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Advertising a product to face a competitor entry: a differential game approach

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Abstract
We analyze a market in which advertising is the dominant marketing tool to create market share. We assume that an incumbent firm dominates the market during an initial stage, and that a new competitor is going to enter the market. In particular, we analyze the different advertising policies that the incumbent firm can adopt, before and after the entry of the rival. We explore three possible behaviours. In the first scenario the firm knows that the competitor will arrive at a given instant. In the second one we assume the original firm to be surprised, in the sense that it does not anticipate the entry of the opponent either because it does not expect the competitor to arrive, or it is not prepared to react before the entry takes place. Finally, in the third scenario, the original firm knows that the competitor will enter at a constant rate. We characterize a differential game model and compare the firms' behaviours in a strategic perspective.

Keywords Differential games · Nash equilibria · Marketing · Market entry

JEL Classification C72 · C73 · L13 · D21

1 Introduction

We consider a market in a given mature product category characterized by the presence of an incumbent firm which faces some other minor firms (see Morgan et al. 2009, Ch. 13). The competition for market share is performed by using advertising as the dominant marketing tool. We formalize such an issue as a monopoly, assuming the

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market share of small competitors to be negligible. In Beatty and Samuelson (2016, Ch. 8 p. 1004) it is argued that a monopoly does not only depend strictly on how much market share a firm controls. They rely upon the fact that the real key to identify it, is the capacity that an incumbent has to exclude competitors from the market or to control the prices. Hence, differently from the traditional behaviour of the courts, that considered monopoly a share of the market between 70 and 90%, according to Beatty and Samuelson, modern antitrust law does not care much on market share, especially if a competitor could easily enter the market any time it wants. We account for this interpretation in a way that the incumbent is able to control the market share by its advertising efforts alone. The economical context, in which the transition takes place, is a mature market where the price is fixed while firms compete on market share through advertising.

After that, we assume that a new strong competitor will enter the market selling a similar product. By *strong* we mean that the presence of the new competitor is not negligible, since it affects the market share, and transforms the monopoly into a duopoly. As a matter of fact, the most common situation of competition is given by an oligopoly, where many firms want to sell similar products against many rival companies. However, we limit to study the duopoly, not only because the classical literature on competition follows such an approach, but also because duopoly is a situation that often appears in some specific markets, between two well-known brands. Some of the most famous examples are the eternal rivalry between Coca Cola and Pepsi Cola, the contention between Polaroid and Kodak about their instant colour cameras or the most recent battle between Apple and Samsung for the supremacy on the world of smart phones and tablets.

Kotler and Armstrong (2018, p. 519) affirm that companies must avoid "competitor myopia", because the probability to lose market share against a latent competitor is higher than the probability to lose it because of a well-known competitor. We prove this statement in a theoretical model by defining the following three different scenarios for the incumbent, according to its behaviour with respect to the competitor's market entry.

Non-surprised: The incumbent is aware of the competitor's entry and about the timing. Consequently, it already adapts to the entry in the monopoly stage.

Surprised: The incumbent is not aware that a competitor is about to enter the market. It uses the optimal advertising policy according to a monopolistic market. After having recognized the competitors market entry (either immediately or with some time lag), the incumbent adapts its advertising policy according to the duopoly.

Stochastic: The incumbent is aware of a competitor that intends to enter the market, but is not sure about the timing. It expects the entry at a certain rate over time and includes this information in deriving the optimal policy.

Note that the three scenarios are basically different in the information about the competitor's entry. In the non-surprised case the incumbent has complete information. In the surprised case the incumbent is not aware of the entry and recognizes this event with a time lag. In the stochastic case probably the most realistic one in our model, the incumbent is aware about a possible competitor, but can expect the arrival to occur at some rate.

The objective of this paper is to investigate the behaviour of a first entrant being in an incumbent position for some time and facing a potential competitor for a subsequent stage. In particular the main purpose is to show how different strategies and information can lead to significant differences in economic terms. This analysis demonstrates the importance of the knowledge about potential competitors and their corresponding market entries. We want to focus on the transition between the monopoly and the duopoly and to study how to consider strategically such transition. The entry of the competitor will verify in any case, and the incumbents perception of such an entry is crucial and makes him farsighted or not.

One of the earliest examples of advertising dealing with market share models is the Vidale–Wolfe model (1957), which described the dynamics of the sales rate, expressed as a fraction of the total market. The optimal advertising path for the Vidale–Wolfe dynamics is provided by Sethi (1973) where the author focuses on the monopolist problem but allows for a stochastic component in the evolution of the market share. Nevertheless no interaction between the advertising policies of the two firms is considered there. Bagwell and Ramey (1988) analyzed the situation of the entry of a competitor in an existing duopoly, considering prices as decision variables and their work lies in the stream of static optimization. Subsequent research in the dynamic optimization context extended the basic frameworks to incorporate competitive advertising, based on the Lanchester model of combat (see in Jørgensen and Zaccour 2004, p. 286). This was followed by Sethi (1983) and Prasad and Sethi (2004) who examine the problem where the market share is determined by stochastic disturbances in addition to advertising expenditures. The former formulates and solves a stochastic optimal control problem in infinite time horizon, and the latter presents a stochastic differential game in infinite time horizon. Sorger (1989) uses a special case of the Lanchester model to obtain a duopoly version of the Sethi model.

In the previous models of the related literature either a monopoly or a duopoly situation was considered. To the best of our knowledge no other paper considers the transition between the stages. In our paper we want to focus on the transition between the two stages and to study how the incumbent may consider strategically such (inevitable) transition. We consider a two-stage (monopoly/duopoly) continuous-time model with the advertising effort as the only control variable. Price and cost are equal between the players, which differs from Bagwell and Ramsey stream of works. We do not consider stochastic disturbances in the dynamics, as in Sethi (1983) and Prasad and Sethi (2004), due to our different research question (here the step from monopoly to duopoly is discussed) and for mathematical tractability.

A similar framework has been analyzed in other contexts, as for example by Eliashberg and Jeuland (1986) in an optimal pricing model and by Kort and Wrzaczek (2015) in a capital accumulation model. Gromova and López-Barrientos (2016) analyze the entry of a competitive firm in a resource extraction model.

We formalize the problem of the monopoly stage in terms of an optimal control model using the advertising effort as the incumbent's control variable. In the duopoly stage two decision makers are interacting implying a differential game formulation. We solve the problem backward starting from the second stage. We derive the Markovian

Nash equilibrium[1] of the game, with the aim of comparing the marketing effect of the three different behaviours for the incumbent. In this attempt we assume that both market shares and effort choices are observable by the players. After that we solve the optimal control problem of the monopoly stage linked to the duopoly problem throughout its salvage function.

The paper is organized as follows: In Sect. 2 we formulate the model in the context of dynamic optimization and differential games. In Sect. 3 we present and solve the problem of the non-surprised firm, and in Sect. 4 we present and solve the problem of the surprised firm together with two different reactions the incumbent may have, also after the competitor's entry and after being surprised. Finally in Sect. 5 the stochastic formulation is studied. For all models we find the optimal advertising strategies, which are compared in Sect. 6 where also an economic interpretation of the results is given. Section 7 compares the optimal profits for a specific parameter choice, and Sect. 8 concludes. The proofs of the theorems are provided in "Appendix".

2 The model

In this section we introduce the model step by step. The time horizon is subdivided by the entry of the competitor into two stages. The time of the entry will be referred to as T^e. We denote the stage before and after T^e by *monopoly* and *duopoly* stage, respectively.

