4 we demonstrate the possible occurrence of multiple equilibria, complex dynamics as well as complex basins. Section 5 concludes our paper.

2 The model

Consider the discrete time neoclassical one-sector growth model as proposed by Böhm and Kaas (2000): following Kaldor (1956, 1957) and Pasinetti (1962), we assume that workers and shareholders have different but constant saving rates, respectively, $s_{\mathrm{w}} \in (0, 1)$ and $s_{\mathrm{r}} \in (0, 1)$). Moreover, shareholders receive the marginal product of capital $f'(k)$, while the total capital income per worker is $kf'(k)$. We assume that the wage rate equals the marginal product of labour, that is

$$w(k) = f(k) - kf'(k). \tag{1}$$

Following Böhm and Kaas (2000), the map describing capital accumulation over time $t \in \mathbb{N}$ is given by

$$k_{t+1} = \phi(k_t) = \frac{1}{1+n}\left[(1-\delta)\,k_t + s_{\mathrm{w}} w(k_t) + s_{\mathrm{r}} k_t f'(k_t)\right], \quad k_t \geq 0\,. \tag{2}$$

Following Capasso et al. (2010), we consider a shifted Cobb–Douglas (SCD) production function that is a continuous non-strictly concave and non-differentiable

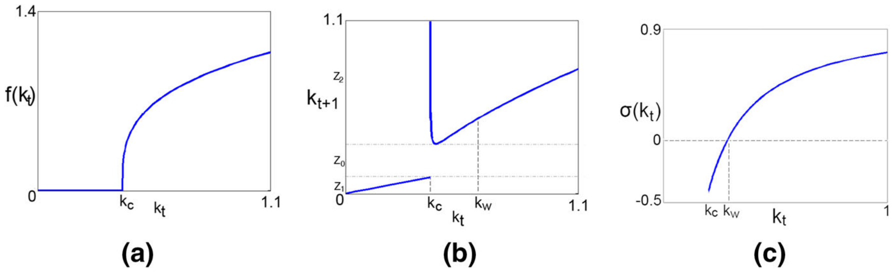

Fig. 1 Common parameter values: $n = 0.3$, $\delta = 0.4$, $s_w = 0.6$ and $s_r = 0.7$. **a** SCD production function. Parameter values: $\alpha = 0.3$, $A = 1.2$ and $k_c = 0.4$. **b** The final map for capital accumulation. Parameter values: $\alpha = 0.5$, $A = 2.122$ and $k_c = 0.4$. **c** The elasticity of substitution. Parameter values: $\alpha = 0.3$, $A = 1.2$ and $k_c = 0.2$

production function stating the existence of a minimum level of capital needed before making returns. This production function is able to describe non-developed countries, since it takes into account the realistic need to establish a basic structure for production (such as machineries and infrastructures) to obtain output. For instance when a country with almost no physical capital is considered, an initial investment is required before production. The SCD production function in its intensive form is given by

$$f(k_t) = \begin{cases} 0 & 0 \leq k_t \leq k_c \\ A(k_t - k_c)^{\alpha} & k_t > k_c \end{cases} \tag{3}$$

where $A > 0$ is the total productivity factor, $0 < \alpha < 1$ is the output elasticity of capital and $k_c \geq 0$ is the critical level of capital per capita delimiting the poverty trap, that is, the minimum capital per capita initial level bringing increasing returns (see Fig. 1a).

Notice that if $k_c \to 0^+$, then $f(k_t)$ approaches the CD production function; therefore, $f(k_t)$ can be considered as a generalisation of the well-known CD production function. Moreover,

$$f'(k_t) = \begin{cases} 0 & 0 < k_t < k_c \\ \alpha A(k_t - k_c)^{\alpha - 1} & k_t > k_c \end{cases}.$$

To assure a nonnegative wage and an economically meaningful framework, we assume that if $f(k) - kf'(k) < 0$, then the resulting wage is equal to zero, hence

$$w(k_t) = \begin{cases} 0 & 0 \leq k_t \leq k_w \\ A(k_t - k_c)^{\alpha - 1}[(1-\alpha)k_t - k_c] & k_t > k_w \end{cases} \tag{4}$$

where $k_w = \frac{k_c}{1-\alpha} > k_c$. Notice that zero wages are not desirable but possible in the economy [see among all Schefold (2005)]. If we take into account Eqs. (2), (3) and

(4), the final one-dimensional map describing capital per capita evolution is given by:

$$k_{t+1} = \phi(k_t) = \begin{cases} \frac{1-\delta}{1+n} k_t & 0 \le k_t \le k_c \\ \frac{1}{1+n}[(1-\delta)k_t + A\alpha s_r k_t (k_t - k_c)^{\alpha-1}] & k_c < k_t \le k_w \\ \frac{1}{1+n}\left\{(1-\delta)\, k_t + A\frac{s_w(k_t-k_c)+\alpha(s_r-s_w)k_t}{(k_t-k_c)^{1-\alpha}}\right\} & k_t > k_w \end{cases}. \tag{5}$$

In order to simplify the exposition, we define function

$$G(k) = \begin{cases} s_r\alpha(k-k_c)^{\alpha-1} & k_c < k \le k_w \\ \frac{s_w(k-k_c)+\alpha(s_r-s_w)k}{k(k-k_c)^{1-\alpha}} & k > k_w \end{cases} \tag{6}$$

and

$$H(k) = G(k) + kG'(k) = \begin{cases} s_r\alpha(\alpha k - k_c)(k-k_c)^{\alpha-2} & k_c < k < k_w \\ \frac{\alpha\{[s_w+(s_r-s_w)\alpha]k - s_r k_c\}}{(k-k_c)^{2-\alpha}} & k > k_w \end{cases}. \tag{7}$$

Then, as long as $k > k_c$, function ϕ may be written in terms of function G as defined in (6), as follows

$$\phi(k) = \frac{1}{1+n}\{(1-\delta)k + AkG(k)\} \tag{8}$$

while

$$\phi'(k) = \frac{1}{1+n}\{1-\delta + AH(k)\} \qquad \forall k > k_c, k \ne k_w \tag{9}$$

where H is defined in (7).

