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Composition Operators on the Bloch Space of Several Complex Variables

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Abstract In this paper, we study the boundedness and compactness of composition operator C_φ on the Bloch space $\beta(\Omega)$, Ω being a bounded homogeneous domain. For $\Omega = B_n$, we give the necessary and sufficient conditions for a composition operator C_φ to be compact on $\beta(B_n)$ or $\beta_0(B_n)$.

Keywords Bloch space, Composition operator, Bergman metric
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1 Introduction

Let Ω be a bounded homogeneous domain in C^n. By $H(\Omega)$ we denote the class of functions holomorphic in Ω. Let $\varphi: \Omega \to \Omega$ be a holomorphic self-map of Ω; for $f \in H(\Omega)$, denote the composition $f \circ \varphi$ by $C_\varphi f$ and call C_φ the composition operator induced by φ. We are concerned here with the question of when C_φ will be a bounded or compact operator on Bloch space $\beta(\Omega)$ or little Bloch space $\beta_0(\Omega)$.

More recently, Madigan and Matheson [1] studied the same problem on the Bloch space $\beta(U)$ and little Bloch space $\beta_0(U)$ on the unit disc U. They proved that C_φ is always bounded on $\beta(U)$ and bounded on $\beta_0(U)$ if and only if $\varphi \in \beta_0(U)$. They also gave the sufficient and necessary conditions that C_φ is compact on $\beta(U)$ or $\beta_0(U)$.

In this paper, we prove that C_φ is always bounded on $\beta(\Omega)$(Theorem 1), where Ω is a bounded homogeneous domain in C^n.

Let B_n be the unit ball of C^n, $\varphi = (\varphi_1, \ldots, \varphi_n): B_n \to B_n$ be the self-map of B_n. We prove that C_φ is bounded on $\beta_0(B_n)$ if and only if $\varphi_j \in \beta_0(B_n)$, $j = 1, \ldots, n$ (Theorem 2). We

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also give the sufficient and necessary conditions that C_φ is compact on $\beta(B_n)$ (Theorem 4) or $\beta_0(B_n)$ (Theorem 7).

In what follows, Ω always denotes a bounded homogeneous domain in C^n, and B_n the unit ball of C^n. φ denotes a holomorphic self-map of Ω or B_n, and C a positive constant not necessarily the same on each occasion.

2 The Boundedness of C_φ

Let $K(z,z)$ be the Bergman kernel function of Ω; the Bergman metric $H_z(u,u)$ on Ω is defined by

$$H_z(u,u) = \sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k,$$

where $z \in \Omega$ and $u = (u_1, \ldots, u_n) \in C^n$.

Following Timoney [2], we say that $f \in H(\Omega)$ is in the Bloch space $\beta(\Omega)$, if $\| f \|_{\beta(\Omega)} = \sup_{z\in\Omega} Q_f(z) < \infty$, where $Q_f(z) = \sup\left\{\frac{|\nabla f(z)u|}{H_z(u,u)^{\frac{1}{2}}} : u \in C^n - \{0\}\right\}$, and $\nabla f(z) = (\frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_n})$, $(\nabla f(z))u = \sum_{j=1}^n \frac{\partial f(z)}{\partial z_j} u_j$.

Our first result is the following

Theorem 1 *For every holomorphic self-map φ of Ω, C_φ is bounded on $\beta(\Omega)$.*

To prove this theorem, we need two lemmas.

Lemma 1 [2, Theorem 2.12] *Let Ω be a bounded homogeneous domain in C^n and $H_z(u,u)$ denote the Bergman metric on Ω. $\varphi : \Omega \to \Omega$ is a holomorphic self-map. Then there exists a constant C depending only on Ω, such that $H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u) \le CH_z(u,u)$ for each $z \in \Omega$, where $J\varphi(z) = \left(\frac{\partial \varphi_j(z)}{\partial z_k}\right)_{1\le j,k\le n}$ denotes the Jacobian matrix of φ, and $J\varphi(z)u$ denotes a vector, whose jth component is $(J\varphi(z)u)_j = \sum_{k=1}^n \frac{\partial \varphi_j(z)}{\partial z_k} u_k$.*

Lemma 2 *Let Ω be a bounded homogeneous domain in C^n, $f \in H(\Omega)$. $\varphi : \Omega \to \Omega$ is a holomorphic self-map. Then there exists a constant C depending only on Ω, such that $Q_{f\circ\varphi}(z) \le CQ_f(\varphi(z))$ for each $z \in \Omega$.*

Proof Using the chain rule, we get

$$\nabla(f\circ\varphi)(z) = (\nabla f)(\varphi(z)) \cdot J\varphi(z)\ . \tag{1}$$

For $u \in C^n - \{0\}$ and $J\varphi(z)u \ne 0$,

$$\frac{\nabla(f\circ\varphi)(z)u}{H_z(u,u)^{\frac{1}{2}}} = \frac{(\nabla f)(\varphi(z)) \cdot J\varphi(z)u}{[H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)]^{\frac{1}{2}}} \cdot \left\{\frac{H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)}{H_z(u,u)}\right\}^{\frac{1}{2}}\ .$$

From Lemma 1, we obtain

$$\frac{|\nabla(f\circ\varphi)(z)u|}{H_z(u,u)^{\frac{1}{2}}} \le C \cdot \frac{|(\nabla f)(\varphi(z)) \cdot J\varphi(z)u|}{[H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)]^{\frac{1}{2}}}. \tag{2}$$

If $u \in C^n - \{0\}$, but $J\varphi(z)u = 0$, then $\nabla(f \circ \varphi)(z)u = 0$ by (1). Thus the inequality (2) implies

$$\begin{aligned} Q_f(z) &= \sup\left\{\frac{|\nabla(f\circ\varphi)(z)u|}{H_z(u,u)^{\frac{1}{2}}} : u \in C^n - \{0\}\right\} \\ &= \sup\left\{\frac{|\nabla(f\circ\varphi)(z)u|}{H_z(u,u)^{\frac{1}{2}}} : u \in C^n - \{0\}, J\varphi(z)u \neq 0\right\} \\ &\le C \sup\left\{\frac{|(\nabla f)(\varphi(z)) \cdot J\varphi(z)u|}{[H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)]^{\frac{1}{2}}} : u \in C^n - \{0\}, J\varphi(z)u \neq 0\right\} \\ &\le CQ_f(\varphi(z)) \ . \end{aligned}$$

The lemma is proved.

Now Theorem 1 is a simple corollary of Lemma 2.

Proof of Theorem 1 Let $f \in \beta(\Omega)$; by Lemma 2, we have

$$||C_\varphi(f)||_{\beta(\Omega)} = ||f\circ\varphi||_{\beta(\Omega)} = \sup_{z\in\Omega} Q_{f\circ\varphi}(z) \le C \cdot \sup_{z\in\Omega} Q_f(\varphi(z)) \le C \cdot ||f||_{\beta(\Omega)}.$$

This means that C_φ is bounded on $\beta(\Omega)$. This completes the proof.

