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REGULAR PAPER

A signed-distance-based approach to importance assessment and multi-criteria group decision analysis based on interval type-2 fuzzy set

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Abstract Interval type-2 fuzzy sets are associated with greater imprecision and more ambiguities than ordinary fuzzy sets. This paper presents a signed-distance-based method for determining the objective importance of criteria and handling fuzzy, multiple criteria group decision-making problems in a flexible and intelligent way. These advantages arise from the method's use of interval type-2 trapezoidal fuzzy numbers to represent alternative ratings and the importance of various criteria. An integrated approach to determine the overall importance of the criteria is also developed using the subjective information provided by decision-makers and the objective information delivered by the decision matrix. In addition, a linear programming model is developed to estimate criterion weights and to extend the proposed multiple criteria decision analysis method. Finally, the feasibility and effectiveness of the proposed methods are illustrated by a group decision-making problem of patient-centered medicine in basilar artery occlusion.

Keywords Interval type-2 fuzzy set · Signed distance · Objective importance · Multiple criteria group decision-making · Linear programming model · Patient-centered medicine

1 Introduction

A type-2 fuzzy set (T2FS) is a membership function represented by a fuzzy set on the interval [0, 1] [64]. T2FSs are more capable than ordinary fuzzy sets of handling imprecision and imperfect information in real-world applications. In addition, T2FS theory provides an intuitive and computationally feasible method for dealing with uncertain and ambiguous properties. Considering the decision-maker's point of view and the circumstances of the multiple criteria decision-making process, we find that subjective opinions and judgments are inherently imprecise and involve many uncertainties, especially when hybrid data, vague concepts, and uncertain data were involved in the decision process [18]. In this respect,

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we represent multiple criteria decisions in terms of T2FSs in this study. Nonetheless, prior scholars have argued against T2FSs by pointing out the difficulties in construction and manipulation [1]. To resolve the difficulties in establishing and handling the secondary membership functions, scholars use interval type-2 fuzzy sets (IT2FSs), which are also known as interval-valued fuzzy sets [48,64]. IT2FSs contain membership values that are crisp intervals, and they are the most widely used of the higher order fuzzy sets because of their relative simplicity [39,59].

The concept of IT2FSs is defined by an interval-valued membership function [48,64]. Many useful methods have been developed to enrich IT2FS theory [8,6,11,26,32,59,65]. Recently, IT2FSs have been applied to several areas, including power systems [50,51], nonlinear systems [3,33], chaotic systems [34], fuzzy logic systems [7,43], and conceptual designs [2]. IT2FSs are especially useful in circumstances in which a crisp degree of membership is difficult to determine. Decision-makers often have a difficult time precisely quantifying their opinions of subjective judgments as a number in an interval [0, 1] during the decision-making process. Thus, the degree of membership is better represented by a higher order fuzzy set than an exact membership grade. A growing number of scholars are becoming interested in developing the methods for multiple criteria decision analysis (MCDA) within the context of IT2FSs.

Chen and Tsao [20] extended the technique for order preference by similarity to an ideal solution (TOPSIS) that is based on IT2FSs. They conducted an experimental analysis on distance measures as well. Yang et al. [63] combined IT2FSs and soft sets to obtain an interval-valued fuzzy soft set. They defined the complement, defined the "and" and "or" operations, proved DeMorgan's associative and distribution laws and applied these laws to a decision-making problem. Chen and Wang [21] developed an interval-valued fuzzy permutation method to solve multi-criteria decision-making problems with IT2FS data and conducted an experimental analysis on cardinal and ordinal evaluations. Lu et al. [36] developed an interval-valued fuzzy linear programming method based on infinite α-cuts and applied this method to the problem of water resource management. Chen and Lee [19] presented an interval type-2 fuzzy TOPSIS method to handle group decision-making problems based on IT2FSs. Vahdani and Hadipour [52] proposed an elimination and choice translating reality (ELECTRE) method based on IT2FSs to solve a problem involving the selection of a maintenance strategy. Vahdani et al. [53] developed an ELECTRE method with interval weights and data to solve multi-criteria decision-making problems. Chen [12] presented a useful method for estimating the importance of the criteria in MCDA and for reducing the leniency bias in MCDA based on IT2FSs. Wei et al. [58] introduced correlation and correlation coefficients for interval-valued intuitionistic fuzzy sets and proposed a multiple attribute decision-making method with incomplete weight information. Chen [13] utilized several score functions based on interval type-2 fuzzy point operators to quantify both optimistic and pessimistic estimations and developed a model to reduce cognitive dissonance based on IT2FSs. Because IT2FS theory is valuable for both modeling imprecision and for its ability to easily reflect the ambiguous nature of subjective judgments, this paper used IT2FSs to capture imprecise or uncertain decision information in the fields that require MCDA.

Wang and Li [55] defined the interval-valued fuzzy numbers (i.e., interval type-2 fuzzy numbers (IT2FNs)) and provided a starting point for real-world applications. Because a decision-maker's method of evaluating alternatives and making decisions is guided by his or her subjective judgments, the decision data used in MCDA can be reasonably considered to be IT2FNs. Nevertheless, processing sophisticated IT2FN data may be difficult or troublesome. A simple and effective method for managing complicated data is needed. To date, the study of decision-making (such as decision-theoretic model and decision-support system) has been, and still is, undertaken in various ways and by various scholars and practitioners working in the area [9,28,35,42]. However, the relevant issues of the MCDA models and methods with the IT2FN data have not yet been thoroughly explored. Accordingly, we intend to construct a new group decision-making method based on signed distances to resolve a fuzzy MCDA problem within the context of IT2FNs. A signed distance (i.e., an oriented distance or directed distance) has often been used to determine the rankings of fuzzy numbers. Combined with the axiom of choice, the concept of signed distances can be extended to the IT2FN environment and helps us develop an interval type-2 fuzzy group decision-making method.

The purpose of this study was to develop a group decision-making method for solving MCDA problems with both alternative evaluations and criterion importance values expressed as IT2FNs to be considered via signed distances. In addition, a method for estimating the criterion weights (non-negative crisp numbers and normalized to sum to one) from the IT2FN preference information using an integrated programming model was further developed to make more practical contributions to decision-making reality. This article is organized as follows. Section 2 briefly reviews the concept of IT2FSs and introduces linguistic variables, which can be converted to IT2FNs. Section 3 defines the signed distance between IT2FNs and discusses some properties of the proposed signed distance. The multi-person MCDA problem with IT2FN data is formulated in Sect. 4. In addition, Sect. 4 establishes an integrated approach to combine the objective and subjective importance values of criteria and develops a ranking procedure based on a signed-distance-based method in the decision context of IT2FNs. If assessing the priority weights of criteria is important to the decision-makers, another version of the MCDA method is provided in Sect. 5 to estimate criterion weights from the IT2FN data. Section 6 illustrates and discusses the proposed methods using a practical clinical medicine problem of patient-centered care, and gives a deep comparative analysis and discussions. Further discussions on the proposed methodology and conclusions are provided in Sect. 7.

2 Preliminaries

Based on the IT2FN framework, we used the interval type-2 trapezoidal fuzzy numbers (IT2TrFNs) to propound the signed-distance-based group decision-making method for MCDA. This section reviews a few relevant definitions and operations of IT2FSs and IT2TrFNs.

Definition 1 Let Int([0, 1]) stand for the set of all closed subintervals of [0, 1]. Let X be an ordinary finite nonempty set. An IT2FS A in the universe of discourse X is an expression given by $A = \{\langle x, \mu_A(x)\rangle \,|x \in X\}$, where the function $\mu_A : X \to \text{Int}([0, 1])$ defines the degree of membership of an element x to A, such that $x \to \mu_A(x) = \left[\mu_A^-(x), \mu_A^+(x)\right]$. All IT2FSs on X are denoted by IT2FS(X).

Definition 2 Let $A \in \text{IT2FS}\,(X)$. If $A(x)$ is a convex set and is defined in a closed and bounded interval, then A is called an IT2FN on X. Let $A(x) = \left[A^L(x), A^U(x)\right]$, where $0 \le A^L(x) \le A^U(x) \le 1,\ x \in X,\ A^L : X \to [0, 1]$, and $A^U : X \to [0, 1]$. All IT2FNs on X are denoted by IT2FN(X).

Definition 3 Let A^L and A^U be a lower and an upper trapezoidal fuzzy numbers, and let h_A^L and h_A^U denote the heights of A^L and A^U. An IT2TrFN A defined on X is represented by the following:

$$A = \left[A^L, A^U\right] = \left[\left(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L\right), \left(a_1^U, a_2^U, a_3^U, a_4^U; h_A^U\right)\right], \tag{1}$$

where $a_1^L \leq a_2^L \leq a_3^L \leq a_4^L,\ a_1^U \leq a_2^U \leq a_3^U \leq a_4^U,\ 0 \leq h_A^L \leq h_A^U \leq 1, a_1^U \leq a_1^L$, and $a_4^L \leq a_4^U$. In addition, $A^L \subset A^U$ (if and only if $\forall x \in X, \mu_{A^L}(x) \leq \mu_{A^U}(x)$), in which the membership functions of A^L and A^U are expressed as the following:

$$A^L(x) = \begin{cases} h_A^L\left(x - a_1^L\right)/\left(a_2^L - a_1^L\right) & \text{for } a_1^L \leq x \leq a_2^L, \\ h_A^L & \text{for } a_2^L \leq x \leq a_3^L, \\ h_A^L\left(a_4^L - x\right)/\left(a_4^L - a_3^L\right) & \text{for } a_3^L \leq x \leq a_4^L, \\ 0 & \text{otherwise;} \end{cases} \tag{2}$$

$$A^U(x) = \begin{cases} h_A^U\left(x - a_1^U\right)/\left(a_2^U - a_1^U\right) & \text{for } a_1^U \leq x \leq a_2^U, \\ h_A^U & \text{for } a_2^U \leq x \leq a_3^U, \\ h_A^U\left(a_4^U - x\right)/\left(a_4^U - a_3^U\right) & \text{for } a_3^U \leq x \leq a_4^U, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Definition 4 The arithmetic operations between the IT2TrFNs $A = \left[\left(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L\right), \left(a_1^U, a_2^U, a_3^U, a_4^U; h_A^U\right)\right]$ and $B = \left[\left(b_1^L, b_2^L, b_3^L, b_4^L; h_B^L\right), \left(b_1^U, b_2^U, b_3^U, b_4^U; h_B^U\right)\right]$ are defined as:

(1) Addition operation

$$\begin{aligned} A \oplus B = \Big[&\left(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; \min\left(h_A^L, h_B^L\right)\right), \\ &\left(a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; \min\left(h_A^U, h_B^U\right)\right)\Big]. \end{aligned} \tag{4}$$

(2) Subtraction operation

$$\begin{aligned} A \Theta B = \Big[&\left(a_1^L - b_4^L, a_2^L - b_3^L, a_3^L - b_2^L, a_4^L - b_1^L; \min\left(h_A^L, h_B^L\right)\right), \\ &\left(a_1^U - b_4^U, a_2^U - b_3^U, a_3^U - b_2^U, a_4^U - b_1^U; \min\left(h_A^U, h_B^U\right)\right)\Big]. \end{aligned} \tag{5}$$

(3) Multiplication operation ($a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$, and b_4^U are positive real numbers)

$$\begin{aligned} A \otimes B = \Big[&\left(a_1^L \times b_1^L, a_2^L \times b_2^L, a_3^L \times b_3^L, a_4^L \times b_4^L; \min\left(h_A^L, h_B^L\right)\right), \\ &\left(a_1^U \times b_1^U, a_2^U \times b_2^U, a_3^U \times b_3^U, a_4^U \times b_4^U; \min\left(h_A^U, h_B^U\right)\right)\Big]. \end{aligned} \tag{6}$$

(4) Division operation ($a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, a_4^U, b_1^L, b_2^L, b_3^L, b_4^L, b_1^U, b_2^U, b_3^U$, and b_4^U are nonzero positive real numbers)

$$\begin{aligned} A \emptyset B = \Big[&\left(a_1^L/b_4^L, a_2^L/b_3^L, a_3^L/b_2^L, a_4^L/b_1^L; \min\left(h_A^L, h_B^L\right)\right), \\ &\left(a_1^U/b_4^U, a_2^U/b_3^U, a_3^U/b_2^U, a_4^U/b_1^U; \min\left(h_A^U, h_B^U\right)\right)\Big]. \end{aligned} \tag{7}$$

(5) Multiplication by an ordinary number

$$\begin{aligned} q \cdot A &= A \cdot q \\ &= \begin{cases} \left[(q \times a_1^L, q \times a_2^L, q \times a_3^L, q \times a_4^L; h_A^L),\right. \\ \quad \left.(q \times a_1^U, q \times a_2^U, q \times a_3^U, q \times a_4^U; h_A^U)\right] & \text{if } q \geq 0, \\ \left[(q \times a_4^L, q \times a_3^L, q \times a_2^L, q \times a_1^L; h_A^L),\right. \\ \quad \left.(q \times a_4^U, q \times a_3^U, q \times a_2^U, q \times a_1^U; h_A^U)\right] & \text{if } q \leq 0. \end{cases} \end{aligned} \tag{8}$$

(6) Division by an ordinary number (q is a nonzero number)

$$\frac{A}{q}=\begin{cases}\left[\left(\frac{a_1^L}{q},\frac{a_2^L}{q},\frac{a_3^L}{q},\frac{a_4^L}{q};h_A^L\right),\left(\frac{a_1^U}{q},\frac{a_2^U}{q},\frac{a_3^U}{q},\frac{a_4^U}{q};h_A^U\right)\right] & \text{if } q>0,\\ \left[\left(\frac{a_4^L}{q},\frac{a_3^L}{q},\frac{a_2^L}{q},\frac{a_1^L}{q};h_A^L\right),\left(\frac{a_4^U}{q},\frac{a_3^U}{q},\frac{a_2^U}{q},\frac{a_1^U}{q};h_A^U\right)\right] & \text{if } q<0.\end{cases} \tag{9}$$

(7) Inverse operation ($a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U$, and a_4^U are nonzero positive real numbers)

$$A^{-1}=\left[\left(1/a_4^L, 1/a_3^L, 1/a_2^L, 1/a_1^L; h_A^L\right), \left(1/a_4^U, 1/a_3^U, 1/a_2^U, 1/a_1^U; h_A^U\right)\right]. \tag{10}$$

(8) ηth power operation ($a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U$, and a_4^U are positive real numbers and η is a natural number excluding 0)

$$A^\eta=\left[\left((a_1^L)^\eta, (a_2^L)^\eta, (a_3^L)^\eta, (a_4^L)^\eta; h_A^L\right), \left((a_1^U)^\eta, (a_2^U)^\eta, (a_3^U)^\eta, (a_4^U)^\eta; h_A^U\right)\right]. \tag{11}$$

(9) Root of order η ($a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U$, and a_4^U are positive real numbers and η is a natural number excluding 0)

$$A^{\frac{1}{\eta}}=\left[\left(\sqrt[\eta]{a_1^L}, \sqrt[\eta]{a_2^L}, \sqrt[\eta]{a_3^L}, \sqrt[\eta]{a_4^L}; h_A^L\right), \left(\sqrt[\eta]{a_1^U}, \sqrt[\eta]{a_2^U}, \sqrt[\eta]{a_3^U}, \sqrt[\eta]{a_4^U}; h_A^U\right)\right]. \tag{12}$$

Note that the operations of multiplication, division, inverse, ηth power, and root of order η produce approximate IT2TrFNs for simple computations.

3 Linguistic ratings and signed distances for IT2TrFNs

This section introduces the concepts of linguistic variables and signed distances to effectively construct and handle IT2TrFN data.

3.1 Linguistic variables for IT2TrFN ratings

In the IT2TrFN context, the alternative evaluations and the criterion importance were expressed as IT2TrFNs. However, it may be difficult to directly collect IT2TrFN data. In fact, most of the decision-makers are constantly making decisions within linguistic environments in practice [45,46,56,60,61]. The fuzzy linguistic approach represents qualitative aspects as linguistic values by means of linguistic variables [56]. Thus, a direct method of survey research can be used to collect linguistic information to construct degrees of membership [31], and these linguistic values are often represented by fuzzy numbers. Thus, the ratings in this paper were considered to be linguistic variables to overcome the difficulty of data collection in an IT2TrFN framework.

For better sensitivity, this study adopted a nine-linguistic-term set, which originates from Chen's 1996 work, to accurately measure variability in responses. In general, the greater the number of scale categories, the finer the discrimination among decision outcomes or criterion importance that is possible. Traditional guidelines suggest that the appropriate number of categories should be seven plus or minus two, that is, between five and nine [23,25,54].

Table 1 Linguistic variables and their corresponding IT2TrFNs

Linguistic terms	Corresponding IT2TrFNs
Absolutely low (AL)	[(0.0, 0.0, 0.0, 0.0; 1.0), (0.0, 0.0, 0.0, 0.0; 1.0)]
Very low (VL)	[(0.0075, 0.0075, 0.015, 0.0525; 0.8), (0.0, 0.0, 0.02, 0.07; 1.0)]
Low (L)	[(0.0875, 0.12, 0.16, 0.1825; 0.8), (0.04, 0.10, 0.18, 0.23; 1.0)]
Medium low (ML)	[(0.2325, 0.255, 0.325, 0.3575; 0.8), (0.17, 0.22, 0.36, 0.42; 1.0)]
Medium (M)	[(0.4025, 0.4525, 0.5375, 0.5675; 0.8), (0.32, 0.41, 0.58, 0.65; 1.0)]
Medium high (MH)	[(0.65, 0.6725, 0.7575, 0.79; 0.8), (0.58, 0.63, 0.80, 0.86; 1.0)]
High (H)	[(0.7825, 0.815, 0.885, 0.9075; 0.8), (0.72, 0.78, 0.92, 0.97; 1.0)]
Very high (VH)	[(0.9475, 0.985, 0.9925, 0.9925; 0.8), (0.93, 0.98, 1.0, 1.0; 1.0)]
Absolutely high (AH)	[(1.0, 1.0, 1.0, 1.0; 1.0), (1.0, 1.0, 1.0, 1.0; 1.0)]

In this paper, sophisticated data of IT2TrFNs were used to develop a group decision-making method. It follows that the MCDA problem in this study was complicated or large-scaled and required more sensitive and precise input data for decision aiding.

Several factors were taken into account in deciding on the number of categories for this study. First, the individual responses of each decision-maker were of interest in group decision-making. In addition, the investigated data were analyzed by the proposed sophisticated fuzzy techniques to deal with the complex nature of the MCDA problem. Therefore, a nine-linguistic-term set was sufficient and necessary in this study. Second, because all decision-makers were involved in the scaling task in a complex MCDA problem and were moderately to highly knowledgeable about the alternative evaluations or the criterion importance, it was appropriate to employ a large number of categories (e.g., nine categories) [38]. Third, faced with the necessity of choosing among non-inferior/non-dominant alternatives, the decision-makers evaluated and provided available information on the alternatives and eventually established a preference order. When a person made a decision based on available alternatives, each of which had certain advantages and disadvantages over the others, varying levels of post-decision dissonance resulted [41]. Because the decision-maker might have doubts and anxieties about the choices, a scale of fine discrimination was required to differentiate the relative attractiveness of the alternatives and to reduce post-decision dissonance further. Following the discussions above, a nine-linguistic-term set was adopted in this paper.