2.1 Preliminary properties

Map ϕ is not negative, defined in $\mathbb{R}_+$ and it is discontinuous in $k_t = k_c$ since $\lim_{k_t\to k_c^-} \phi(k_t) = \frac{1-\delta}{1+n}k_c$ and $\lim_{k_t\to k_c^+} \phi(k_t) = +\infty$, while it is continuous in k_w being $\lim_{k_t\to k_w^-} \phi(k_t) = \lim_{k_t\to k_w^+} \phi(k_t) = \frac{1}{1+n}\left[\frac{1-\delta}{1-\alpha}k_c + As_r\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha}\right]$. Furthermore, $\lim_{k_t\to +\infty} \phi(k_t) = +\infty$ (see Fig. 1b). Observe that, for well-developed countries, those characterised by levels of capital per capita high enough, i.e. $k_t > k_c$, the higher the difference between workers and shareholders saving rates, the higher the capital per capita at time $t+1$.

Recall that the elasticity of substitution between production factors (which measures the ease with which capital and labour can be substituted in production) for nonlinear and twice differentiable functions is defined as follows [see Sato and Hoffman (1968)]:

$$\sigma(k_t) = -\frac{f'(k_t)[f(k_t) - f'(k_t)k_t]}{f(k_t)f''(k_t)k_t} \tag{10}$$

while it is assumed to be $\sigma = +\infty$ for linear production functions. Being

$$f''(k_t) = A\alpha(\alpha-1)(k_t-k_c)^{\alpha-2}, \qquad k_t > k_c$$

the elasticity of substitution between production factors for the SCD can be easily calculated and it is given by

$$\sigma(k_t) = \begin{cases} +\infty & 0 < k_t < k_\mathrm{c} \\ 1 - \frac{k_\mathrm{c}}{(1-\alpha)k_t} & k_t > k_\mathrm{c} \end{cases}.$$

Observe that $f(k_t)$ belongs to the class of variable elasticity of substitution (VES) production functions, as $\sigma(k_t)$ depends on the level of capital per capita k_t. Moreover, $\sigma(k_t)$ is discontinuous in $k_t = k_\mathrm{c}$ being $\lim_{k_t \to k_\mathrm{c}^+} \sigma(k_t) = \frac{-\alpha}{1-\alpha} \neq +\infty$ while $\lim_{k_t \to +\infty} \sigma(k_t) = 1$. Notice that if $k_t > k_\mathrm{w} > k_\mathrm{c}$, then $\sigma(k_t) > 0$, whereas if $k_\mathrm{c} < k_t < k_\mathrm{w}$, then $\sigma(k_t) < 0$ (see Fig. 1c). Notice also that σ is always smaller than 1 for $k_t > k_\mathrm{c}$. As far as the sign of σ is concerned, we observe that even if a negative elasticity of substitution between production factors is not conventional, several production functions in the literature show negative elasticity of substitution [see Prywas (1986), Andrikopoulos et al. (1989), Thompson and Taylor (1995), Nguyen and Streitwieser (1997), Stern (2004), Hamilton et al. (2005) and Jurgen (2014)]. For instance, as suggested by Paterson (2012), a negative elasticity of substitution can occur if complementary inputs are considered. Therefore, a negative elasticity of substitution between production factors for $k_\mathrm{c} < k_t < k_\mathrm{w}$ suggests that, in the early stages of production, immediately outside the poverty trap, capital and labour are complementary and not replaceable.

For $k_t \leq k_\mathrm{c}$, map ϕ is a linear function passing through the origin with slope $m = \frac{1-\delta}{1+n}$. Note that m is positive and smaller than 1; moreover, it increases as δ or n decreases. Firstly, we compute the derivative for map ϕ that is given by

$$\phi'(k_t) = \begin{cases} \frac{1-\delta}{1+n} & k_t < k_\mathrm{c} \\ \frac{1}{1+n}\left\{1 - \delta + \alpha A \left[\frac{s_\mathrm{r}(\alpha k_t - k_\mathrm{c})}{(k_t - k_\mathrm{c})^{2-\alpha}}\right]\right\} & k_\mathrm{c} < k_t < k_\mathrm{w} \\ \frac{1}{1+n}\left\{1 - \delta + \alpha A \left[\frac{s_\mathrm{r}(\alpha k_t - k_\mathrm{c}) + (1-\alpha)s_\mathrm{w} k_t}{(k_t - k_\mathrm{c})^{2-\alpha}}\right]\right\} & k_t > k_\mathrm{w} \end{cases}. \tag{11}$$

Notice that ϕ is non-differentiable in k_w and, hence, if ϕ admits an attractor A and $k_\mathrm{w} \in A$, its stability must be discussed separately. With regard to the behaviour of map ϕ, for sufficiently high levels of capital per capita we observe that $\forall k > k_\mathrm{c}$ function ϕ presents a turning point, i.e. a minimum point, as stated in the following proposition.

Proposition 1 *Function ϕ given by (5) is unimodal for $k_t > k_\mathrm{c}$ with minimum point $k_{\min}$.*

Proof Recall function ϕ and ϕ' may be written, respectively, as in (8) and (9). Let $k > k_\mathrm{c}$, $k \neq k_\mathrm{w}$, then $\phi'(k) = 0$ iff $H(k) = \frac{\delta-1}{A}$. Hence, the turning points of ϕ different from k_w are solutions of

$$H(k) = \frac{\delta - 1}{A}. \tag{12}$$

Function $H(k)$ is such that $\lim_{k\to k_c^+} H(k) = -\infty$; moreover,

$$H'(k) = \begin{cases} \frac{s_r\alpha(\alpha-1)(\alpha k-2k_c)}{(k-k_c)^{3-\alpha}} & k_c < k < k_w \\ \frac{\alpha(1-\alpha)\{(2s_r-s_w)k_c-[(1-\alpha)s_w+\alpha s_r]k\}}{(k-k_c)^{3-\alpha}} & k > k_w \end{cases}.$$

Assume $k_p = \frac{s_r k_c}{\alpha s_r+(1-\alpha)s_w}$ and $z = s_r(2\alpha-1)\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$.

