AStA (2008) 92: 29–34
DOI 10.1007/s10182-008-0054-5

ORIGINAL PAPER

On the existence of unbiased estimators for the portfolio weights obtained by maximizing the Sharpe ratio

Wolfgang Schmid · Taras Zabolotskyy

Received: 15 February 2007 / Accepted: 2 November 2007 / Published online: 6 February 2008

Abstract We consider the problem of estimating the portfolio weights obtained by maximizing the Sharpe ratio. Assuming that the underlying asset returns are independent and multivariate normally distributed, Okhrin and Schmid (J. Econom. 134:235–256, 2006) showed that the frequently used sample estimators of these weights do not have a first moment. This paper proves that an unbiased estimator of the Sharpe ratio portfolio weights does not exist at all. Moreover, we show that there is no asymptotically unbiased estimator of these weights within the family of estimators which are bounded by cylinder functions.

Keywords Optimal portfolio weights · Unbiased estimator · Asymptotically unbiased estimator · Sharpe ratio optimal weights

1 Introduction

People who invest their wealth in stocks must decide how much of that money to allocate into a particular stock. The choice of the portfolio weights, i.e. the relative proportions of the wealth invested in each stock, is one of the most important questions within the asset allocation process and one of the central topics of financial literature. In a pioneering paper Markowitz (1952) introduced the mean-variance principle for portfolio selection. This paper was the starting point for an intensive discussion of this problem. The utility maximization is another frequently applied approach to explain the investor's decision process in asset management (see von Neumann and Morgenstern 1944; Merton 1969). Both procedures lead, in a certain sense, to optimal portfolio weights.

W. Schmid (✉) · T. Zabolotskyy
Department of Statistics, European University, PO Box 1786, 15207 Frankfurt (Oder), Germany
e-mail: schmid@euv-frankfurt-o.de

This paper focuses exclusively on the portfolio weights obtained by maximizing the Sharpe ratio. The Sharpe ratio is one of the most popular performance measures for portfolio selection (e.g., Cochrane 1999; MacKinley and Pastor 2000; Jobson and Korkie 1981). It is based on a portfolio that consists of $n \geq 1$ risky assets and one riskless investment with return r. Let $\boldsymbol{\mu}$ denote the mean vector of the risky asset returns and $\boldsymbol{\Sigma}$ their covariance matrix. Let $\mathbf{w}$ be the vector of the relative proportions of the investor's wealth invested in the risky assets. The Sharpe ratio is defined as the normalized excess return of the portfolio and it is given by

$$\mathrm{SR} = \frac{\mathbf{w}'(\boldsymbol{\mu} - r\mathbf{1})}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}}. \tag{1}$$

Here $\mathbf{1}$ denotes the vector whose elements are all equal to 1.

Now the Sharpe ratio is maximized with respect to $\mathbf{w}$ under the restriction $\mathbf{w}'\mathbf{1} = 1$. There is an explicit solution of this maximization problem and it is given by

$$\mathbf{w}_S = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}, \tag{2}$$

provided that $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) \neq 0$.

These weights are equal to the weights of the so-called tangency portfolio, which can be characterized as the portfolio with the maximum Sharpe ratio of all portfolios with risky assets (cf. Campbell et al. 1997, p. 188). The weights are based on a criterion that cannot be justified by the expected utility theory.

Note that negative weights and weights larger than 1 are not excluded. The "short selling" corresponds to the situation when the investor sells the asset he or she does not own. This leads to negative portfolio weights. Usually it is implemented by borrowing the asset from another institution and selling it on the market. A value of a weight higher than 1 implies that the investor sold other assets short in order to achieve more than 100% stake of this asset in the portfolio (cf. Farrell 1997). If we pose the restriction that the portfolio weights belong to the interval between zero and one, then an explicit formula of the optimal portfolio weights cannot be derived and the numerical Kuhn–Tucker optimization is required.