The linguistic variables can be converted to IT2TrFNs, as depicted in Table 1. There were nine translations of linguistic terms into IT2TrFNs, and each of nine-point interval linguistic terms is corresponding to the nine-member linguistic term set developed by [10]. Based on Table 1, the linguistic terms were easily converted to IT2TrFNs, including lower trapezoidal fuzzy numbers [17] and upper trapezoidal fuzzy numbers [16,17,57]. The height of the lower trapezoidal fuzzy numbers was modified to 0.80 according to [57], except for the responses AL (absolutely low) and AH (absolutely high).

As indicated in Table 1, the uncertain linguistic problems in MCDA were handled using IT2TrFN values. Although the expression of IT2TrFNs seems very complicated, the mathematical manipulation and the computational process are substantially simple and effective via the employment of signed distances between IT2TrFNs. In addition, IT2TrFNs are associated with greater imprecision and more ambiguities than type-1 fuzzy numbers. Decision-making information provided by the decision-maker is often imprecise or uncertain, due to lack of data, time pressure, or the decision-maker's limited attention and information-processing capabilities. For these reasons, we can rationally justify that the employment of the predefined linguistic set of IT2TrFNs is a flexible method capable of tackling such MCDA problems and modeling second-order uncertainties [29].

3.2 Signed distance of IT2TrFNs

The concept of signed distances was extended to develop a new method for obtaining objective criteria importance values and for solving MCDA problems with IT2TrFN data. According to the axiom of choice, alternatives that were closer to the ideal were preferred to those that were farther away. The rationale of human choice is to be as close as possible to the perceived ideal. Thus, the basic principle of the proposed method was that the chosen alternative should have the shortest distance to the ideal solution. In the context of IT2TrFNs, this method employed a signed-distance-based approach to identify the separation measures of each alternative from the ideal solution. In the literature, the concept of signed distances can be used to determine rankings of fuzzy numbers. Linear scale normalization was conducted to transform the various IT2TrFN scales into a single scale between zero and one. Then, the position of the ideal solution could be defined as [(1,1,1,1;1), (1,1,1,1;1)], which was placed on the y-axis at $x = 1$.

Let A and B be two IT2TrFNs defined on the universe of discourse X, where $A = \left[A^L, A^U\right] = \left[\left(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L\right), \left(a_1^U, a_2^U, a_3^U, a_4^U; h_A^U\right)\right]$ and $B = \left[B^L, B^U\right] = \left[\left(b_1^L, b_2^L, b_3^L, b_4^L; h_B^L\right), \left(b_1^U, b_2^U, b_3^U, b_4^U; h_B^U\right)\right]$. In the following description, which was motivated by the idea proposed in [22], the signed distance of an IT2TrFN from the y-axis at $x = 1$ was determined. The proofs of Propositions 1 and 2 as well as Properties 1–4 have been provided in the author's previous researches [14,15].

Proposition 1 *Let the level 1 fuzzy number $\tilde{1}_1$ map onto the y-axis at $x = 1$. The signed distances from A^L and A^U to $\tilde{1}_1$ are the following:*

$$d\left(A^L, \tilde{1}_1\right) = \frac{1}{4}\left(a_1^L + a_2^L + a_3^L + a_4^L - 4\right), \tag{13}$$

$$d\left(A^U, \tilde{1}_1\right) = \frac{1}{4}\left(a_1^U + a_2^U + a_3^U + a_4^U - 4\right). \tag{14}$$

Property 1 If both $A^L = \left(a^L, a^L, a^L, a^L; h_A^L\right)$ and $A^U = \left(a^U, a^U, a^U, a^U; h_A^U\right)$, then the absolute value of the signed distances in Proposition 1 is identical to the Hamming distance between the corresponding ordinary number (a^L or a^U) and $\tilde{1}_1$.

Property 2 A^L is located at $\tilde{1}_1$ if and only if $d\left(A^L, \tilde{1}_1\right) = 0$. Both A^L andA^U are located at $\tilde{1}_1$ if and only if $d\left(A^U, \tilde{1}_1\right) = 0$.

Proposition 2 *The signed distance from A to $\tilde{1}_1$ is the following:*

$$d(A, \tilde{1}_1) = \frac{1}{8}\Bigg(a_1^L + a_2^L + a_3^L + a_4^L + 4a_1^U + 2a_2^U + 2a_3^U + 4a_4^U + 3\left(a_2^U + a_3^U - a_1^U - a_4^U\right)\frac{h_A^L}{h_A^U} - 16\Bigg). \tag{15}$$

when $0 < h_A^L = h_A^U \leq 1$,

$$d\left(A, \tilde{1}_1\right) = \frac{1}{8}\left(a_1^L + a_2^L + a_3^L + a_4^L + a_1^U + 5a_2^U + 5a_3^U + a_4^U - 16\right). \tag{16}$$

Property 3 A is located at $\tilde{1}_1$ if and only if $d\left(A, \tilde{1}_1\right) = 0$, where $a_1^L = a_2^L = a_3^L = a_4^L = a_1^U = a_2^U = a_3^U = a_4^U = 1$.

Property 4 Let C be an IT2TrFN defined on X, where $C = [(1, 1, 1, 1; 1), (1, 1, 1, 1; 1)]$. The IT2TrFN A is closer to the IT2TrFN C than the IT2TrFN B if and only if $d(A, \tilde{1}_1) > d(B, \tilde{1}_1)$.

Because the signed distances $d\left(A, \tilde{1}_1\right)$ and $d\left(B, \tilde{1}_1\right)$ are real numbers, they satisfy linear ordering. Additionally, one of the following three conditions must hold: $d\left(A, \tilde{1}_1\right) > d\left(B, \tilde{1}_1\right)$, $d\left(A, \tilde{1}_1\right) = d\left(B, \tilde{1}_1\right)$, or $d\left(A, \tilde{1}_1\right) < d\left(B, \tilde{1}_1\right)$. It follows that the signed distance based on IT2TrFNs satisfies the law of trichotomy as indicated in Definition 5. By utilizing the signed distances, a new MCDA method was developed to order the priorities of various alternatives.

Definition 5 According to the principle of "the higher the better", the ranking of A and B by the signed distances $d\left(A, \tilde{1}_1\right)$ and $d\left(B, \tilde{1}_1\right)$ on X can be defined as the following:

(1) $d\left(A, \tilde{1}_1\right) > d\left(B, \tilde{1}_1\right)$ if and only if $A \succ B$;

(2) $d\left(A, \tilde{1}_1\right) = d\left(B, \tilde{1}_1\right)$ if and only if $A \sim B$;

(3) $d\left(A, \tilde{1}_1\right) < d\left(B, \tilde{1}_1\right)$ if and only if $A \prec B$.

4 MCDA method based on IT2TrFNs

This section first describes a decision environment based on IT2TrFNs for MCDA problems that was constructed by aggregating multiple decision-makers' opinions. It then presents an integrated approach to determine criterion importance according to the subjective preference information and the objective decision-relevant information. A new method suitable for handling IT2TrFN data is also proposed as a method for estimating the objective importance of criteria. A ranking procedure based on a signed-distance-based method was developed for group decision-making, and an effective algorithm was developed to solve the MCDA problem in the context of IT2TrFNs.

4.1 Decision environment defined over IT2TrFNs

For an MCDA problem, the evaluations of each alternative with respect to each criterion and the grades of importance for decision criteria can be expressed using IT2TrFNs. Suppose that there is a non-inferior/non-dominant set of alternatives, $A = \{A_1, A_2, \ldots, A_m\}$, where m is the number of alternatives. Each alternative is assessed based on n criteria, which are denoted by $X = \{x_1, x_2, \ldots, x_n\}$. The criterion set X can be divided into two sets, X_b and X_c, where X_b denotes a collection of benefit criteria (i.e., larger values of x_j indicate a greater preference), X_c denotes a collection of cost criteria (i.e., smaller values of x_j indicate a greater preference), $X_b \cap X_c = \phi$, and $X_b \cup X_c = X$.

Assume that $E = \{E_1, E_2, \ldots, E_K\}$ is the set of decision-makers involved in the decision process. Based on the linguistic variables in Table 1, each decision-maker constructs positive IT2TrFNs using linguistic terms to estimate the subjective importance values of the criteria and to evaluate the alternatives for each criterion according to his/her experience and preference. Let $A_{ij}^{k} = \left[A_{ij}^{kL}, A_{ij}^{kU}\right]$ be an evaluation value of the alternative $A_i \in A$ with respect to each criterion $x_j \in X$ $(j = 1, 2, \ldots, n)$ provided by the kth decision-maker $(k = 1, 2, \ldots, K)$. A_{ij}^{kL} and A_{ij}^{kU} denote the lower extreme and the upper extreme of the IT2TrFN A_{ij}^{k}, respectively, where $A_{ij}^{kL} = \left(a_{1ij}^{kL}, a_{2ij}^{kL}, a_{3ij}^{kL}, a_{4ij}^{kL}; h_{A_{ij}}^{kL}\right)$, $A_{ij}^{kU} = \left(a_{1ij}^{kU}, a_{2ij}^{kU}, a_{3ij}^{kU}, a_{4ij}^{kU}; h_{A_{ij}}^{kU}\right)$, and $A_{ij}^{kL} \subset A_{ij}^{kU}$. Note that $0 \le a_{1ij}^{kL} \le a_{2ij}^{kL} \le a_{3ij}^{kL} \le a_{4ij}^{kL} \le 1$ and $0 \le a_{1ij}^{kU} \le a_{2ij}^{kU} \le a_{3ij}^{kU} \le a_{4ij}^{kU} \le 1$. It, therefore, follows that the rating of alternative A_i with respect to criterion x_j can be calculated as the following:

$$\begin{aligned} A_{ij} &= \frac{1}{K} \cdot \left([A_{ij}^{1L}, A_{ij}^{1U}] \oplus [A_{ij}^{2L}, A_{ij}^{2U}] \oplus \cdots \oplus [A_{ij}^{KL}, A_{ij}^{KU}]\right) \\ &= \left[\left(\frac{\sum_{k=1}^{K} a_{1ij}^{kL}}{K}, \frac{\sum_{k=1}^{K} a_{2ij}^{kL}}{K}, \frac{\sum_{k=1}^{K} a_{3ij}^{kL}}{K}, \frac{\sum_{k=1}^{K} a_{4ij}^{kL}}{K}; \min_k h_{A_{ij}}^{kL}\right),\right. \\ &\qquad \left.\left(\frac{\sum_{k=1}^{K} a_{1ij}^{kU}}{K}, \frac{\sum_{k=1}^{K} a_{2ij}^{kU}}{K}, \frac{\sum_{k=1}^{K} a_{3ij}^{kU}}{K}, \frac{\sum_{k=1}^{K} a_{4ij}^{kU}}{K}; \min_k h_{A_{ij}}^{kU}\right)\right]. \end{aligned} \tag{17}$$

Let us denote $a_{1ij}^{L} = \Sigma_{k=1}^{K} a_{1ij}^{kL}/K$, $a_{2ij}^{L} = \Sigma_{k=1}^{K} a_{2ij}^{kL}/K$, $a_{3ij}^{L} = \Sigma_{k=1}^{K} a_{3ij}^{kL}/K$, $a_{4ij}^{L} = \Sigma_{k=1}^{K} a_{4ij}^{kL}/K$, $a_{1ij}^{U} = \Sigma_{k=1}^{K} a_{1ij}^{kU}/K$, $a_{2ij}^{U} = \Sigma_{k=1}^{K} a_{2ij}^{kU}/K$, $a_{3ij}^{U} = \Sigma_{k=1}^{K} a_{3ij}^{kU}/K$, $a_{4ij}^{U} = \Sigma_{k=1}^{K} a_{4ij}^{kU}/K$, $h_{A_{ij}}^{L} = \min_k h_{A_{ij}}^{kL}$, and $h_{A_{ij}}^{U} = \min_k h_{A_{ij}}^{kU}$ for brevity. The evaluation of alternative A_i on criterion x_j can then be expressed as the following:

$$A_{ij} = \left[A_{ij}^{L}, A_{ij}^{U}\right] = \left[\left(a_{1ij}^{L}, a_{2ij}^{L}, a_{3ij}^{L}, a_{4ij}^{L}; h_{A_{ij}}^{L}\right), \left(a_{1ij}^{U}, a_{2ij}^{U}, a_{3ij}^{U}, a_{4ij}^{U}; h_{A_{ij}}^{U}\right)\right], \tag{18}$$

where $0 \le a_{1ij}^{L} \le a_{2ij}^{L} \le a_{3ij}^{L} \le a_{4ij}^{L} \le 1$, $0 \le a_{1ij}^{U} \le a_{2ij}^{U} \le a_{3ij}^{U} \le a_{4ij}^{U} \le 1$, $0 \le h_{A_{ij}}^{L} \le h_{A_{ij}}^{U} \le 1$, $a_{1ij}^{U} \le a_{1ij}^{L}$, $a_{4ij}^{L} \le a_{4ij}^{U}$, and $A_{ij}^{L} \subset A_{ij}^{U}$. It follows that the decision-matrix D is defined in the following way:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \end{array}. \tag{19}$$

The characteristics of the alternative A_i can be represented by the IT2TrFN in the following way:

$$\begin{aligned} A_i &= \left\{\left\langle x_1, [A_{i1}^{L}, A_{i1}^{U}]\right\rangle, \left\langle x_2, [A_{i2}^{L}, A_{i2}^{U}]\right\rangle, \ldots, \left\langle x_n, [A_{in}^{L}, A_{in}^{U}]\right\rangle\right\} \\ &= \left\{\left\langle x_j, [A_{ij}^{L}, A_{ij}^{U}]\right\rangle \middle| x_j \in X, j = 1, 2, \ldots, n\right\}, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{20}$$

In a similar manner, the IT2TrFN can be used to express the subjective importance for various decision criteria during the decision-maker's evaluation process. Based on the linguistic terms, an IT2TrFN W^{k}, which is provided by the kth decision-maker and is defined on the universe of discourse X, is an object of the following form:

$$\begin{aligned} W^k &= \left\{\left\langle x_1, [W_1^{kL}, W_1^{kU}]\right\rangle, \left\langle x_2, [W_2^{kL}, W_2^{kU}]\right\rangle, \ldots, \left\langle x_n, [W_n^{kL}, W_n^{kU}]\right\rangle\right\} \\ &= \left\{\left\langle x_j, [W_j^{kL}, W_j^{kU}]\right\rangle \middle| x_j \in X, j = 1, 2, \ldots, n\right\}, \quad k = 1, 2, \ldots, K, \end{aligned} \tag{21}$$

where $W_j^{kL} = \left(w_{1j}^{kL}, w_{2j}^{kL}, w_{3j}^{kL}, w_{4j}^{kL}; h_{W_j}^{kL}\right)$, $W_j^{kU} = \left(w_{1j}^{kU}, w_{2j}^{kU}, w_{3j}^{kU}, w_{4j}^{kU}; h_{W_j}^{kU}\right)$, and $W_j^{kL} \subset W_j^{kU}$. Note that $0 \le w_{1j}^{kL} \le w_{2j}^{kL} \le w_{3j}^{kL} \le w_{4j}^{kL} \le 1$ and $0 \le w_{1j}^{kU} \le w_{2j}^{kU} \le w_{3j}^{kU} \le w_{4j}^{kU} \le 1$. It follows that the average subjective importance grade of criterion x_j for all decision-makers is calculated as follows:

$$\begin{aligned} W_j &= \frac{1}{K} \cdot \left(\left[W_j^{1L}, W_j^{1U}\right] \oplus \left[W_j^{2L}, W_j^{2U}\right] \oplus \cdots \oplus \left[W_j^{KL}, W_j^{KU}\right]\right) \\ &= \left[\left(\frac{\sum_{k=1}^K w_{1j}^{kL}}{K}, \frac{\sum_{k=1}^K w_{2j}^{kL}}{K}, \frac{\sum_{k=1}^K w_{3j}^{kL}}{K}, \frac{\sum_{k=1}^K w_{4j}^{kL}}{K}; \min_k h_{W_j}^{kL}\right),\right. \\ &\quad \left.\left(\frac{\sum_{k=1}^K w_{1j}^{kU}}{K}, \frac{\sum_{k=1}^K w_{2j}^{kU}}{K}, \frac{\sum_{k=1}^K w_{3j}^{kU}}{K}, \frac{\sum_{k=1}^K w_{4j}^{kU}}{K}; \min_k h_{W_j}^{kU}\right)\right]. \end{aligned} \tag{22}$$

Let us denote $w_{1j}^L = \Sigma_{k=1}^K w_{1j}^{kL}/K$, $w_{2j}^L = \Sigma_{k=1}^K w_{2j}^{kL}/K$, $w_{3j}^L = \Sigma_{k=1}^K w_{3j}^{kL}/K$, $w_{4j}^L = \Sigma_{k=1}^K w_{4j}^{kL}/K$, $w_{1j}^U = \Sigma_{k=1}^K w_{1j}^{kU}/K$, $w_{2j}^U = \Sigma_{k=1}^K w_{2j}^{kU}/K$, $w_{3j}^U = \Sigma_{k=1}^K w_{3j}^{kU}/K$, $w_{4j}^U = \Sigma_{k=1}^K w_{4j}^{kU}/K$, $h_{W_j}^L = \min_k h_{W_j}^{kL}$, and $h_{W_j}^U = \min_k h_{W_j}^{kU}$ for brevity. It then follows that the subjective importance of criterion x_j can be expressed as follows:

$$W_j = \left[W_j^L, W_j^U\right] = \left[\left(w_{1j}^L, w_{2j}^L, w_{3j}^L, w_{4j}^L; h_{W_j}^L\right), \left(w_{1j}^U, w_{2j}^U, w_{3j}^U, w_{4j}^U; h_{W_j}^U\right)\right], \tag{23}$$

where $0 \le w_{1j}^L \le w_{2j}^L \le w_{3j}^L \le w_{4j}^L \le 1$, $0 \le w_{1j}^U \le w_{2j}^U \le w_{3j}^U \le w_{4j}^U \le 1$, $0 \le h_{W_j}^L \le h_{W_j}^U \le 1$, $w_{1j}^U \le w_{1j}^L$, and $w_{4j}^L \le w_{4j}^U$.

4.2 The MCDA method using a signed-distance-based approach

Based on the proposed signed distance of IT2TrFNs, a new MCDA method was developed to handle IT2TrFN data. As stated previously, the linguistic variables can be described by IT2TrFNs (Table 1), and the MCDA problem can be concisely expressed in a matrix format (D) after aggregating the decision-makers' opinions. In addition, linear scale normalization was used to transform the various criteria values to ensure that the best criterion value was located in the level 1 fuzzy number $\tilde{1}_1$.

