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A note on portfolio selection and stochastic dominance

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Abstract This note provides new and simpler conditions ensuring that, when one portfolio dominates another via stochastic dominance, a decision maker prefers the first one. The conditions are derived for the case of third-order stochastic dominance and for the general case of *N*th-order stochastic dominance.

Keywords Portfolio · Stochastic dominance · Third-order stochastic dominance · *N*th-order stochastic dominance

JEL Classification G11 · D81

1 Introduction

Many studies in financial decision-making concern optimal portfolio selection. A specific issue in the field is portfolio choice in the case of stochastic dominance. The literature particularly focuses on conditions on preferences which ensure that, when one portfolio dominates another via stochastic dominance, a decision maker prefers the first one.

Well-known results (Whitmore 1970; Porter and Gaumnitz 1972) show that, in the case of third-order stochastic dominance (TSD), given the usual assumption of "non-satiation" (meaning a positive first derivative of decision maker's utility function), the dominating portfolio is preferred to the dominated portfolio when the decision maker's utility function exhibits a negative second derivative and a positive third derivative. Generalization to *N*th-order stochastic dominance (NSD) shows that the dominating

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portfolio is preferred if the derivatives of decision maker's utility function alternate in sign until the derivative of order N, with positive sign for odd derivatives and negative sign for even derivatives (see Ingersoll 1987; Levy 2006).

Using the recent analysis by Menegatti (2014, 2015) and under standard regularity assumptions on the utility function, this note provides new and simpler conditions for this problem. In particular, when considering TSD, for the dominating portfolio to be preferred to the dominated one, we require only a positive third derivative of the utility function for every level of wealth, without any assumption on the sign of the second derivative being introduced. Similarly, in the case of NSD, the condition ensuring that the dominating portfolio is preferred by the decision maker involves only the sign of the Nth-order derivative (which, for every level of wealth, must be positive if N is odd and negative if N is even) and does not involve the signs of all the derivatives of orders from 2 to $N-1$.

The paper proceeds as follows. Section 2 derives the results for the case of TSD. Section 3 derives the results for the case of NSD. Finally Sect. 4 briefly concludes.

2 Third-order stochastic dominance

We consider a decision maker whose preferences are described by the utility function $U(x)$ defined over R^+, where x represents wealth. We introduce the usual assumption of non-satiation, requiring that utility is increasing in wealth ($\frac{U(x)}{x} > 0\ \forall x \in R^+$). We also introduce the regularity assumption that marginal utility is bounded for wealth tending to infinity (i.e. $\lim_{x\to+\infty} \frac{dU(x)}{dx} \neq +\infty$). This assumption simply requires that marginal utility does not explode when consumption becomes extremely high. Also note that this assumption does not require marginal utility to be decreasing, and is compatible with increasing marginal utility,[1] and that it is satisfied by all the most frequently used utility functions (such as linear, quadratic, CRRA, CARA utilities).

We assume that the decision maker compares two different risky portfolios represented by the two random variables $\tilde{y}$ and $\tilde{z}$, which are defined over the interval $[a, b]$ where $a, b \in R^+$. Also assume that F and G denote the cumulative distribution functions for these random variables.

Assume now that portfolio $\tilde{y}$ dominates portfolio $\tilde{z}$ via third-order stochastic dominance (TSD).[2] This assumption is usually formalized using the following definition:

Definition 1 The random portfolio $\tilde{y}$ dominates the random portfolio $\tilde{z}$ via third-order stochastic dominance (TSD) if $\int_a^m \int_a^q F(s)\mathrm{d}s\mathrm{d}q \leq \int_a^m \int_a^q G(s)\mathrm{d}s\mathrm{d}q$.

A well-known result on stochastic dominance and preferences (see Whitmore 1970; Porter and Gaumnitz 1972; Levy 2006) states that:

Lemma 1 *The following two statements are equivalent:*

[1] Conversely, the assumption is satisfied by all utility functions which are strictly concave, as is often assumed in economic models.

[2] It is worth noting that TSD is related to portfolios skewness. For a presentation of this relationship see, for instance, Levy (2006).