Because in practice $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are both unknown, they have to be estimated from the data. Trying to reduce the bias of the Sharpe ratio estimator, Jobson and Korkie (1980) proposed an approximately unbiased estimator and provided a test for comparing the Sharpe ratios of two and more portfolios. Recently several papers have been published dealing with the influence of parameter estimation on the portfolio weights (e.g., Britten-Jones 1999; Okhrin and Schmid 2006, 2007, 2008). Okhrin and Schmid (2006, 2008) analyzed the behavior of several estimators of optimal portfolio weights for finite and infinite sample size as well. One of their most interesting results concerns the portfolio weights obtained by the Sharpe ratio. If the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated by the mean and the sample covariance matrix, they proved that the first moment of the corresponding estimator of the Sharpe portfolio weights does not exist at all. Consequently the estimator is neither unbiased nor asymptotically unbiased. This is a hint that it should be carefully used in practice and that interpretation of the results is difficult.

Of course, an obvious idea is to look for an estimator of the Sharpe ratio portfolio weights that has nice statistical properties like, e.g., unbiasedness. In this paper it is proved that an unbiased estimator of the portfolio weights obtained from the Sharpe ratio does not exist. It has to be emphasized that there are several well-known situations in statistics where there is no unbiased estimator of a parameter like, e.g., in the case of density estimation. Here we analyze the problem in more detail. We also deal with the problem of whether there exists an asymptotically unbiased estimator of the Sharpe portfolio weights. This problem turns out to be much harder, and we have not been able to solve it completely. Here we obtain a partial answer. We show that there is no asymptotically unbiased estimator within the family of estimators which are bounded by cylinder functions.

The paper is organized as follows. In Sect. 2 we present our main results about the nonexistence of an unbiased estimator of the Sharpe ratio portfolio weights (Theorem 1) and the nonexistence of an asymptotically unbiased estimator within a certain family of estimators (Theorem 2). All proofs are given in the Appendix.

2 Estimators of the Sharpe ratio portfolio weights

We consider a portfolio consisting of n risky assets and one riskless asset. Let $\boldsymbol{\mu}$ denote the mean of the returns within the portfolio and $\boldsymbol{\Sigma}$ be their covariance matrix. Then the Sharpe ratio optimal portfolio weights are given by $\mathbf{w}_S$ as defined in (1). The unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated using the random sample $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N$ of the returns. The random vector $\mathbf{X}_t$ denotes the returns of the assets at time t. Throughout the paper it is always assumed that the variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N$ are independent and multivariate normally distributed.

First, it is proved that there is no unbiased estimator of $\mathbf{w}_S$.

Theorem 1 *Let $\mathbf{X}_t$, $t = 1, \ldots, N$, be independent and multivariate normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let $\boldsymbol{\Sigma}$ be positive definite. Then there is no unbiased estimator of $\mathbf{w}_S$, i.e., there is no estimator $\mathbf{T}$ of $\mathbf{w}_S$ such that $\mathbf{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{T}) = \mathbf{w}_S(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for all $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ with $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) \neq 0$.*

The proof of Theorem 1 is based on a contradiction. This behavior can be explained by the fact that $\mathbf{w}_S$ is not defined for $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) = 0$. Thus it is possible to choose a sequence $\{\boldsymbol{\mu}_m\}$ such that $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_m - r\mathbf{1}) \to 0$ and $\mathbf{w}_S(\boldsymbol{\mu}_m, \boldsymbol{\Sigma}) \to \infty$ as $m \to \infty$, but for which $\lim_{m\to\infty} \mathbf{E}_{\boldsymbol{\mu}_m,\boldsymbol{\Sigma}}(\mathbf{T})$ exists.

Next we are dealing with the question of whether there is an asymptotically unbiased estimator of the Sharpe ratio portfolio weights. This problem turns out to be much harder, and we have not found its solution. However, we can show that such estimator does not exist within a certain family of functions.