In [2], Timoney proved that for $f \in H(B_n)$ and $z \in B_n$, $Q_f(z) \approx (1-|z|^2)|\nabla f(z)|$. This means that there exist two positive constants A and B, such that

$$AQ_f(z) \le (1-|z|^2)|\nabla f(z)| \le BQ_f(z) \tag{3}$$

for every $z \in B_n$.

Thus, $f \in H(B_n)$ is in $\beta(B_n)$ if and only if

$$\sup_{z\in B_n} (1-|z|^2)|\nabla f(z)| < \infty \ .$$

In [3], Timoney defined that $f \in H(B_n)$ is in $\beta_0(B_n)$, called the little Bloch space, if $\lim_{|z|\to 1} Q_f(z) = 0$, namely

$$\lim_{|z|\to 1} (1-|z|^2)|\nabla f(z)| = 0 \ . \tag{4}$$

He also proved the following result in [3]: Let $\mathcal{D}$ be a bounded symmetric domain in C^n and not holomorphically equivalent to B_n. If $f \in \beta(\mathcal{D})$ and $Q_f(z) \to 0$ as $z \to \partial\mathcal{D}$, then f is a constant.

Due to this reason, for any bounded symmetric domain $\mathcal{D}$ in C^n, Timoney defined the little Bloch space $\beta_0(\mathcal{D})$ as follows : $\beta_0(\mathcal{D})$ is the closure in the Banach space $\beta(\mathcal{D})$ of the polynomial functions on $\mathcal{D}$. But when $\mathcal{D} = B_n$, this definition is equivalent to (4) (see [3], p. 8).

The following theorem gives the characterization of φ that C_φ is bounded on $\beta_0(B_n)$.

Theorem 2 *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of B_n. Then C_φ is bounded on $\beta_0(B_n)$ if and only if $\varphi_j \in \beta_0(B_n)$, $j = 1, \ldots, n$.*

Proof Let C_φ be bounded on $\beta_0(B_n)$. It is easy to see that for every $j = 1, \ldots, n$, the functions $f_j(z) = z_j$ belong to $\beta_0(B_n)$. Therefore $\varphi_j = f_j \circ \varphi = C_\varphi f_j \in \beta_0(B_n)$. This proves that the condition is necessary.

Conversely, let $\varphi_j \in \beta_0(B_n)$ for every $j = 1, \ldots, n$ and $f \in \beta_0(B_n)$. By the definition of $\beta_0(B_n)$, for a given $\epsilon > 0$, there exists a $\delta > 0$, such that $Q_f(z) < \epsilon$ if $|z|^2 > 1 - \delta$. It follows from Lemma 2 that $Q_{f\circ\varphi}(z) \le C\epsilon$ if $|\varphi(z)|^2 > 1-\delta$. Thus inequality (3) gives

$$(1-|z|^2)|\nabla(f\circ\varphi)(z)| \le C\epsilon \tag{5}$$

if $|\varphi(z)|^2 > 1-\delta$.

On the other hand, if $|\varphi(z)|^2 \le 1-\delta$, applying chain rule and Cauchy inequality,

$$\begin{aligned}(1-|z|^2)|\nabla(f\circ\varphi)(z)| \le & (1-|z|^2)|(\nabla f)(\varphi(z))|\cdot(|\nabla\varphi_1(z)|+\cdots+|\nabla\varphi_n(z)|)\\ < & \frac{(1-|\varphi(z)|^2)|(\nabla f)(\varphi(z))|}{\delta}\cdot(1-|z|^2)(|\nabla\varphi_1(z)|+\cdots+|\nabla\varphi_n(z)|)\\ \le & \frac{\| f \|_{\beta(\Omega)}}{\delta}(1-|z|^2)(|\nabla\varphi_1(z)|+\cdots+|\nabla\varphi_n(z)|).\end{aligned}$$

Since $\varphi_j \in \beta_0(B_n),\ j = 1, \ldots, n$, the right side of the above inequality tends to 0 as $|z| \to 1$. Combining the above statements together, we get $(1-|z|^2)|\nabla(f\circ\varphi)(z)| \to 0$ as $|z| \to 1$. This completes the proof of Theorem 2.

3 The Compactness of C_φ

In this section, we first give a sufficient condition that C_φ is a compact composition operator on $\beta(\Omega)$, then we will prove that this sufficient condition is also necessary for $\beta(B_n)$. To do this, we need the following useful lemma.

Lemma 3 *C_φ is compact on $\beta(\Omega)$ if and only if for any bounded sequence $\{f_k\}$ in $\beta(\Omega)$ which converges to* 0 *uniformly on compact subset of Ω, we have $\| f_k\circ\varphi \|_{\beta(\Omega)} \to 0$, as $k \to \infty$.*

Proof Let B denote the closed unit ball in $\beta(\Omega)$ and suppose $\{f_k\}$ is a sequence of functions in B. We wish to show that the image sequence $\{C_\varphi f_k\}$ has a convergent subsequence.

Fix $z_0 \in \Omega$, without loss of generality, suppose $f_k(z_0) = 0,\ \| f_k \|_{\beta(\Omega)} \le 1,\ k = 1, 2, \ldots$. For given $r > 0$, take $z \in \Omega$ with $\rho(z, z_0) \le r$, where $\rho(z, z_0)$ denotes the Bergman distance between z_0 and z. Let $\gamma : [0, 1] \to \Omega$ be a geodesic (in the Bergman metric) with $\gamma(0) = z_0$ and $\gamma(1) = z$. Then

$$\begin{aligned}|f_k(z)| &= |f_k(z)-f_k(z_0)| = |f_k(\gamma(1))-f_k(\gamma(0))|\\ &\le \int_0^1 |(f_k\circ\gamma)^{'}(t)|dt = \int_0^1 |(\nabla f_k)(\gamma(t))\gamma^{'}(t)|dt\\ &\le \int_0^1 Q_{f_k}(\gamma(t))H_{\gamma(t)}(\gamma'(t),\gamma'(t))^{\frac{1}{2}}dt \le \int_0^1 H_{\gamma(t)}(\gamma'(t),\gamma'(t))^{\frac{1}{2}}dt\\ &= \rho(z,z_0) \le r\ .\end{aligned}$$

Since every compact subset K of Ω is contained in $\{z \in \Omega : \rho(z, z_0) \le r\}$ for some $r > 0$, it follows that the sequence $\{f_k\}$ is uniformly bounded on every compact subset of Ω. Montel's theorem picks out a subsequence $\{f_{k_l}\}$ that converges uniformly on compact subsets of Ω to a holomorphic function g. We claim that $g \in \beta(\Omega)$. Indeed, for each $z \in \Omega$,

$$Q_g(z) = \lim_{l\to\infty} Q_{f_{k_l}}(z) \le \sup \| f_{k_l} \|_\beta \le 1 \ .$$

By the definition of $\beta(\Omega)$, we get $g \in \beta(\Omega)$ and $\| g \|_{\beta(\Omega)} \le 1$. Thus the sequence $\{f_{k_l} - g\}$ is bounded in $\beta(\Omega)$ and $f_{k_l} - g \to 0$ uniformly on compact subset of Ω. The hypothesis of the lemma ensures that $\| C_\varphi(f_{k_l} - g) \|_{\beta(\Omega)} \to 0$ as desired.