2.1 Monopoly stage

Let us denote by Firm 1 the incumbent firm and let $x(t)$ be its market share at time t. Following the model structure as proposed by Sethi (1983), the dynamics of the market share is

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0. \tag{1}$$

Firm 1's control variable $u_1(t) \geq 0$ represents its advertising effort. Parameter $\rho > 0$, originally introduced by Vidale and Wolfe (1957), denotes the advertising efficiency in terms of market share of the two firms, and $\delta > 0$, called *churn* as in Prasad et al. (2012), captures phenomena as product obsolescence, forgetting, lack of market differentiation, lack of information, variety seeking and brand switching (see Tsai and Chen 2010 for details). The nonlinear effect of advertising is a variant of the Vidale and Wolfe model introduced by Sethi in (1983). It can be explained as an additional process of word-of-mouth communication between the individuals comprising the sold portion and those comprising the unsold portion. For a detailed explanation we refer to Dockner et al. (2000, p. 287). Let us assume the following constraint for the initial market share, $x_0 \in [0, 1]$.

[1] Note that the Markovian Nash equilibrium in differential games is equivalently called feedback Nash equilibrium in some texts.

The objective function of the incumbent equals

$$\max_{u_1 \geq 0} \int_0^T e^{-rt}(mx(t) - cu_1^2(t))\mathrm{d}t + e^{-rT}S(x(T)), \tag{2}$$

where $m > 0$ is the unit profit margin, $c > 0$ the cost parameter and $r > 0$ the discounting rate. Note that the cost parameter can have any positive value in general, it is fixed to $c = 1$ and therefore suppressed for the rest of the paper, since only the ratio between m and c matters in the solution.

T denotes the time horizon for Firm 1. The value of T depends on the scenario (non-surprised, surprised, stochastic) we are looking at.

In the non-surprised case, Firm 1 exactly knows the entry time of the competitor (referred to as Firm 2 from now on), so that $T = T^e$. In this case the monopoly model differs from the one in Sethi (1983) because it has a finite time horizon and the salvage function. The salvage function $S(x(T^e))$ represents the value of the state at the end of the time horizon T^e. The link between this salvage function and the initial state of the duopoly stage is justified by the Bellman's optimality principle and will be explained later on.

In the surprised case Firm 1 is not aware of any competitor and believes that the monopoly will hold on forever, i.e. $T = +\infty$. In this case the monopoly model coincides with the deterministic issue of the one in Sethi (1983).

In the stochastic case Firm 1 expects a competitor at a certain rate (for details we refer to Sect. 5), i.e. T is a random variable exponentially distributed with parameter γ. In this case the monopoly model is substantially different with respect to the one in Sethi (1983) because it does not have stochastic dynamics, the stochasticity in the objective function.

2.2 Duopoly stage

At T^e Firm 2 enters the market with an advertising effort $u_2(t) \geq 0$, and the duopoly stage begins. A new state variable $y(t)$ represents the market share of Firm 2, so that the initial condition for y equals the remaining market share, i.e. $y(T^e) = 1 - x(T^e)$. That means that the new Firm 2 has overtaken the market potential of all remaining firms of the monopoly stage, either by overtaking all firms or by a merger. This assumption implies $x(t) + y(t) = 1$ for $t \in [T^e, +\infty)$. We also assume that both firms are symmetric with respect to all parameters. This does not alter the qualitative nature of the results, and it permits to avoid difficult equations. We are interested in comparing the different behaviours that the same incumbent (Firm 1) can have in case it is either surprised or not. A comparison between Firm 1 and Firm 2 is not the focus of our study.

The model structure (after using $x(t) + y(t) = 1$) is the one proposed by Prasad and Sethi (2004) with the dynamics

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \rho u_2(t)\sqrt{x(t)} - \delta(2x(t) - 1), \quad x(T^e) = \lim_{t \to T^e-} x(t). \tag{3}$$

The objective functions of the two firms are analogous. But in contrast to the monopoly stage the time horizon is infinite, i.e.

$$\begin{aligned} &\text{Firm 1:} \qquad \max_{u_1 \geq 0} \int_{T^e}^{+\infty} e^{-rt}\big(mx(t) - u_1^2(t)\big)\mathrm{d}t \\ &\text{Firm 2:} \qquad \max_{u_2 \geq 0} \int_{T^e}^{+\infty} e^{-rt}\big(m(1 - x(t)) - u_2^2(t)\big)\mathrm{d}t \end{aligned} \tag{4}$$

Having presented the dynamics and the objective functions of the monopoly and duopoly stage, the next step is to distinguish accurately the three scenarios already briefly described in the introduction: non-surprised, surprised, stochastic. Here the differences arise due to the assumed information corresponding to entry time T^e and the type of strategic interaction of the opponent.

In the following sections we determine the optimal advertising strategy for the three cases and we compare them in order to analyze the marketing effect of the three different types of behaviour. The superscripts M and D denote the monopoly and duopoly stages. The subscript iNS $(i = 1, 2)$ denotes that the given advertising policy corresponds to the Markov perfect Nash equilibrium solution for Firm i, and the " $*$" denotes that the policy is the optimal one.

3 Non-surprised behaviour

In this scenario we assume that Firm 1 has complete information about the entry of Firm 2, in particular about its entry time, which is assumed to be exogenous. Firm 2 has complete information that Firm 1 perfectly anticipates the entry. Thus we introduce the following assumption.

(A1) $T^e > 0$ is exogenous. Both firms have complete information about T^e and about the opponents optimal policy.

This scenario can be seen as a benchmark case, i.e. *complete information for everybody*. Even if it is not the most realistic case to happen in reality, it is important to learn what would have been the best behaviour in case of complete information and how much has been lost in terms of the objective function.

Using assumption A1 the model reads

- **Monopoly stage**:

$$\text{Firm 1:} \quad \max_{u_1 \geq 0} \int_0^{T^e} e^{-rt}(mx(t) - u_1^2(t))\mathrm{d}t + e^{-rT^e} S(x(T^e)) \tag{5}$$

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0 > 0. \tag{6}$$

- **Duopoly stage**:

$$\text{Firm 1:} \quad \max_{u_1 \geq 0} \int_{T^e}^{+\infty} e^{-rt}(mx(t) - u_1^2(t))\mathrm{d}t \tag{7}$$

$$\text{Firm 2:} \quad \max_{u_2 \geq 0} \int_{T^e}^{+\infty} e^{-rt}(m(1 - x(t)) - u_2^2(t))\mathrm{d}t \tag{8}$$

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \rho u_2(t)\sqrt{x(t)} - \delta(2x(t) - 1), \tag{9}$$

$$x(T^e) = \lim_{t \to T^e -} x(t). \tag{10}$$

At T^e Firm 2 enters as second decision maker, implying the treatment as a differential game. The initial condition of the duopoly stage is exactly the market share of Firm 1 at T^e, and the rest of the market has been taken up by Firm 2. Considering Firm 1 as an incumbent during the first stage, we are assuming that the effect of the other firms in the market can be negligible in terms of market share both before and after the entry of Firm 2.

Since we are interested in a realistic solution without commitment, we derive the Markov perfect Nash equilibrium (see Dockner et al. 2000, chapter 4), where the optimal policies of both players are functions of the state.[2]

The salvage value function of Firm 1 equals the value function of the second stage, i.e. $S(x(T^e)) = V_1(x(T^e)) := \int_{T^e}^{+\infty} e^{-rt}(mx(t) - (u_1^*(t))^2)\mathrm{d}t$. The standard method of deriving a Markov perfect Nash equilibrium is the Hamilton–Jacobi–Bellman (HJB) approach, in which a guess of the value functions (depending on parameters, which have to be derived) of both players is used. Thus by this method we obtain directly the value function for Firm 1, which can be used for the salvage value function of the monopoly stage.