We consider so-called cylinder functions (e.g., Skorohod 1974). A function $\phi : \mathbb{R}^\infty \to \mathbb{R}$ is called a cylinder function if for some finite dimensional subspace L of $\mathbb{R}^\infty$ it is $\mathcal{B}^{\mathrm{L}}$ measurable, i.e., it has the form $\phi(\mathbf{x}) = \phi_{\mathrm{L}}(\mathrm{P}_{\mathrm{L}}\mathbf{x})$. Here ϕ_{L} is some $\mathcal{B}^{\mathrm{L}}$ measurable function defined on L and P_{L} is the projection operator. We define the class $\mathbf{Z}(P)$ as the set of all sequences $\{\mathrm{f}_N\}$ with $\mathrm{f}_N : \mathbb{R}^\infty \to \mathbb{R}$, f_N is $(\mathbb{R}^\infty, \mathcal{B}^\infty)$

measurable and integrable with respect to a measure P on $(\mathbb{R}^\infty, \mathcal{B}^\infty)$, and $|\mathrm{f}_N(\mathbf{x})| \le \phi(\mathbf{x})$ for all N, where ϕ is a cylinder function.

In Theorem 2 the sequence of estimators $\{T_N\}$ of the Sharpe ratio portfolio weights is assumed to be bounded by cylinder functions. In order to interchange a limit with an integral it is frequently demanded in measure theory that the sequence of functions is bounded. Here we need more. The upper bound is a cylinder function. This is a purely technical assumption. It must be noted that in the present case the number of integrals increases with N as well and therefore we need a stronger condition. In principle, this assumption controls the influence of newly incoming information.

Theorem 2 *Let* $\mathbf{X}_t$, $t \ge 1$, *be independent and multivariate normally distributed random vectors with mean* $\boldsymbol{\mu}$ *and covariance matrix* $\boldsymbol{\Sigma}$. *Then there is no asymptotically unbiased estimator* $\{\mathbf{T}_N\}$ *of* $\mathbf{w}_S$ *of the form* $\mathbf{T}_N = (\mathrm{f}_{1,N}(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots), \ldots, \mathrm{f}_{n,N}(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots))'$ *with* $\{\mathrm{f}_{i,N}\} \in \mathbf{Z}(P^*)$ *for* $i = 1, \ldots, n$ *and* P^* *is the measure on* $\mathbb{R}^\infty$ *generated by the Gaussian distribution with parameters* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$.

The proof of Theorem 2 is again based on a contradiction. The main problem in proving the nonexistence of an asymptotically unbiased estimator is that Lemma 1 cannot be applied without any restrictions on the estimator because the upper bound converges to infinity if N tends to infinity.

3 Conclusions

This paper proves that there is no unbiased estimator of the portfolio weights obtained from the Sharpe ratio. This result is a generalization of a result by Okhrin and Schmid (2006), who showed that the estimator of the Sharpe ratio portfolio weights based on the sample mean and the sample covariance matrix has no first moment. Because the Sharpe ratio is widely applied in practice it is a warning that this quantity should be carefully interpreted. It has to be emphasized that it is not unusual in statistics to have no unbiased estimator. The most famous example provides the problem of density estimation. It appears to us that in the present case the problem is more difficult. We have been able to prove that there is no asymptotically unbiased estimator within a certain family of functions. The main problem about the existence of an asymptotically unbiased estimator remains open.

Appendix

The proofs of Theorems 1 and 2 are based on Lemma 1. We use the notation $\mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to denote an n-variate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Lemma 1 *Let* $\mathbf{X}_1,\ldots,\mathbf{X}_N$ *be independent and identically distributed random vectors with* $\mathbf{X}_i \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Suppose that* $\mathrm{T} = \mathrm{T}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ *is a real valued function with* $\mathbf{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(|\mathrm{T}|) < \infty$ *for all* $\boldsymbol{\mu}, \boldsymbol{\Sigma}$. *Let* $\{b_m\} \subset \mathbb{R}$ *be a bounded sequence. Then the sequence* $\mathbf{E}_{\boldsymbol{\mu}+b_m\boldsymbol{\Sigma}^{1/2}\mathbf{1},\boldsymbol{\Sigma}}(|\mathrm{T}|)$ *is bounded as well.*