4.3 Normalization of the decision matrix and subjective importance

The initial data with respect to each criterion can be normalized to the maximum criterion values for benefit criteria and the minimum criterion values for cost criteria. Let $a_j^+ = \max_{i_1} a_{4i_1j}^U$ (for $x_j \in X_b$) and $a_j^- = \min_{i_1} a_{1i_1j}^U$ (for $x_j \in X_c$). The transformed outcome of A_{ij} is denoted by the following:

$$\bar{A}_{ij} = \left[\bar{A}_{ij}^L, \bar{A}_{ij}^U\right] = \left[\left(\bar{a}_{1ij}^L, \bar{a}_{2ij}^L, \bar{a}_{3ij}^L, \bar{a}_{4ij}^L; \bar{h}_{A_{ij}}^L\right), \left(\bar{a}_{1ij}^U, \bar{a}_{2ij}^U, \bar{a}_{3ij}^U, \bar{a}_{4ij}^U; \bar{h}_{A_{ij}}^U\right)\right], \tag{24}$$

where

$$\left[\left(\bar{a}^L_{1ij}, \bar{a}^L_{2ij}, \bar{a}^L_{3ij}, \bar{a}^L_{4ij}; \bar{h}^L_{A_{ij}}\right), \left(\bar{a}^U_{1ij}, \bar{a}^U_{2ij}, \bar{a}^U_{3ij}, \bar{a}^U_{4ij}; \bar{h}^U_{A_{ij}}\right)\right]$$
$$= \begin{cases} \left[\left(\frac{a^L_{1ij}}{a^+_j}, \frac{a^L_{2ij}}{a^+_j}, \frac{a^L_{3ij}}{a^+_j}, \frac{a^L_{4ij}}{a^+_j}; h^L_{A_{ij}}\right), \left(\frac{a^U_{1ij}}{a^+_j}, \frac{a^U_{2ij}}{a^+_j}, \frac{a^U_{3ij}}{a^+_j}, \frac{a^U_{4ij}}{a^+_j}; h^U_{A_{ij}}\right)\right] & \text{if } x_j \in X_b, \\ \left[\left(\frac{a^-_j}{a^L_{4ij}}, \frac{a^-_j}{a^L_{3ij}}, \frac{a^-_j}{a^L_{2ij}}, \frac{a^-_j}{a^L_{1ij}}; h^L_{A_{ij}}\right), \left(\frac{a^-_j}{a^U_{4ij}}, \frac{a^-_j}{a^U_{3ij}}, \frac{a^-_j}{a^U_{2ij}}, \frac{a^-_j}{a^U_{1ij}}; h^U_{A_{ij}}\right)\right] & \text{if } x_j \in X_c. \end{cases} \tag{25}$$

Then, the normalized decision matrix, D^N, is constructed as follows:

$$D^N = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{bmatrix} [\bar{A}^L_{11}, \bar{A}^U_{11}] & [\bar{A}^L_{12}, \bar{A}^U_{12}] & \cdots & [\bar{A}^L_{1n}, \bar{A}^U_{1n}] \\ [\bar{A}^L_{21}, \bar{A}^U_{21}] & [\bar{A}^L_{22}, \bar{A}^U_{22}] & \cdots & [\bar{A}^L_{2n}, \bar{A}^U_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\bar{A}^L_{m1}, \bar{A}^U_{m1}] & [\bar{A}^L_{m2}, \bar{A}^U_{m2}] & \cdots & [\bar{A}^L_{mn}, \bar{A}^U_{mn}] \end{bmatrix} \end{array}. \tag{26}$$

The subjective importance of the decision criteria can be normalized using a similar procedure. Let $w^+ = \max_j w^U_{4j}$. It then follows that the normalized subjective importance of criterion x_j is denoted by the following:

$$\overline{W}_j = \left[\overline{W}^L_j, \overline{W}^U_j\right] = \left[\left(\bar{w}^L_{1j}, \bar{w}^L_{2j}, \bar{w}^L_{3j}, \bar{w}^L_{4j}; \bar{h}^L_{W_j}\right), \left(\bar{w}^U_{1j}, \bar{w}^U_{2j}, \bar{w}^U_{3j}, \bar{w}^U_{4j}; \bar{h}^U_{W_j}\right)\right], \tag{27}$$

where

$$\left[\left(\bar{w}^L_{1j}, \bar{w}^L_{2j}, \bar{w}^L_{3j}, \bar{w}^L_{4j}; \bar{h}^L_{W_j}\right), \left(\bar{w}^U_{1j}, \bar{w}^U_{2j}, \bar{w}^U_{3j}, \bar{w}^U_{4j}; \bar{h}^U_{W_j}\right)\right]$$
$$= \left[\left(\frac{w^L_{1j}}{w^+}, \frac{w^L_{2j}}{w^+}, \frac{w^L_{3j}}{w^+}, \frac{w^L_{4j}}{w^+}; h^L_{W_j}\right), \left(\frac{w^U_{1j}}{w^+}, \frac{w^U_{2j}}{w^+}, \frac{w^U_{3j}}{w^+}, \frac{w^U_{4j}}{w^+}; h^U_{W_j}\right)\right]. \tag{28}$$

4.3.1 Objective approach to criterion importance

When multiple criteria are considered, it is essential to appropriately measure the criterion importance grades. In this paper, both the subjective and the objective assessments of criterion importance were involved in the proposed MCDA method. The subjective assessment of criterion importance reflects the decision-maker's cultural, psychological, societal, and environmental backgrounds. Thus, a direct explication through interviews or questionnaire surveys (e.g., the linguistic variables in Table 1) can articulate the decision-maker's preferences based on the subjective importance of the criteria. The objective assessment of criterion importance depends on the actual decision situation. If more decision-relevant messages are emitted by a criterion, then the criterion is more salient in the decision-making process. In other words, if the alternative ratings are more distinct and differentiated with respect to a given criterion, then the contrast intensity of this criterion is larger, and the amount of decision information transmitted by this criterion is greater (and vice versa). In this respect, the objective importance of each criterion was measured with the deviation defined in the STEP method (STEM) [5] described in this study.

The STEM is a progressive method for multi-objective decision problems, and its properties include a displaced ideal and preference dependency. To determine relative deviations, the maximal IT2TrFN value and minimal IT2TrFN value of each column in D^N must be identified and regarded as positive- and negative-ideal anchor values, respectively. Because the signed distance based on IT2TrFNs as defined in Definition 5 satisfies the law of trichotomy, a comparison of the IT2TrFN values can be drawn via the signed distance to $\tilde{1}_1$. According to Proposition 2, the signed distance from the normalized outcome $\bar{A}_{ij}$ to $\tilde{1}_1$ is calculated as follows:

$$d\left(\bar{A}_{ij}, \tilde{1}_1\right) = \frac{1}{8}\left(\bar{a}^L_{1ij} + \bar{a}^L_{2ij} + \bar{a}^L_{3ij} + \bar{a}^L_{4ij} + 4\bar{a}^U_{1ij} + 2\bar{a}^U_{2ij} + 2\bar{a}^U_{3ij} + 4\bar{a}^U_{4ij} + 3\left(\bar{a}^U_{2ij} + \bar{a}^U_{3ij} - \bar{a}^U_{1ij} - \bar{a}^U_{4ij}\right)\frac{\bar{h}^L_{A_{ij}}}{\bar{h}^U_{A_{ij}}} - 16\right). \tag{29}$$

The positive-ideal anchor value $\bar{A}_{+j}$ and the negative-ideal anchor value $\bar{A}_{-j}$ of each criterion $x_j \in X$ can be computed from the following expressions:

$$\bar{A}_{+j} = \left\{\bar{A}_{ij} \,|\max_i d\left(\bar{A}_{ij}, \tilde{1}_1\right)\right\}, \tag{30}$$

$$\bar{A}_{-j} = \left\{\bar{A}_{ij} \left|\min_i d\left(\bar{A}_{ij}, \tilde{1}_1\right)\right.\right\}. \tag{31}$$

According to the definition in the STEM, the relative objective importance ϖ'_j of criterion x_j is placed into the deviations in the following way:

$$\varpi'_j = \Psi'_j \varnothing \left(\Psi'_1 \oplus \Psi'_2 \oplus \cdots \oplus \Psi'_n\right), \tag{32}$$

where

$$\Psi'_j = \left[\left(\bar{A}_{+j} \Theta \bar{A}_{-j}\right) \varnothing \bar{A}_{+j}\right] \otimes \left(\bar{A}^2_{1j} \oplus \bar{A}^2_{2j} \oplus \cdots \oplus \bar{A}^2_{mj}\right)^{-\frac{1}{2}}. \tag{33}$$

However, two problems follow from the above definitions. First, when the normalized outcomes $\bar{A}_{ij}$ are the same for all $A_i \in A$ with respect to a specific criterion x_j, the variation of the IT2TrFN values in x_j should be equal to zero, and criterion x_j does not transmit any useful information. Thus, $\bar{A}_{+j} \Theta \bar{A}_{-j} = \tilde{0}_{\bar{h}} (= \left[\left(0, 0, 0, 0; \bar{h}^L_{A^+_j}\right), \left(0, 0, 0, 0; \bar{h}^U_{A^+_j}\right)\right]), \varpi'_j = \tilde{0}_{\bar{h}}$, and the criterion x_j can be removed from further decision consideration at that time. Nevertheless, $\bar{A}_{+j} \Theta \bar{A}_{-j} \neq \tilde{0}_{\bar{h}}$ in the context of IT2TrFNs. If it is assumed that $\bar{A}_{+j} = \bar{A}_{-j} = \left[\left(\bar{a}^L_{1ij}, \bar{a}^L_{2ij}, \bar{a}^L_{3ij}, \bar{a}^L_{4ij}; \bar{h}^L_{A_{ij}}\right), \left(\bar{a}^U_{1ij}, \bar{a}^U_{2ij}, \bar{a}^U_{3ij}, \bar{a}^U_{4ij}; \bar{h}^U_{A_{ij}}\right)\right]$, then the following can be written:

$$\bar{A}_{+j} \Theta \bar{A}_{-j} = \left[\left(\bar{a}^L_{1ij} - \bar{a}^L_{4ij}, \bar{a}^L_{2ij} - \bar{a}^L_{3ij}, \bar{a}^L_{3ij} - \bar{a}^L_{2ij}, \bar{a}^L_{4ij} - \bar{a}^L_{1ij}; \bar{h}^L_{A_{ij}}\right), \left(\bar{a}^U_{1ij} - \bar{a}^U_{4ij}, \bar{a}^U_{2ij} - \bar{a}^U_{3ij}, \bar{a}^U_{3ij} - \bar{a}^U_{2ij}, \bar{a}^U_{4ij} - \bar{a}^U_{1ij}; \bar{h}^U_{A_{ij}}\right)\right] \neq \tilde{0}_{\bar{h}}. \tag{34}$$

The second problem is the consistency of the normalization procedures used in this study. The IT2TrFN values of alternatives for each criterion and the subjective importance of criteria use a linear scale normalization procedure to transform all values in a proportional way. If a different way of normalizing the Ψ'_j values in (32) is adopted, there are computational problems inherent to the presence of different scales within the decision environment. For sensibility in our amalgamation methods, the same linear scale normalization was consistently used in the STEM.

The signed distances were used to solve the first problem mentioned above. Let $\left|d\left(\bar{A}_{+j}, \tilde{1}_1\right)\right| = \left|\max_i d\left(\bar{A}_{ij}, \tilde{1}_1\right)\right|$ be the lower anchor value, and let $\left|d\left(\bar{A}_{-j}, \tilde{1}_1\right)\right| = \left|\min_i d\left(\bar{A}_{ij}, \tilde{1}_1\right)\right|$ be the upper anchor value. It follows that $\bar{A}_{+j}$ and $\bar{A}_{-j}$ in the first term of

(33) were replaced by $\left|d\left(\bar{A}_{-j},\tilde{1}_1\right)\right|$ and $\left|d\left(\bar{A}_{+j},\tilde{1}_1\right)\right|$, respectively. For consistency in the amalgamation method and to overcome the second problem, the linear scale transformation was used to compute the objective importance of the criteria. Thus, the formula in (33) was modified to the following:

$$\Psi_j=\left(\frac{\left|d\left(\bar{A}_{-j},\tilde{1}_1\right)\right|-\left|d\left(\bar{A}_{+j},\tilde{1}_1\right)\right|}{\left|d\left(\bar{A}_{-j},\tilde{1}_1\right)\right|}\right)\cdot\left(\bar{A}_{1j}^2\oplus\bar{A}_{2j}^2\oplus\cdots\oplus\bar{A}_{mj}^2\right)^{-\frac{1}{2}},\tag{35}$$

where

$$\begin{aligned}&\left(\bar{A}_{1j}^2\oplus\bar{A}_{2j}^2\oplus\cdots\oplus\bar{A}_{mj}^2\right)^{-\frac{1}{2}}\\&=\left[\left(1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{4ij}^L\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{3ij}^L\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{2ij}^L\right)^2},\right.\right.\\&\qquad\left.1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{1ij}^L\right)^2};\min_i\bar{h}_{A_{ij}}^L\right),\\&\quad\left(1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{4ij}^U\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{3ij}^U\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{2ij}^U\right)^2},\right.\\&\qquad\left.\left.1\Big/\sqrt{\textstyle\sum_{i=1}^m\left(\bar{a}_{1ij}^U\right)^2};\min_i\bar{h}_{A_{ij}}^U\right)\right].\end{aligned}\tag{36}$$

For brevity, Ψ_j is defined as $\left[\left(\psi_{1j}^L,\psi_{2j}^L,\psi_{3j}^L,\psi_{4j}^L;h_{\Psi_j}^L\right),\left(\psi_{1j}^U,\psi_{2j}^U,\psi_{3j}^U,\psi_{4j}^U;h_{\Psi_j}^U\right)\right]$. Let $\psi^+=\max_j\psi_{4j}^U$. By modifying (32) in the STEM, the relative objective importance $\varpi_j=\left[\varpi_j^L,\varpi_j^U\right]$ of criterion x_j can be defined as follows:

$$\varpi_j=\frac{1}{\psi^+}\cdot\Psi_j=\left[\left(\frac{\psi_{1j}^L}{\psi^+},\frac{\psi_{2j}^L}{\psi^+},\frac{\psi_{3j}^L}{\psi^+},\frac{\psi_{4j}^L}{\psi^+},h_{\Psi_j}^L\right),\left(\frac{\psi_{1j}^U}{\psi^+},\frac{\psi_{2j}^U}{\psi^+},\frac{\psi_{3j}^U}{\psi^+},\frac{\psi_{4j}^U}{\psi^+},h_{\Psi_j}^U\right)\right].\tag{37}$$

In (35) and (37), ϖ_j is the normalized objective importance of criterion x_j, and the IT2TrFN value of ϖ_j is dependent on the variation of criterion x_j from its anchor values. As the variation becomes larger, the emphasis on that particular criterion increases (and vice versa). This relationship demonstrates the criterion dependency and anchor dependency.

4.3.2 Integrated approach to subjective and objective importances

The objective importance of a criterion is arbitrary in the sense that it may bear little resemblance to the decision-maker's subjective importance. Because the subjective evaluation of criterion importance assigned by the decision-makers exhibits preference dependency, an integrated approach is needed to combine the subjective and objective importances of a criterion into a single value so that the synthetic evaluation of alternatives can effectively proceed.

A linear combination of the subjective and objective importances was employed to produce an overall importance of the decision criterion that reflected both the subjective and the objective factors. Recall that the objective importance of criterion x_j is given by $\varpi_j =$

$\left[\varpi_j^L, \varpi_j^U\right] = \left[\left(\varpi_{1j}^L, \varpi_{2j}^L, \varpi_{3j}^L, \varpi_{4j}^L; h_{\varpi_j}^L\right), \left(\varpi_{1j}^U, \varpi_{2j}^U, \varpi_{3j}^U, \varpi_{4j}^U; h_{\varpi_j}^U\right)\right]$ and that the subjective importance of criterion x_j is $\overline{W}_j = \left[\overline{W}_j^L, \overline{W}_j^U\right] = \left[\left(\bar{w}_{1j}^L, \bar{w}_{2j}^L, \bar{w}_{3j}^L, \bar{w}_{4j}^L; \bar{h}_{W_j}^L\right),\right.$ $\left.\left(\bar{w}_{1j}^U, \bar{w}_{2j}^U, \bar{w}_{3j}^U, \bar{w}_{4j}^U; \bar{h}_{W_j}^U\right)\right]$. Let τ be a coefficient that reflects the decision-makers' preferences concerning the relative worth of the subjective criterion importance to the objective criterion importance, where $\tau \in [0, 1]$. The overall importance of criterion x_j is defined by the following linear combination:

$$\overline{\overline{W}}_j = \tau \cdot \overline{W}_j \oplus (1 - \tau) \cdot \varpi_j. \tag{38}$$

For brevity, the overall importance $\overline{\overline{W}}_j$ is defined as follows:

$$\overline{\overline{W}}_j = \left[\overline{\overline{W}}_j^L, \overline{\overline{W}}_j^U\right] = \left[\left(\overline{\overline{w}}_{1j}^L, \overline{\overline{w}}_{2j}^L, \overline{\overline{w}}_{3j}^L, \overline{\overline{w}}_{4j}^L; \overline{\overline{h}}_{W_j}^L\right), \left(\overline{\overline{w}}_{1j}^U, \overline{\overline{w}}_{2j}^U, \overline{\overline{w}}_{3j}^U, \overline{\overline{w}}_{4j}^U; \overline{\overline{h}}_{W_j}^U\right)\right], \tag{39}$$

where $\overline{\overline{w}}_{lj}^L = \tau \cdot \bar{w}_{lj}^L + (1-\tau) \cdot \varpi_{lj}^L$ and $\overline{\overline{w}}_{lj}^U = \tau \cdot \bar{w}_{lj}^U + (1-\tau) \cdot \varpi_{lj}^U$ for $l \in \{1, 2, 3, 4\}$, $\overline{\overline{h}}_{W_j}^L = \min\left(\bar{h}_{W_j}^L, h_{\varpi_j}^L\right)$, and $\overline{\overline{h}}_{W_j}^U = \min\left(\bar{h}_{W_j}^U, h_{\varpi_j}^U\right)$.

When $\tau = 1$, the overall importance $\overline{\overline{W}}_j = \overline{W}_j$ for all $x_j \in X$. In addition, when $\tau = 0$, the overall importance $\overline{\overline{W}}_j = \varpi_j$ for all $x_j \in X$. If the decision-maker difficultly articulated the preference structure directly affecting the τ value, we assumed that τ =0.5 for convenience. Furthermore, the determination of parameter τ can be offered as an adaptation mechanism for future empirical applications. Thus, accounting for what values of parameter τ are suitable for integrating subjective and objective importances in empirical studies remains a topic for future research.