Since the salvage value function is crucial for the solution of the monopoly stage, the model has to be solved backward, i.e. first solving the duopoly stage (and obtaining the salvage value function of the monopoly stage), second solving the monopoly stage.

Theorem 1 (Non-surprised—duopoly stage) *The duopoly stage has a Markov perfect Nash equilibrium in which the optimal advertising efforts are*

$$u_{1NS}^{*D}(x(t)) = \frac{\sqrt{(r + 2\delta)^2 + 3m\rho^2} - (r + 2\delta)}{3\rho}\sqrt{1 - x(t)}, \tag{11}$$

$$u_{2NS}^{*D}(x(t)) = \frac{\sqrt{(r + 2\delta)^2 + 3m\rho^2} - (r + 2\delta)}{3\rho}\sqrt{x(t)}. \tag{12}$$

The market shares converge to $(1/2, 1/2)$ *as* $t \to +\infty$.

The value functions of the two firms are equal to

$$V_1(x) = \alpha + \beta x, \tag{13}$$

$$V_2(x) = \alpha + \beta(1 - x), \tag{14}$$

where α *and* β *are real parameters whose values are specified in "Appendix".*

Proof The derivation of this result is standard, and it is presented in "Appendix", as well as all the proofs of the other theorems.

[2] To identify a Markov perfect Nash equilibrium of the infinite horizon game, note that the model is autonomous. Then it is plausible to look for a stationary Markov perfect equilibrium in which advertising strategies and value functions depend only on the state variable (see Dockner et al. 2000, p. 294).

Note that this solution is not necessarily the unique Markov perfect Nash equilibrium since the standard method used in the proof assumes the choice of value functions. In principle also other value functions could satisfy the HJB equations.

Note that if the market share x of Firm 1 increases, then its optimal advertising effort is reduced; on the other hand, the optimal advertising effort of Firm 2 increases, in order to catch up. So $u^{*D}_{1NS}(x(t))$ is monotonically decreasing while $u^{*D}_{2NS}(x(t))$ is monotonically increasing in x, in fact

$$\frac{\partial u^{*D}_{1NS}}{\partial x} < 0 \quad \text{and} \quad \frac{\partial u^{*D}_{2NS}}{\partial x} > 0. \tag{15}$$

The advertising efforts u^{*D}_{iNS}, $i = 1, 2$, are decreasing in ρ and m, whereas decreasing in c, δ and r. When the effectiveness ρ of the advertising increases, then the amount of advertising increases.

However, this reasoning can not be applied when m, i.e. the sales volume multiplied by the per-unit profit margin, increases or the discount rate r decreases. In these cases, the wasteful advertising is increased, but the size of the pie is increased too. Finally the churn parameter δ reduces competitive intensity.

Considering next the monopolistic stage, we recall that the incumbent problem is a finite time horizon optimal control problem, which will be solved with Pontryagin's Maximum Principle. The salvage value function $S(x(T))$ represents the value of the objective function in the final state of the monopoly stage. This can be considered as the connection point between both stage. In the derivation of the Markov perfect Nash equilibrium (see "Appendix") we use a linear guess for the value function. Therefore,

$$S(x(T^e)) = V_1(x(T^e)) := \int_{T^e}^{+\infty} e^{-rt}(mx(t) - u_1^2(t))\mathrm{d}t = \alpha + \beta x(T^e), \tag{16}$$

where α and $\beta > 0$ are given in (36) and (37). □

Theorem 2 (Non-surprised—Monopoly stage) *Let (A1) hold. Then the optimal advertising effort of the non-surprised incumbent in the monopoly stage is*

$$u^{*M}_{1NS}(t) = \frac{N(t)}{\rho}\sqrt{1 - x(t)}, \tag{17}$$

where

$$N(t) = R - U - 1 + \frac{e^{R(T_e - t)}(-2(R+U) - \beta\rho^2)}{2(R-U) - \beta\rho^2} > 0$$

and

$$U = \delta + r, \quad R = \sqrt{U^2 + m\rho^2}.$$

Finally, summing up the behaviour in the non-surprised case, Firm 1 behaves according to Theorem 2 in the monopoly stage. Since the entry time T^e of Firm 2 is known, Firm 1 derives an optimal control model with a finite time horizon and the corresponding transversality condition derived from (16) (see also "Appendix"). In the duopoly stage both firms behave symmetrically according to Theorem 1.

With this result it is possible to completely derive the optimal advertising policy of Firm 1 over the whole time horizon, i.e. from 0 up to infinity.

4 Surprised behaviour

In the surprised behaviour we assume that Firm 1 either is not aware that a second firm is about to enter the market, or it is not able to take it into account, due to internal constraints/limits. Thus it continues to behave as it still were an incumbent, i.e. solving its own infinite time horizon optimal control model without anticipating (neither explicitly nor stochastically) any change of the market structure. For this case we assume

(A2) During the monopoly stage Firm 1 has no information about the entry of Firm 2. Right after T^e Firm 2 behaves according to the Markov perfect Nash equilibrium.

As a result, the optimization problem of Firm 1 in the monopoly stage is an infinite horizon optimal control problem. It does not take into account that a rival company can begin to produce and sell a similar product. Taking $T = +\infty$ in (2) we get

$$\text{Firm 1:} \quad \max_{u_1 \geq 0} \int_0^{+\infty} e^{-rt}(mx(t) - u_1^2(t))\mathrm{d}t \tag{18}$$

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0 > 0. \tag{19}$$

The following theorem derives the optimal advertising strategy for the surprised incumbent.

Theorem 3 (Surprised—monopoly stage) *Let (A2) hold. The incumbent optimal advertising effort is*

$$u_{1S}^{*M}(t) = \frac{R - U}{\rho}\sqrt{1 - x(t)}, \tag{20}$$

where R and U are the same as in Theorem 2

Inserting (20) in (1) we obtain an analytic expression for the incumbent's market share $x_{eS} := x(T^e)$ at the competitor's entry T^e, i.e.

$$x_{eS} = 1 - e^{-(R-r)T^e}(1 - x_0) - \frac{1 - e^{-(R-r)T^e}}{R - r}\delta. \tag{21}$$

This is the market share of Firm 1 at the entry of the competitor (beginning of the duopoly stage) for each of the following two subcases, which are dealt with in the following.

After the entry of Firm 2 the market structure changes to a duopoly. At that instant Firm 1 might either suddenly realize that such an event has taken place and so that it reacts, or it might need some time to adapt to the new situation, and it changes its advertising policy with a certain time lag. In the latter case Firm 2 in turn might react either before the incumbent reaction or after it. To account for these additional possibilities we assume two more subcases:

Surprised immediate reaction: After Firm 2's market entry Firm 1 immediately reacts and behaves according to the Markov perfect Nash equilibrium.
Surprised time lag: After Firm 2's market entry Firm 1 does not react immediately, but with some time lag, more specifically at $T^R > T^e$ Firm 1 reacts according to the Markov perfect Nash equilibrium.

Summing up, T^e denotes Firm 2's market entry and at time $T^R \geq T^e$ Firm 1 reacts (according to the Markov perfect Nash equilibrium).

4.1 Surprised immediate reaction

At time T^e Firm 1 realizes that the competitor has entered the market. Thus its advertising effort for the duopoly stage is set according to the Markov perfect Nash equilibrium, i.e. $u_{1NS}^{*D}(x(t))$ given in (11). The competitor advertises à la Markov perfect Nash too, i.e. $u_{2NS}^{*D}(x(t))$ as in (12). The incumbent's market share at t, for $t \in (T^e, +\infty)$, can be obtained by integrating (3) (from T^e to t) with initial condition $x(T^e) = x_{eS}$ in (21).