Conversely we assume that C_φ is compact, which means that $C_\varphi(B)$ is a relatively compact subset of $\beta(\Omega)$. We are further given a sequence $\{f_k\}$ that lies in rB(the ball of radius r) and converges to zero uniformly on compact subsets of Ω. We have to show that $\| C_\varphi f_k \|_{\beta(\Omega)} \to 0$, and for this it suffices to show that the zero function is the unique limit point of the sequence $\{C_\varphi f_k\}$ (for the Bloch norm). Since the set $\{C_\varphi f_k\}$ is relatively compact, there must be a function $f_0 \in \beta(\Omega)$ with $\lim_{k\to\infty} \| C_\varphi f_k - f_0 \|_{\beta(\Omega)} = 0$. Fix $z_0 \in \Omega$, for each $z \in \Omega$ with $\rho(z, z_0) \le r$; without loss of generality, suppose $f_0(z_0) = 0$. Using the same method as in the proof of the sufficiency, we get

$$|C_\varphi f_k(z) - f_0(z) - (C_\varphi f_k(z_0) - f_0(z_0))| \le \| C_\varphi f_k - f_0 \|_{\beta(\Omega)} \cdot r \ .$$

Therefore

$$|C_\varphi f_k(z) - f_0(z)| \le \| C_\varphi f_k - f_0 \|_{\beta(\Omega)} \cdot r + |C_\varphi f_k(z_0)| \ ,$$

but $C_\varphi f_k \to 0$ uniformly on compact subsets of Ω. Therefore, $f_0(z) = 0$ for each $z \in \Omega$. We finish the proof.

The following theorem gives a sufficient condition that C_φ is compact on $\beta(\Omega)$.

Theorem 3 *If $\varphi : \Omega \to \Omega$ is a holomorphic self-map, then C_φ is compact on $\beta(\Omega)$ if for every $\epsilon > 0$, there exists a $\delta > 0$, such that $\frac{H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)}{H_z(u,u)} < \epsilon$ for all $u \in C^n - \{0\}$ whenever* $\mathrm{dist}(\varphi(z), \partial\Omega) < \delta$.

Proof By Lemma 3, it is enough to show that if $\{f_k\}$ is a bounded sequence in $\beta(\Omega)$ which converges to 0 uniformly on compact subsets of Ω, then $\| f_k \circ \varphi \|_{\beta(\Omega)} \to 0$.

Let $M = \sup_k \| f_k \|_{\beta(\Omega)}$. For given $\epsilon > 0$, there exists a $\delta > 0$, such that $\frac{H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)}{H_z(u,u)} < \left(\frac{\epsilon}{M}\right)^2$ for all $u \in C^n - \{0\}$ if $\mathrm{dist}(\varphi(z), \partial\Omega) < \delta$. By Lemma 2,

$$Q_{f_k\circ\varphi}(z) \le Q_{f_k}(\varphi(z)) \cdot \sup \left\{ \left[\frac{H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)}{H_z(u,u)} \right]^{\frac{1}{2}} \ : \ u \in C^n - \{0\} \right\}.$$

It follows that $Q_{f_k\circ\varphi}(z) < \epsilon$, if $\mathrm{dist}(\varphi(z), \partial\Omega) < \delta$.

On the other hand, it is easy to see that

$$\inf \left\{ (H_w(u,u))^{\frac{1}{2}} : \ |u| = 1, \ \mathrm{dist}(w, \partial\Omega) \ge \delta \right\} = m > 0 \ .$$

So

$$\frac{|(\nabla f_k)(w)u|}{(H_w(u,u))^{\frac{1}{2}}} \le \frac{|(\nabla f_k)(w)| \cdot |u|}{(H_w(u,u))^{\frac{1}{2}}} = \frac{|(\nabla f_k)(w)|}{(H_w(\frac{u}{|u|}, \frac{u}{|u|}))^{\frac{1}{2}}} \le \frac{|(\nabla f_k)(w)|}{m}$$

if $\text{dist}(w,\partial\Omega)\geq\delta$. Now the hypothesis that $\{f_k\}$ converges to zero uniformly on any compact subset of Ω implies $Q_{f_k}(w)\to 0$ uniformly for $\text{dist}(w,\partial\Omega)\geq\delta$ as $k\to\infty$. From this and Lemma 1, for large enough k, $Q_{f_k\circ\varphi}(z)<\epsilon$ if $\text{dist}(\varphi(z),\partial\Omega)\geq\delta$. Hence, $\| f_k\circ\varphi\|_{\beta(\Omega)}<\epsilon$ for large k. The proof ends.

The following theorem shows that if $\Omega=B_n$, the condition of Theorem 3 is also necessary. We conjecture that for a general bounded homogeneous domain Ω, the condition of Theorem 3 is still necessary and sufficient, but the proof of necessity is so difficult that we cannot give it.

Theorem 4 *Let $\varphi: B_n\to B_n$ be a holomorphic self-map. Then C_φ is compact on $\beta(B_n)$ if and only if for every $\epsilon>0$, there exists a $\delta>0$, whenever* $\text{dist}(\varphi(z),\partial B_n)<\delta$, *such that*

$$\frac{H_{\varphi(z)}(J\varphi(z)u,J\varphi(z)u)}{H_z(u,u)}<\epsilon \tag{6}$$

for all $u\in C^n-\{0\}$.

Proof We only need to prove that condition (6) is necessary.

Now assume that condition (6) fails, then there exists a sequence $\{z^j\}$in B_n with $|\varphi(z^j)|\to 1$ as $j\to\infty$, $u^j\in C^n-\{0\}$, and an $\epsilon_0>0$, such that

$$\frac{H_{\varphi(z^j)}(J\varphi(z^j)u^j,J\varphi(z^j)u^j)}{H_{z^j}(u^j,u^j)}\geq\epsilon_0 \tag{7}$$

for all $j=1,2,\ldots$.

Using the condition (7), we will construct a sequence function $\{f_j\}$ satisfying the following three conditions:

(i) $\{f_j\}$ is a bounded sequence in $\beta(B_n)$;

(ii) $\{f_j\}$ tends to zero uniformly on any compact subset of B_n;

(iii) $||C_\varphi f_j||_\beta\not\to 0$, as $j\to\infty$.