4.3.3 Ranking procedure with signed distances

Considering the differences in the importance values of various criteria, the weighted normalized criterion value of A_{ij} is computed as follows:

$$\begin{aligned}\overline{\overline{A}}_{ij} &= \overline{\overline{W}}_j \otimes \bar{A}_{ij} = \left[\overline{\overline{W}}_j^L, \overline{\overline{W}}_j^U\right] \otimes \left[\bar{A}_{ij}^L, \bar{A}_{ij}^U\right] \\ &= \left[\left(\overline{\overline{w}}_{1j}^L \times \bar{a}_{1ij}^L, \overline{\overline{w}}_{2j}^L \times \bar{a}_{2ij}^L, \overline{\overline{w}}_{3j}^L \times \bar{a}_{3ij}^L, \overline{\overline{w}}_{4j}^L \times \bar{a}_{4ij}^L; \min\left(\overline{\overline{h}}_{W_j}^L, \bar{h}_{A_{ij}}^L\right)\right),\right. \\ &\quad \left.\left(\overline{\overline{w}}_{1j}^U \times \bar{a}_{1ij}^U, \overline{\overline{w}}_{2j}^U \times \bar{a}_{2ij}^U, \overline{\overline{w}}_{3j}^U \times \bar{a}_{3ij}^U, \overline{\overline{w}}_{4j}^U \times \bar{a}_{4ij}^U; \min\left(\overline{\overline{h}}_{W_j}^U, \bar{h}_{A_{ij}}^U\right)\right)\right].\end{aligned} \tag{40}$$

For brevity, we denote the following:

$$\overline{\overline{A}}_{ij} = \left[\overline{\overline{A}}_{ij}^L, \overline{\overline{A}}_{ij}^U\right] = \left[\left(\overline{\overline{a}}_{1ij}^L, \overline{\overline{a}}_{2ij}^L, \overline{\overline{a}}_{3ij}^L, \overline{\overline{a}}_{4ij}^L; \overline{\overline{h}}_{A_{ij}}^L\right), \left(\overline{\overline{a}}_{1ij}^U, \overline{\overline{a}}_{2ij}^U, \overline{\overline{a}}_{3ij}^U, \overline{\overline{a}}_{4ij}^U; \overline{\overline{h}}_{A_{ij}}^U\right)\right]. \tag{41}$$

The weighted normalized decision matrix D^W can then be defined as the following:

$$D^W = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{bmatrix} \left[\overline{\overline{A}}_{11}^L, \overline{\overline{A}}_{11}^U\right] & \left[\overline{\overline{A}}_{12}^L, \overline{\overline{A}}_{12}^U\right] & \cdots & \left[\overline{\overline{A}}_{1n}^L, \overline{\overline{A}}_{1n}^U\right] \\ \left[\overline{\overline{A}}_{21}^L, \overline{\overline{A}}_{21}^U\right] & \left[\overline{\overline{A}}_{22}^L, \overline{\overline{A}}_{22}^U\right] & \cdots & \left[\overline{\overline{A}}_{2n}^L, \overline{\overline{A}}_{2n}^U\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\overline{\overline{A}}_{m1}^L, \overline{\overline{A}}_{m1}^U\right] & \left[\overline{\overline{A}}_{m2}^L, \overline{\overline{A}}_{m2}^U\right] & \cdots & \left[\overline{\overline{A}}_{mn}^L, \overline{\overline{A}}_{mn}^U\right] \end{bmatrix} \end{array} \quad (42)$$

The specification of the ideal solution in this paper was predetermined. All of the values based on IT2TrFNs were between zero and one. The criterion values of the ideal solution can therefore be reasonably defined as [(1,1,1,1;1), (1,1,1,1;1)]. The ideal solution, denoted as A^*, is defined as follows:

$$A^* = \left\{\left\langle x_j, [(1, 1, 1, 1; 1), (1, 1, 1, 1; 1)]\right\rangle \middle| x_j \in X\right\}. \quad (43)$$

The signed distance from each alternative to A^* can be calculated by applying Proposition 2, as in the following expression:

$$\begin{aligned} d_i^* &= \sum_{j=1}^{n} d\left(\overline{\overline{A}}_{ij}, \tilde{1}_1\right) \\ &= \sum_{j=1}^{n} \frac{1}{8}\left[\overline{\overline{a}}_{1ij}^L + \overline{\overline{a}}_{2ij}^L + \overline{\overline{a}}_{3ij}^L + \overline{\overline{a}}_{4ij}^L + 4\overline{\overline{a}}_{1ij}^U + 2\overline{\overline{a}}_{2ij}^U + 2\overline{\overline{a}}_{3ij}^U + 4\overline{\overline{a}}_{4ij}^U\right. \\ &\quad \left. + 3\left(\overline{\overline{a}}_{2ij}^U + \overline{\overline{a}}_{3ij}^U - \overline{\overline{a}}_{1ij}^U - \overline{\overline{a}}_{4ij}^U\right) \cdot \frac{\overline{\overline{h}}_{A_{ij}}^L}{\overline{\overline{h}}_{A_{ij}}^U} - 16\right], \end{aligned} \quad (44)$$

where $i = 1, 2, \ldots, m$. In addition, the normalized signed distance from each alternative to A^* was computed as follows:

$$\begin{aligned} \bar{d}_i^* &= \frac{1}{2n}\sum_{j=1}^{n} d\left(\overline{\overline{A}}_{ij}, \tilde{1}_1\right) \\ &= \frac{1}{16n}\sum_{j=1}^{n} \left[\overline{\overline{a}}_{1ij}^L + \overline{\overline{a}}_{2ij}^L + \overline{\overline{a}}_{3ij}^L + \overline{\overline{a}}_{4ij}^L + 4\overline{\overline{a}}_{1ij}^U + 2\overline{\overline{a}}_{2ij}^U + 2\overline{\overline{a}}_{3ij}^U + 4\overline{\overline{a}}_{4ij}^U\right. \\ &\quad \left. + 3\left(\overline{\overline{a}}_{2ij}^U + \overline{\overline{a}}_{3ij}^U - \overline{\overline{a}}_{1ij}^U - \overline{\overline{a}}_{4ij}^U\right) \cdot \frac{\overline{\overline{h}}_{A_{ij}}^L}{\overline{\overline{h}}_{A_{ij}}^U} - 16\right], \end{aligned} \quad (45)$$

where $i = 1, 2, \ldots, m$. According to Property 4, the signed distance $\bar{d}_i^*$ represents the relative nearness from an alternative A_i to the ideal solution for $i = 1, 2, \ldots, m$. By applying Definition 5, the alternatives can be ranked by their corresponding signed distances to the ideal solution. Let us denote $\left|\bar{d}_i^*\right|$ as the closeness coefficient of A_i to the ideal solution

$(0 \leq \left|\bar{d}_i^*\right| \leq 1)$. It follows that the preference order of the alternatives is ranked in ascending order of $\left|\bar{d}_i^*\right|$, and the alternative with the smallest $\left|\bar{d}_i^*\right|$ value is the best choice.

The algorithm of the proposed MCDA method with IT2TrFN data is summarized in the following series of successive steps:

Step 1: Form a committee of decision-makers, and identify the evaluation criteria and generate feasible alternatives.

Step 2: Select appropriate linguistic variables to grade each criterion's importance and establish linguistic ratings for the alternatives with respect to each criterion.

Step 3: Ask the decision-makers to use the linguistic weighting variables and the linguistic rating variables (Table 1) to assess the subjective importance of the criteria and evaluate the alternatives using each criterion. Then, convert the linguistic evaluations into IT2TrFNs.

Step 4: Pool the decision-makers' opinions to obtain the aggregate rating A_{ij} of alternative A_i on criterion x_j and the aggregate subjective importance W_j of criterion x_j. Then, construct the decision matrix D and the subjective importance W of all criteria.

Step 5: Establish the normalized decision matrix D^N and the normalized subjective importance $\overline{W}$ of the criteria.

Step 6: Conduct the objective approach to criterion importance.

- Step 6-1: Calculate the signed distance from normalized outcomes to $\tilde{1}_1$.
- Step 6-2: Identify the positive- and negative-ideal anchor values of each criterion. Then, the lower and upper anchor values can be obtained.
- Step 6-3: Compute the variation of each criterion from its anchor values.
- Step 6-4: Determine the normalized objective importance ϖ of each criterion.

Step 7: Set the value of the parameter τ to determine the overall importance $\overline{\overline{W}}_j$ of the criteria using the integrated approach.

Step 8: Construct the weighted normalized decision matrix D^W.

Step 9: Derive the normalized signed distances from each alternative to the ideal solution A^* and determine the closeness coefficient $\left|\bar{d}_i^*\right|$ of each alternative.

Step 10: Rank the priority of all alternatives in order of ascending $\left|\bar{d}_i^*\right|$.

5 Estimating criterion weights in the MCDA with IT2TrFNs

An IT2TrFN is a powerful means for expressing information regarding the decision-makers' preferences for criteria. As stated in Sect. 4, the subjective importance, objective importance, and overall importance of each criterion were expressed as IT2TrFNs. However, if assessing the criteria priority weights (non-negative crisp numbers and normalized to sum to one) derived from the IT2TrFN preference information is an important issue for the decision-makers, another version of the MCDA method should be developed to estimate criterion weights from the IT2TrFN data. In this section, a useful method with integrated programming models, motivated by the treatments given by [62], to estimate the importance weights of criteria in the proposed MCDA method is presented.

The distribution of the weights plays a crucial role in most MCDA problems. The weights are non-negative numbers and are independent from the measurement units of the criteria. There is no objection to considering normalized weights, so the criterion weights should be normalized to sum to one. Leaving the sum of the criterion weights unconstrained leads to unequal scales of aggregated weights. Moreover, it is difficult to facilitate a straightforward comparison of alternatives. Considering that it is generally accepted to consider non-negative normalized weights, the sum of the criterion weights would be restricted to unity in the following proposed MCDA method.

Recall that an IT2TrFN $\overline{\overline{W}}_j$ defined on X denotes the overall importance grade of the criterion x_j. $\overline{\overline{W}}_j = \left[\overline{\overline{W}}_j^L, \overline{\overline{W}}_j^U\right] = \left[\left(\overline{\overline{w}}_{1j}^L, \overline{\overline{w}}_{2j}^L, \overline{\overline{w}}_{3j}^L, \overline{\overline{w}}_{4j}^L; \overline{\overline{h}}_{W_j}^L\right), \left(\overline{\overline{w}}_{1j}^U, \overline{\overline{w}}_{2j}^U, \overline{\overline{w}}_{3j}^U, \overline{\overline{w}}_{4j}^U; \overline{\overline{h}}_{W_j}^U\right)\right]$, where $\overline{\overline{W}}_j^L$ and $\overline{\overline{W}}_j^U$ are the lower trapezoidal fuzzy number and the upper trapezoidal fuzzy number, respectively, about $\overline{\overline{W}}_j$. Note that $0 \leq \overline{\overline{w}}_{1j}^L \leq \overline{\overline{w}}_{2j}^L \leq \overline{\overline{w}}_{3j}^L \leq \overline{\overline{w}}_{4j}^L \leq 1$, $0 \leq \overline{\overline{w}}_{1j}^U \leq \overline{\overline{w}}_{2j}^U \leq \overline{\overline{w}}_{3j}^U \leq \overline{\overline{w}}_{4j}^U \leq 1$, $\overline{\overline{w}}_{1j}^U \leq \overline{\overline{w}}_{1j}^L$, and $\overline{\overline{w}}_{4j}^L \leq \overline{\overline{w}}_{4j}^U$. Because $\overline{\overline{w}}_{1j}^U \leq \overline{\overline{w}}_{1j}^L$ and $\overline{\overline{w}}_{4j}^U \geq \overline{\overline{w}}_{4j}^L$, it is reasonable to assume that the decision-makers' weight ω_j of each criterion $x_j \in X$ lies in the closed interval $\left[\overline{\overline{w}}_{1j}^U, \overline{\overline{w}}_{4j}^U\right]$. Let $\omega_j^L = \overline{\overline{w}}_{1j}^U$ and $\omega_j^U = \overline{\overline{w}}_{4j}^U$; in addition, let $0 \leq \omega_j^L \leq \omega_j \leq \omega_j^U \leq 1$. It then follows that the interval $\left[\overline{\overline{w}}_{1j}^U, \overline{\overline{w}}_{4j}^U\right]$ is denoted by $\left[\omega_j^L, \omega_j^U\right]$.

Because the criterion weights must be normalized to sum to one, the conditions of $\sum_{j=1}^n \omega_j^L \leq 1$ and $\sum_{j=1}^n \omega_j^U \geq 1$ are required to determine the weights, $\omega_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$, that satisfy $\sum_{j=1}^n \omega_j = 1$. However, it is possible that $\sum_{j=1}^n \omega_j^L > 1$ or $\sum_{j=1}^n \omega_j^U < 1$, neither of which is permitted by the constraint $\sum_{j=1}^n \omega_j = 1$. In this case, it follows that there are no feasible solutions for the criterion weights. To overcome this difficulty, the condition $\omega_j^L \leq \omega_j \leq \omega_j^U$ is relaxed by introducing the deviation variables e_j^- and e_j^+, which are defined as follows:

$$\omega_j^L - e_j^- \leq \omega_j \leq \omega_j^U + e_j^+, \quad \text{for } j = 1, 2, \ldots, n, \tag{46}$$

where e_j^- and e_j^+ are both non-negative real numbers. If both e_j^- and e_j^+ are equal to zero, then (46) reduces to $\omega_j^L \leq \omega_j \leq \omega_j^U$. These deviation variables can be useful buffers in the case that $\sum_{j=1}^n \omega_j^L \leq 1$ or $\sum_{j=1}^n \omega_j^U \geq 1$ does not hold.

For smaller values of the deviation variables e_j^- and e_j^+, the criterion weights ω_j are closer to the interval value $\left[\omega_j^L, \omega_j^U\right]$. Furthermore, if the deviation variables are close to zero, then there is no gross violation of the necessary conditions. Therefore, the multiple objective optimization model is established as the following:

$$[\text{M1}] \quad \begin{array}{l} \min \left\{e_1^-, e_2^-, \cdots, e_n^-, e_1^+, e_2^+, \ldots, e_n^+\right\} \\ \text{s.t.} \begin{cases} \omega_j + e_j^- \geq \omega_j^L \ (j = 1, 2, \ldots, n), \\ \omega_j - e_j^+ \leq \omega_j^U \ (j = 1, 2, \ldots, n), \\ e_j^-, e_j^+, \omega_j \geq 0 \ (j = 1, 2, \ldots, n), \\ \sum_{j=1}^n \omega_j = 1. \end{cases} \end{array} \tag{47}$$

These different e_j^- and e_j^+ $(j = 1, 2, \ldots, n)$ values can be integrated in the minimax sense. Thus, the model in [M1] can be transformed into a single-objective programming model:

$$[\text{M2}]\quad \begin{array}{l} \min\{\lambda_1\} \\ \text{s.t.} \begin{cases} e_j^- \le \lambda_1 & (j = 1, 2, \ldots, n), \\ e_j^+ \le \lambda_1 & (j = 1, 2, \ldots, n), \\ \omega_j + e_j^- \ge \omega_j^L & (j = 1, 2, \ldots, n), \\ \omega_j - e_j^+ \le \omega_j^U & (j = 1, 2, \ldots, n), \\ e_j^-, e_j^+, \omega_j \ge 0 & (j = 1, 2, \ldots, n), \\ \sum_{j=1}^n \omega_j = 1. \end{cases} \end{array} \tag{48}$$

The optimal deviation values $\breve{e}_j^-$ and $\breve{e}_j^+$ for each criterion can then be determined by solving the programming problem of (48). If both $\breve{e}_j^-$ and $\breve{e}_j^+$ are equal to zero, then the corresponding weight of criterion x_j is consistent with the interval values. Otherwise, the weight is inconsistent with the interval values. The criterion weights are within the range of $\omega_j^L \le \omega_j \le \omega_j^U (j = 1, 2, \ldots, n)$ in the consistent case and are within the range of $\omega_j^L - \breve{e}_j^- \le \omega_j \le \omega_j^U + \breve{e}_j^+ (j = 1, 2, \ldots, n)$ in the inconsistent case.

If the criterion weights $\omega_j \ (j = 1, 2, \ldots, n)$ are used in the MCDA method using a signed-distance-based approach, then the weighted normalized criterion value of A_{ij} is given by the following:

$$\begin{aligned} \overline{\overline{A}}'_{ij} = \omega_j \cdot \bar{A}_{ij} = \Big[&\Big(\omega_j \times \bar{a}_{1ij}^L, \omega_j \times \bar{a}_{2ij}^L, \omega_j \times \bar{a}_{3ij}^L, \omega_j \times \bar{a}_{4ij}^L; h_{A_{ij}}^L\Big), \\ &\Big(\omega_j \times \bar{a}_{1ij}^U, \omega_j \times \bar{a}_{2ij}^U, \omega_j \times \bar{a}_{3ij}^U, \omega_j \times \bar{a}_{4ij}^U; h_{A_{ij}}^U\Big)\Big]. \end{aligned} \tag{49}$$

For brevity, let us denote the following:

$$\overline{\overline{A}}'_{ij} = \left[\overline{\overline{A}}'^L_{ij}, \overline{\overline{A}}'^U_{ij}\right] = \left[\left(\overline{\overline{a}}'^L_{1ij}, \overline{\overline{a}}'^L_{2ij}, \overline{\overline{a}}'^L_{3ij}, \overline{\overline{a}}'^L_{4ij}; h^L_{A_{ij}}\right), \left(\overline{\overline{a}}'^U_{1ij}, \overline{\overline{a}}'^U_{2ij}, \overline{\overline{a}}'^U_{3ij}, \overline{\overline{a}}'^U_{4ij}; h^U_{A_{ij}}\right)\right]. \tag{50}$$

Next, the weighted normalized decision matrix D'^W is constructed as follows:

$$D^W = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{bmatrix} \left[\overline{\overline{A}}'^L_{11}, \overline{\overline{A}}'^U_{11}\right] & \left[\overline{\overline{A}}^L_{12}, \overline{\overline{A}}^U_{12}\right] & \cdots & \left[\overline{\overline{A}}'^L_{1n}, \overline{\overline{A}}'^U_{1n}\right] \\ \left[\overline{\overline{A}}'^L_{21}, \overline{\overline{A}}'^U_{21}\right] & \left[\overline{\overline{A}}'^L_{22}, \overline{\overline{A}}'^U_{22}\right] & \cdots & \left[\overline{\overline{A}}'^L_{2n}, \overline{\overline{A}}'^U_{2n}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\overline{\overline{A}}'^L_{m1}, \overline{\overline{A}}'^U_{m1}\right] & \left[\overline{\overline{A}}'^L_{m2}, \overline{\overline{A}}'^U_{m2}\right] & \cdots & \left[\overline{\overline{A}}'^L_{mn}, \overline{\overline{A}}'^U_{mn}\right] \end{bmatrix} \end{array} \tag{51}$$

The normalized signed distance from each alternative to A^* is computed as follows:

$$\begin{aligned} \bar{d}_i'^* &= \frac{1}{2n} \sum_{j=1}^n d\left(\overline{\overline{A}}'_{ij}, \tilde{1}_1\right) \\ &= \frac{1}{16n} \sum_{j=1}^n \Bigg[\overline{\overline{a}}'^L_{1ij} + \overline{\overline{a}}'^L_{2ij} + \overline{\overline{a}}'^L_{3ij} + \overline{\overline{a}}'^L_{4ij} + 4\overline{\overline{a}}'^U_{1ij} + 2\overline{\overline{a}}'^U_{2ij} + 2\overline{\overline{a}}'^U_{3ij} + 4\overline{\overline{a}}'^U_{4ij} \\ &\quad + 3\left(\overline{\overline{a}}'^U_{2ij} + \overline{\overline{a}}'^U_{3ij} - \overline{\overline{a}}'^U_{1ij} - \overline{\overline{a}}'^U_{4ij}\right) \cdot \frac{h^L_{A_{ij}}}{h^U_{A_{ij}}} - 16 \Bigg] \end{aligned}$$

$$= \frac{1}{16n}\left[\sum_{j=1}^{n}\left(\bar{a}_{1ij}^{L}+\bar{a}_{2ij}^{L}+\bar{a}_{3ij}^{L}+\bar{a}_{4ij}^{L}+\left(4-3\cdot\frac{h_{A_{ij}}^{L}}{h_{A_{ij}}^{U}}\right)\bar{a}_{1ij}^{U}+\left(2+3\cdot\frac{h_{A_{ij}}^{L}}{h_{A_{ij}}^{U}}\right)\bar{a}_{2ij}^{U}\right.\right.$$
$$\left.\left.+\left(2+3\cdot\frac{h_{A_{ij}}^{L}}{h_{A_{ij}}^{U}}\right)\bar{a}_{3ij}^{U}+\left(4-3\cdot\frac{h_{A_{ij}}^{L}}{h_{A_{ij}}^{U}}\right)\bar{a}_{4ij}^{U}\right)\cdot\omega_j-16n\right]. \quad (52)$$

where $i = 1, 2, \ldots, m$. The closeness coefficient of alternative A_i is given by $\left|\bar{d}_i'^{*}\right|$.