4.2 Surprised time lag

After T^e Firm 1 continues to advertise with surprised monopoly intensity $u_{1S}^{*M}(t)$ given in (20). This is not realized by Firm 2, that already behaves à la Markov perfect Nash given by $u_{2NS}^{*D}(x, t)$ as in (12). At $T^R > T^e$ Firm 1 reacts to the new market situation and changes its strategy also to the Markov perfect Nash equilibrium given by $u_{1NS}^{*D}(x, t)$ given in (11). The incumbent's market share can be obtained as before by using the corresponding advertising policies for both firms.

During the inertia stage the players do not have a rational behaviour, neither the incumbent (that is surprised) nor by the competitor, which continues to use its Nash strategy even though the incumbent is not playing à la Nash. Possibly the competitor will react and adopt a best response to the surprised behaviour of the incumbent in order to force him go to a lower market share. This will obviously penalize Firm 1; so that we omit such an analysis because it will definitely not bring Firm 1 to the greater profit.

5 Stochastic behaviour

The probably most realistic way to consider the entry of a potential competitor is a stochastic approach, in which the incumbent assumes a certain entry rate. For this case we make the following assumption

(A3) Let T^e be a random variable, which follows an exponential distribution with parameter γ.

Consequently, γ can be understood as the conditional probability density for the entry of Firm 2 at t, given that Firm 2 has not already entered the market.

Having this definition in mind, Firm 1 considers the following stochastic optimal control problem with random time horizon

$$\max_{u_1 \geq 0} \quad \mathbb{E}\left[\int_0^{T^e} e^{-rt}(mx(t) - u_1^2(t))\mathrm{d}t + e^{-rT^e} S(x(T^e))\right] \tag{22}$$

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0 > 0. \tag{23}$$

In contrast to the non-surprised case (see Sect. 3) Firm 1 does not know T^e, but it can expect it at the rate γ. This fact is taken into account by using a random time horizon. It means that Firm 1 behaves according to that optimal control model as long as Firm 2 has not entered. If Firm 2 enters the market (at T^e), it adapts its strategy immediately and follows the Markov perfect Nash equilibrium solution. Therefore again the value function of the duopoly stage [i.e. the value function of the Markov perfect Nash equilibrium presented in (16)] is used for the salvage value function.

This stochastic optimal control problem can be reformulated equivalently as a deterministic infinite time horizon optimal control problem. Firstly used in the seminal work by Kamien and Schwartz (1970), this trick has been formalized by Boukas et al. (1990), as follows (we are already making use of $S(x) = V_1(x)$)

$$J(u(\cdot)) = \int_0^{+\infty} \left\{\int_0^t e^{-rs}(mx(s) - u_1^2(s))\mathrm{d}s + e^{-rt} V_1(x(t))\right\} e^{-\gamma t}\gamma \mathrm{d}t, \tag{24}$$

where $e^{-\gamma t}\gamma$ is the probability that the switch has still not occurred times the probability at which it does occur exactly at t. Integrating by parts we obtain the following optimal control problem for Firm 1

$$\max_{u_1 \geq 0} \quad \int_0^{+\infty} e^{-(r+\gamma)t}(mx(t) - u_1^2(t) + \gamma V_1(x(t)))\mathrm{d}t \tag{25}$$

$$\dot{x}(t) = \rho u_1(t)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0 > 0. \tag{26}$$

The following theorem presents the optimal behaviour of Firm 1 as long as Firm 2 has not entered the market (i.e. optimal behaviour for $t < T^e$).

Theorem 4 (Stochastic—monopoly stage) *Let (A3) hold. Firm 1's optimal advertising effort is given by*

$$u_{1St}^{*M}(t) = \frac{K}{\rho}\sqrt{1 - x(t)}, \tag{27}$$

where

$$K = \sqrt{(U + \gamma)^2 + \rho^2(m + \gamma\beta)} - (U + \gamma) \tag{28}$$

and

$$\beta = \frac{\sqrt{(r+2\delta)^2 + 3m\rho^2} - (r+2\delta)}{3\rho^2/2}. \tag{29}$$

The steady state is

$$\hat{x}_{St} = 1 - \frac{\delta}{\sqrt{(U+\gamma)^2 + \rho^2(m+\gamma\beta)} - (r+\gamma)}.$$

The stochastic case can be considered as an intermediate case between being surprised and being non-surprised. It depends on the value of γ.

If $\gamma \to 0$, Firm 1 assumes that Firm 2 never enters the market. Thus the optimal solution equals that of the surprised case if Firm 2 would never enter the market. This is reasonable, since in the surprised case it is assumed that Firm 2 never enters the model and that the monopoly maintains forever. If $\gamma \to 1$, the behaviour of the stochastic case is not similar to the behaviour of the non-surprised case: in fact, even if the firm knows that the competitor will enter certainly in the market, it does not know the exact instant at which this will happen.

After Firm 2's entry, ($t \geq T^e$), both firms are in duopoly and follow the Markov perfect Nash equilibrium strategy. The optimal behaviour is presented in Theorem 1.

6 Solutions' comparison

In this section we analyze and emphasize how the different awareness about the entry of a competitor can lead to significantly different advertising strategies. More precisely, the incumbents firms perceptions of the arrival of the opponent are different in the three scenarios, i.e. deterministic (surprised or non-surprised) or stochastic. The comparison is performed among the advertising efforts, the state trajectories and the steady states of Firm 1.

Comparison of advertising efforts

Let us first consider the monopolistic stage, the advertising efforts in the three scenarios are

$$u_{1S}^{*M}(x(t)) = \frac{R-U}{\rho}\sqrt{1-x_M(t)},$$

$$u_{1NS}^{*M}(x(t)) = \frac{N(t)}{\rho}\sqrt{1-x_N(t)},$$

$$u_{1St}^{*M}(x(t)) = \frac{-(U+\gamma) + \sqrt{(U+\gamma)^2 + \rho^2(m+\gamma\beta)}}{\rho}\sqrt{1-x_S(t)}.$$

Note that the controls, considered in their Markov perfect form as function of the market share, are decreasing and concave since ($j \in \{S, NS, St\}$)

$$\frac{\partial u_{1j}^{*M}}{\partial x} < 0 \quad \text{and} \quad \frac{\partial^2 u_{1j}^{*M}}{\partial x^2} < 0.$$

It means that during the monopoly stage the optimal advertising efforts are decreasing in the market share. When the market share tends to 1, i.e. the whole market, no advertising is necessary. The smaller the market share, the higher the optimal advertising expenditures.

Furthermore the following Proposition holds.

Proposition 1 *Given a fixed level of the market share x, during the monopoly stage, the surprised advertising expenditures are higher than the ones in the non-surprised case and higher than the ones of the stochastic case, i.e.*

$$u_{1NS}^{*M}(x) \leq u_{1S}^{*M}(x), \quad \textit{and} \quad u_{1St}^{*M}(x) \leq u_{1S}^{*M}(x). \tag{30}$$

Comparison of state trajectories

The non-surprised firm is farsighted, and it perfectly predicts the exact moment at which the second firm enters in the market. During the monopoly, its market share is smaller than the market share perceived by the surprised firm. However, it is completely prepared for the duopoly.

During the monopoly stage the following Proposition holds.

Proposition 2 *Given the same initial market share $x(0) = x_0$ for all three cases, the following relation holds during the monopoly stage.*

$$x_{NS}(t) \leq x_S(t), \quad \textit{and} \quad x_{St}(t) \leq x_S(t), \qquad \forall\ t \in [0, T^e].$$

Furthermore, the greater γ, in $x_{St}(t)$ the smaller market share of the stochastic case with respect to the market share perceived by the surprised firm.[3]

The surprised firm spoils its energy during the monopoly stage and increases its market share, which turns out to be higher both with respect to the one of the non-surprised firm and to the one of the stochastic case.