This contradicts the compactness of C_φ by Lemma 3.

To construct the sequence of functions $\{f_j\}$, we first assume that

$$\varphi(z^j)=r_je_1,\quad j=1,2,\ldots. \tag{8}$$

where $e_1=(1,0,\ldots,0)\in C^n$. Denote $J\varphi(z^j)u^j=w^j$ and write $w^j=(w_1^j,\tilde{w}^j)$ where $\tilde{w}^j=(w_2^j,\ldots,w_n^j)\in C^{n-1}$. We will construct the functions according to two different cases:

1° If for some j, $\frac{|\tilde{w}^j|}{\sqrt{1-r_j^2}}\leq\frac{|w_1^j|}{1-r_j^2}$, then set

$$f_j(z)=\log(1-e^{-a(1-r_j)}z_1)-\log(1-z_1)\ , \tag{9}$$

where $a>0$ is any positive number.

2° If for some j, $\frac{|\tilde{w}^j|}{\sqrt{1-r_j^2}}>\frac{|w_1^j|}{1-r_j^2}$, then set

$$f_j(z)=(e^{-i\theta_2^j}z_2+\cdots+e^{-i\theta_n^j}z_n)\left\{(1-e^{-a(1-r_j)}z_1)^{-\frac{1}{2}}-(1-z_1)^{-\frac{1}{2}}\right\}, \tag{10}$$

where $a>0$ and $\theta_k^j=\arg w_k^j$, $k=2,\ldots,n$. If $w_k^j=0$ for some k, replace the corresponding term $e^{-i\theta_k^j}z_k$ by 0.

For the functions defined by (9), it is easy to check that $\{||f_j||_\beta\}$ is bounded and $\{f_j\}$ converges to 0 uniformly on compact subsets of B_n. We now prove that $||C_\varphi f_j||_\beta \not\to 0$. In fact, by (7)

$$\begin{aligned}||C_\varphi f_j||_\beta &= ||f_j \circ \varphi||_\beta \geq Q_{f_j\circ\varphi}(z^j) \geq \frac{|\nabla(f\circ\varphi)(z^j)u^j|}{[H_{z^j}(u^j,u^j)]^{\frac{1}{2}}} \\ &= \frac{|(\nabla f_j)(\varphi(z^j))J\varphi(z^j)u^j|}{\{H_{\varphi(z^j)}(J\varphi(z^j)u^j, J\varphi(z^j)u^j)\}^{\frac{1}{2}}} \cdot \left\{\frac{H_{\varphi(z^j)}(J\varphi(z^j)u^j, J\varphi(z^j)u^j)}{H_{z^j}(u^j,u^j)}\right\}^{\frac{1}{2}} \\ &\geq \sqrt{\epsilon_0} \cdot \frac{|(\nabla f_j)(r_je_1)w^j|}{(H_{r_je_1}(w^j,w^j))^{\frac{1}{2}}} \ . \end{aligned} \tag{11}$$

It is well known that the Bergman metric of B_n is

$$H_z(u,u) = \frac{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}{(1-|z|^2)^2} \ .$$

Therefore

$$H_{r_je_1}(w^j,w^j) = \frac{(1-r_j^2)(|w_1^j|^2 + |\tilde{w}^j|^2) + r_j^2|w_1^j|^2]}{(1-r_j^2)^2} = \frac{|\tilde{w}^j|^2}{1-r_j^2} + \frac{|w_1^j|^2}{(1-r_j^2)^2}.$$

So

$$H^{\frac{1}{2}}_{r_je_1}(w^j,w^j) \leq \frac{|\tilde{w}^j|}{\sqrt{1-r_j^2}} + \frac{|w_1^j|}{1-r_j^2} \leq \frac{2|w_1^j|}{1-r_j^2} \ .$$

Now (11) gives

$$\begin{aligned}||C_\varphi f_j||_\beta &\geq \sqrt{\epsilon_0}\cdot\frac{1-r_j^2}{|w_1^j|}\cdot|(\nabla f_j)(r_je_1)w^j| \\ &= \frac{\sqrt{\epsilon_0}}{2}\cdot(1-r_j^2)\left|\frac{\partial f_j(r_je_1)}{\partial z_1}\right| \\ &= \frac{\sqrt{\epsilon_0}}{2}\cdot(1-r_j^2)\left|\frac{1}{1-r_j} - \frac{e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right| \\ &\geq \frac{\sqrt{\epsilon_0}}{2}\cdot\left|1 - \frac{(1-r_j)e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right| \ . \end{aligned}$$

Since

$$\lim_{j\to\infty}\left[1 - \frac{(1-r_j)e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right] = \frac{a}{1+a} \neq 0 \ ,$$

so $||C_\varphi f_j||_\beta \not\to 0$ and $\{f_j\}$ satisfy the conditions (i),(ii) and (iii).

For the functions defined by (10), since

$$\frac{\partial f_j(z)}{\partial z_1} = \frac{1}{2}(e^{-i\theta_2^j}z_2 + \cdots + e^{-i\theta_n^j}z_n)\left(\frac{e^{-a(1-r_j)}}{(1-e^{-a(1-r_j)}z_1)^{\frac{3}{2}}} + \frac{1}{(1-z_1)^{\frac{3}{2}}}\right),$$

$$\frac{\partial f_j(z)}{\partial z_k} = e^{-i\theta_k^j}\left(\frac{1}{(1-e^{-a(1-r_j)}z_1)^{\frac{1}{2}}} - \frac{1}{(1-z_1)^{\frac{1}{2}}}\right), \quad k = 2,\ldots,n \ ,$$

hence

$$(1-|z|^2)\left|\frac{\partial f_j(z)}{\partial z_1}\right| \le C(1-|z|^2)(|z_2|+\cdots+|z_n|)\cdot\frac{1}{(1-|z_1|)^{\frac{3}{2}}}$$
$$\le C(1-|z|^2)(1-|z_1|^2)^{\frac{1}{2}}\cdot\frac{1}{(1-|z_1|)^{\frac{3}{2}}}\le C$$

and

$$(1-|z|^2)\left|\frac{\partial f_j(z)}{\partial z_k}\right| \le C(1-|z|^2)\cdot\frac{1}{(1-|z_1|)^{\frac{1}{2}}}\le C,\quad k=2,\dots,n\ .$$