The best choice is the one that is the closest to the ideal solution determined by min $\left|\bar{d}_i'^{*}\right|$, for which the criterion importance weights are not precisely known. Considering the criterion weights without exact values but with interval numbers, the optimal value of the closeness coefficient for alternative A_i with respect to the ideal solution can be measured by a linear programming model with normalized signed distances. Because there are m alternatives in the set A, a total of m linear programming models must be solved to provide m optimal closeness coefficients. Although the optimal weight vector for each alternative can be computed, these optimal weights may be different in general. Thus, the corresponding optimal values of the closeness coefficients for all m alternatives cannot be compared. Considering that the decision-makers cannot easily or evidently judge the preference relations among all of the non-dominant alternatives, it is reasonable to assume that all non-dominant alternatives are of equal importance. Thus, the m linear programming models can be aggregated with one programming model. A multiple objective optimization model is constructed as follows:

$$\text{[M3]}\quad \begin{aligned}&\min\left\{\left|\bar{d}_1'^{*}\right|,\left|\bar{d}_2'^{*}\right|,\ldots,\left|\bar{d}_m'^{*}\right|\right\}=\min\left\{-\bar{d}_1'^{*},-\bar{d}_2'^{*},\ldots,-\bar{d}_m'^{*}\right\}\\ &\text{s.t.}\begin{cases}\omega_j+e_j^{-}\geq\omega_j^{L}\ (j=1,2,\ldots,n),\\ \omega_j-e_j^{+}\leq\omega_j^{U}\ (j=1,2,\ldots,n),\\ e_j^{-},e_j^{+},\omega_j\geq 0\ (j=1,2,\ldots,n),\\ \sum_{j=1}^{n}\omega_j=1.\end{cases}\end{aligned} \quad (53)$$

By utilizing the minimax operator, the model in [M3] can be integrated into the following single-objective programming model:

$$\text{[M4]}\quad \begin{aligned}&\min\{\lambda_2\}\\ &\text{s.t.}\begin{cases}-\bar{d}_i'^{*}\leq\lambda_2 & (i=1,2,\ldots,m),\\ \omega_j+e_j^{-}\geq\omega_j^{L} & (j=1,2,\ldots,n),\\ \omega_j-e_j^{+}\leq\omega_j^{U} & (j=1,2,\ldots,n),\\ e_j^{-},e_j^{+},\omega_j\geq 0 & (j=1,2,\ldots,n),\\ \sum_{j=1}^{n}\omega_j=1.\end{cases}\end{aligned} \quad (54)$$

In general, the optimal solutions of the criterion weights in [M1] and [M3] are different. Thus, it is not possible to derive a unique criterion weight vector to compute the optimal closeness coefficient of each alternative. To determine a consistent weight vector based on the models in [M1] and [M3], the following integrated multi-objective programming model is constructed:

$$
[\mathrm{M5}] \quad \begin{array}{l} \min \left\{e_1^-, e_2^-, \ldots, e_n^-, e_1^+, e_2^+, \ldots, e_n^+\right\} \\ \min \left\{-\bar{d}_1^{\prime *}, -\bar{d}_2^{\prime *}, \ldots, -\bar{d}_m^{\prime *}\right\} \\ \text{s.t.} \begin{cases} \omega_j + e_j^- \geq \omega_j^L & (j = 1, 2, \ldots, n), \\ \omega_j - e_j^+ \leq \omega_j^U & (j = 1, 2, \ldots, n), \\ e_j^-, e_j^+, \omega_j \geq 0 & (j = 1, 2, \ldots, n), \\ \sum_{j=1}^n \omega_j = 1. \end{cases} \end{array} \tag{55}
$$

In a similar manner to that presented in [M2] and [M4], the model in [M5] can be transformed into the following single-objective optimization model by the linear equal-weighted summation method:

$$
[\mathrm{M6}] \quad \begin{array}{l} \min \{\lambda_1 + \lambda_2\} \\ \text{s.t.} \begin{cases} -\bar{d}_i^{\prime *} \leq \lambda_2 & (i = 1, 2, \ldots, m), \\ e_j^- \leq \lambda_1 & (j = 1, 2, \ldots, n), \\ e_j^+ \leq \lambda_1 & (j = 1, 2, \ldots, n), \\ \omega_j + e_j^- \geq \omega_j^L & (j = 1, 2, \ldots, n), \\ \omega_j - e_j^+ \leq \omega_j^U & (j = 1, 2, \ldots, n), \\ e_j^-, e_j^+, \omega_j \geq 0 & (j = 1, 2, \ldots, n), \\ \sum_{j=1}^n \omega_j = 1. \end{cases} \end{array} \tag{56}
$$

The minimization model in [M6] has m variables corresponding to the signed distances of the alternatives, n weight variables, $2n$ deviation variables, 1 linear equality constraint, $m + 7n$ linear inequality constraints, and a linear objective function. By solving [M6], it is possible to obtain the optimal values of the signed distances $\breve{\bar{d}}_i^{\prime *}$ $(i = 1, 2, \ldots, m)$, the optimal weight vector, $\breve{\omega} = \left(\breve{\omega}_1, \breve{\omega}_2, \ldots, \breve{\omega}_n\right)$, and the optimal deviation values $\breve{e}_j^-$ and $\breve{e}_j^+$ $(j = 1, 2, \ldots, n)$. Then, the optimal closeness coefficient of alternative A_i is derived from $\left|\breve{\bar{d}}_i^{\prime *}\right|$. The ranking of all alternatives is presented along with the optimal values of the closeness coefficients for individual alternatives to clarify the best alternative. A smaller value of $\left|\breve{\bar{d}}_i^{\prime *}\right|$ indicates a better alternativeA_i. Thus, the m alternatives can be ranked by increasing $\left|\breve{\bar{d}}_i^{\prime *}\right|$ for all $A_i \in A$.

The MCDA method using IT2TrFNs to estimate criterion weights is summarized in the following series of successive steps:

Steps $1'$–$7'$: See Steps 1–7 of the algorithm in Sect. 4.2.
Step $8'$: Establish the weighted normalized decision matrix D'^W.
Step $9'$: Determine the normalized signed distances $\bar{d}_i^{\prime *}$ from each alternative to the ideal solution A^*.
Step $10'$: Construct the integrated single-objective optimization model in [M6] and solve for the optimal values of the signed distances and the optimal criterion weights.
Step $11'$: Calculate the closeness coefficient $\left|\breve{\bar{d}}_i^{\prime *}\right|$ of each alternative. Then, rank the priority order of all alternatives by increasing $\left|\breve{\bar{d}}_i^{\prime *}\right|$.

In the following, a real-world case study, exploring medical decision-making at Chang Gung Memorial Hospital in Taiwan, is analyzed to demonstrate the feasibility of the proposed methods.

6 Applications to patient-centered medicine

Patient-centered care, which is a fundamental component of practicing integrative medicine, places the patient at the center of the delivery of care, improves the continuity of care, and enhances the integration of health professionals and patients [44]. The purpose of the empirical study was to develop a multi-person, multi-criteria model for patient-centered decision-making within interval type-2 fuzzy environment. In addition, the studied case was from the Division of Cerebrovascular Disease of Department of Neurology at Linkou Medical Center of Chang Gung Memorial Hospital in Taiwan.

6.1 Problem background

Patient-centered care is health care that meets and responds to patients' wants, needs, and preferences and where patients are autonomous and able to decide treatment and care for themselves [4,24,30]. Patient-centered care treats the patient as a unique individual [47]. As [49] indicated, patient-centered care means putting the patient in the center, not in the middle, of care services. Because patient-centered care considers the patient as a whole person with physical, psychological, and social needs, an increasing number of modern healthcare systems are rapidly adopting a patient-centered approach to care [44]. Till now, patient-centered care has now made it to center stage in discussions of quality [27]. Patient-centered care provides patients and relatives (family and friends) with abundant opportunities to be informed and involved in medical decision-making processes [40,44]. Patient- and family-centered care has developed along with the realization that psychosocial factors and health beliefs have an impact on well-being. Moreover, families and affected persons will be able to articulate medical treatment preferences as they relate to personal definitions of quality of life [30].

Patient-centered care assumes that the patients and relatives are qualified to decide their own needs and expectations and that they are able to make decisions and choices about what they need and want [37]. However, collaborative decision information provided by patients, relatives, and healthcare providers is inherently imprecise and involves many uncertainties. IT2FSs have a greater ability than type-1 fuzzy sets to handle imprecision and imperfect information in real-world therapeutic applications. In light of patient-centeredness, this paper applied the proposed signed-distance-based method for handling a collaborative decision-making problem in which individual assessments are provided as IT2TrFNs.

The case is from Chang Gung Memorial Hospital in Taiwan. The patient, Mr. Peng, was an 82-year-old widowed male with a history of hypertension. Mr. Peng was a retired government employee with two sons and one daughter, who were all married with children. Due to complaints of physical discomfort, Mr. Peng was brought to the hospital by his eldest son and daughter-in-law, who live in the same residence. While waiting for his examination results and diagnosis, his condition deteriorated and he fell into a coma when the definitive diagnosis was made. His younger son and daughter rushed to the hospital upon being informed by their older brother.

Because Mr. Peng was unconscious, the attending physician explained the diagnosis of basilar artery occlusion to his family members. Basilar artery occlusion is an acute cerebrovascular disease caused by a complete or partial occlusion of the basilar artery. Basilar artery occlusion is characterized by a gradual disturbance in consciousness, rapid progression, and a critical and poor prognosis. The attending physician assessed the patient's medical history and current physical conditions and provided the four treatment options: intravenous thrombolysis (A_1), intra-arterial thrombolysis (A_2), antiplatelet treatment (A_3), and heparinization (A_4).

To have the patient's family members fully understand the advantages and disadvantages of each treatment, the physician provided additional information based on several criteria, including survival rate (x_1), severity of the complications (x_2), probability of a cure (x_3), expense (x_4), and self-care capacity (x_5). The survival rate is the probability of survival between the start of the surgery and the postoperative period. The complications involved mostly intra-operative or postoperative complications. Given all of the possible complications, the severity of the most serious complication was considered. The probability of a cure is the probability of both symptom alleviation and the cure of the disease. The expense includes the expenditure associated with the treatment and hospitalizations. Self-care capacity indicates the prognosis of the patient's self-care capacity (i.e., eating, bathing, and using the toilet without assistance).

The physician described the four treatment methods using the five criteria, as summarized in the following:

About intravenous thrombolysis (A_1):

(1) A very high survival rate.
(2) The possibility of an intracerebral hemorrhage as a complication.
(3) A 60 % probability of a cure if recanalization is achieved.
(4) A greater out-of-pocket expense, even though the procedure is covered by the patient's health benefits.
(5) The prognosis for the patient's self-care capacity is less than average.

About intra-arterial thrombolysis (A_2):

(1) A very high survival rate.
(2) The possibility of an intracerebral hemorrhage as a complication. The probability is a little higher than that of A_1.
(3) A very high probability of a cure.
(4) The procedure is not covered by health insurance and is expensive (about NT$200K) compared with the out-of-pocket expenses under health benefits.
(5) A moderate prognosis for the patient's self-care capacity.

About antiplatelet treatment (A_3):

(1) A moderate survival rate.
(2) The possibility of progressive stroke as a complication, which may aggravate and prolong the disease course.
(3) A very low or near-zero probability of a cure.
(4) Health insurance covers most of the expenses, with a very low out-of-pocket expense.
(5) The worst prognosis for the patient's self-care capacity.

About heparinization (A_4):

(1) A high survival rate.
(2) The possibility of an intracerebral hemorrhage as a complication, but with a lower severity compared with A_1 and A_2.

Table 2 The therapeutic ratings and criterion importance values evaluated by the decision-makers

Criteria	Treatment options	Decision makers		
		E_1 (eldest son)	E_2 (younger son)	E_3 (daughter)
x_1 (survival rate)	A_1	AH	VH	H
	A_2	VH	VH	AH
	A_3	MH	M	MH
	A_4	MH	H	H
	Importance	H	VH	AH
x_2 (severity of the complications)	A_1	ML	M	L
	A_2	MH	H	MH
	A_3	ML	ML	M
	A_4	L	ML	ML
	Importance	ML	M	MH
x_3 (probability of a cure)	A_1	MH	MH	H
	A_2	AH	VH	VH
	A_3	AL	VL	VL
	A_4	ML	M	L
	Importance	H	H	VH
x_4 (expense)	A_1	ML	ML	L
	A_2	AH	AH	H
	A_3	L	VL	VL
	A_4	M	MH	ML
	Importance	AH	M	H
x_5 (self-care capacity)	A_1	L	ML	ML
	A_2	MH	ML	M
	A_3	AL	VL	VL
	A_4	VL	ML	L
	Importance	AH	H	H

(3) A relatively low probability of a cure.
(4) Low coverage by the patient's health insurance and moderately higher out-of-pocket expenses.
(5) A poor prognosis for the patient's self-care ability.

6.2 Illustrative application of the proposed methods

This section illustrates the implementation of our proposed methods using the medical decision-making problem. Mr. Peng's three children, eldest son, younger son, and daughter, are the three decision-makers E_1, E_2, and E_3, respectively. There are four treatment options available, including intravenous thrombolysis (A_1), intra-arterial thrombolysis (A_2), antiplatelet treatment (A_3), and heparinization (A_4). The set of all alternatives is denoted by $A = \{A_1, A_2, A_3, A_4\}$. The three decision-makers considered the five criteria, including survival rate (x_1), severity of the complications (x_2), probability of a cure (x_3), expense (x_4), and self-care capacity (x_5). Here, x_2 and x_4 are cost criteria, whereas the remaining criteria are benefit ones. The set of evaluative criteria is denoted by $X = \{x_1, x_2, \ldots, x_5\}$, with $X_b = \{x_1, x_3, x_5\}$ and $X_c = \{x_2, x_4\}$. (Note that Step 1 had been completed.)

According to Step 2, the three decision-makers used the linguistic rating variables (Table 1) to evaluate the four alternatives and to identify the subjective importance values for the five criteria. The results are presented in Table 2. Then, these linguistic evaluations were converted into IT2TrFNs by applying Step 3.

Table 3 The results of the aggregated ratings of alternatives and subjective importance values of criteria in D

	A_{ij}^L					A_{ij}^U				
	a_{1ij}^L	a_{2ij}^L	a_{3ij}^L	a_{4ij}^L	$h_{A_{ij}}^L$	a_{1ij}^U	a_{2ij}^U	a_{3ij}^U	a_{4ij}^U	$h_{A_{ij}}^U$
A_{11}	0.9100	0.9333	0.9592	0.9667	0.8000	0.8833	0.9200	0.9733	0.9900	1.0000
A_{12}	0.2408	0.2758	0.3408	0.3692	0.8000	0.1767	0.2433	0.3733	0.4333	1.0000
A_{13}	0.6942	0.7200	0.8000	0.8292	0.8000	0.6267	0.6800	0.8400	0.8967	1.0000
A_{14}	0.1842	0.2100	0.2700	0.2992	0.8000	0.1267	0.1800	0.3000	0.3567	1.0000
A_{15}	0.1842	0.2100	0.2700	0.2992	0.8000	0.1267	0.1800	0.3000	0.3567	1.0000
A_{21}	0.9650	0.9900	0.9950	0.9950	0.8000	0.9533	0.9867	1.0000	1.0000	1.0000
A_{22}	0.6942	0.7200	0.8000	0.8292	0.8000	0.6267	0.6800	0.8400	0.8967	1.0000
A_{23}	0.9650	0.9900	0.9950	0.9950	0.8000	0.9533	0.9867	1.0000	1.0000	1.0000
A_{24}	0.9275	0.9383	0.9617	0.9692	0.8000	0.9067	0.9267	0.9733	0.9900	1.0000
A_{25}	0.4283	0.4600	0.5400	0.5717	0.8000	0.3567	0.4200	0.5800	0.6433	1.0000
A_{31}	0.5675	0.5992	0.6842	0.7158	0.8000	0.4933	0.5567	0.7267	0.7900	1.0000
A_{32}	0.2892	0.3208	0.3958	0.4275	0.8000	0.2200	0.2833	0.4333	0.4967	1.0000
A_{33}	0.0050	0.0050	0.0100	0.0350	0.8000	0.0000	0.0000	0.0133	0.0467	1.0000
A_{34}	0.0342	0.0450	0.0633	0.0958	0.8000	0.0133	0.0333	0.0733	0.1233	1.0000
A_{35}	0.0050	0.0050	0.0100	0.0350	0.8000	0.0000	0.0000	0.0133	0.0467	1.0000
A_{41}	0.7383	0.7675	0.8425	0.8683	0.8000	0.6733	0.7300	0.8800	0.9333	1.0000
A_{42}	0.1842	0.2100	0.2700	0.2992	0.8000	0.1267	0.1800	0.3000	0.3567	1.0000
A_{43}	0.2408	0.2758	0.3408	0.3692	0.8000	0.1767	0.2433	0.3733	0.4333	1.0000
A_{44}	0.4283	0.4600	0.5400	0.5717	0.8000	0.3567	0.4200	0.5800	0.6433	1.0000
A_{45}	0.1092	0.1275	0.1667	0.1975	0.8000	0.0700	0.1067	0.1867	0.2400	1.0000
	W_j^L					W_j^U				
	w_{1j}^L	w_{2j}^L	w_{3j}^L	w_{4j}^L	$h_{W_j}^L$	w_{1j}^U	w_{2j}^U	w_{3j}^U	w_{4j}^U	$h_{W_j}^U$
W_1	0.9100	0.9333	0.9592	0.9667	0.8000	0.8833	0.9200	0.9733	0.9900	1.0000
W_2	0.4283	0.4600	0.5400	0.5717	0.8000	0.3567	0.4200	0.5800	0.6433	1.0000
W_3	0.8375	0.8717	0.9208	0.9358	0.8000	0.7900	0.8467	0.9467	0.9800	1.0000
W_4	0.7283	0.7558	0.8075	0.8250	0.8000	0.6800	0.7300	0.8333	0.8733	1.0000
W_5	0.8550	0.8767	0.9233	0.9383	0.8000	0.8133	0.8533	0.9467	0.9800	1.0000

In Step 4, the ratings of the alternatives with respect to each criterion and the subjective importance values of the criteria were obtained with (17) and (22), respectively. Table 3 shows the aggregated rating A_{ij} of alternative A_i on criterion x_j and summarizes the aggregated subjective importance W_j of criterion x_j. With these, it was possible to establish the decision matrix D and the subjective importance W of the criteria.