That means that the incumbent holds the highest market share during the monopoly stage if it is completely uninformed. The more information it has the lower the market share is. The reason for that is that the competing firm can advertise more efficiently if the market share of the incumbent is high, while the incumbents' own advertising efforts are less efficient.

Comparison of steady states

Even though the objective of the players is the profit maximization and though the steady states will not be reached in finite time, it turns out interesting to compare the steady states during the both stages.

The non-surprised, surprised and the stochastic steady states during the monopoly stage are, respectively,

$$\hat{x}_S = \hat{x}_{NS} = 1 - \frac{\delta}{R - r}, \qquad \hat{x}_{St} = 1 - \frac{\delta}{R_{St} - (r + \gamma)} < \hat{x}_S,$$

[3] Note that in fact, the more γ increases the more the firm becomes less surprised.

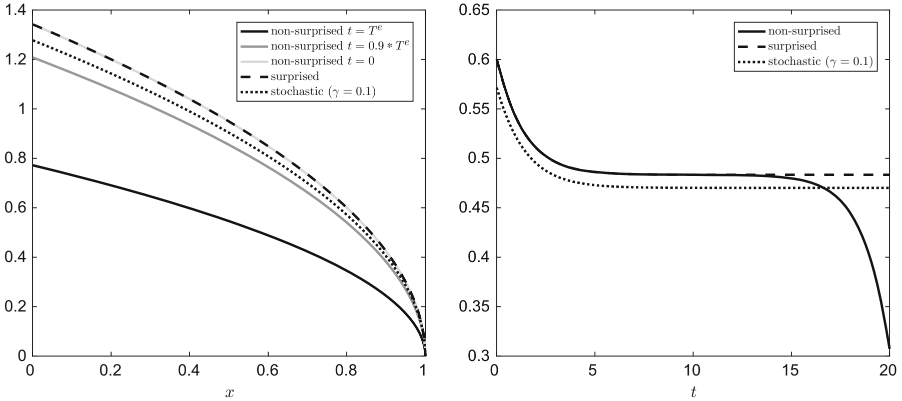

Fig. 1 Optimal advertising efforts of Firm 1 in the monopoly stage depending on the state value (left panel) and over time (right panel)

where

$$R = \sqrt{U^2 + m\rho^2}, \quad R_{St} = \sqrt{(U + \gamma)^2 + \rho^2(m + \gamma\beta)}.$$

The steady state of the duopoly is obviously equal to $1/2$ because of the symmetry of the two firms.

The surprised and the stochastic cases differ in the presence of rate of arrival term γ, and they coincide when γ tends to 0. In fact in this case the probability that a switch will happen turns out to be negligible, and this is exactly the situation of the surprised monopoly. Let γ be different from 0, then the following theorem holds. The steady states of the surprised and the non-surprised monopoly coincide, as the corresponding optimal control problems only differ in the time horizon.

Numerical example

Having derived the relations of optimal advertising efforts and optimal trajectories analytically, we now plot an example for the following parameter set:

$$\delta = 0.1, \quad m = 2.5, \quad \rho = 0.5, \quad r = 0.03, \quad T^e = 20.$$

In Fig. 1 we plot the optimal advertising efforts and in Figs. 2 and 3 the optimal state trajectories of all different cases. We assume the initial market share of the incumbent as $x_0 = 0.8$.

In the left panel of Fig. 1 the advertising efforts are plotted for all three cases depending on the market share. Considering the non-surprised case in the left panel one also has to look on different t, since the advertising efforts depends on $N(t)$ (and thus on t). It is obvious that the strategy is nearly equal to the surprised case, but deviates when it approaches T^e. The stochastic case is always below the surprised case, but higher than the non-surprised when it approaches T^e. On the right panel the advertising efforts are plotted over time ($t \in [0, T^e]$), and we see a similar thing. The stochastic case is below the surprised one. And the non-surprised is equal (i.e.

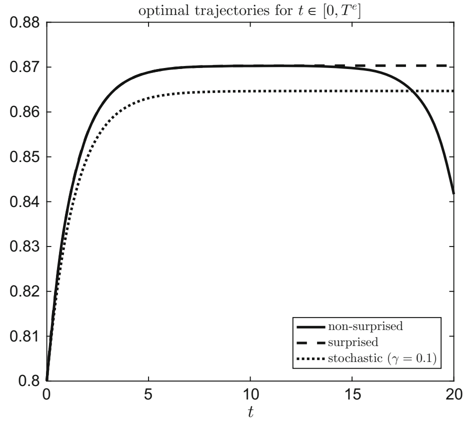

Fig. 2 Optimal trajectories of Firm 1 before Firm 2's market entry over time ($t \in [0, T^e]$)

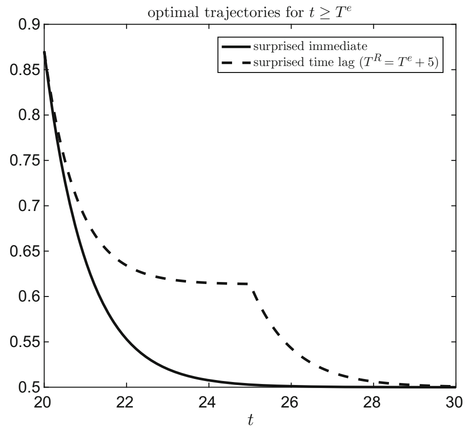

Fig. 3 Optimal trajectories of the surprised case of Firm 1 after Firm 2's market entry over time ($t \in [T^e, T^R]$)

nearly equal) to the non-surprised one for a long time and deviates at the end. Both relationships have been analytically derived in Proposition 1.

In Fig. 2 we plot the stage before Firm 2's market entry, i.e. $t \in [0, T^e]$. In the monopoly case we can prove that $\dot{x} = N - (N + \delta)x$ and therefore there exist a unique peak in the optimal trajectory at $x = \frac{N}{N+\delta}$, as shown in Fig. 2.

As we have shown in Theorem 2 the market share of the surprised case is the highest, whereas a relation beyond the non-surprised and the stochastic cannot be shown. It turns out that the optimal market share of the non-surprised case follows the surprised one (which converges to the steady state in the long run) until $t \approx 15$ and anticipates then the market entry of Firm 2 (decreases). The optimal trajectory of the stochastic case converges to the steady state of the stochastic case, which is smaller than the ones of the surprised and of the non-surprised cases. However, between $t = 15$ and $t = T^e$ the anticipation of the non-surprised case is so strong that the optimal trajectory of the non-surprised case gets below the one of the stochastic case. This confirms the result of Theorem 2, i.e. the two trajectories of the non-surprised and the stochastic case cannot be ranked pointwise.

In Fig. 3 we plot the optimal trajectories of the surprised case after Firm 2's entry. The non-surprised case and the stochastic case are both straightforward since in both cases the firm behaves according to the Markov perfect Nash equilibrium and approaches the steady state $\hat{x}^D_{NS} = 0.5$. In the surprised case we see that the optimal market share of Firm 1 is much greater in case of a time lag than without time lag (in the figure we have used $T^R = T^e + 5 = 25$). The reason is due to the asymmetry in the information. Firm 2 has full information and behaves immediately according to the Markov perfect Nash equilibrium, whereas Firm 1 has a lack of the information on the market structure (or is unable to react immediately) and does not include strategic interaction. Thus Firm 1 has too high advertising efforts. Note, that the time lag scenario is not sustainable for a long time-stage since it is not a game theoretic equilibrium, meaning that it relies on the assumption that firm 1 cannot react immediately on Firm 2's market entry. After this short period until T^R either Firm 1 will react (by behaving according to the Markov perfect Nash equilibrium) or Firm 2 will react on Firm 1's behaviour (possibly playing its best response). We implemented the first idea rather than the second one, because it is more realistic that Firm 1 realizes the new market situation and adapts the behaviour than that Firm 2 realizes Firm 1's time lag and switches to another strategy which is also not a game theoretic equilibrium. In principle a couple of different reactions are possible and interesting, but may be analyzed in an extension.