We first consider the solutions of Eq. (12) for $k \in (k_c, k_w)$:

(i) for $\alpha > \frac{1}{2}$, function $H(k) < 0$ iff $k_c < k < \frac{k_c}{\alpha}$, therefore $H(k)$ can intersect the constant and negative function $v = \frac{\delta-1}{A}$ only in the interval $I_1 = \left(k_c, \frac{k_c}{\alpha}\right)$. Moreover, $H'(k) > 0 \forall k \in I_1$ and $\lim_{k\to \frac{k_c}{\alpha}^-} H(k) = 0$. So that $H(k) = v$ has always one solution;

(ii) for $\alpha \leq \frac{1}{2}$, function $H(k) < 0 \forall k \in (k_c, k_w) = I_2$, $H'(k) > 0 \forall k \in I_2$ and $\lim_{k\to k_w} H(k) = z$, so that $H(k) = v$ has one solution in the interval I_2 if $v \leq z$.

We now consider the solutions of Eq. (12) for $k > k_w$:

(iii) for $\alpha < \frac{s_r-s_w}{2s_r-s_w}$ and $s_r > s_w$, function $H(k) < 0$ for $k \in (k_w, k_p) = I_3$, moreover, $\lim_{k\to k_w^+} H(k) = z + (1-\alpha)s_w\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$, $\lim_{k\to k_p^-} H(k) = 0$ and $H'(k) > 0 \forall k \in I_3$ so that $H(k) = v$ has one solution in the interval I_3 if $v \geq z + (1-\alpha)s_w\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$.

Since $z < z+(1-\alpha)s_w\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$, cases (ii) and (iii) cannot occur simultaneously, and hence for $\alpha < \frac{s_r-s_w}{2s_r-s_w}$ there may exist, at most, one turning point. If condition (i) holds, then $k_{min} < \frac{k_c}{\alpha}$, if condition (ii) holds, then $k_{min} < k_w$, while if condition (iii) holds, then $k_{min} \in (k_w, k_p)$.

Consider now $k = k_w$ and notice that, when conditions (i), (ii) or (iii) do not hold, Eq. (12) has no solution. Nevertheless, for these parameter values $\lim_{k\to k_w^-} H'(k) < 0$ and $\lim_{k\to k_w^+} H'(k) > 0$ and hence $\phi(k)$ is unimodal with minimum point $k_{min} = k_w$. □

Information about the position of the minimum point of map ϕ deriving from Proposition 1 is summarised in the following table (where $v = \frac{\delta-1}{A}$, $z = s_r(2\alpha-1)\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$, $p = (1-\alpha)s_w\left(\frac{\alpha}{1-\alpha}k_c\right)^{\alpha-1}$ and $k_p = \frac{s_r k_c}{\alpha s_r+(1-\alpha)s_w}$).

Parameter values	Minimum point
$\alpha > \frac{1}{2}$	$k_{min} \in \left(k_c, \frac{k_c}{\alpha}\right)$
$\alpha \leq \frac{1}{2}$ and $v \leq z$	$k_{min} \in (k_c, k_w)$
$\alpha < \frac{s_r-s_w}{2s_r-s_w}$, $s_r > s_w$ and $v \geq z + p$	$k_{min} \in (k_w, k_p)$
All others	$k_{min} = k_w$

Notice that when $k_{min} = k_w$ the turning point is not differentiable.

3 Equilibria and local dynamics

In this section, we consider the question of the existence of steady states of system (5) and then we discuss about their local stability.

3.1 Existence of steady states

The problem of finding the number of steady states is not trivial, considering the high number of parameters. As a general result, map ϕ always admits one fixed point characterised by zero capital per capita, i.e. $k = 0$ is a fixed point for any choice of parameter values as stated in the following Proposition.

Proposition 2 *Map ϕ as given by (5) always has the fixed point $k = 0$ that is locally stable.*

Proof $k_t = 0$ is a solution of equation $k_t = \phi(k_t)$ for all parameter values. Moreover, $\phi'(0) \in (0, 1)$. □

Steady states which are economically interesting are those characterised by positive capital per worker. As previously underlined, ϕ is a discontinuous map. Moreover, no positive fixed point exists for $0 < k_t \leq k_c$, being $0 < \frac{1-\delta}{1+n} < 1$. In order to determine the positive fixed points of ϕ with $k_t > k_c$, we consider function G as given by (6) and consider that the positive steady states of map ϕ are the solutions of equation

$$G(k_t) = \frac{n+\delta}{A}. \tag{13}$$

For what in concerns the number of positive steady states of the Solow growth model with SCD and differential savings the following proposition holds.

Proposition 3 *Consider ϕ as given by (5). Define $g = \frac{n+\delta}{A}$ and $k_M = k_c \frac{(2-\alpha)s_w + \sqrt{\alpha s_w[(4s_r - 3s_w)\alpha - 4(s_r - s_w)]}}{2(1-\alpha)[s_w + (s_r - s_w)\alpha]}$.*

(i) *Assume $s_r \geq s_w$. Then ϕ has one positive fixed point given by $k^* > k_c$. Moreover,*
   (a) *if $g \geq G(k_w)$, $k^* \leq k_w$;*
   (b) *if $g < G(k_w)$, $k^* > k_w$.*

(ii) *Assume $s_r < s_w$. Then*
   (a) *if $g > G(k_M)$, there exists one positive fixed point given by $k_c < k^* < k_w$;*
   (b) *if $g = G(k_M)$, there exist two positive fixed points given by $k_c < k_1 < k_w$ and $k_2 = k_M > k_w$;*
   (c) *if $G(k_w) < g < G(k_M)$, there exist three positive fixed points given by $k_1 \in (k_c, k_w)$, $k_2 \in (k_w, k_M)$ and $k_3 \in (k_M, +\infty)$;*
   (d) *if $g = G(k_w)$, there exist two fixed points given by $k_1 = k_w$ and $k_2 > k_M$;*
   (e) *if $g < G(k_w)$, there exists one positive fixed point given by $k^* > k_M$.*

Proof Being $\frac{1-\delta}{1+n} < 1$, for all $0 < k_t \leq k_c$ map ϕ does not intercept the main diagonal. Function G is such that $G(k_t) > 0 \forall k_t > k_c$, furthermore, $\lim_{k_t \to k_c^+} G(k) =$

$+\infty$ while $\lim_{k_t\to\infty} G(k_t) = 0$. $G(k_t)$ is continuous in k_w, being $\lim_{k_t\to k_w^-} G(k_t) = \lim_{k_t\to k_w^+} G(k_t) = G(k_w) = \alpha^\alpha s_r \left(\frac{k_c}{1-\alpha}\right)^{\alpha-1}$. We distinguish between the following cases.