6.2.1 Illustration of the MCDA method with signed distances

Let us recall that $X_b = \{x_1, x_3, x_5\}$ and $X_c = \{x_2, x_4\}$. From the data in Table 3, it was known that $a_1^+ = 1.0000$, $a_2^- = 0.1267$, $a_3^+ = 1.0000$, $a_4^- = 0.0133$, $a_5^+ = 0.6433$, and $w^+ = 0.9900$. Based on Table 3, (25), and (28), the normalized decision matrix D^N and the

Table 4 The results of the normalized decision matrix D^N, the normalized subjective importance $\overline{W}$, and the signed distance $d\left(\overline{A}_{ij}, \tilde{1}_1\right)$

	$\overline{A}_{ij}^L$					$\overline{A}_{ij}^U$					$d\left(\overline{A}_{ij}, \tilde{1}_1\right)$
	$\overline{a}_{1ij}^L$	$\overline{a}_{2ij}^L$	$\overline{a}_{3ij}^L$	$\overline{a}_{4ij}^L$	$\overline{h}_{A_{ij}}^L$	$\overline{a}_{1ij}^U$	$\overline{a}_{2ij}^U$	$\overline{a}_{3ij}^U$	$\overline{a}_{4ij}^U$	$\overline{h}_{A_{ij}}^U$	
$\overline{A}_{11}$	0.9100	0.9333	0.9592	0.9667	0.8000	0.8833	0.9200	0.9733	0.9900	1.0000	−0.1129
$\overline{A}_{12}$	0.3432	0.3718	0.4594	0.5262	0.8000	0.2924	0.3394	0.5208	0.7170	1.0000	−1.1124
$\overline{A}_{13}$	0.6942	0.7200	0.8000	0.8292	0.8000	0.6267	0.6800	0.8400	0.8967	1.0000	−0.4789
$\overline{A}_{14}$	0.0445	0.0493	0.0633	0.0722	0.8000	0.0373	0.0443	0.0739	0.1050	1.0000	−1.8779
$\overline{A}_{15}$	0.2863	0.3264	0.4197	0.4651	0.8000	0.1970	0.2798	0.4663	0.5545	1.0000	−1.2522
$\overline{A}_{21}$	0.9650	0.9900	0.9950	0.9950	0.8000	0.9533	0.9867	1.0000	1.0000	1.0000	−0.0235
$\overline{A}_{22}$	0.1528	0.1584	0.1760	0.1825	0.8000	0.1413	0.1508	0.1863	0.2022	1.0000	−1.6622
$\overline{A}_{23}$	0.9650	0.9900	0.9950	0.9950	0.8000	0.9533	0.9867	1.0000	1.0000	1.0000	−0.0235
$\overline{A}_{24}$	0.0137	0.0138	0.0142	0.0143	0.8000	0.0134	0.0137	0.0144	0.0147	1.0000	−1.9719
$\overline{A}_{25}$	0.6658	0.7151	0.8394	0.8887	0.8000	0.5545	0.6529	0.9016	1.0000	1.0000	−0.4455
$\overline{A}_{31}$	0.5675	0.5992	0.6842	0.7158	0.8000	0.4933	0.5567	0.7267	0.7900	1.0000	−0.7166
$\overline{A}_{32}$	0.2964	0.3201	0.3950	0.4381	0.8000	0.2551	0.2924	0.4472	0.5759	1.0000	−1.2458
$\overline{A}_{33}$	0.0050	0.0050	0.0100	0.0350	0.8000	0.0000	0.0000	0.0133	0.0467	1.0000	−1.9765
$\overline{A}_{34}$	0.1388	0.2101	0.2956	0.3889	0.8000	0.1079	0.1814	0.3994	1.0000	1.0000	−1.3298
$\overline{A}_{35}$	0.0078	0.0078	0.0155	0.0544	0.8000	0.0000	0.0000	0.0207	0.0726	1.0000	−1.9634
$\overline{A}_{41}$	0.7383	0.7675	0.8425	0.8683	0.8000	0.6733	0.7300	0.8800	0.9333	1.0000	−0.3911
$\overline{A}_{42}$	0.4235	0.4693	0.6033	0.6878	0.8000	0.3552	0.4223	0.7039	1.0000	1.0000	−0.8366
$\overline{A}_{43}$	0.2408	0.2758	0.3408	0.3692	0.8000	0.1767	0.2433	0.3733	0.4333	1.0000	−1.3855
$\overline{A}_{44}$	0.0233	0.0246	0.0289	0.0311	0.8000	0.0207	0.0229	0.0317	0.0373	1.0000	−1.9449
$\overline{A}_{45}$	0.1697	0.1982	0.2591	0.3070	0.8000	0.1088	0.1659	0.2902	0.3731	1.0000	−1.5360

	$\overline{W}_j^L$					$\overline{W}_j^U$				
	$\overline{w}_{1j}^L$	$\overline{w}_{2j}^L$	$\overline{w}_{3j}^L$	$\overline{w}_{4j}^L$	$\overline{h}_{W_j}^L$	$\overline{w}_{1j}^U$	$\overline{w}_{2j}^U$	$\overline{w}_{3j}^U$	$\overline{w}_{4j}^U$	$\overline{h}_{W_j}^U$
$\overline{W}_1$	0.9192	0.9427	0.9689	0.9765	0.8000	0.8922	0.9293	0.9831	1.0000	1.0000
$\overline{W}_2$	0.4326	0.4646	0.5455	0.5775	0.8000	0.3603	0.4242	0.5859	0.6498	1.0000
$\overline{W}_3$	0.8460	0.8805	0.9301	0.9453	0.8000	0.7980	0.8553	0.9563	0.9899	1.0000
$\overline{W}_4$	0.7357	0.7634	0.8157	0.8333	0.8000	0.6869	0.7374	0.8417	0.8821	1.0000
$\overline{W}_5$	0.8636	0.8856	0.9326	0.9478	0.8000	0.8215	0.8619	0.9563	0.9899	1.0000

normalized subjective importance $\overline{W}$ of the criteria can be constructed according to Step 5. Table 4 shows the normalized rating $\bar{A}_{ij}$ of alternative A_i on criterion x_j and summarizes the normalized importance $\overline{W}_j$ of criterion x_j.

Next, Step 6 was applied to acquire the objective importance values of the criteria. Based on Step 6-1, the signed distances from the normalized outcomes $\bar{A}_{ij}$ to $\tilde{1}_1$ for all $A_i \in A$ and $x_j \in X$ were obtained, as shown in Table 4. In Step 6-2, the positive- and negative-ideal anchor values of each criterion were found by evaluating $\max_i d\left(\bar{A}_{ij}, \tilde{1}_1\right)$ and $\min_i d\left(\bar{A}_{ij}, \tilde{1}_1\right)$, respectively. It followed that the lower anchor values $\left|d\left(\bar{A}_{+j},\tilde{1}_1\right)\right|$ and the upper anchor values $\left|d\left(\bar{A}_{-j},\tilde{1}_1\right)\right|$ for $A_i \in A$ were given by $\left|d\left(\bar{A}_{+1},\tilde{1}_1\right)\right| = 0.0235$, $\left|d\left(\bar{A}_{+2},\tilde{1}_1\right)\right| = 0.8366$, $\left|d\left(\bar{A}_{+3},\tilde{1}_1\right)\right| = 0.0235$, $\left|d\left(\bar{A}_{+4},\tilde{1}_1\right)\right| = 1.3298$, $\left|d\left(\bar{A}_{+5},\tilde{1}_1\right)\right| = 0.4455$, $\left|d\left(\bar{A}_{-1},\tilde{1}_1\right)\right| = 0.7166$, $\left|d\left(\bar{A}_{-2},\tilde{1}_1\right)\right| = 1.6622$, $\left|d\left(\bar{A}_{-3},\tilde{1}_1\right)\right| = 1.9765$, $\left|d\left(\bar{A}_{-4},\tilde{1}_1\right)\right| = 1.9719$, and $\left|d\left(\bar{A}_{-5},\tilde{1}_1\right)\right| = 1.9634$.

The relative variation of each criterion from its anchor values was computed in Step 6-3. Let us take Ψ_1 as an example:

$$
\begin{aligned}
\Psi_1 = {} & \left(\frac{\left|d\left(\bar{A}_{-1},\tilde{1}_1\right)\right| - \left|d\left(\bar{A}_{+1},\tilde{1}_1\right)\right|}{\left|d\left(\bar{A}_{-1},\tilde{1}_1\right)\right|}\right)\cdot\Bigg[\Bigg(1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{4i1}^{L}\right)^2},\\
& 1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{3i1}^{L}\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{2i1}^{L}\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{1i1}^{L}\right)^2};\min_i \bar{h}_{A_{i1}}^{L}\Bigg),\\
& \Bigg(1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{4i1}^{U}\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{3i1}^{U}\right)^2},1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{2i1}^{U}\right)^2},\\
& 1\Big/\sqrt{\textstyle\sum_{i=1}^{4}\left(\bar{a}_{1i1}^{U}\right)^2};\min_i \bar{h}_{A_{i1}}^{U}\Bigg)\Bigg]\\
= {} & \left[(0.5414, 0.5504, 0.5781, 0.5968; 0.8), (0.5188, 0.5365, 0.5927, 0.6262; 1)\right],
\end{aligned}
$$

where, for instance,

$$
\psi_{1j}^{L} = \left(\frac{0.7166 - 0.0235}{0.7166}\right)\cdot\left(\frac{1}{\sqrt{0.9667^2 + 0.9950^2 + 0.7158^2 + 0.8683^2}}\right) = 0.5415.
$$

The relative variations of the other criteria were the following:

$$
\begin{aligned}
\Psi_2 &= \left[(0.5030, 0.5690, 0.7125, 0.7773; 0.8), (0.3616, 0.4964, 0.7836, 0.9118; 1)\right],\\
\Psi_3 &= \left[(0.7334, 0.7477, 0.7874, 0.8147; 0.8), (0.6998, 0.7274, 0.8081, 0.8559; 1)\right],\\
\Psi_4 &= \left[(0.8201, 1.0710, 1.4960, 2.1964; 0.8), (0.3235, 0.7987, 1.7261, 2.7877; 1)\right],\\
\Psi_5 &= \left[(0.7360, 0.7940, 0.9536, 1.0386; 0.8), (0.6416, 0.7321, 1.0598, 1.2919; 1)\right].
\end{aligned}
$$

Let $\psi^{+} = \max(0.6262, 0.9118, 0.8559, 2.7877, 1.2919) = 2.7877$. According to Step 6-4, the normalized objective importance ϖ of the criteria was obtained as follows:

$$
\begin{aligned}
\varpi_1 &= \left[\left(\frac{0.5414}{2.7877},\frac{0.5504}{2.7877},\frac{0.5781}{2.7877},\frac{0.5968}{2.7877},0.8\right),\left(\frac{0.5188}{2.7877},\frac{0.5365}{2.7877},\frac{0.5927}{2.7877},\frac{0.6262}{2.7877},1\right)\right]\\
&= \left[(0.1942, 0.1974, 0.2074, 0.2141; 0.8), (0.1861, 0.1925, 0.2126, 0.2246; 1)\right],\\
\varpi_2 &= \left[(0.1804, 0.2041, 0.2556, 0.2788; 0.8), (0.1297, 0.1781, 0.2811, 0.3271; 1)\right],\\
\varpi_3 &= \left[(0.2631, 0.2682, 0.2825, 0.2922; 0.8), (0.2510, 0.2609, 0.2899, 0.3070; 1)\right],\\
\varpi_4 &= \left[(0.2942, 0.3842, 0.5366, 0.7879; 0.8), (0.1160, 0.2865, 0.6192, 1.0000; 1)\right],\\
\varpi_5 &= \left[(0.2640, 0.2848, 0.3421, 0.3726; 0.8), (0.2302, 0.2626, 0.3802, 0.4634; 1)\right].
\end{aligned}
$$

The priority order ($\varpi_4 > \varpi_5 > \varpi_3 > \varpi_2 > \varpi_1$ because $d\left(\varpi_4,\tilde{1}_1\right) > d\left(\varpi_5,\tilde{1}_1\right) > d\left(\varpi_3,\tilde{1}_1\right) > d\left(\varpi_2,\tilde{1}_1\right) > d\left(\varpi_1,\tilde{1}_1\right)$) via the normalized objective importance values of

Table 5 The results of the overall importance $\overline{\overline{W}}$ for each criterion

	$\overline{\overline{W}}_j^L$					$\overline{\overline{W}}_j^U$				
	$\overline{\overline{w}}_{1j}^L$	$\overline{\overline{w}}_{2j}^L$	$\overline{\overline{w}}_{3j}^L$	$\overline{\overline{w}}_{4j}^L$	$\overline{\overline{h}}_{W_j}^L$	$\overline{\overline{w}}_{1j}^U$	$\overline{\overline{w}}_{2j}^U$	$\overline{\overline{w}}_{3j}^U$	$\overline{\overline{w}}_{4j}^U$	$\overline{\overline{h}}_{W_j}^U$
(1) $\tau = 0.75$										
$\overline{\overline{W}}_1$	0.7380	0.7564	0.7785	0.7859	0.8000	0.7157	0.7451	0.7905	0.8062	1.0000
$\overline{\overline{W}}_2$	0.3696	0.3995	0.4730	0.5028	0.8000	0.3027	0.3627	0.5097	0.5691	1.0000
$\overline{\overline{W}}_3$	0.7003	0.7274	0.7682	0.7820	0.8000	0.6613	0.7067	0.7897	0.8192	1.0000
$\overline{\overline{W}}_4$	0.6253	0.6686	0.7459	0.8220	0.8000	0.5442	0.6247	0.7861	0.9116	1.0000
$\overline{\overline{W}}_5$	0.7137	0.7354	0.7850	0.8040	0.8000	0.6737	0.7121	0.8123	0.8583	1.0000
(2) $\tau = 0.5$										
$\overline{\overline{W}}_1$	0.5567	0.5701	0.5882	0.5953	0.8000	0.5392	0.5609	0.5979	0.6123	1.0000
$\overline{\overline{W}}_2$	0.3065	0.3344	0.4006	0.4282	0.8000	0.2450	0.3012	0.4335	0.4885	1.0000
$\overline{\overline{W}}_3$	0.5546	0.5744	0.6063	0.6188	0.8000	0.5245	0.5581	0.6231	0.6485	1.0000
$\overline{\overline{W}}_4$	0.5150	0.5738	0.6762	0.8106	0.8000	0.4015	0.5120	0.7305	0.9411	1.0000
$\overline{\overline{W}}_5$	0.5638	0.5852	0.6374	0.6602	0.8000	0.5259	0.5623	0.6683	0.7267	1.0000
(3) $\tau = 0.25$										
$\overline{\overline{W}}_1$	0.3755	0.3837	0.3978	0.4047	0.8000	0.3626	0.3767	0.4052	0.4185	1.0000
$\overline{\overline{W}}_2$	0.2435	0.2692	0.3281	0.3535	0.8000	0.1874	0.2396	0.3573	0.4078	1.0000
$\overline{\overline{W}}_3$	0.4088	0.4213	0.4444	0.4555	0.8000	0.3878	0.4095	0.4565	0.4777	1.0000
$\overline{\overline{W}}_4$	0.4046	0.4790	0.6064	0.7993	0.8000	0.2587	0.3992	0.6748	0.9705	1.0000
$\overline{\overline{W}}_5$	0.4139	0.4350	0.4897	0.5164	0.8000	0.3780	0.4124	0.5242	0.5950	1.0000

the criteria was obviously different from the priority order ($\overline{W}_1 > \overline{W}_5 > \overline{W}_3 > \overline{W}_4 > \overline{W}_2$ because $d\left(\overline{W}_1, \tilde{1}_1\right) > d\left(\overline{W}_5, \tilde{1}_1\right) > d\left(\overline{W}_3, \tilde{1}_1\right) > d\left(\overline{W}_4, \tilde{1}_1\right) > d\left(\overline{W}_2, \tilde{1}_1\right)$) via the normalized subjective importance. Although the criterion x_1 (survival rate) possessed the highest subjective importance, its objective importance was the lowest because it did not provide much information in the given decision situation.

In Step 7, the overall importance $\overline{\overline{W}}_j$ of the criteria was determined based on both the subjective and the objective importance values. In this example, τ was set to 1, 0.75, 0.5, 0.25, and 0 for comparison. Recall that $\overline{\overline{W}}_j = \overline{W}_j$ when $\tau = 1$ and $\overline{\overline{W}}_j = \varpi_j$ when $\tau = 0$. The computation results in the cases of $\tau = 0.75$, 0.5, and 0.25 are listed in Table 5. Comparing the values of $\overline{\overline{W}}$ and $\overline{W}$, we can see that the largest, $\overline{W}_1$, was offset by the lowest ϖ_1. The criterion x_1 did not have the highest overall importance; moreover, the value of $\overline{\overline{W}}_1$ fell as the τ value decreased.

In Step 8, the weighted normalized decision matrix D^W was established, as shown in Tables 6, 7, 8, 9, and 10. From the overall importance of each criterion in Table 5, the weighted normalized criterion value of A_{ij} was determined using (40). The weighted results for $\tau = 1, 0.75, 0.5, 0.25$, and 0 are presented in Tables 6, 7, 8, 9, and 10, respectively.