This proves that $\{||f_j||_\beta\}$ is bounded. It is clear that $\{f_j\}$ converges to 0 uniformly on compact subsets of B_n. Finally we prove that $||C_\varphi f_j||_\beta \not\to 0$ as $j\to\infty$. Now

$$\begin{aligned}||C_\varphi f_j||_\beta &\ge \sqrt{\epsilon_0}\cdot\frac{|(\nabla f_j)(r_je_1)w^j|}{(H_{r_je_1}(w^j,w^j))^{\frac{1}{2}}}\\
&\ge \frac{\sqrt{\epsilon_0}}{2}\cdot\frac{\sqrt{1-r_j^2}}{|\tilde{w}^j|}\cdot|(\nabla f_j)(r_je_1)w^j|\\
&= \frac{\sqrt{\epsilon_0}}{2}\cdot\frac{\sqrt{1-r_j^2}}{|\tilde{w}^j|}\cdot\left|\frac{\partial f_j(r_je_1)}{\partial z_2}w_2^j+\cdots+\frac{\partial f_j(r_je_1)}{\partial z_n}w_n^j\right|\\
&= \frac{\sqrt{\epsilon_0}}{2}\cdot\frac{\sqrt{1-r_j^2}}{|\tilde{w}^j|}\cdot\left|(1-e^{-a(1-r_j)}r_j)^{-\frac{1}{2}}-(1-r_j)^{-\frac{1}{2}}\right|(|w_2^j|+\cdots+|w_n^j|)\\
&\ge \frac{\sqrt{\epsilon_0}}{2}\cdot\left|1-\left(\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}\right)^{\frac{1}{2}}\right|\ .\end{aligned}$$

Since $\lim_{j\to\infty}\left(\frac{1-r_j}{1-e^{-a(1-r_j)}r_j}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{1+a}}$, so $||C_\varphi f_j||_\beta\not\to 0$.

In a general situation, if $\varphi(z^j)\ne r_je_1$, we use the unitary transformation U_j to make $\varphi(z^j)=r_je_1U_j$, for $j=1,2,\dots$. A direct computation gives

$$\nabla(f\circ U)(z)=(\nabla f)(zU)U',\quad H_{zU}(w,w)=H_z(\overline{U'}w,\overline{U'}w),$$

$$\frac{|\nabla g_j(\varphi(z^j))w^j|}{(H_{\varphi(z^j)}(w^j,w^j))^{\frac{1}{2}}}=\frac{|\nabla(g_j\circ U_j)(r_je_1)\overline{U_j'}w^j|}{H_{r_je_1}(\overline{U'}w^j,\overline{U'}w^j)}\ .$$

Now $g_j=f_j\circ U_j^{-1}$, $j=1,2,\dots$, is the desired function sequence. The proof is completed.

When $n=1$, the Bergman metric of the unit disc U is $H_z(u,u)=\frac{|u|^2}{(1-|z|^2)^2}$, $z\in U,\ u\in C$. Hence

$$\frac{H_{\varphi(z)}(\varphi'(z)u,\varphi'(z)u)}{H_z(u,u)}=\left\{\frac{1-|z|^2}{1-|\varphi(z)|^2}\right\}^2|\varphi'(z)|^2,$$

where φ is a holomorphic self-map from U to U. Thus, from Theorem 4 we obtain Theorem 2 in [1].

Theorem 4 gives the sufficient and necessary condition that C_φ is compact on $\beta(B_n)$. But Condition (6) is expressed by the Bergman metric of B_n, and is not convenient for application. The following sufficient conditions or necessary conditions are perhaps more convenient for application.

Theorem 5 *Let $\varphi : B_n \to B_n$ be a holomorphic self-map and $\varphi = (\varphi_1, \ldots, \varphi_n)$. If for every $\epsilon > 0$, there exists a $\delta > 0$, such that*

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}(|\nabla\varphi_1(z)| + \cdots + |\nabla\varphi_n(z)|) < \epsilon$$

whenever $|\varphi(z)| > \delta$, then C_φ is compact on $\beta(B_n)$.

Proof Using the chain rule, we get

$$(1-|z|^2)\nabla(f\circ\varphi)(z) = (1-|z|^2)(\nabla f)(\varphi(z)) \cdot J\varphi(z).$$

Using the Cauchy inequality on the right side of the above equality, we obtain

$$\begin{aligned}(1-|z|^2)|\nabla(f\circ\varphi)(z)| &\le (1-|z|^2)\sqrt{(|\nabla\varphi_1(z)|^2 + \cdots + |\nabla\varphi_n(z)|^2)}|(\nabla f)(\varphi(z))| \\ &\le \frac{(1-|z|^2)(|\nabla\varphi_1| + \cdots + |\nabla\varphi_n|)}{1-|\varphi(z)|^2} \cdot [(1-|\varphi(z)|^2)|(\nabla f)(\varphi(z))|].\end{aligned}$$

The remaining proof is similar to that of Theorem 3, and we omit it. This completes the proof.

Theorem 6 *Let $\varphi : B_n \to B_n$ be a holomorphic self-map and $\varphi = (\varphi_1, \ldots, \varphi_n)$. If C_φ is compact on $\beta(B_n)$, then for every $\epsilon > 0$, there exists a $\delta > 0$, such that*

$$(1-|z|^2)\left(\frac{|\nabla\varphi_1(z)|}{1-|\varphi_1(z)|^2} + \cdots + \frac{|\nabla\varphi_n(z)|}{1-|\varphi_n(z)|^2}\right) < \epsilon \tag{12}$$

whenever $|\varphi(z)| > \delta$.

Proof Assume (12) is false, then there exist a sequence $\{z^k\} \subset B_n$ and some $\epsilon_0 > 0$, such that

$$(1-|z^k|^2)\left(\frac{|\nabla\varphi_1(z^k)|}{1-|\varphi_1(z^k)|^2} + \cdots + \frac{|\nabla\varphi_n(z^k)|}{1-|\varphi_n(z^k)|^2}\right) \ge \epsilon_0, \tag{13}$$

for all k and as $k \to \infty, w^k = \varphi(z^k) \to \xi \in \partial B_n$. Let $\xi = (\xi_1, \cdots, \xi_n)$. We first prove that there exists some index j with $|\xi_j| = 1$ and $\xi_k = 0,\ k \ne j$. In fact, if $|\xi_i| < 1$ for all $i = 1, \ldots, n$, since

$$H_z(u,u) \le \frac{|u|^2}{(1-|z|^2)^2}, \quad H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u) \ge \frac{|J\varphi(z)u|^2}{1-|\varphi(z)|^2},$$

by Lemma 1, we get $\frac{(1-|z|^2)^2|(J\varphi(z))u|^2}{(1-|\varphi(z)|^2)|u|^2} \le C$ for all $u \in C^n - \{0\}$. Set $u_i = e_i = (0, \ldots, 1, \ldots, 0)$, the ith coordinate being 1 and the others 0, $i = 1, \ldots, n$. Then $(1-|z|^2)|\nabla\varphi_i(z)| \le C \cdot \sqrt{1-|\varphi(z)|^2}$, $i = 1, \ldots, n$. This implies that

$$\frac{(1-|z^k|^2)|\nabla\varphi_i(z^k)|}{1-|\varphi_i(z^k)|^2} \le C \cdot \frac{\sqrt{1-|\varphi(z^k)|^2}}{1-|\varphi_i(z^k)|^2} \to 0 \tag{14}$$

as $k \to \infty$, $i = 1, \ldots, n$. This contradicts (13).