Proof It holds that

$$\begin{aligned}
&\mathbf{E}_{\boldsymbol{\mu}+b_m\boldsymbol{\Sigma}^{1/2}\mathbf{1},\boldsymbol{\Sigma}}\left(|\mathrm{T}|\right)\\
&\quad=\frac{1}{(2\pi)^{nN/2}}\int_{\mathbb{R}^{nN}}\left|\mathrm{T}\left(\boldsymbol{\Sigma}^{1/2}\mathbf{y}_1,\ldots,\boldsymbol{\Sigma}^{1/2}\mathbf{y}_N\right)\right|\\
&\qquad\times\exp\left(-\frac{1}{2}\sum_{i=1}^{N}\left(\mathbf{y}_i-\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}-b_m\mathbf{1}\right)'\left(\mathbf{y}_i-\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}-b_m\mathbf{1}\right)\right)\mathrm{d}\mathbf{y}_1\cdots\mathrm{d}\mathbf{y}_N.
\end{aligned}$$

Suppose that $a\le b_m\le b$. Let $\mathrm{h}(\lambda,\mathbf{y})=(\mathbf{y}-\lambda\mathbf{1})'(\mathbf{y}-\lambda\mathbf{1})$ with $\lambda\in[a,b]$. Then

$$\mathrm{h}(\lambda,\mathbf{y})=n\left(\lambda-\frac{\mathbf{y}'\mathbf{1}}{n}\right)^2+\mathbf{y}'\mathbf{y}-\frac{(\mathbf{y}'\mathbf{1})^2}{n}.$$

Because

$$\begin{aligned}
(\lambda-u)^2&\ge\begin{cases}(u-b)^2 & \text{for } u>b,\\ (u-b)^2-(a-b)^2 & \text{for } a\le u\le b,\\ (u-a)^2 & \text{for } u<a\end{cases}\\
&\ge\min\left\{(u-a)^2,(u-b)^2\right\}-(a-b)^2
\end{aligned}$$

it follows that

$$\mathrm{h}(\lambda,\mathbf{y})\ge\min\left\{\mathrm{h}(a,\mathbf{y}),\mathrm{h}(b,\mathbf{y})\right\}-(a-b)^2.$$

Applying this result we get that

$$\begin{aligned}
&\frac{1}{(2\pi)^{nN/2}}\int_{\mathbb{R}^{nN}}\left|\mathrm{T}\left(\boldsymbol{\Sigma}^{1/2}\mathbf{y}_1,\ldots,\boldsymbol{\Sigma}^{1/2}\mathbf{y}_N\right)\right|\\
&\qquad\times\exp\left(-\frac{1}{2}\sum_{i=1}^{N}\left(\mathbf{y}_i-\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}-b_m\mathbf{1}\right)'\left(\mathbf{y}_i-\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}-b_m\mathbf{1}\right)\right)\mathrm{d}\mathbf{y}_1\cdots\mathrm{d}\mathbf{y}_N\\
&\quad\le\exp\left(\frac{N}{2}(a-b)^2\right)\left(\mathbf{E}_{\boldsymbol{\mu}+a\boldsymbol{\Sigma}^{1/2}\mathbf{1},\boldsymbol{\Sigma}}\left(|\mathrm{T}|\right)+\mathbf{E}_{\boldsymbol{\mu}+b\boldsymbol{\Sigma}^{1/2}\mathbf{1},\boldsymbol{\Sigma}}\left(|\mathrm{T}|\right)\right)<\infty.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1