Table 6 The results of the weighted normalized decision matrix D^W ($\tau = 1$)

	$\overline{\overline{A}}{}^{L}_{ij}$					$\overline{\overline{A}}{}^{U}_{ij}$				
	$\overline{\overline{a}}{}^{L}_{1ij}$	$\overline{\overline{a}}{}^{L}_{2ij}$	$\overline{\overline{a}}{}^{L}_{3ij}$	$\overline{\overline{a}}{}^{L}_{4ij}$	$\overline{\overline{h}}{}^{L}_{A_{ij}}$	$\overline{\overline{a}}{}^{U}_{1ij}$	$\overline{\overline{a}}{}^{U}_{2ij}$	$\overline{\overline{a}}{}^{U}_{3ij}$	$\overline{\overline{a}}{}^{U}_{4ij}$	$\overline{\overline{h}}{}^{U}_{A_{ij}}$
$\overline{\overline{A}}_{11}$	0.8365	0.8798	0.9294	0.9440	0.8000	0.7881	0.8550	0.9569	0.9900	1.0000
$\overline{\overline{A}}_{12}$	0.1485	0.1727	0.2506	0.3039	0.8000	0.1054	0.1440	0.3051	0.4659	1.0000
$\overline{\overline{A}}_{13}$	0.5873	0.6340	0.7441	0.7838	0.8000	0.5001	0.5816	0.8033	0.8876	1.0000
$\overline{\overline{A}}_{14}$	0.0327	0.0376	0.0516	0.0602	0.8000	0.0256	0.0327	0.0622	0.0926	1.0000
$\overline{\overline{A}}_{15}$	0.2472	0.2891	0.3914	0.4408	0.8000	0.1618	0.2412	0.4459	0.5489	1.0000
$\overline{\overline{A}}_{21}$	0.8870	0.9333	0.9641	0.9716	0.8000	0.8505	0.9169	0.9831	1.0000	1.0000
$\overline{\overline{A}}_{22}$	0.0661	0.0736	0.0960	0.1054	0.8000	0.0509	0.0640	0.1092	0.1314	1.0000
$\overline{\overline{A}}_{23}$	0.8164	0.8717	0.9254	0.9406	0.8000	0.7607	0.8439	0.9563	0.9899	1.0000
$\overline{\overline{A}}_{24}$	0.0101	0.0105	0.0116	0.0119	0.8000	0.0092	0.0101	0.0121	0.0130	1.0000
$\overline{\overline{A}}_{25}$	0.5750	0.6333	0.7828	0.8423	0.8000	0.4555	0.5627	0.8622	0.9899	1.0000
$\overline{\overline{A}}_{31}$	0.5216	0.5649	0.6629	0.6990	0.8000	0.4401	0.5173	0.7144	0.7900	1.0000
$\overline{\overline{A}}_{32}$	0.1282	0.1487	0.2155	0.2530	0.8000	0.0919	0.1240	0.2620	0.3742	1.0000
$\overline{\overline{A}}_{33}$	0.0042	0.0044	0.0093	0.0331	0.8000	0.0000	0.0000	0.0127	0.0462	1.0000
$\overline{\overline{A}}_{34}$	0.1021	0.1604	0.2411	0.3241	0.8000	0.0741	0.1338	0.3362	0.8821	1.0000
$\overline{\overline{A}}_{35}$	0.0067	0.0069	0.0145	0.0516	0.8000	0.0000	0.0000	0.0198	0.0719	1.0000
$\overline{\overline{A}}_{41}$	0.6786	0.7235	0.8163	0.8479	0.8000	0.6007	0.6784	0.8651	0.9333	1.0000
$\overline{\overline{A}}_{42}$	0.1832	0.2180	0.3291	0.3972	0.8000	0.1280	0.1791	0.4124	0.6498	1.0000
$\overline{\overline{A}}_{43}$	0.2037	0.2428	0.3170	0.3490	0.8000	0.1410	0.2081	0.3570	0.4289	1.0000
$\overline{\overline{A}}_{44}$	0.0171	0.0188	0.0236	0.0259	0.8000	0.0142	0.0169	0.0267	0.0329	1.0000
$\overline{\overline{A}}_{45}$	0.1466	0.1755	0.2416	0.2910	0.8000	0.0894	0.1430	0.2775	0.3693	1.0000

In Step 9, the ideal solution A^* was defined as in (43) because the IT2TrFNs in the weighted normalized decision matrix D^W were positive; their ranges belonged to the closed interval [0, 1]. Next, the normalized signed distances from each alternative to A^* were calculated using (45). Consider the case of $\tau = 0.25$ for example. The closeness coefficient of alternative A_i from the ideal solution was computed as follows: $\left|\bar{d}_1^*\right| = 0.7341$, $\left|\bar{d}_2^*\right| = 0.6876$, $\left|\bar{d}_3^*\right| = 0.8447$, and $\left|\bar{d}_4^*\right| = 0.8079$. In Step 10, the ranking of the four treatment options was established from the closeness coefficients: $A_2 \succ A_1 \succ A_4 \succ A_3$. Thus, the best treatment option was A_2. The detailed results for various τ values are summarized in Table 11. We can observe that the value of τ did not perceptibly influence the ranking of the alternatives in this practical example. As indicated in Table 11, the most appropriate treatment option for Mr. Peng was A_2 (intra-arterial thrombolysis).

6.2.2 Illustration of the weight-assessing method

If the three decision-makers (E_1, E_2, and E_3) would like to know the estimation of criterion weights to have more information in decision-making, they could use the weight-assessing

Table 7 The results of the weighted normalized decision matrix D^W ($\tau = 0.75$)

	$\overline{\overline{A}}_{ij}^L$					$\overline{\overline{A}}_{ij}^U$				
	$\overline{\overline{a}}_{1ij}^L$	$\overline{\overline{a}}_{2ij}^L$	$\overline{\overline{a}}_{3ij}^L$	$\overline{\overline{a}}_{4ij}^L$	$\overline{\overline{h}}_{A_{ij}}^L$	$\overline{\overline{a}}_{1ij}^U$	$\overline{\overline{a}}_{2ij}^U$	$\overline{\overline{a}}_{3ij}^U$	$\overline{\overline{a}}_{4ij}^U$	$\overline{\overline{h}}_{A_{ij}}^U$
$\overline{\overline{A}}_{11}$	0.6716	0.7059	0.7467	0.7597	0.8000	0.6322	0.6855	0.7694	0.7981	1.0000
$\overline{\overline{A}}_{12}$	0.1268	0.1485	0.2173	0.2646	0.8000	0.0885	0.1231	0.2655	0.4080	1.0000
$\overline{\overline{A}}_{13}$	0.4861	0.5237	0.6146	0.6484	0.8000	0.4144	0.4806	0.6633	0.7346	1.0000
$\overline{\overline{A}}_{14}$	0.0278	0.0330	0.0472	0.0593	0.8000	0.0203	0.0277	0.0581	0.0957	1.0000
$\overline{\overline{A}}_{15}$	0.2043	0.2400	0.3295	0.3739	0.8000	0.1327	0.1992	0.3788	0.4759	1.0000
$\overline{\overline{A}}_{21}$	0.7122	0.7488	0.7746	0.7820	0.8000	0.6823	0.7352	0.7905	0.8062	1.0000
$\overline{\overline{A}}_{22}$	0.0565	0.0633	0.0832	0.0918	0.8000	0.0428	0.0547	0.0950	0.1151	1.0000
$\overline{\overline{A}}_{23}$	0.6758	0.7201	0.7644	0.7781	0.8000	0.6304	0.6973	0.7897	0.8192	1.0000
$\overline{\overline{A}}_{24}$	0.0086	0.0092	0.0106	0.0118	0.8000	0.0073	0.0086	0.0113	0.0134	1.0000
$\overline{\overline{A}}_{25}$	0.4752	0.5259	0.6589	0.7145	0.8000	0.3736	0.4649	0.7324	0.8583	1.0000
$\overline{\overline{A}}_{31}$	0.4188	0.4532	0.5326	0.5625	0.8000	0.3531	0.4148	0.5745	0.6369	1.0000
$\overline{\overline{A}}_{32}$	0.1095	0.1279	0.1868	0.2203	0.8000	0.0772	0.1061	0.2279	0.3277	1.0000
$\overline{\overline{A}}_{33}$	0.0035	0.0036	0.0077	0.0274	0.8000	0.0000	0.0000	0.0105	0.0383	1.0000
$\overline{\overline{A}}_{34}$	0.0868	0.1405	0.2205	0.3197	0.8000	0.0587	0.1133	0.3140	0.9116	1.0000
$\overline{\overline{A}}_{35}$	0.0056	0.0057	0.0122	0.0437	0.8000	0.0000	0.0000	0.0168	0.0623	1.0000
$\overline{\overline{A}}_{41}$	0.5449	0.5805	0.6559	0.6824	0.8000	0.4819	0.5439	0.6956	0.7524	1.0000
$\overline{\overline{A}}_{42}$	0.1565	0.1875	0.2854	0.3458	0.8000	0.1075	0.1532	0.3588	0.5691	1.0000
$\overline{\overline{A}}_{43}$	0.1686	0.2006	0.2618	0.2887	0.8000	0.1169	0.1719	0.2948	0.3550	1.0000
$\overline{\overline{A}}_{44}$	0.0146	0.0164	0.0216	0.0256	0.8000	0.0113	0.0143	0.0249	0.0340	1.0000
$\overline{\overline{A}}_{45}$	0.1211	0.1458	0.2034	0.2468	0.8000	0.0733	0.1181	0.2357	0.3202	1.0000

method in the signed-distance-based approach. Let ω_j $(j = 1, 2, \ldots, 5)$ be the weight of criterion x_j and $\omega_j \in \left[\omega_j^L, \omega_j^U\right]$. Let $\omega_j^L = \overline{\overline{w}}_{1j}^U$ and $\omega_j^U = \overline{\overline{w}}_{4j}^U$. Take $\tau = 0.25$ as an example again. From Table 5, it is known that $\omega_1^L = 0.3626$, $\omega_2^L = 0.1874$, $\omega_3^L = 0.3878$, $\omega_4^L = 0.2587$, $\omega_5^L = 0.3780$, $\omega_1^U = 0.4185$, $\omega_2^U = 0.4078$, $\omega_3^U = 0.4777$, $\omega_4^U = 0.9705$, and $\omega_5^U = 0.5950$. We noted that $\sum_{j=1}^5 \omega_j^L = 1.5745 > 1$ and $\sum_{j=1}^5 \omega_j^U = 2.8695 > 1$.

The weighted normalized criterion value of A_{ij} was obtained with (49). Then, the weighted normalized decision matrix D'^W was correspondingly constructed by (51) in Step 8'. In Step 9', the normalized signed distance from each alternative to A^* was computed using Table 4 and (52) as follows:

$$\bar{d}_1'^* = \frac{1}{80}\left(15.0970\omega_1 + 7.1005\omega_2 + 12.1688\omega_3 + 0.9771\omega_4 + 5.9827\omega_5 - 80\right),$$

$$\bar{d}_2'^* = \frac{1}{80}\left(15.8118\omega_1 + 2.7025\omega_2 + 15.8118\omega_3 + 0.2246\omega_4 + 12.4360\omega_5 - 80\right),$$

$$\bar{d}_3'^* = \frac{1}{80}\left(10.2669\omega_1 + 6.0334\omega_2 + 0.1882\omega_3 + 5.3616\omega_4 + 0.2927\omega_5 - 80\right),$$

Table 8 The results of the weighted normalized decision matrix D^W ($\tau = 0.5$)

	$\overline{\overline{A}}^L_{ij}$					$\overline{\overline{A}}^U_{ij}$				
	$\overline{\overline{a}}^L_{1ij}$	$\overline{\overline{a}}^L_{2ij}$	$\overline{\overline{a}}^L_{3ij}$	$\overline{\overline{a}}^L_{4ij}$	$\overline{\overline{h}}^L_{A_{ij}}$	$\overline{\overline{a}}^U_{1ij}$	$\overline{\overline{a}}^U_{2ij}$	$\overline{\overline{a}}^U_{3ij}$	$\overline{\overline{a}}^U_{4ij}$	$\overline{\overline{h}}^U_{A_{ij}}$
$\overline{\overline{A}}_{11}$	0.5066	0.5321	0.5642	0.5755	0.8000	0.4763	0.5160	0.5819	0.6062	1.0000
$\overline{\overline{A}}_{12}$	0.1052	0.1243	0.1840	0.2253	0.8000	0.0716	0.1022	0.2258	0.3503	1.0000
$\overline{\overline{A}}_{13}$	0.3850	0.4136	0.4850	0.5131	0.8000	0.3287	0.3795	0.5234	0.5815	1.0000
$\overline{\overline{A}}_{14}$	0.0229	0.0283	0.0428	0.0585	0.8000	0.0150	0.0227	0.0540	0.0988	1.0000
$\overline{\overline{A}}_{15}$	0.1614	0.1910	0.2675	0.3071	0.8000	0.1036	0.1573	0.3116	0.4030	1.0000
$\overline{\overline{A}}_{21}$	0.5372	0.5644	0.5853	0.5923	0.8000	0.5140	0.5534	0.5979	0.6123	1.0000
$\overline{\overline{A}}_{22}$	0.0468	0.0530	0.0705	0.0781	0.8000	0.0346	0.0454	0.0808	0.0988	1.0000
$\overline{\overline{A}}_{23}$	0.5352	0.5687	0.6033	0.6157	0.8000	0.5000	0.5507	0.6231	0.6485	1.0000
$\overline{\overline{A}}_{24}$	0.0071	0.0079	0.0096	0.0116	0.8000	0.0054	0.0070	0.0105	0.0138	1.0000
$\overline{\overline{A}}_{25}$	0.3754	0.4185	0.5350	0.5867	0.8000	0.2916	0.3671	0.6025	0.7267	1.0000
$\overline{\overline{A}}_{31}$	0.3159	0.3416	0.4024	0.4261	0.8000	0.2660	0.3123	0.4345	0.4837	1.0000
$\overline{\overline{A}}_{32}$	0.0908	0.1070	0.1582	0.1876	0.8000	0.0625	0.0881	0.1939	0.2813	1.0000
$\overline{\overline{A}}_{33}$	0.0028	0.0029	0.0061	0.0217	0.8000	0.0000	0.0000	0.0083	0.0303	1.0000
$\overline{\overline{A}}_{34}$	0.0715	0.1206	0.1999	0.3152	0.8000	0.0433	0.0929	0.2918	0.9411	1.0000
$\overline{\overline{A}}_{35}$	0.0044	0.0046	0.0099	0.0359	0.8000	0.0000	0.0000	0.0138	0.0528	1.0000
$\overline{\overline{A}}_{41}$	0.4110	0.4376	0.4956	0.5169	0.8000	0.3630	0.4095	0.5262	0.5715	1.0000
$\overline{\overline{A}}_{42}$	0.1298	0.1569	0.2417	0.2945	0.8000	0.0870	0.1272	0.3051	0.4885	1.0000
$\overline{\overline{A}}_{43}$	0.1335	0.1584	0.2066	0.2285	0.8000	0.0927	0.1358	0.2326	0.2810	1.0000
$\overline{\overline{A}}_{44}$	0.0120	0.0141	0.0195	0.0252	0.8000	0.0083	0.0117	0.0232	0.0351	1.0000
$\overline{\overline{A}}_{45}$	0.0957	0.1160	0.1652	0.2027	0.8000	0.0572	0.0933	0.1939	0.2711	1.0000

$$\bar{d}_4'^{*} = \frac{1}{80}\left(12.8712\omega_1 + 9.3075\omega_2 + 4.9156\omega_3 + 0.4409\omega_4 + 3.7119\omega_5 - 80\right).$$

Consider the case of $\tau = 0.25$ for example. In Step 10', the single-objective optimization model in (56) was constructed to estimate the importance weights of the criteria:

$$\min\{\lambda_1 + \lambda_2\}$$
$$\text{s.t.}\begin{cases} -\frac{1}{80}\left(15.0970\omega_1 + 7.1005\omega_2 + 12.1688\omega_3 + 0.9771\omega_4 + 5.9827\omega_5 - 80\right) \le \lambda_2, \\ -\frac{1}{80}\left(15.8118\omega_1 + 2.7025\omega_2 + 15.8118\omega_3 + 0.2246\omega_4 + 12.4360\omega_5 - 80\right) \le \lambda_2, \\ -\frac{1}{80}\left(10.2669\omega_1 + 6.0334\omega_2 + 0.1882\omega_3 + 5.3616\omega_4 + 0.2927\omega_5 - 80\right) \le \lambda_2, \\ -\frac{1}{80}\left(12.8712\omega_1 + 9.3075\omega_2 + 4.9156\omega_3 + 0.4409\omega_4 + 3.7119\omega_5 - 80\right) \le \lambda_2, \\ e_1^- \le \lambda_1, e_2^- \le \lambda_1, e_3^- \le \lambda_1, e_4^- \le \lambda_1, e_5^- \le \lambda_1, e_1^+ \le \lambda_1, e_2^+ \le \lambda_1, e_3^+ \le \lambda_1, e_4^+ \le \lambda_1, e_5^+ \le \lambda_1, \\ w_1 + e_1^- \ge 0.3626,\ w_2 + e_2^- \ge 0.1874,\ w_3 + e_3^- \ge 0.3878,\ w_4 + e_4^- \ge 0.2587,\ w_5 + e_5^- \ge 0.3780, \\ w_1 - e_1^+ \le 0.4185,\ w_2 - e_2^+ \le 0.4078,\ w_3 - e_3^+ \le 0.4777,\ w_4 - e_4^+ \le 0.9705,\ w_5 - e_5^+ \le 0.5950, \\ e_1^- \ge 0, e_2^- \ge 0, e_3^- \ge 0, e_4^- \ge 0, e_5^- \ge 0, e_1^+ \ge 0, e_2^+ \ge 0, e_3^+ \ge 0, e_4^+ \ge 0, e_5^+ \ge 0, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, w_5 \ge 0, \\ w_1 + w_2 + w_3 + w_4 + w_5 = 1. \end{cases}$$

Table 9 The results of the weighted normalized decision matrix D^W ($\tau = 0.25$)

	$\overline{\overline{A}}_{ij}^{L}$					$\overline{\overline{A}}_{ij}^{U}$				
	$\overline{\overline{a}}_{1ij}^{L}$	$\overline{\overline{a}}_{2ij}^{L}$	$\overline{\overline{a}}_{3ij}^{L}$	$\overline{\overline{a}}_{4ij}^{L}$	$\overline{\overline{h}}_{A_{ij}}^{L}$	$\overline{\overline{a}}_{1ij}^{U}$	$\overline{\overline{a}}_{2ij}^{U}$	$\overline{\overline{a}}_{3ij}^{U}$	$\overline{\overline{a}}_{4ij}^{U}$	$\overline{\overline{h}}_{A_{ij}}^{U}$
$\overline{\overline{A}}_{11}$	0.3417	0.3581	0.3816	0.3912	0.8000	0.3203	0.3466	0.3944	0.4143	1.0000
$\overline{\overline{A}}_{12}$	0.0836	0.1001	0.1507	0.1860	0.8000	0.0548	0.0813	0.1861	0.2924	1.0000
$\overline{\overline{A}}_{13}$	0.2838	0.3033	0.3555	0.3777	0.8000	0.2430	0.2785	0.3835	0.4284	1.0000
$\overline{\overline{A}}_{14}$	0.0180	0.0236	0.0384	0.0577	0.8000	0.0096	0.0177	0.0499	0.1019	1.0000
$\overline{\overline{A}}_{15}$	0.1185	0.1420	0.2055	0.2402	0.8000	0.0745	0.1154	0.2444	0.3299	1.0000
$\overline{\overline{A}}_{21}$	0.3624	0.3799	0.3958	0.4027	0.8000	0.3457	0.3717	0.4052	0.4185	1.0000
$\overline{\overline{A}}_{22}$	0.0372	0.0426	0.0577	0.0645	0.8000	0.0265	0.0361	0.0666	0.0825	1.0000
$\overline{\overline{A}}_{23}$	0.3945	0.4171	0.4422	0.4532	0.8000	0.3697	0.4041	0.4565	0.4777	1.0000
$\overline{\overline{A}}_{24}$	0.0055	0.0066	0.0086	0.0114	0.8000	0.0035	0.0055	0.0097	0.0143	1.0000
$\overline{\overline{A}}_{25}$	0.2756	0.3111	0.4111	0.4589	0.8000	0.2096	0.2693	0.4726	0.5950	1.0000
$\overline{\overline{A}}_{31}$	0.2131	0.2299	0.2722	0.2897	0.8000	0.1789	0.2097	0.2945	0.3306	1.0000
$\overline{\overline{A}}_{32}$	0.0722	0.0862	0.1296	0.1549	0.8000	0.0478	0.0701	0.1598	0.2349	1.0000
$\overline{\overline{A}}_{33}$	0.0020	0.0021	0.0044	0.0159	0.8000	0.0000	0.0000	0.0061	0.0223	1.0000
$\overline{\overline{A}}_{34}$	0.0562	0.1006	0.1793	0.3108	0.8000	0.0279	0.0724	0.2695	0.9705	1.0000
$\overline{\overline{A}}_{35}$	0.0032	0.0034	0.0076	0.0281	0.8000	0.0000	0.0000	0.0109	0.0432	1.0000
$\overline{\overline{A}}_{41}$	0.2772	0.2945	0.3351	0.3514	0.8000	0.2441	0.2750	0.3566	0.3906	1.0000
$\overline{\overline{A}}_{42}$	0.1031	0.1263	0.1979	0.2431	0.8000	0.0666	0.1012	0.2515	0.4078	1.0000
$\overline{\overline{A}}_{43}$	0.0984	0.1162	0.1515	0.1682	0.8000	0.0685	0.0996	0.1704	0.2070	1.0000
$\overline{\overline{A}}_{44}$	0.0094	0.0118	0.0175	0.0249	0.8000	0.0054	0.0091	0.0214	0.0362	1.0000
$\overline{\overline{A}}_{45}$	0.0702	0.0862	0.1269	0.1585	0.8000	0.0411	0.0684	0.1521	0.2220	1.0000

Next, the integrated programming model was solved to obtain the optimal weight vector, $\breve{\omega} = (0.2477, 0.0725, 0.2729, 0.1438, 0.2631)$; the optimal deviation values, $\breve{e}_1^- = \breve{e}_2^- = \breve{e}_3^- = \breve{e}_4^- = \breve{e}_5^- = 0.1149$, $\breve{e}_1^+ = \breve{e}_2^+ = \breve{e}_3^+ = \breve{e}_4^+ = \breve{e}_5^+ = 0$; the optimal signed distances, $\breve{\bar{d}}_1^{\prime *} = -0.8839$, $\breve{\bar{d}}_2^{\prime *} = -0.8534$, $\breve{\bar{d}}_3^{\prime *} = -0.9515$, $\breve{\bar{d}}_4^{\prime *} = -0.9219$; and the optimal objective value, 1.0664. In Step 11', the optimal ranking of the four treatment options was determined by the values of $\left|\breve{\bar{d}}_i^{\prime *}\right|$: $A_2 \succ A_1 \succ A_4 \succ A_3$. The detailed results for various τ values are summarized in Table 12. As indicated in this table, the criterion weights are moderately different in the cases of various τ values. The rankings of the weights were given by $\breve{\omega}_1 > \breve{\omega}_5 > \breve{\omega}_3 > \breve{\omega}_4 > \breve{\omega}_2$, $\breve{\omega}_1 > \breve{\omega}_5 > \breve{\omega}_3 > \breve{\omega}_4 > \breve{\omega}_2$, $\breve{\omega}_1 > \breve{\omega}_5 > \breve{\omega}_3 > \breve{\omega}_4 > \breve{\omega}_2$, $\breve{\omega}_3 > \breve{\omega}_5 > \breve{\omega}_1 > \breve{\omega}_4 > \breve{\omega}_2$, and $\breve{\omega}_3 > \breve{\omega}_5 > \breve{\omega}_1 > \breve{\omega}_2 > \breve{\omega}_4$ for $\tau = 1, 0.75, 0.5, 0.25$, and 0, respectively. Although the distributions of importance weights of the criteria showed different patterns in various settings of the τ values, the obtained rankings of the four treatment options were the same ($A_2 \succ A_1 \succ A_4 \succ A_3$). Therefore, A_2 was the best choice.