7 The optimal profits

Profit is the objective of the players involved in the game. So that, in order to compare the three different behaviours for the incumbent, we must compute its optimal objective function evaluated with its optimal strategy in the three studied scenarios. This can be done by substituting the optimal strategies and the related state functions in (5), (7) and in (18).

Due to the long analytical formulations of the optimal profit, we performed a numerical simulation. Here we present some results which have been obtained by using the same parameter values as in the previous section, i.e. $\delta = 0.1$, $m = 2.5$, $\rho = 0.5$, $r = 0.03$, $x_0 = 0.8$, nevertheless several simulations with different scenarios confirm that the solution is not sensitive to the value of the parameters.

Let us first consider the non-surprised case, i.e. the benchmark case. Figure 4 plots the profit of Firm 1, i.e. objective function (5). Since the profits of the incumbent are higher in the first stage compared to the competitive situation of the second stage (higher market share in the first stage), the profit is increasing in T^e.

Next we turn to the surprised case. Here Firm 1 behaves equally in both cases of the first stage (surprised immediate reaction, surprised time lag), but differently in the second one (compare also Fig. 3). The left panel of Fig. 5 shows the profit of Firm 1 in the first stage depending on T^e. Naturally it turns out that it is strictly increasing and has a concave shape.

The differentiation of the surprised immediate and the time lag subcases occurs at Firm 2's market entry at T^e. The profit of the second stage adds additionally to the already earned profit of the first stage (left panel of Fig. 5). Recalling that the surprised

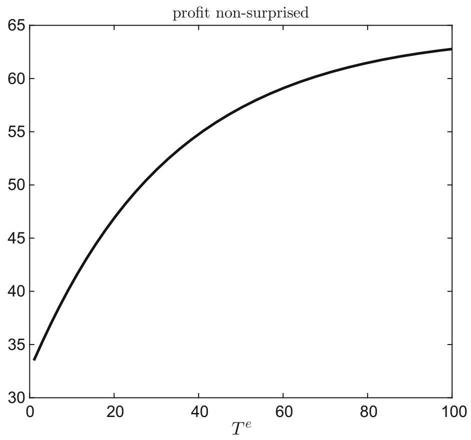

Fig. 4 Optimal profit of Firm 1 in the non-surprised case depending on T^e

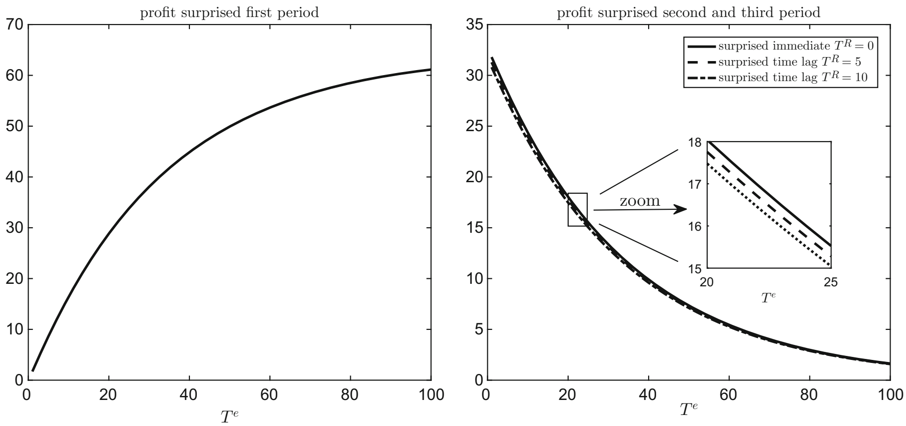

Fig. 5 Profit of Firm 1 in the surprised case depending on T^e for the first period (left panel) and the second and third period (right panel)

immediate case is a special case of the surprised time lag case with a zero time lag, we can compare the three cases $T^R = T^e$, $T^R = T^e + 5$ and $T^R = T^e + 10$ plotted in the right panel of Fig. 5 (a zoom is included for a better visualization of the differences between the three cases). The numerical results imply that for Firm 1 it is better the shorter the time lag is, i.e. the earlier it reacts correctly to the new market situation the better it will be. A time lag means that Firm 1 behaves as a monopolist although the market is already a duopoly (Firm 2 already adapted the Markov perfect Nash equilibrium strategy). Therefore, advertising efforts are too high and too expensive, which implies lower profits. This asymmetry of the information is not an equilibrium in a game theoretic sense.

Finally, the profit of the stochastic case is derived. Figure 6 plots the profit of the incumbent during the first stage depending on T^e for $\gamma = 0.1$ (solid line) and $\gamma = 0.5$ (dashed line). The steady-state market share (of the stochastic case) depends negatively on γ, i.e. the earlier firm 2 is expected to enter the smaller the steady-state market

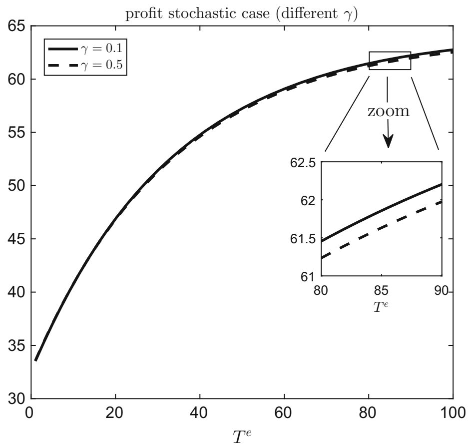

Fig. 6 Optimal profit of Firm 1 in the stochastic case depending on T^e for different γ

Table 1 Relative profit loss on the model parameters

	T^e	δ	m	ρ	T^R
Surprised immediate	↘	↗	↘	↘	
Surprised time lag	↘	?	?	?	↗
Stochastic ($\gamma = 0.1$)	↗	↗	↗	↗	

share is. The relation between the two cases is not monotonous, which can be seen by comparing the two zooms in Fig. 6. For very low T^e the profit derived with $\gamma = 0.5$ is better, whereas for higher T^e the profit with $\gamma = 0.1$ is better. This is implied by the fact that an early (late) expected market entry of Firm 2, i.e. low (high) T^e, is reflected by a high (low) γ.

Deriving and comparing the resulting total profits (i.e. sum of the profits of the first, time lag and second stage; $T^e = 20$, $T^R = 5$ and $T^R = 10$, and $\gamma = 0.1$ and $\gamma = 0.2$) allow for a couple of insights:

– The incumbent earns the highest profits in the non-surprised case, which is implied by the model setup (complete information about the entry of Firm 2), followed by the stochastic and the surprised case. The stochastic case has the advantage that Firm 2's entry is expected at a certain entry rate γ implying also a higher profit compared to the surprised case.
– Considering the profit of the monopoly stage only reveals that the higher profit of the non-surprised case is earned before Firm 2 enters the market. Briefly before T^e it is optimal for the incumbent to reduce the advertising efforts (see again Fig. 2) and to skim the profit of the high market share. Mathematically this comes from the transversality condition (43) of the non-surprised case that gives a lower value for the adjoint variable λ (compared to the surprised and the stochastic case).

Finally we derive the relative profit loss (i.e. lower profit compared to the non-surprised case) of the surprised immediate, surprised time lag and stochastic case for varying parameters. The results are summarized in Table 1.