(i) If $s_r \geq s_w$, $G(k_t)$ is strictly decreasing, since $G'(k_t) \leq 0$ and $G'(k_t) = 0$ in one point at most. Hence, $G(k_t)$ intersects the positive and constant function $g = \frac{n+\delta}{A}$ in a unique value $k^* > k_c$.

(ii) If $s_r < s_w$, $\exists k_M > k_w$ such that $G'(k_t) < 0$ for $k_c < k_t < k_w \vee k_t > k_M$ and $G'(k_t) > 0$ for $k_w < k_t < k_M$. The local minimum and maximum points of function G are given by k_w and $k_M = k_c \frac{(2-\alpha)s_w+\sqrt{\alpha s_w[(4s_r-3s_w)\alpha-4(s_r-s_w)]}}{2(1-\alpha)[s_w+(s_r-s_w)\alpha]}$, respectively. Hence, if $\frac{n+\delta}{A} > G(k_M)$ or $\frac{n+\delta}{A} < G(k_w)$, then equation $G(k_t)$ intersects the positive and constant function $g = \frac{n+\delta}{A}$ in a unique positive value $k_t = k^* > k_c$. Whereas, if $G(k_w) < \frac{n+\delta}{A} < G(k_M)$, then equation $G(k_t) = g$ admits three positive solutions, k_1, k_2, k_3, where $k_1 \in (k_c, k_w)$, $k_2 \in (k_w, k_M)$, $k_3 > k_M$. For $g = G(k_w)$ a border collision bifurcation occurs with the merging of the fixed point with the kink point of ϕ, while for $g = G(k_M)$ a fold bifurcation occurs since the constant function g is tangent to G in the maximum point and it intersects function G in a second point $k^* < k_w$.

□

Notice that the case denoted $(ii.b)$ corresponds to a smooth fold bifurcation of the map, while the case denoted $(ii.d)$ corresponds to a border collision fold bifurcation of the map in the kink point k_w. In both cases, two fixed points are generated.

Taking into account Proposition 3, the Solow growth model with differential savings and shifted Cobb–Douglas production function always admits the equilibrium $k = 0$. Moreover, multiple equilibria can exist: up to three positive fixed points are exhibited, depending on the parameter values (see Fig. 2). Note that the necessary condition for the existence of more than one positive equilibrium is $s_r < s_w$. Moreover, more than one positive fixed point can emerge for sufficiently high values of the output elasticity of capital α. Note that these results agree with those obtained by Brianzoni et al. (2012b) considering the Revankar (Revankar (1971)) VES production function: up to three positive fixed points may emerge if the elasticity of substitution is smaller than one and workers save more than shareholders. Differently, when a CES production function is considered, at most two positive fixed point may emerge [see Brianzoni et al. (2007)].

We want to highlight how the output elasticity of capital α and the difference between saving rates influence the number of steady states. To this purpose, we define $\Delta_s = s_r - s_w$, $\Delta_s \in (-s_w, 1 - s_w)$. Taking into account the conditions related to the existence and number of fixed points stated in Proposition 3, it is possible to describe how the number of fixed points varies as the output elasticity of capital α or the difference between saving rates Δ_s changes. To this scope, we fix all the parameter values but α and Δ_s and we consider several parameters combinations (Δ_s, α) taken on the set $\Omega = [-s_w, 1 - s_w] \times [0, 1]$. Define

$$C_1 = \{(\alpha, \Delta_s) \in \Omega : s_r - s_w = 0\} \tag{14}$$

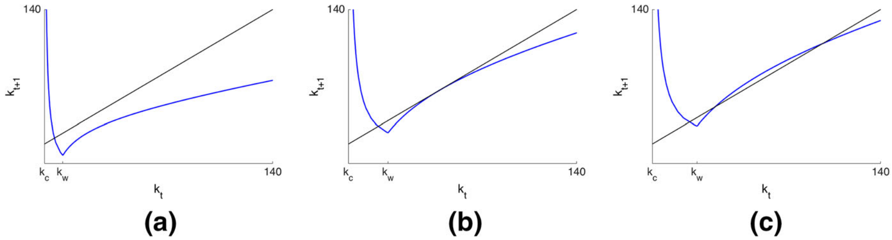

Fig. 2 Map ϕ and its positive fixed points for $k_t > k_c$ in the case of $s_r < s_w$ for the following parameter values: $\delta = 0.65$, $s_w = 0.45$, $s_r = 0.25$, $n = 0.45$, $A = 100$, $k_c = 44$. **a** One positive fixed point for $\alpha = 0.15$, **b** two positive fixed points for $\alpha = 0.275$, **c** three positive fixed points for $\alpha = 0.4$

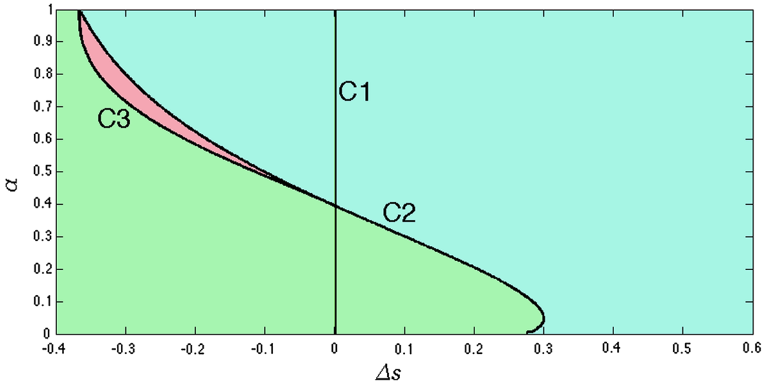

Fig. 3 Parameter values: $\delta = 0.05$, $s_w = 0.4$, $n = 0.05$, $A = 3$, $k_c = 20$. Number of fixed points according to Proposition 3. In the blue region, there is 1 positive fixed point (cases $(i.b)$ and $(ii.e)$), in the green region there is 1 positive fixed point (cases $(i.a)$ and $(ii.a)$), in the red region there are 3 positive fixed points (case $(ii.c)$). Curves C_1, C_2 and C_3 are defined in Eqs. (14), (15) and (16), respectively

$$C_2 = \left\{ (\alpha, \Delta_s) \in \Omega : \frac{n+\delta}{A} - G(k_w) = 0 \right\} \tag{15}$$

$$C_3 = \left\{ (\alpha, \Delta_s) \in \Omega : G(k_M) - \frac{n+\delta}{A} = 0, s_r < s_w \right\} \tag{16}$$

then curves C_1, C_2 and C_3 separate the plane Ω into three regions, each of which contains parameter values corresponding to a case stated in Proposition 3.