Table 10 The results of the weighted normalized decision matrix D^W ($\tau = 0$)

	$\overline{\overline{A}}_{ij}^L$					$\overline{\overline{A}}_{ij}^U$				
	$\overline{\overline{a}}_{1ij}^L$	$\overline{\overline{a}}_{2ij}^L$	$\overline{\overline{a}}_{3ij}^L$	$\overline{\overline{a}}_{4ij}^L$	$\overline{\overline{h}}_{A_{ij}}^L$	$\overline{\overline{a}}_{1ij}^U$	$\overline{\overline{a}}_{2ij}^U$	$\overline{\overline{a}}_{3ij}^U$	$\overline{\overline{a}}_{4ij}^U$	$\overline{\overline{h}}_{A_{ij}}^U$
$\overline{\overline{A}}_{11}$	0.1767	0.1842	0.1989	0.2070	0.8000	0.1644	0.1771	0.2069	0.2224	1.0000
$\overline{\overline{A}}_{12}$	0.0619	0.0759	0.1174	0.1467	0.8000	0.0379	0.0604	0.1464	0.2345	1.0000
$\overline{\overline{A}}_{13}$	0.1826	0.1931	0.2260	0.2423	0.8000	0.1573	0.1774	0.2435	0.2753	1.0000
$\overline{\overline{A}}_{14}$	0.0131	0.0189	0.0340	0.0569	0.8000	0.0043	0.0127	0.0458	0.1050	1.0000
$\overline{\overline{A}}_{15}$	0.0756	0.0930	0.1436	0.1733	0.8000	0.0453	0.0735	0.1773	0.2570	1.0000
$\overline{\overline{A}}_{21}$	0.1874	0.1954	0.2064	0.2130	0.8000	0.1774	0.1899	0.2126	0.2246	1.0000
$\overline{\overline{A}}_{22}$	0.0276	0.0323	0.0450	0.0509	0.8000	0.0183	0.0269	0.0524	0.0661	1.0000
$\overline{\overline{A}}_{23}$	0.2539	0.2655	0.2811	0.2907	0.8000	0.2393	0.2574	0.2899	0.3070	1.0000
$\overline{\overline{A}}_{24}$	0.0040	0.0053	0.0076	0.0113	0.8000	0.0016	0.0039	0.0089	0.0147	1.0000
$\overline{\overline{A}}_{25}$	0.1758	0.2037	0.2872	0.3311	0.8000	0.1276	0.1715	0.3428	0.4634	1.0000
$\overline{\overline{A}}_{31}$	0.1102	0.1183	0.1419	0.1533	0.8000	0.0918	0.1072	0.1545	0.1774	1.0000
$\overline{\overline{A}}_{32}$	0.0535	0.0653	0.1010	0.1221	0.8000	0.0331	0.0521	0.1257	0.1884	1.0000
$\overline{\overline{A}}_{33}$	0.0013	0.0013	0.0028	0.0102	0.8000	0.0000	0.0000	0.0039	0.0143	1.0000
$\overline{\overline{A}}_{34}$	0.0408	0.0807	0.1586	0.3064	0.8000	0.0125	0.0520	0.2473	1.0000	1.0000
$\overline{\overline{A}}_{35}$	0.0021	0.0022	0.0053	0.0203	0.8000	0.0000	0.0000	0.0079	0.0336	1.0000
$\overline{\overline{A}}_{41}$	0.1434	0.1515	0.1747	0.1859	0.8000	0.1253	0.1405	0.1871	0.2096	1.0000
$\overline{\overline{A}}_{42}$	0.0764	0.0958	0.1542	0.1918	0.8000	0.0461	0.0752	0.1979	0.3271	1.0000
$\overline{\overline{A}}_{43}$	0.0634	0.0740	0.0963	0.1079	0.8000	0.0444	0.0635	0.1082	0.1330	1.0000
$\overline{\overline{A}}_{44}$	0.0069	0.0095	0.0155	0.0245	0.8000	0.0024	0.0066	0.0196	0.0373	1.0000
$\overline{\overline{A}}_{45}$	0.0448	0.0564	0.0886	0.1144	0.8000	0.0250	0.0436	0.1103	0.1729	1.0000

Finally, the patient's family decided to adopt intra-arterial thrombolysis (A_2) as the treatment. Via surgery, a catheter was inserted into the femoral artery in the thigh and placed at the thrombus that had caused the thrombolytic occlusion. A direct and rapid therapeutic effect was achieved through this approach, and the dose of thrombolytic agents was lower than that of intravenous thrombolysis. Mr. Peng became conscious after the treatment and is still undergoing rehabilitation.

6.3 Comparative analysis and discussions

Most of the present research concerning MCDA with IT2FSs has only considered the subjective importance values of the criteria. Based on the subjective importance values $\overline{W}$ of the criteria, the solution results by using the signed-distance-based MCDA method can be found in the case of $\tau = 1$ in Table 11, where $\left|\bar{d}_1^*\right| = 0.4445$, $\left|\bar{d}_2^*\right| = 0.3384$, $\left|\bar{d}_3^*\right| = 0.7201$, $\left|\bar{d}_4^*\right| = 0.6002$, and $A_2 \succ A_1 \succ A_4 \succ A_3$. Although the subjective importance values of criteria assigned by the decision-makers can exhibit preference dependence, the

Table 11 The comparative results yielded by the signed-distance-based MCDA method

	$\tau = 1$ (subjective importance only)		$\tau = 0.75$		$\tau = 0.5$		$\tau = 0.25$		$\tau = 0$ (objective importance only)	
	$\overline{d}_i^*$	$\lvert\overline{d}_i^*\rvert$	$\overline{d}_i^*$	$\lvert\overline{d}_i^*\rvert$	$\overline{d}_i^*$	$\lvert\overline{d}_i^*\rvert$	$\overline{d}_i^*$	$\lvert\overline{d}_i^*\rvert$	$\overline{d}_i^*$	$\lvert\overline{d}_i^*\rvert$
A_1	−0.4445	0.4445	−0.5410	0.5410	−0.6376	0.6376	−0.7341	0.7341	−0.8306	0.8306
A_2	−0.3384	0.3384	−0.4548	0.4548	−0.5712	0.5712	−0.6876	0.6876	−0.8040	0.8040
A_3	−0.7201	0.7201	−0.7616	0.7616	−0.8031	0.8031	−0.8447	0.8447	−0.8862	0.8862
A_4	−0.6002	0.6002	−0.6694	0.6694	−0.7386	0.7386	−0.8079	0.8079	−0.8771	0.8771
Priority order	$A_2 \succ A_1 \succ A_4 \succ A_3$		$A_2 \succ A_1 \succ A_4 \succ A_3$		$A_2 \succ A_1 \succ A_4 \succ A_3$		$A_2 \succ A_1 \succ A_4 \succ A_3$		$A_2 \succ A_1 \succ A_4 \succ A_3$	

Table 12 The comparative results yielded by the weight-assessing method

	$\tau = 1$ (subjective weights only)	$\tau = 0.75$	$\tau = 0.5$	$\tau = 0.25$	$\tau = 0$ (objective weights only)
$\breve{\omega}_1$	0.3426	0.3170	0.2915	0.2477	0.2246
$\breve{\omega}_2$	0.0000	0.0000	0.0000	0.0725	0.1782
$\breve{\omega}_3$	0.2483	0.2626	0.2767	0.2729	0.2510
$\breve{\omega}_4$	0.1373	0.1455	0.1537	0.1438	0.1160
$\breve{\omega}_5$	0.2718	0.2749	0.2781	0.2631	0.2302
$\breve{e}_1^-$	0.5497	0.3987	0.2478	0.1149	0.0000
$\breve{e}_2^-$	0.3603	0.3027	0.2450	0.1149	0.0000
$\breve{e}_3^-$	0.5497	0.3987	0.2478	0.1149	0.0000
$\breve{e}_4^-$	0.5497	0.3987	0.2478	0.1149	0.0000
$\breve{e}_5^-$	0.5497	0.3987	0.2478	0.1149	0.0000
$\breve{e}_1^+$	0.0000	0.0000	0.0000	0.0000	0.0000
$\breve{e}_2^+$	0.0000	0.0000	0.0000	0.0000	0.0000
$\breve{e}_3^+$	0.0000	0.0000	0.0000	0.0000	0.0000
$\breve{e}_4^+$	0.0000	0.0000	0.0000	0.0000	0.0000
$\breve{e}_5^+$	0.0000	0.0000	0.0000	0.0000	0.0000
$\left\lvert \breve{\bar{d}}_1'^{*} \right\rvert$	0.8756	0.8779	0.8802	0.8839	0.8850
$\left\lvert \breve{\bar{d}}_2'^{*} \right\rvert$	0.8406	0.8423	0.8440	0.8534	0.8639
$\left\lvert \breve{\bar{d}}_3'^{*} \right\rvert$	0.9453	0.9479	0.9506	0.9515	0.9485
$\left\lvert \breve{\bar{d}}_4'^{*} \right\rvert$	0.9163	0.9193	0.9224	0.9219	0.9164
Objective value	1.4949	1.3467	1.1984	1.0664	0.9485
Priority order	$A_2 \succ A_1 \succ A_4 \succ A_3$	$A_2 \succ A_1 \succ A_4 \succ A_3$	$A_2 \succ A_1 \succ A_4 \succ A_3$	$A_2 \succ A_1 \succ A_4 \succ A_3$	$A_2 \succ A_1 \succ A_4 \succ A_3$

intrinsic information generated by a given decision situation is not involved in assessing subjective criterion importance, and the criterion dependence in the given situation is thus ignored. This study modified the STEM to develop a signed-distance-based method to determine objective importance values of the criteria.

In this study, the approach to estimate objective criterion importance is measured with the deviation defined in the STEM approach. However, in comparison with the traditional STEM [5], several problems follow from the STEM definitions in the context of IT2TrFNs. The signed distances and linear scale transformation were used to solve the problems and to modify the STEM method in this paper. Based on the objective importance ϖ_j of each criterion, the solution results by using the signed-distance-based MCDA method can be found in the case of $\tau = 0$ in Table 11, where $\left|\bar{d}_1^*\right| = 0.8306$, $\left|\bar{d}_2^*\right| = 0.8040$, $\left|\bar{d}_3^*\right| = 0.8862$, $\left|\bar{d}_4^*\right| = 0.8771$, and $A_2 \succ A_1 \succ A_4 \succ A_3$. Furthermore, an approach was proposed to identify the overall importance by combining the subjective and objective information in this paper. Note that the subjective and objective importance values of the criteria are the special cases of our proposed overall importance measures. In addition to the ranking results computed from the subjective and objective importance values, we could incorporate the overall importance of each criterion into the proposed MCDA methods for comprehensive decision aiding.

On the other hand, it is important to know how to measure the criterion importance weights correctly when multiple criteria are simultaneously considered in a decision problem. This paper presented a useful method for estimating the importance weights of criteria with IT2TrFN data. Considering that the criteria may be assigned unduly high or low ratings, some deviation variables were introduced to mitigate the effects that overestimated and underestimated ratings have on criterion importance. Given the two objectives of maximal closeness coefficients and minimal deviation values, an integrated programming model was proposed to estimate the optimal weights for the criteria and the corresponding closeness coefficient values for alternative rankings.

In regard to the proposed integrated programming model for estimating the importance weights, the solution results on the basis of subjective weights can be found in the case of $\tau = 1$ in Table 12, where $\breve{\omega}_1 = 0.3426, \breve{\omega}_2 = 0.0000, \breve{\omega}_3 = 0.2483, \breve{\omega}_4 = 0.1373, \breve{\omega}_5 = 0.2718, \left|\breve{\bar{d}}_1'^*\right| = 0.8756, \left|\breve{\bar{d}}_2'^*\right| = 0.8406, \left|\breve{\bar{d}}_3'^*\right| = 0.9453, \left|\breve{\bar{d}}_4'^*\right| = 0.9163$, and $A_2 \succ A_1 \succ A_4 \succ A_3$. In contrast, the solution results on the basis of objective weights can be found in the case of $\tau = 0$ in Table 12, where $\breve{\omega}_1 = 0.2246, \breve{\omega}_2 = 0.1782, \breve{\omega}_3 = 0.2510, \breve{\omega}_4 = 0.1160, \breve{\omega}_5 = 0.2302, \left|\breve{\bar{d}}_1'^*\right| = 0.8850, \left|\breve{\bar{d}}_2'^*\right| = 0.8639, \left|\breve{\bar{d}}_3'^*\right| = 0.9485, \left|\breve{\bar{d}}_4'^*\right| = 0.9164$, and $A_2 \succ A_1 \succ A_4 \succ A_3$. More specifically, it is significant that the overall importance of each criterion can be replaced by the subjective or objective importance in the presented weight-assessment method. If the decision-makers are only seeking the subjective criterion weights, then let the overall weight ω_j' of each $x_j \in X$ lie in the closed interval $\left[\omega_j'^L, \omega_j'^U\right]$, where $\omega_j'^L = \overline{W}_{1j}^U$ and $\omega_j'^U = \overline{W}_{4j}^U$. If the decision-makers merely want to obtain the objective criterion weights, then let the objective weight ω_j'' of each criterion $x_j \in X$ lie in the closed interval $\left[\omega_j''^L, \omega_j''^U\right]$, where $\omega_j''^L = \varpi_{1j}^U$ and $\omega_j''^U = \varpi_{4j}^U$. As a word, in addition to the overall weights, our proposed weight-assessment method can be applied to the subjective weights and to the objective weights. Moreover, the approaches to estimating subjective or objective weights of the criteria are the special cases in our proposed method for assessing the overall criterion weights.

7 Conclusions

This paper established an MCDA method for handling subjective and objective information and for estimating criterion weights via signed distances in the context of an interval-valued fuzzy framework. When subjective human judgments are involved, a two-valued logic approach is rarely employed to assess each criterion. On the contrary, a multi-valued logic approach for the expression of fuzzy linguistic variables more closely resembles the uncertainty of human thinking. If traditional binary numerals are used to explain psychological uncertainty and ambiguity, there is tendency for this method to be mistakenly over-applied. The utilization of fuzzy linguistic variables during the assessment of group decision-making reduces the pressure felt by the decision-makers during the assessment and closely resembles the value judgment perceived by the decision-makers. Via the presentation of fuzzy linguistic variables, decision-makers have the flexibility to provide appropriate assessment values for each criterion. This study used IT2TrFNs as the performance assessment values used by the decision-makers. Compared to type-1 trapezoidal fuzzy numbers, the IT2TrFNs within a value range can further represent the deep-seated uncertainty manifested by the decision-makers. As the assessment criteria become more complex and abstract, IT2TrFNs become more suitable as objective and quantitative tools.

This research has made important contributions to the current literature in a number of significant aspects. First, instead of a complicated computational process for coping with IT2TrFN data, a useful decision analysis method was developed based on the concept of signed distances. Second, our approach determined the objective importance of criteria by using the signed-distance-based deviations within the IT2TrFN decision environment. Third, based on the subjective preference information provided by the decision-makers and the objective information emitted by the decision matrix, an integrated approach was established to determine the overall importance of criteria and to then incorporate it in the MCDA. Fourth, an integrated programming model was constructed to solve for the optimal criteria importance weights and to replace the IT2TrFN format of criteria importance because non-negative normalized weights are generally accepted and widely used. Finally, through a real-world medical decision-making problem, we demonstrated that the proposed MCDA method with the signed-distance-based approach was easy to employ and produced actionable results for decision analysis. In summary, this paper has charted the landscape of IT2TrFNs within multiple criteria, decision-making environments. More specifically, a method was developed for generating a signed-distance-based approach for decision analysis and weight assessment.

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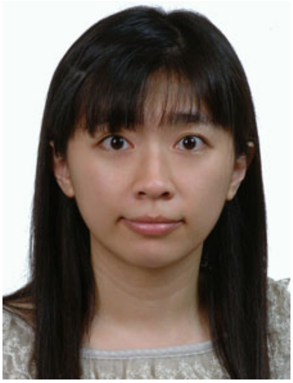
Author Biography



Ting-Yu Chen received her BS degree in transportation engineering and management, MS degree in civil engineering, and PhD degree in traffic and transportation from National Chiao Tung University in Taiwan. In 1998, she joined the Department of Business Administration at Chang Gung University as an Assistant Professor. From 2001 to 2011, she was an Associate Professor in the Department of Industrial and Business Management at the same university. Since August 2011, she has been a Professor in the Graduate Institute of Business and Management at Chang Gung University. Her current activities are focused on multiple criteria decision analysis and fuzzy decision-making methods.