Without loss of generality, we now assume that $\xi = (\xi_1, \ldots, \xi_n) = (1, 0, \ldots, 0)$. Since $\xi_2 = \cdots = \xi_n = 0$, it follows from (14) that

$$\frac{(1-|z^k|^2)|\nabla\varphi_i(z^k)|}{1-|\varphi_i(z^k)|^2} \to 0, \quad \text{as } k \to \infty,\ i = 2, \ldots, n\ ,$$

and for large k, $\frac{(1-|z^k|^2)|\nabla\varphi_1(z^k)|}{1-|\varphi_1(z^k)|^2} > \frac{\epsilon_0}{2}$. Let $f_k(z) = \log\frac{1}{1-\overline{w_1^k}z_1}$. Then $f_k(z)$ converges to $f_0 = \log\frac{1}{1-z_1}$ uniformly on compact subsets of B_n and $\| f_k \|_\beta$ is bounded. But

$$\begin{aligned}\|C_\varphi f_k - C_\varphi f_0\|_\beta &= \left\|\log\frac{1}{1-\overline{w_1^k}\varphi_1} - \log\frac{1}{1-\varphi_1}\right\|_\beta \\ &\geq (1-|z^k|^2)|\nabla\varphi_1(z^k)| \cdot \left|\frac{\overline{w_1^k}}{1-|w_1^k|^2} - \frac{1}{1-w_1^k}\right| \\ &= \frac{(1-|z^k|^2)|\nabla\varphi_1(z^k)|}{1-|w_1^k|^2} > \frac{\epsilon_0}{2}\ .\end{aligned}$$

This contradicts the compactness of C_φ by Lemma 3. This completes the proof.

For $n=1$, from Theorems 5 and 6, we obtain Theorem 2 in [1] again.

Now we turn to the discussion of the compactness of C_φ on $\beta_0(B_n)$. First we introduce a useful lemma.

Lemma 4 *A closed set K in $\beta_0(B_n)$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1^-}\sup_{f\in K}(1-|z|^2)|\nabla f(z)| = 0\ . \tag{15}$$

Proof Assume K is compact on $\beta_0(B_n)$. Then for every $\epsilon > 0$, there exist finite open sets of $\beta_0(B_n)$,

$$U(f_i,\epsilon) = \left\{f\in\beta_0(B_n)\ :\ \|f_i - f\|_\beta < \frac{\epsilon}{2}\right\}\ ,\quad i=1,\ldots,l,$$

such that $K\subset\bigcup_{i=1}^l U(f_i,\epsilon)$. Since $f_i\in\beta_0(B_n)$, $i=1,\ldots,l$, it implies that there exists an r, $0<r<1$, whenever $|z|>r$, such that $(1-|z|^2)|\nabla f_i(z)| < \frac{\epsilon}{2}$ for $i=1,\ldots,l$. If $f\in K$, then for some f_{i_0}, $1\leq i_0\leq l$, with $\|f-f_{i_0}\|_\beta < \frac{\epsilon}{2}$. Therefore, when $|z|>r$,

$$(1-|z|^2)|\nabla f(z)| \leq \|f-f_{i_0}\|_\beta + (1-|z|^2)|\nabla f_{i_0}(z)| < \epsilon\ .$$

This establishes (15).

Consider the other side, if K is a bounded set on $\beta_0(B_n)$ and satisfies (15). Let $\{f_k\}$ be a function sequence in K. By the Montel theorem, there exists a subsequence $\{f_{k_l}\}$ of $\{f_k\}$ which converges to some holomorphic function f uniformly on compact subset of B_n. Due to the same reason $\left\{\frac{\partial f_{k_l}}{\partial z_i}\right\}$, $i=1,\ldots,n$, converges to $\frac{\partial f}{\partial z_i}$ uniformly on compact subset of B_n. By (15), for every $\epsilon>0$, there exists an r, $0<r<1$, whenever $|z|>r$, such that for all $g\in K$, $(1-|z|^2)|\nabla g(z)| < \frac{\epsilon}{2}$. It follows that $(1-|z|^2)|\nabla f(z)| < \frac{\epsilon}{2}$ if $|z|>r$. Since $\{f_{k_l}\}$ converges to f uniformly on $|z|\leq r$ and $\left\{\frac{\partial f_{k_l}}{\partial z_i}\right\}$ $(i=1,\ldots,n)$ converges to $\frac{\partial f}{\partial z_i}$ uniformly on $|z|\leq r$, therefore, for enough large l, $\sup_{|z|\leq r}(1-|z|^2)(|\nabla f(z)-\nabla f_{k_l}(z)|)\leq\frac{\epsilon}{2}$. Combining the discussions together, we get $\lim_{l\to\infty}\|f_{k_l}-f\|_\beta = 0$ and so K is compact on $\beta_0(B_n)$. The proof ends.

Similarly to Theorem 4, we have

Theorem 7 *Let $\varphi\ :\ B_n\to B_n$ be a holomorphic self-map. Then C_φ is compact on $\beta_0(B_n)$ if and only if*

$$\lim_{|z|\to 1^-}\frac{H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)}{H_z(u,u)} = 0 \tag{16}$$

for all $u \in C^n - \{0\}$ *uniformly.*

Proof We first prove that Condition (16) is sufficient. Since

$$\frac{H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)}{H_z(u,u)} \geq \frac{(1-|z|^2)^2}{1-|\varphi(z)|^2} \cdot \frac{|J\varphi(z)u|^2}{|u|^2} \geq (1-|z|^2)^2 |J\varphi(z)\xi|^2,$$

where $\xi = \frac{u}{|u|} \in \partial B_n$, it follows from (16) that $\lim_{|z|\to 1^-}(1-|z|^2)^2|J\varphi(z)\xi|^2 = 0$ for all $\xi \in \partial B_n$ uniformly. Let $\xi = e_j$. We obtain $\lim_{|z|\to 1^-}(1-|z|^2)|\nabla\varphi_j(z)| = 0$. Namely $\varphi_j \in \beta_0(B_n)$, $j = 1, \ldots, n$. Here φ_j is the jth component of φ. Thus $C_\varphi f \in \beta_0(B_n)$ for each $f \in \beta_0(B_n)$ by Theorem 2. Let

$$A = \{f \in \beta_0(B_n) : \ ||f||_\beta \leq 1\}, \quad K = \{C_\varphi f : \ f \in A\}.$$

Then K is a closed set in $\beta_0(B_n)$. By Lemma 4, we only need to prove

$$\lim_{|z|\to 1^-} \sup_{||f||_\beta \leq 1} (1-|z|^2)|\nabla(f\circ\varphi)(z)| = 0 . \tag{17}$$

By (16), for given ϵ, there exists an r, $0 < r < 1$, such that

$$\frac{\nabla(f\circ\varphi)(z)u}{H_z(u,u)^{\frac{1}{2}}} \leq \epsilon \frac{|(\nabla f)(\varphi(z)) \cdot J\varphi(z)u|}{[H_{\varphi(z)}(J\varphi(z)u, J\varphi(z)u)]^{\frac{1}{2}}}$$

for all $u \in C^n - \{0\}$, whenever $|z| > r$. Thus $Q_{f\circ\varphi}(z) < \epsilon Q_f(\varphi(z)) \leq \epsilon ||f||_\beta$. Now (17) follows from (3) immediately.