T^e: The higher time horizon of the monopoly stage, the lower is the relative profit loss for the surprised immediate and surprised time-lag case. This is due to the turnpike property, i.e. a finite time solution (of an optimal control model) approaches the steady state within the time horizon and gets away at the end. For the stochastic case that does not apply since the incumbent approaches a different equilibrium during the monopoly stage. Thus the relative profit loss increases with T^e.

δ: The higher the δ, the higher the relative profit loss for the surprised immediate and the stochastic case will be. A higher δ means that the market share is lost much faster and that it is more difficult to adapt for a new unexpected market situation. In the surprised time-lag scenario the dependence is ambiguous and depends also on the T^R.

m: If the unit profit margin increases, then the relative profit loss decreases in the non-surprised immediate case since the current market share becomes more important in contrast to anticipation. In the stochastic case the dependence is the other way around. The profit margin pushes the stochastic equilibrium further away from the non-surprised one and thus increases the relative profit loss. In the surprised time-lag scenario the dependence is ambiguous.

ρ: The dependence on ρ is equal to the dependence of m. The reason is that if the advertising efficiency is higher, then the incumbent can react better (i.e. cheaper due to quadratic marketing costs) on the new unexpected market situation.

T^R: The profit of the surprised time-lag scenario is relatively smaller with respect to the one of the non-surprised scenario the longer the time lag is (i.e. the higher T^R). In this stage Firm 1 still behaves as it were a monopolist, whereas Firm 2 already adapts the Markov perfect Nash equilibrium strategy. This asymmetry implies that Firm 1's advertising efforts are too costly (i.e. as high expenditures as in the first stage, but less effective), and therefore it loses profit. The longer the time lag takes, the more profit is lost.

8 Conclusions

We have considered three different behaviours that an incumbent firm can adopt facing the entry of a competitor in its market. The firm might not be aware, or perfectly predict or assume at some rate γ such an event.

The topic of this paper is to study how these behaviours influence his total profits, and the results confirm that the anticipation of Firm 2's market entry increases the profit of the incumbent. So that this confirms the importance of a farsighted attitude and of the necessity for the incumbent to be constantly updated about the market he is dealing in. Even if this means having a lower market share. Furthermore a comparison between immediate and lagged setting has been performed and, in case the event of the entry of the competitor takes the incumbent by surprise, it ends up convenient to react as soon as possible. This is due to the high competition and the lower profit of the second stage which carries over to the first stage. The stochastic case is an intermediate case, but even if γ tends to 1, the firm has an incomplete information because it does not know the exact instant at which the switch will happen, and in this way its market share decreases.

Of course the proposed model can be extended in many directions. First of all in the surprised case a best response strategy of Firm 2 should be considered, as already briefly discussed in Sect. 6. Secondly, the model should be extended to oligopoly markets. We expect that the result will carry over for a small number of firms, but get smaller the more firms are involved in the market.

In addition it is important to look at the case of asymmetric firms, which can lead to strategically different results. Further on it is important to incorporate other decision variables such as price. A realistic approach for the stochastic case would be to consider a rate γ depending on the R&D investments of the competitor or T^e as a decision variable set by a third player, e.g. the governor.

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A Proof of Theorem 1

The non-surprised case in duopoly stage corresponds to the symmetric case presented and solved in Prasad and Sethi (2004). The optimal Markovian advertising decisions are then

$$u_1^*(x) = V_1'(x)\rho\sqrt{1-x}/2, \tag{31}$$
$$u_2^*(x) = -V_2'(x)\rho\sqrt{x}/2. \tag{32}$$

where the value functions have the following forms:

$$V_1 = \alpha_1 + \beta_1 x, \quad V_2 = \alpha_2 + \beta_2(1-x) \tag{33}$$

and the optimal coefficients $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$ are determined by solving the following equations

$$r\alpha = \beta^2\rho^2/4 + \beta\delta, \tag{34}$$
$$r\beta = m - 3\beta^2\rho^2/4 - 2\beta\delta. \tag{35}$$

There exist two solutions for β with opposite sign, the negative one has to be rejected because it would make (31) negative, and this would not satisfy the non-negativity constraint for the control. So that we obtain

$$\alpha = \frac{(r-\delta)(r+2\delta-\sqrt{(r+2\delta)^2+3m\rho^2}) + 6m\rho^2/4}{18\rho^2 r/4}, \tag{36}$$
$$\beta = \frac{\sqrt{(r+2\delta)^2+3m\rho^2}-(r+2\delta)}{3\rho^2/2}. \tag{37}$$

It is easy to check that $\beta > 0$ represents the weight of the final goodwill and, as similarly to α, it depends on the parameters ρ, m, δ and r. The optimal Markovian advertising decisions give (11) and (12).

In order to find the steady state we put

$$\dot{x}(t) = \beta\rho^2(1 - x(t))/2 - \beta\rho^2 x(t)/2 - \delta(2x(t) - 1)) = 0 \tag{38}$$

that gives $\hat{x}^D_{NS} = 0.5$, and symmetrically for player 2. Equation (38) is linear in x with a negative coefficient, and therefore the obtained steady state is stable. So that the market share converges to the steady state for any initial value. Moreover, since $\ddot{x} = -2(\beta\rho^2/2 + \delta)\dot{x}$, the market share is either increasing and concave (if $x_0 < x_{SS}$) or decreasing and convex, (if $x_0 > x_{SS}$) and the trajectory cannot have multiple peaks.

Following Dockner et al. (2000, p. 100) the Markovian Nash equilibrium is subgame perfect, and therefore called Markov perfect.

B Proof of Theorem 2

The current value Hamiltonian function associated to the non-surprised incumbent problem in the monopoly stage is

$$H(x, u_1, \lambda, t) = mx - u_1^2 + \lambda(\rho u_1\sqrt{1 - x} - \delta x). \tag{39}$$

Maximizing the Hamiltonian function w.r.t. control we obtain

$$u^M_{1NS}(t) = \frac{\lambda(t)\rho\sqrt{1 - x(t)}}{2}. \tag{40}$$

Since the Hamiltonian function is strictly concave w.r.t. control $u^M_{1NS}(t)$ turns out to be a unique Maximum. Thus the advertising decision satisfying the Maximum Principle conditions is

$$u^{*M}_{1NS}(t) = \max\left\{0, \frac{\lambda(t)\rho\sqrt{1 - x(t)}}{2}\right\}. \tag{41}$$

The costate equation is

$$\begin{aligned}\dot{\lambda}(t) &= -\frac{\partial H}{\partial x}(t) + r\lambda(t)\\ &= -m + \lambda(t)\rho u_1(t)\frac{1}{2\sqrt{1 - x(t)}} + \lambda(t)\delta + r\lambda(t)\\ &= -m + \frac{\rho^2\lambda^2(t)}{4} + (\delta + r)\lambda(t)\\ &= \frac{\rho^2}{2}\lambda^2(t) + U\lambda(t) - m,\end{aligned} \tag{42}$$

where (41) has been substituted and $U = \delta + r$, $R = \sqrt{U^2 + m\rho^2}$ has been used. Recalling the salvage value function we obtain the following transversality condition

$$\lambda(T) = \frac{\partial S(x(T))}{\partial x} = \beta > 0 \tag{43}$$

which can be used to solve the above costate equation, i.e.

$$\lambda(t) = \frac{2\left(\beta\rho^2\left((R+U)e^{R(t-\mathrm{Te})} + R - U\right) + 2(R-U)(R+U)\left(1 - e^{R(t-\mathrm{Te})}\right)\right)}{\rho^2\left(\beta\rho^2 + e^{R(t-\mathrm{Te})}\left(2(R-U) - \beta\rho^2\right) + 2(R+U)\right)}. \tag{44}$$

Observe that because of the transversality condition the costate function is positive at the final time and whenever it is null its derivative is negative, in fact

$$\dot{\lambda}(t)|_{\lambda(t)=0} = -m < 0,$$

so that it follows that $\lambda(t)$ is positive for all $t \in [0, T]$. Therefore the optimal control $u_{1NS}^{*M}(t)$ in (41) is strictly positive for all t and it becomes (17).