The three regions are depicted in Fig. 3: the points on the right of curve C_1 verify the condition of the case (i), while the left region contains the parameter values related to case (ii). Curve C_2 verifies condition $(ii.d)$, while the curve C_3 verifies condition $(ii.b)$. Notice that the existence of positive fixed points is due to high values of parameter α combined with low values of parameter Δ_s. The border collision bifurcation occurs in C_2 while the smooth fold bifurcation is caused in $C3$. In the red region, three fixed points exist, two of them generated via bifurcation.

3.2 Stability of steady states

We now discuss about the local stability of the steady states of map ϕ. As far as the local stability of the steady state $k = 0$ is concerned, the following proposition holds.

Proposition 4 *Let ϕ be as given by (5). Then the equilibrium $k = 0$ is always locally stable.*

Proof Note that $\lim_{k_t \to 0^+} \phi'(k_t) = \frac{1-\delta}{1+n} \in [0, 1)$ and consequently the origin is a locally stable fixed point for map ϕ. □

Notice that for all the initial conditions $k_0 \leq k_c$, map ϕ behaves as a contraction map and the iterates monotonically converge to $k = 0$. Therefore, we define the poverty trap as a situation in which, at the initial time, the capital per capita level is not high enough, i.e. $k_0 \leq k_c$, and such that the economy will not survive in the long term. The interval $\mathcal{B}_0 = [0, k_c)$ is the immediate basin of the origin and, when it coincide with the global basin of $k = 0$, the long-run behaviour for any initial condition $k_0 < k_c$ is predictable. This result diverges from those obtained by using a CES or VES production function [see Brianzoni et al. (2007, 2009, 2012b) and Grassetti et al. (2015)], since a poverty trap exists [see also Capasso et al. (2010) and Brianzoni et al. (2012a, 2015)]. Notice that CES and VES production functions properly describe developed economies but are not able to capture the vicious circle of poverty that typically characterises non-developed countries, whereas the SCD production function makes it possible to consider this phenomenon. Thus, the presence of a poverty trap threatens the possibility of economic growth: economies starting from a low level of physical capital may be captured by the poverty trap and, consequently, the dynamic of physical capital may converge to zero. Note that for a small displacement from the stable equilibrium $k = 0$, the time trend of the relative displacement is $T_r = (\frac{1-\delta}{1+n})^t$. Therefore, if an economy lies in the poverty trap, a higher depreciation rate of capital or a higher labour force growth rate causes a faster return to the steady state characterised by zero capital per capita.

As the long-term dynamics produced by the model are known for all the initial capital per worker less than the threshold value k_c, we now focus on the growth patterns concerning sufficiently high initial states (i.e. $k_0 > k_c$). Due to the complexity of the map, a complete definition about the local stability of the positive hyperbolic steady states is not possible. The following Remark summarises results about the existence of stable fixed points and conditions on parameters under which ϕ' is negative and, hence, complex dynamics may arise.

Remark 1 Consider ϕ as given by (5) and recall Proposition 3.

(i) Assume $s_r \geq s_w$. If $k^* > k_{min}$, the equilibrium k^* is locally stable. Otherwise $\phi'(k^*) < 0$.

(ii) Assume $s_r < s_w$.

   (a) Consider $g > G(k_M)$. Then, if $k^* > k_{min}$ the equilibrium k^* is locally stable. Otherwise $\phi'(k^*) < 0$.

   (b) Consider $G(k_w) < g < G(k_M)$ then the fixed point $k_3 > k_M > k_{min}$ is always locally stable while the fixed point $k_w > k_2 > k_M$ is always unstable.

Furthermore if $k_1 > k_{\min}$, the equilibrium k_1 is always locally stable, whereas, if $k_1 < k_{\min}$, then $\phi'(k_1) < 0$.

(c) Assume $g < G(k_{\mathrm{w}})$. Then the equilibrium k^* is locally stable.

Proof Observe that for any fixed point k_f it follows $\phi'(k_f) = 1 + \frac{A}{1+n} k_f G'(k_f)$. Being $G'(k^*) < 0$ for cases (i), $(ii.a)$ and $(ii.c)$ of Proposition 3, then $\phi'(k^*) = 1 + \frac{A}{1+n} k^* G'(k^*) < 1$. Moreover, ϕ is unimodal with minimum point $k_{\min}$, so that if $k^* > k_{\min}$ then $\phi'(k^*) \in (0, 1)$, whereas, if $k^* < k_{\min}$, $\phi'(k^*) < 0$. In case $(ii.c)$ k_1 is stable if $k_1 > k_{\min}$, while $\phi'(k_1) < 0$ for $k_1 < k_{\min}$. Moreover, being $\phi(k)$ strictly increasing $\forall k > k_{\min}$ and $G'(k_2) > 0$ then $\phi'(k_2) > 1$, it follows that the fixed point k_2 is unstable, while the fixed point k_3, $k_3 > k_2 > k_{\min}$ is locally stable. □

Notice that multiple equilibria coexist and hence multistability phenomena may occur. Therefore, the global analysis of basins is mathematically significant (complex basins may exist) and particularly economically relevant, since it makes it possible to answer one of the fundamental questions concerning developing countries and poverty trap: is it possible for an economy with a sufficiently high capital per capita level, i.e. $k_0 > k_{\mathrm{c}}$, to avoid the poverty trap?