(a) *The random portfolio $\tilde{y}$ dominates the random portfolio $\tilde{z}$ via TSD;*
(b) $E[U(\tilde{y})] \geq E[U(\tilde{z})]$ *for any function* $U(x)$ *such that* $\frac{d^2U(x)}{dx^2} < 0 \; \forall x \in [a, b]$ *and* $\frac{d^3U(x)}{dx^3} > 0 \; \forall x \in [a, b]$.

The traditional result in the case of TSD stated in Lemma 1 shows that, given the assumption of non-satiation, the dominating portfolio $\tilde{y}$ is preferred to the dominated portfolio $\tilde{z}$ when, in the interval $[a, b]$, the second derivative of the utility function is negative and the third derivative is positive.

Given these results we can now derive new conditions ensuring that the dominating portfolio is preferred by the decision maker by using a recent result obtained by Menegatti (2014). Menegatti (2014) show that, under the assumptions on $U(x)$ introduced above, we have:

Lemma 2 *If* $U(x)$ *exhibits* $\frac{d^3U(x)}{dx^3} > 0 \; \forall x \in R^+$ *then it also exhibits* $\frac{d^2U(x)}{dx^2} < 0 \; \forall x \in R^+$

Proof See Proposition 1 (b) by Menegatti (2014, p. 615).

Lemma 2 thus shows that if the utility function exhibits a positive third derivative for every level of wealth, then it also exhibits a negative second derivative for every level of wealth.

By using Lemmas 1 and 2 together, we immediately obtain that:

Proposition 1 *The following two statements are equivalent:*

(a) *The random portfolio $\tilde{y}$ dominates the random portfolio $\tilde{z}$ via TSD;*
(b) $E[U(\tilde{y})] \geq E[U(\tilde{z})]$ *for any function* $U(x)$ *such that* $\frac{d^3U(x)}{dx^3} > 0 \; \forall x \in R^+$.

Proof By Lemma 2, Statement (b) in Proposition 1 is equivalent to Statement (b) in Lemma 1. This, together with Lemma 1, proves the proposition.

The result in Proposition 1 shows that, in the case of TSD, the dominating portfolio $\tilde{y}$ is preferred to the dominated portfolio $\tilde{z}$ when the third derivative of the decision maker's utility function is positive for every level of wealth. No assumptions on the second derivative are required.

3 *N*th-order stochastic dominance

We consider a decision maker whose preferences satisfy the assumptions introduced in Sect. 2.

We assume again that the decision maker compares two different risky portfolios represented by the two random variables $\tilde{y}$ and $\tilde{z}$, which are defined over the interval $[a, b]$ where $a, b \in R^+$. Also assume that F and G denote the cumulative distribution functions for these random variables. Define $F^{(0)}(m) \equiv F(m)$ and $F^{(j)}(m) \equiv \int_a^m F^{(j-1)}(m)\mathrm{d}t$ for $j \geq 1$ and similarly define $G^{(0)}(m)$ and $G^{(i)}(m)$ for $i \geq 1$.

Assume now that portfolio $\tilde{y}$ dominates portfolio $\tilde{z}$ via Nth-order stochastic dominance. This assumption is usually formalized using the following definition:

Definition 2 The random portfolio $\tilde{y}$ dominates the random portfolio $\tilde{z}$ via Nth-order stochastic dominance (NSD) if $F^{(n-1)}(m) \leq G^{(n-1)}(m)$ $\forall m \in [a, b]$ and $F^{(k)}(b) \leq G^{(k)}(b)$ for $k = 1, 2, \ldots, N-2$.

A well-known result on stochastic dominance and preferences (see Ingersoll 1987; Levy 2006) states that:

Lemma 3 *The following two statements are equivalent:*

(a) *The random portfolio $\tilde{y}$ dominates the random portfolio $\tilde{z}$ via NSD;*
(b) $E[U(\tilde{y})] \geq E[U(\tilde{z})]$ *for any function $U(x)$ such that* $(-1)^{(n+1)} \frac{d^n U(x)}{dx^n} > 0$ $\forall x \in [a, b]$ *for* $n = 2, \ldots, N$.