Observe that the maximized Hamiltonian H^*

$$H^*(x, \lambda, t) = mx(t) - \frac{\lambda^2(t)\rho^2(1-x(t))}{4} + \frac{\lambda^2(t)\rho^2(1-x(t))}{2} - \lambda(t)\delta x(t), \tag{45}$$

turns out to be linear with respect to x, as well as the salvage function, so the sufficiency Arrow Theorem holds and the Maximum Principle conditions are also sufficient for optimality. Moreover, note that (41) and (44) imply that the solution of the optimal control problem is unique.

C Proof of Theorem 3

The current value Hamiltonian function associated to the surprised incumbent is

$$H(x, u_1, \lambda, t) = mx - u_1^2 + \lambda(\rho u_1\sqrt{1-x} - \delta x). \tag{46}$$

Analogously to the proof before we are able to derive the optimal control value, which is

$$u_{1S}^{*M}(t) = \max\left\{0, \frac{\lambda(t)\rho\sqrt{1-x(t)}}{2}\right\}. \tag{47}$$

Strict concavity of the Hamiltonian implies that $u_{1S}^{*M}(t)$ is a unique Maximum.

For the costate equation we obtain [substituting (47) in (48) and $U = \delta + r$]

$$\begin{aligned}\dot{\lambda}(t) &= -\frac{\partial H}{\partial x}(t) + r\lambda(t) = -m + \lambda(t)\rho u_1(t)\frac{1}{2\sqrt{1-x(t)}} + \lambda(t)\delta + r\lambda(t)\\ &= \rho^2\lambda(t)^2/4 + U\lambda(t) - m.\end{aligned} \tag{48}$$

Since the model has infinite time horizon the limiting transversality condition reads

$$\lim_{t\to\infty} e^{-rt}\lambda(t) = 0. \tag{49}$$

The steady state of the canonical system $(\dot{x}, \dot{\lambda})$ can uniquely be derived, i.e.

$$\hat{x}_S = 1 - \frac{\delta}{R - U + \delta}$$
$$\hat{\lambda}_S = \frac{-U + \sqrt{U^2 + m\rho^2}}{\rho^2/2} = \frac{(R - U)2}{\rho^2}, \tag{50}$$

where $R = \sqrt{U^2 + m\rho^2}$ and $\hat{x}_S \in [0, 1]$. Its corresponding Jacobian matrix (evaluated in the steady state) is

$$J(x, \lambda) = \begin{pmatrix} \lambda\rho^2/2 + \delta + r & 0 \\ \rho^2(1 - x)/2 & -\rho^2\lambda/2 - \delta. \end{pmatrix} \tag{51}$$

In the steady state $\det(J(\hat{x}_S, \hat{\lambda}_S)) < 0$ holds, therefore $(\hat{x}_S, \hat{\lambda}_S)$ is a saddle point. According to standard optimal control theory the optimal solution follows the stable trajectory of the (unique) saddle point equilibrium (see Grass et al. 2008, section 3).

Analogously to the proof before the maximized Hamiltonian is linear in x, thus the Arrow sufficiency Theorem holds and implies that the conditions of the Maximum Principle are sufficient. Furthermore the solution of the optimal control problem is unique.

D Proof of Theorem 4

Recalling that $V_1(x(t)) = \alpha + \beta x(t)$, the Pontryagin's Maximum Principle for the infinite time horizon stochastic problem (25) and (26) gives the current value Hamiltonian function

$$H(x, u_1, \lambda, t) = mx - u_1^2 + \gamma(\alpha + \beta x) + \lambda(\rho u_1\sqrt{1 - x} - \delta x). \tag{52}$$

Analogously to the proof of Theorem 2 before we are able to derive the optimal control value, which is

$$u_{1St}^{*M}(t) = \frac{\lambda(t)\rho\sqrt{1 - x(t)}}{2}. \tag{53}$$

Strict concavity of the Hamiltonian implies that $u_{1St}^{*M}(t)$ is a unique Maximum.

The costate equation we obtain

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t) + (r + \gamma)\lambda(t)$$
$$= -m - \gamma\beta + \lambda(t)\rho u_1(t)\frac{1}{2\sqrt{1 - x(t)}} + \lambda(t)(\delta + r + \gamma). \tag{54}$$

with the limiting transversality condition $\lim_{t\to\infty} e^{-rt}\lambda(t)$.

Analogously to the proof before the unique steady-state solution can be derived, which equals

$$\hat{x}_{St} = 1 - \frac{\delta}{-Q + \sqrt{Q^2 + \rho^2(m + \gamma\beta)} + \delta}$$
$$\hat{\lambda}_{St} = \frac{-Q + \sqrt{Q^2 + \rho^2(m + \gamma\beta)}}{\rho^2/2} \tag{55}$$

where $Q = r + \delta + \gamma$. The associated Jacobian matrix (evaluated in the steady state) is

$$J(x, \lambda) = \begin{pmatrix} \rho^2\lambda/2 + Q & 0 \\ \rho^2(1-x)/2 & -\rho^2\lambda/2 - \delta \end{pmatrix}$$

and its determinant is negative, hence $(\hat{x}_{St}, \hat{\lambda}_{St})$ is a saddle point (see Grass et al. 2008, section 2). The solution can exactly be found as in the case of the surprised monopoly ("Appendix C").

Analogously to the proof before the maximized Hamiltonian is linear in x, thus the Arrow sufficiency Theorem holds and implies that the conditions of the Maximum Principle are sufficient. Furthermore the solution of the optimal control problem is unique.

E Proof of Proposition 1

Let be $x_{St}(t) = x_S(t) = x_{NS}(t) = x$, It can be easily proved that $N(t) < R - U$, for all t, so that the first inequality in (30) trivially holds.

The second condition is equivalent to

$$-(U + \gamma) + \sqrt{(U + \gamma)^2 + (m + \gamma\beta)\rho^2} < -U + \sqrt{U^2 + m\rho^2}, \tag{56}$$

and it is verified if and only if

$$m > \beta(\beta\rho^2/4 + \delta + r). \tag{57}$$

Recalling the value of β

$$\beta = \frac{\sqrt{(r + 2\delta)^2 + 3m\rho^2} - (r + 2\delta)}{3\rho^2/2}$$

(57) becomes

$$3m\rho^2 - (2r + \delta)\left(\sqrt{3m\rho^2 + (r + 2\delta)^2} - (r + 2\delta)\right) > 0, \tag{58}$$

which is always verified.

F Proof of Proposition 2

From the initial conditions we have that

$$x_{NS}(0) = x_S(0) = x_0$$

and

$$\dot{x}_{NS}(0) = N(0)(1 - x_0) - \delta x_0 < (R - U)(1 - x_0) - \delta x_0 = \dot{x}_S(0).$$

So that $x_{NS}(t) < x_S(t)$ for any $t \in [0, \epsilon[$ with $\epsilon \geq 0$. We can prove that as soon as the two functions coincide at a given instant $\tilde{t} > 0$ than at that point their derivatives satisfy the inequality

$$\dot{x}_{NS}(\tilde{t}) = N(\tilde{t})(1 - x_0) - \delta x_0 < (R - U)(1 - x_0) - \delta x_0 = \dot{x}_S(\tilde{t}),$$

are therefore we can conclude that $x_{NS}(t) < x_S(t)$ for any $t \in [0, T^e]$.

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