Conversely, if C_φ is compact on $\beta_0(B_n)$, but (16) fails, then there exist sequences $\{z^j\} \subset B_n$, $\{u^j\} \subset C^n - \{0\}$ and an $\epsilon_0 > 0$, such that $|z^j| \to 1$ and

$$\frac{H_{\varphi(z^j)}(J\varphi(z^j)u^j, J\varphi(z^j)u^j)}{H_{z^j}(u^j, u^j)} \geq \epsilon_0 \tag{18}$$

for all $j = 1, 2, \ldots$.

We will use (18) to construct a sequence functions $\{f_j\}$, such that

(i) $f_j \in \beta_0(B_n)$, $j = 1, 2, \ldots$.

(ii) $||f_j||_\beta \leq 1$, $j = 1, 2, \ldots$.

(iii) $(1-|z^j|^2)|\nabla(f_j\circ\varphi)(z^j)| \geq \epsilon_0$, $j = 1, 2, \ldots$.

This means that C_φ is not compact on $\beta_0(B_n)$ by Lemma 4.

Similarly to the proof of Theorem 4, we may assume that $\varphi(z^j) = r_j e_1$, $j = 1, 2, \ldots$, and denote $w^j = J\varphi(z^j)u^j$, $j = 1, 2, \ldots$. The following two cases will be considered.

Case 1 $|\varphi(z^j)| \leq \rho < 1$, as $|z^j| \to 1^-$.

Let $f_j(z) = e^{-i\theta_1^j}z_1 + \cdots + e^{-i\theta_n^j}z_n$, $j = 1, 2, \ldots$, where $\theta_k^j = \arg w_k^j$, $k = 1, \ldots, n$. If $w_k^j = 0$ for some k, replace the corresponding term $e^{-i\theta_k^j}z_k$ by 0.

It is easy to see that Conditions (i) and (ii) are satisfied. To prove (iii), we note that

$$H_{r_j e_1}(w^j, w^j) = \frac{|\tilde{w}^j|^2}{1-r_j^2} + \frac{|w_1^j|^2}{(1-r_j^2)^2} \leq \frac{1}{(1-\rho^2)^2} \cdot |w^j|^2 .$$

Here $w^j = (w_1^j, \tilde{w}^j)$, $\tilde{w}^j = (w_2^j, \dots, w_n^j) \in C^{n-1}$. Thus

$$\begin{aligned}
(1-|z^j|^2)|\nabla(f_j\circ\varphi)(z^j)| &\geq CQ_{f_j\circ\varphi}(z^j)\\
&\geq C\frac{|\nabla(f\circ\varphi)(z^j)u^j|}{[H_{z^j}(u^j,u^j)]^{\frac{1}{2}}}\\
&\geq C\sqrt{\epsilon_0}\frac{|(\nabla f_j)(\varphi(z^j))J\varphi(z^j)u^j|}{\left\{H_{\varphi(z^j)}(J\varphi(z^j)u^j, J\varphi(z^j)u^j)\right\}^{\frac{1}{2}}}\\
&= C\sqrt{\epsilon_0}\cdot\frac{|(\nabla f_j)(r_je_1)w^j|}{(H_{r_je_1}(w^j,w^j))^{\frac{1}{2}}}\\
&\geq C\sqrt{\epsilon_0}\frac{1-\rho^2}{|w^j|}\cdot\left|\frac{\partial f_j(r_je_1)}{\partial z_1}w_1^j+\cdots+\frac{\partial f_j(r_je_1)}{\partial z_n}w_n^j\right|\\
&= C\sqrt{\epsilon_0}(1-\rho^2)\cdot\frac{1}{|w^j|}(|w_1^j|+\cdots+|w_n^j|)\\
&\geq C\sqrt{\epsilon_0}(1-\rho^2)\ .
\end{aligned}$$

This proves (iii).

Case 2 $|\varphi(z^j)| \to 1$ as $|z^j| \to 1$.

Similarly to Theorem 4, we construct $\{f_j\}$ according to the following two different situations:

1° If for some j, $\frac{|\tilde{w}^j|}{\sqrt{1-r_j^2}} \leq \frac{|w_1^j|}{1-r_j^2}$, then we let $f_j(z) = \frac{1}{2}\log(1-e^{-a(1-r_j)}z_1)$, $a > 0$. For fixed j,

$$(1-|z|^2)|\nabla f_j(z)| = \frac{1}{2}(1-|z|^2)\left|\frac{e^{-a(1-r_j)}}{1-e^{-a(1-r_j)}r_j}\right| \leq \frac{1}{2}(1-|z|^2)\frac{1}{1-e^{-a(1-r_j)}r_j} \to 0,\ |z|\to 1^-.$$

So $f_j \in \beta_0(B_n)$ for each j. On the other hand,

$$(1-|z|^2)|\nabla f_j(z)| \leq \frac{1}{2}(1-|z|^2)\cdot\frac{1}{1-|z_1|} \leq 1\ .$$

So $||f_j||_{\beta(B_n)} \leq 1$. Thus $\{f_j\}$ satisfies (i) and (ii). The proof of (iii) is similar to that of Theorem 4, we omit it here.

2° If for some j, $\frac{|\tilde{w}^j|}{\sqrt{1-r_j^2}} > \frac{|w_1^j|}{1-r_j^2}$, then we let

$$f_j(z) = \frac{4}{\sqrt{n}}(e^{-i\theta_2^j}z_2+\cdots+e^{-i\theta_n^j}z_n)(1-e^{-a(1-r_j)}z_1)^{-\frac{1}{2}},$$

where θ_k^j, $k = 2,\dots,n$, are the same as in Case 1. The proof that f_j satisfies (i), (ii) and (iii) is similar to that in 1° and Theorem 4, we omit the details. The proof is completed.

Similarly to Theorems 5 and 6, we also have two theorems which are convenient for confirming or negating the compactness of C_φ on $\beta_0(B_n)$.