Further considerations on the nature of the fixed points, their basins and their behaviour are debated in the following section.

4 Global dynamics and numerical experiments

In this section, we analyse the qualitative asymptotic properties of map ϕ by using both numerical simulations and analytical tools. Note that the map may show complex dynamics if a fixed point is located on the decreasing branch of ϕ (see Remark 1). In order to consider the possibility of complex attractors to emerge, we analyse the case in which $k_{\min} > \phi(k_{\min})$. Since the analytic form of function ϕ is complicated, we cannot analytically describe this condition and the dynamic behaviour needs to be analysed by numerical simulations.

4.1 Complex attractors and multistability

Recall from Remark 1 that, if a fixed point is placed in the interval $(k_{\mathrm{c}}, k_{\min})$, it may be locally stable or unstable and hence a more complex attractor A may appear around it. The following proposition states the existence of a trapping interval for map ϕ.

Proposition 5 *Consider ϕ as given in (5). Assume $k_{\min} > \phi(k_{\min}) > k_{\mathrm{c}}$ and, if three positive fixed points exist, assume $k_2 > \phi^2(k_{\min})$. Then the set $J = [\phi(k_{\min}), \phi^2(k_{\min})]$ is trapping.*

Being ϕ unimodal, it admits a trapping set J under the conditions of Proposition 5.

Since J is trapping set, then if a complex attractor A exists, it must belong to it. Furthermore, A must attract the trajectory starting from the turning point $k_{\min}$. As Sushko et al. (2005) demonstrated, when $k_{\mathrm{w}} \in J$, a second attractor B (related to images of k_{w}) could exist in J. However, several numerical experiment did not show

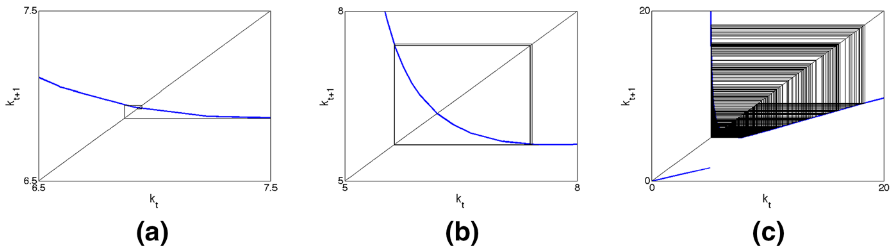

Fig. 4 Staircase diagram of ϕ being $n = 0.3$, $\delta = 0.6$, $s_{\text{w}} = 0.25$, $s_{\text{r}} = 0.5$, $A = 5$, $k_{\text{c}} = 5$ and i.c. $k_0 = k_{\min}$ for different values of α. **a** $\alpha = 0.5$, stable fixed point. **b** $\alpha = 0.4$, stable 2-period cycle. **c** $\alpha = 0.35$ complex attractor

the existence of a second attractor; therefore, in the following we will focus on the case in which only one attractor exists in J. Recall that if $k_{\min} > \phi(k_{\min})$, then the eigenvalue of the fixed point placed on the decreasing branch of map ϕ (if map ϕ is differentiable in that point) is negative and hence it may lose stability only via period-doubling bifurcation. Notice that subsequent bifurcations may be of the border collision type.

In Fig. 4, we show three different staircase diagrams of map ϕ with the initial condition $k_0 = k_{\min}$ and satisfying the hypothesis given in Proposition 5. Therefore, the attractor belongs to the trapping set J. In panel (a), a stable fixed point is presented for $\alpha = 0.5$. In panel (b), a stable cycle C_2 of period 2 is reached for $\alpha = 0.4$. Complexity emerges as the parameter α decreases and a complex attractor is visible in panel (c) for $\alpha = 0.35$. In order to discuss the bifurcations leading to chaos within the trapping interval J defined in Proposition 5, we take into account the role of the difference between saving propensities and elasticity of substitution. Figure 5a contains the sequence of bifurcations of map ϕ as parameter Δ_{s} is moved while Fig. 5b shows the asymptotic dynamics versus the bifurcation parameter α.

In both the diagrams, complex dynamics arise. Note that, since $\sigma(k_t) = 1 - \frac{k_{\text{c}}}{(1-\alpha)k_t}$, then $\sigma(k_t) < 1$ for all parameter values when $k_t > k_{\text{c}}$, confirming that complex dynamics may arise if the elasticity of substitution is smaller than one, as demonstrated by extended literature [see, among all, Brianzoni et al. (2007, 2009, 2012a, b, 2015)]. Notice that two contact bifurcations may occur: $\phi(k_{\min}) = k_{\text{c}}$ and $k_2 = \phi^2(k_{\min})$. When $\phi(k_{\min}) < k_{\text{c}}$, trajectories from $k > k_{\text{c}}$ may reach $k = 0$ which is otherwise impossible. While, when $k_2 < \phi^2(k_{\min})$, more trajectories from $k < k_2$ reach the highest fixed point k_3. In Fig. 5, Δ_{s}^* and $\alpha = \alpha^*$ are the values at which the contact bifurcation $k_{\text{c}} = \phi(k_{\min})$ occurs. The poverty trap attracts trajectories if Δ_{s} is sufficiently low; therefore, an economic policy that increases the saving propensity of shareholders could avoid the vicious circle of poverty for the economy.