The traditional result in the case of NSD, stated in Lemma 3, shows that the dominating portfolio $\tilde{y}$ is preferred to the dominated portfolio $\tilde{z}$ when, in the interval $[a, b]$, the derivatives of the utility function until order N alternate in sign with positive odd derivatives and negative even derivatives.

Given these results, we can now derive new conditions ensuring that the dominating portfolio is preferred by the decision maker by using a recent result by Menegatti (2015). Menegatti (2015) shows that, under the assumptions on $U(x)$ introduced in the first paragraph of Section Two, we have:

Lemma 4 *If the utility function $U(x)$ exhibits* $(-1)^{N+1} \frac{d^N U(x)}{dx^N} > 0$ $\forall x \in R^+$ *then it also exhibits* $(-1)^{(n+1)} \frac{d^n U(x)}{dx^n} > 0$ $\forall x \in R^+$ *for* $n = 2, \ldots, N$.

Proof See Proposition 2 by Menegatti (2015, p. 679).

Lemma 4 thus shows that if the Nth-order derivative of the utility function is positive when N is odd and negative when N is even for every level of wealth, then the same utility function has derivatives of orders from 2 to N alternating in sign (with positive odd derivatives and negative even derivatives) for every level of wealth.

Using the result stated in Lemma 4 together with the traditional result on portfolio choice stated in Lemma 3, we immediately prove that:

Proposition 2 *The following two statements are equivalent:*

(a) *The random portfolio $\tilde{y}$ dominates the random portfolio $\tilde{z}$ via NSD;*
(b) $E[U(\tilde{y})] \geq E[U(\tilde{x})]$ *for any function $U(x)$ such that* $(-1)^{N+1} \frac{d^N U(x)}{dx^N} > 0$ $\forall x \in R^+$.

Proof By Lemma 4, Statement (b) in Proposition 2 is equivalent to Statement (b) in Lemma 3. This, together with Lemma 3, proves the proposition.

The result in Proposition 2 shows that, in the case of NSD, the dominating portfolio $\tilde{y}$ is preferred to the dominated portfolio $\tilde{z}$ when the Nth-order derivative of the utility function has the appropriate sign (positive when N is odd and negative when N is even) for every level of wealth. No assumptions on the derivatives of orders from 2 to $N-1$ are required.

4 Conclusion

Under standard regularity assumptions on the utility function, the results in Sects. 2 and 3 provide new and simpler conditions ensuring that a first portfolio dominating a second portfolio via stochastic dominance of different order is preferred by a decision maker.

In the case of TSD, the condition derived in Sect. 2 requires the third derivative of the utility function to be positive. Unlike the well-known result in the previous literature, no assumption on the sign of the second derivative is introduced. The cost of removing the requirement on the second derivative is that the requirement on the third derivative must be introduced for every level of wealth and not only in the interval $[a, b]$.

Also note that the new condition derived is relevant not only for the analysis of portfolio choice but also for the possible connection with a recent strand of research literature in risk theory. The assumption that an agent has a negative second derivative of the utility function is usually called "risk aversion." The assumption that agent has a positive third derivative of the utility function is called either "downside risk aversion" (Menezes et al. 1980) or "prudence" (Kimball 1990). An increasing emphasis is put in recent risk theory on results obtained under the assumption of downside risk aversion/prudence without introducing the assumption of risk aversion. On this, see, for instance, Nocetti (2015).

In the case of NSD, the condition derived in Section Three requires that the Nth-order derivative of the decision maker's utility function has the appropriate sign: positive if N is odd and negative if N is even, for every level of wealth. Unlike result in the previous literature, no assumptions on the signs of the derivatives of orders from 2 to $N - 1$ are introduced. The condition introduced is thus clearly less demanding in terms of specific features required for decision maker preferences. The cost of obtaining this simplification is again that the requirement on the Nth-order derivative must be introduced for every level of wealth and not only in the interval $[a, b]$.

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