Theorem 8 *Let $\varphi : B_n \to B_n$ is a holomorphic self-map and $\varphi = (\varphi_1,\dots,\varphi_n)$, Then C_φ is compact on $\beta_0(B_n)$ if*

$$\lim_{|z|\to 1^-}\frac{1-|z|^2}{1-|\varphi(z)|^2}(|\nabla\varphi_1(z)|+\cdots+|\nabla\varphi_n(z)|) = 0\ . \tag{19}$$

Proof It is easy to see from (19) that $\varphi_j \in \beta_0(B_n)$ for every $j = 1, \ldots, n$. Hence $C_\varphi f \in \beta_0(B_n)$ for each $f \in \beta_0(B_n)$. Let A, K be the sets as in Theorem 7. Then

$$\begin{aligned}
&\sup_{||f||_\beta \le 1} (1-|z|^2)|\nabla(f\circ\varphi)(z)| \\
&\le \frac{(1-|z|^2)(|\nabla\varphi_1| + \cdots + |\nabla\varphi_n|)}{1-|\varphi(z)|^2} \sup_{||f||_\beta \le 1} \{(1-|\varphi(z)|^2)|(\nabla f)(\varphi(z))|\} \\
&\le \frac{(1-|z|^2)(|\nabla\varphi_1| + \cdots + |\nabla\varphi_n|)}{1-|\varphi(z)|^2} \to 0\ , \quad |z| \to 1^-\ .
\end{aligned}$$

Now Lemma 4 gives the desired result. This completes the proof.

Theorem 9 *Let $\varphi : B_n \to B_n$ be a holomorphic self-map and $\varphi = (\varphi_1, \ldots, \varphi_n)$. If C_φ is compact on $\beta_0(B_n)$, then*

$$\lim_{|z|\to 1^-} (1-|z|^2)\left(\frac{|\nabla\varphi_1(z)|}{1-|\varphi_1(z)|^2} + \cdots + \frac{|\nabla\varphi_n(z)|}{1-|\varphi_n(z)|^2}\right) = 0. \tag{20}$$

Proof Let C_φ be compact on $\beta_0(B_n)$, but (20) fail. This means that there exist a sequence $z^j \subset B_n$ and an $\epsilon_0 > 0$, such that $|z^j| \to 1^-$ as $j \to \infty$ and

$$(1-|z^j|^2)\left(\frac{|\nabla\varphi_1(z^j)|}{1-|\varphi_1(z^j)|^2} + \cdots + \frac{|\nabla\varphi_n(z^j)|}{1-|\varphi_n(z^j)|^2}\right) \ge \epsilon_0\ . \tag{21}$$

We first prove that $|\varphi(z^j)| \to 1^-$ as $|z^j| \to 1^-$, $j \to \infty$. If not, there exists a $\rho < 1$ with $|\varphi(z^j)| \le \rho$, $j = 1, 2, \ldots$. Thus (21) yields

$$(1-|z^j|^2)\left(|\nabla\varphi_1(z^j)| + \cdots + |\nabla_n(z^j)|\right) \ge \epsilon_0(1-\rho^2)\ . \tag{22}$$

Since $C_\varphi f \in \beta_0(B_n)$ for every $f \in \beta_0(B_n)$, this implies $\varphi_j \in \beta_0(B_n)$ by Theorem 2. This is a contradiction to (22). Let $\varphi(z^j) \to \xi \in \partial B_n$, as $|z^j| \to 1^-$. We have proved in Theorem 6 that $\xi = e^{i\theta}e_k$ for some k; without loss of generality, we assume that $\xi = e^{i\theta}e_1$. Now (21) implies

$$\frac{(1-|z^j|^2)|\nabla\varphi_1(z^j)|}{1-|\varphi_1(z^j)|^2} > \frac{\epsilon_0}{2} \tag{23}$$

for large enough j.

Denote $\varphi_1(z^j) = w_1^j$ and let $f_j(z) = \frac{1}{2}\log(1-\overline{w}_1^j z_1)$, $j = 1, 2, \ldots$. It is easy to check that every $f_j \in \beta_0(B_n)$ and $||f_j||_\beta \le 1$, $j = 1, 2, \ldots$. A direct computation gives

$$\nabla(f_j\circ\varphi)(z^j) = \frac{1}{2}\cdot\frac{-\overline{w}_1^j}{1-|\varphi_1(z^j)|^2}\nabla\varphi_1(z^j)\ . \tag{24}$$

It follows from (23), (24) and $|w_1^j| \to 1^-$ as $j \to \infty$ that

$$(1-|z^j|^2)|\nabla(f_j\circ\varphi)(z^j)| = \frac{|w_1^j|}{2}\cdot\frac{(1-|z^j|^2)|\nabla\varphi_1(z^j)|}{1-|\varphi_1(z^j)|^2} > \frac{\epsilon_0}{5}$$

for large enough j. This shows that C_φ is not compact on $\beta_0(B_n)$ by Lemma 4. The theorem is proved.

It is easy to see that (19) implies (20), but the two conditions are not equivalent. For example, let

$$\varphi(z) = (\varphi_1(z), \varphi_2(z)) = \left(\frac{1}{2}\left(z_1 + \frac{1}{\sqrt{2}}\right), \frac{1}{2}\left(z_2 + \frac{1}{\sqrt{2}}\right)\right).$$

It is easy to check that φ is a holomorphic self-map of B_2, $\nabla\varphi_1(z) = (\frac{1}{2}, 0)$, $\nabla\varphi_2(z) = (0, \frac{1}{2})$ and

$$|\varphi_1(z)|^2 = \frac{1}{4}\left|z_1 + \frac{1}{\sqrt{2}}\right|^2 \leq \frac{3}{4}, \quad |\varphi_2(z)|^2 = \frac{1}{4}\left|z_2 + \frac{1}{\sqrt{2}}\right|^2 \leq \frac{3}{4}\,.$$

So

$$(1 - |z|^2)\left(\frac{|\nabla\varphi_1(z)|}{1 - |\varphi_1(z)|^2} + \frac{|\nabla\varphi_2(z)|}{1 - |\varphi_2(z)|^2}\right) \leq 2(1 - |z|^2) \to 0\,, \quad |z| \to 1^-.$$

Condition (20) is satisfied.

On the other hand, letting $z_1 = z_2 = \frac{1}{\sqrt{2}}r$, then

$$\begin{aligned} &\frac{1 - |z|^2}{1 - |\varphi(z)|^2}(|\nabla\varphi_1(z)| + |\nabla\varphi_2(z)|) \\ &= \frac{1}{2}(1 - |z|^2) \cdot \left[1 - \frac{1}{4}\left(\left|z_1 + \frac{1}{\sqrt{2}}\right|^2 + \left|z_2 + \frac{1}{\sqrt{2}}\right|^2\right)\right]^{-1} \\ &= \frac{2(1 - r^2)}{(1 - r)(3 + r)} \to 1, \qquad r \to 1^-. \end{aligned}$$

This shows that Condition (19) fails.

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