Two questions arise: whether different initial conditions give rise to trajectories converging to different attractors and why, in both bifurcation diagrams, a value Δ_{s}^* or α^* is observed that closes the bifurcation cascade from above and before the convergence of the trajectories to the poverty trap. To answer to these questions, notice that if $\alpha > \alpha^*$ and α is sufficiently close to α^*, then complex dynamics emerge if the difference between workers and shareholders is big enough. This result is in line with those obtained by Brianzoni et al. (2007, 2009, 2012b). Note also that the elasticity of substitution increases as parameter α decreases and hence fluctuations arise when

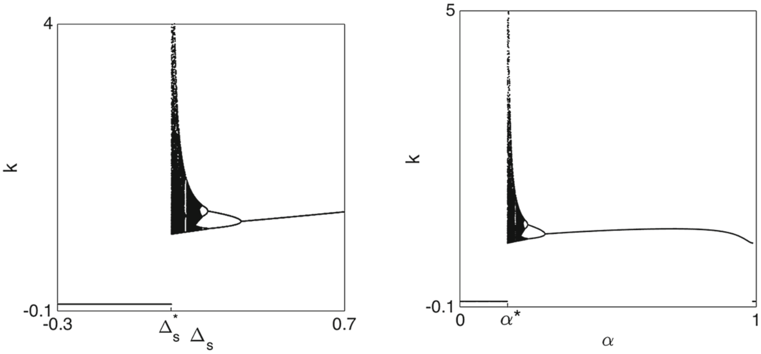

Fig. 5 Parameter values $n = s_w = 0.3$, $\delta = 0.2$, $A = k_c = 1$ and $k_0 = k_{min}$. **a** Bifurcation diagram of ϕ w.r.t. Δ_s being $\alpha = 0.2$. **b** Bifurcation diagram of ϕ w.r.t. α being $s_r = 0.47$

the elasticity of substitution between production factors is sufficiently high (but still smaller than one).

With regard to the first issue, notice that if $s_r < s_w$ and $G(k_w) < \frac{n+\delta}{A} < G(k_M)$, then from 2 to 3 attractors may coexist: the fixed point characterised by zero capital per capita, the positive fixed point k_3 and the attractor A previously described. As long as conditions in Proposition 5 hold, A may be a cycle or a more complex set. Note that these results differ from those obtained with a VES production function where, if a multistability phenomenon appears, attractors are only positive fixed points. Differently from previous literature, if the SCD production function is considered, the Solow–Swan growth model with differential savings properly describes the long-run dynamics of non-developed, developing and developed countries. In particular, it is able to describe three different long-run behaviours: convergence to the poverty trap, convergence to cycle or a more complex set in which the economy alternates boom and bust periods, and, finally, convergence to a positive capital per capita value. Figure 6 shows a multistability phenomenon for map ϕ [with regard to multistability, see Bischi et al. (2000), Brianzoni et al. (2012a, 2015) and Sushko et al. (2005)]: given the same parameter values and two different initial conditions, in panel (a) a stable 2-period cycle is presented, while panel (b) depicts the coexisting attracting fixed point. This is, hence, the case in which three different attractors coexist. The two basins $\mathcal{B}(J)$ and $\mathcal{B}(k_3)$ are separated by the pre-images of k_2. As long as $\phi^2(k_{min}) < k_2$ the basin $\mathcal{B}(k_3)$ consists only of two intervals. A bifurcation in the basins' structure occurs when $k_2 = \phi^2(k_{min})$ and, for $k_2 \leq \phi^2(k_{min})$, almost every trajectory converges to k_3. A second bifurcation occurs when $k_c = \phi(k_{min})$, after which the two basins $\mathcal{B}(k_3)$ and $\mathcal{B}(0)$ may have a fractal structure or a simple structure depending on the dynamics existing in J, as long as it is invariant. Given the considerations done until now, the second question is clearly worth answering. This will be done in the next section.

4.2 Complex basins

We now describe the structure of the basins of attraction for the given attractors in order to show that, when more than one attractor exits, the initial condition becomes crucial for the long-run behaviour of the economy, determining the possibility for a

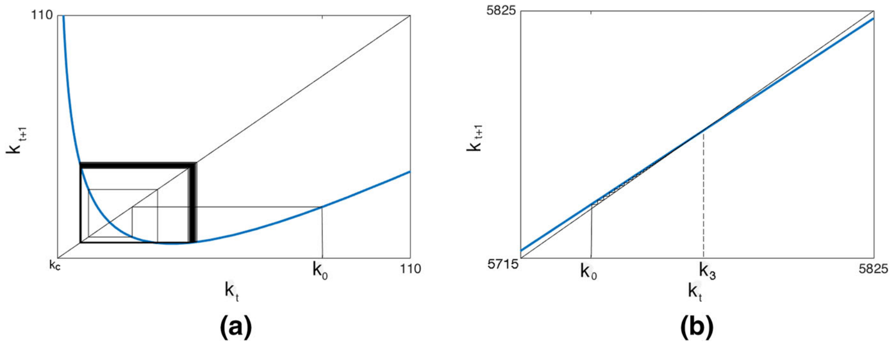

Fig. 6 Staircase diagrams. The same map ϕ is depicted in two different intervals in order to show both attractors. Parameter values $\delta = 0.1$, $s_w = 0.9$, $s_r = 0.05$, $\alpha = 0.75$, $A = 14$, $k_c = 90$. **a** Stable 2-period cycle. **b** Stable fixed point

country to reach a positive level of capital per capita or to fall into the "vicious circle of poverty".

Function ϕ is unimodal for $k > k_c$. Moreover, several numerical experiments show that $\phi(k_{min}) > \phi(k_c)$, therefore $\phi(k_{min})$ and $\phi(k_c)$ separate the set $\mathbb{R}_+$ in three subsets: $Z_1 = (0, \phi(k_c))$, $Z_0 = (\phi(k_c), \phi(k_{min}))$ and $Z_2 = (\phi(k_{min}), +\infty)$ whose points have, respectively, one zero and two rank-1 pre-images (see Fig. 1b), so that ϕ is a $Z_1 - Z_0 - Z_2$ map [with regard to pre-images, see Makrooni et al. (2015)]. As previously shown, map ϕ may have three attractors: the fixed point $k = 0$ characterised by zero capital per capita, the attractor $A \in J$ and a positive fixed point k_3. Notice that if k_3 is an attractor (i.e. 4 equilibria exist), then k_2 is the unstable fixed point. In this case, complex basins may emerge depending on parameter values, as it will be described. Regarding the global basin of the origin notice that, as long as $\phi(k_{min}) > k_c$, it corresponds to the immediate basin. Differently, when $\phi(k_{min}) < k_c$ then the immediate basins have other pre-images, leading to a wider global basin. In cases for which $k_c < \phi(k_{min})$ an initial condition $k_0 > k_c$ can never decrease too much and the transition to a poverty trap cannot occur, thus the poverty trap can be avoided. Although, there can be up to two positive equilibria, in which case the economy can switch from the lower lever to the higher one.