The proof of Theorem 1 is based on a contradiction. It is assumed that there exists an unbiased estimator $\mathbf{T}=(\mathrm{T}_1,\ldots,\mathrm{T}_n)'$ of $\mathbf{w}_S=(\mathrm{w}_{1s},\ldots,\mathrm{w}_{ns})'$, i.e. it holds that $\mathbf{E}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{T}(\mathbf{X}_1,\ldots,\mathbf{X}_N))=\mathbf{w}_S(\boldsymbol{\mu},\boldsymbol{\Sigma})$ for all $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ with $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-r\mathbf{1})\neq0$. The idea of the proof is to choose a sequence $\boldsymbol{\mu}_m=(\mu_{1m},\ldots,\mu_{nm})'$ such that $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_m-r\mathbf{1})\neq0$ for all $m\in\mathbb{N}$, $\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_m-r\mathbf{1})\underset{m\to\infty}{\to}0$ and $\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_m-r\mathbf{1})\underset{m\to\infty}{\to}\mathbf{q}\neq\mathbf{0}$. Note that it is easy to find such a sequence. We can take, e.g., $\boldsymbol{\mu}_m=r\mathbf{1}+\tau+a_m\boldsymbol{\Sigma}^{1/2}\mathbf{1}$ with $a_m=\frac{(-1)^m}{m}$. The vector $\mathbf{q}$ can be taken as an arbitrary vector unequal to zero but the sum of its components must be equal to 0. Then we can choose $\tau = \mathbf{\Sigma q}$.

Suppose that $\mathbf{q} = (q_1, \ldots, q_n)'$ and $q_r \neq 0$. Next we consider the sequence of expectations $\mathbf{E}_{\mu_m, \mathbf{\Sigma}}(\mathrm{T}_r(\mathbf{X}_1, \ldots, \mathbf{X}_N))$. Because of Lemma 1 the sequence is bounded. Thus we can choose a convergent subsequence $\mathbf{E}_{\mu_{m_k}, \mathbf{\Sigma}}(\mathrm{T}_r(\mathbf{X}_1, \ldots, \mathbf{X}_N))$. This leads to a contradiction because $\mathrm{w}_{rs}(\mu_{m_k}, \mathbf{\Sigma}) \underset{k\to\infty}{\to} \pm\infty$.

Proof of Theorem 2

As in the proof of Theorem 1 it is assumed that there is a sequence $\{\mathbf{T}_N\}$ of the form $\mathbf{T}_N = (\mathrm{f}_{1,N}, \ldots, \mathrm{f}_{n,N})'$ such that for all $i = 1, \ldots, n$, $\mathrm{f}_{i,N}$ is lying in $\mathbf{Z}(P^*)$ and

$$\lim_{N\to\infty} \mathbf{E}_{\mu,\mathbf{\Sigma}}\big(\mathbf{T}_N(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots)\big) = \mathbf{w}_S(\mu, \mathbf{\Sigma}) \quad \text{for all } \mu, \mathbf{\Sigma} \text{ with } \mathbf{1}'\mathbf{\Sigma}(\mu - r\mathbf{1}) \neq 0.$$

We choose the sequence $\{\mu_m\}$ as in the proof of Theorem 1. Then it holds that

$$\lim_{N\to\infty} \mathbf{E}_{\mu_m,\mathbf{\Sigma}}\big(\mathbf{T}_N(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots)\big) = \mathbf{w}_S(\mu_m, \mathbf{\Sigma}).$$

Next we show that $\lim_{N\to\infty} \mathbf{E}_{\mu_m,\mathbf{\Sigma}}(\mathbf{T}_N(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots))$ is bounded in m and thus there is a convergent subsequence. This leads to a contradiction.

Because

$$\mathbf{E}_{\mu_m,\mathbf{\Sigma}}\big(\big|\mathbf{T}_N(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots)\big|\big) \leq \mathbf{E}_{\mu_m,\mathbf{\Sigma}}\big(\phi(\mathbf{X}_1, \ldots)\big)\mathbf{1}$$

it follows that $\lim_{N\to\infty} \mathbf{E}_{\mu_m,\mathbf{\Sigma}}(|\mathbf{T}_N(\mathbf{X}_1, \ldots, \mathbf{X}_N, \mathbf{0}, \ldots)|)$ is bounded and thus the proof is finished.

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