We start by assuming $\phi(k_{min}) > k_{min}$, i.e. the turning point of map ϕ is above the main diagonal and two attractors coexist: the fixed point $k = 0$ and the positive fixed point k_3. In this case, the basins of attraction are simple: the boundary that separates the basin of attraction $\mathcal{B}(0)$ of the attractor $k = 0$ from the basin of attraction $\mathcal{B}(k_3)$ is point k_c (see Fig. 7a). Thus, as long as a parameter is moved (for instance s_r), the non-differentiable point k_w crosses the value $\phi(k_w)$, a border collision bifurcation occurs, two fixed points are created and the attractor $A \in J$, given for instance by a fixed point, may appear. After this bifurcation, if some parameter still varies (for instance s_r decreases), a period-doubling bifurcation takes place and the attractor A becomes a 2-period cycle (see Fig. 7b). Notice that, after the bifurcation, three basins of attraction exist: $\mathcal{B}(0)$, $\mathcal{B}(k_3)$ and the basin $\mathcal{B}(A)$ of the emerged attractor A. In particular $\mathcal{B}(0) = [0, k_c]$, while the basin of attraction of k_3 is now made of unconnected portions: the immediate basin $(k_2, +\infty)$ and the unconnected portion $(k_c, (k_2)_{-1})$, where $(k_2)_{-1}$ is the pre-image of k_2, therefore $\mathcal{B}(k_3) = (k_c, (k_2)_{-1}) \cup (k_2, +\infty)$. The basin of

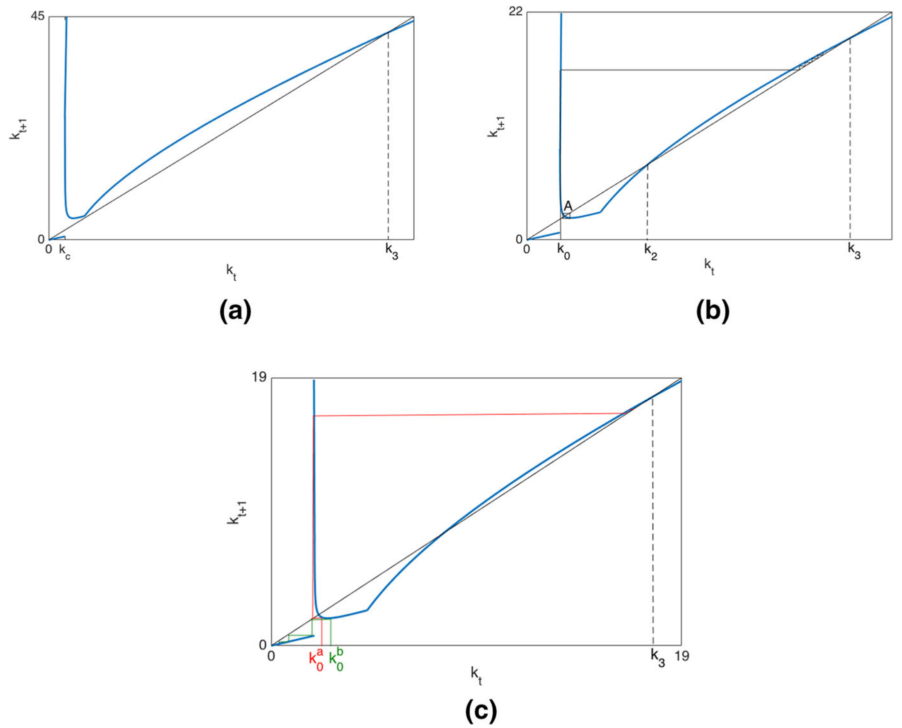

Fig. 7 Parameter values: $\delta = 0.2$, $n = 1.25$ $s_\mathrm{w} = 0.85$, $\alpha = 0.55$, $A = 15$, $k_\mathrm{c} = 2$. **a** $s_\mathrm{r} = 0.3$; **b** $s_\mathrm{r} = 0.1$; **c** $s_\mathrm{r} = 0.085$

attraction of A is given by $\mathcal{B}(A) = ((k_2)_{-1}, k_2)$. Notice that, immediately after the critical level of capital k_c, the initial conditions generate trajectories converging to the upper equilibrium k_3, as shown in Fig. 7b. A further decrease in parameter s_r moves the minimum point $\phi(k_{\min})$ downward, the attractor A loses stability and its basin of attraction $\mathcal{B}(A)$ disappears. Moreover, two contact bifurcations occur: first $\phi^2(k_{\min}) = k_2$ and then $k_\mathrm{c} = \phi(k_{\min})$. After the global bifurcations, non-connected portions of $\mathcal{B}(0)$ and $\mathcal{B}(k_3)$ appear (see panel (c)). In this case, economic policy trying to raise capital per capita in order to reach the positive equilibrium level k_3 may fail and economies may be captured by the poverty trap, as shown in Fig. 7c: different and close initial conditions generate trajectories converging either to the fixed point characterised by zero capital per capita (i.c. k_0^b) or to the positive fixed point (i.c. k_0^a).

5 Conclusions

In this paper we investigated a Solow–Swan growth model with differential saving rates between workers and shareholders [see Böhm and Kaas (2000), Kaldor (1956, 1957) and Pasinetti (1962)] using the shifted Cobb–Douglas production function [see Capasso et al. (2010)], a VES technology that properly describes non-developed, developing and developed economies. The results of our analysis show that fluctuations or even chaotic patterns can be exhibited by the model, confirming the results obtained by Brianzoni et al. (2007, 2009, 2012a, b, 2015): cycles and more complex dynamics may arise if the elasticity of substitution between production factors is smaller than one. As in Brianzoni et al. (2012a), the system may converge to the poverty trap since the origin is always a locally stable fixed point. Furthermore up to three positive fixed points may exist. The model presents multistability phenomena since, if shareholders save more than workers, three positive attractors may exist: a fixed point characterised by zero capital per capita (so-called vicious circle of poverty), a more complex attractor (cycle or chaotic set) and a positive fixed point. The model may present complex basins so that economic policies may fail and economies may be captured by the poverty trap even when a country with sufficiently high level of capital per capita is considered.

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