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Calderón–Zygmund-Type Operators on Weighted Weak Hardy Spaces over $\mathbb{R}^n$

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Abstract We introduce certain Calderón–Zygmund-type operators and discuss their boundedness on spaces such as weighted Lebesgue spaces, weighted weak Lebesgue spaces, weighted Hardy spaces and weighted weak Hardy spaces. The sharpness of some results is also investigated.

Keywords Calderón–Zygmund operator, Lebesgue space, Weak Lebesgue space, Hardy space, Weak Hardy space, Atom, Weight
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1 Introduction

Calderón–Zygmund operators and their generalizations on the Euclidean space $\mathbb{R}^n$ have been extensively studied [1–6]. In particular, Yabuta [6] introduced certain θ-type Calderón–Zygmund operators to facilitate his study of certain classes of pseudo-differential operators (cf Coifman and Meyer [1]). The results in this paper are of three kinds. First, Theorem 1 gives several equivalent conditions for a θ-type Calderón–Zygmund operator to be bounded on $L^2(\mathbb{R}^n)$. Second, we show that our θ-type Calderón–Zygmund operators are bounded on spaces like $L^p_\omega(\mathbb{R}^n)$, weighted weak Lebesgue spaces, weighted Hardy spaces and weighted weak Hardy spaces (see Theorems 2–5). We note that our results are closely related to others recent work on weighted weak Hardy spaces [7–12]. Third, we weaken θ-type Calderón–Zygmund operators

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to semi-(θ, p_0)-type Calderón–Zygmund operators and prove in Theorem 6 that such operators are bounded on the power-weighted $L^p(\mathbb{R}^n)$. In Theorem 7, we prove that our operators are even bounded on the power-weighted Hardy space and the power-weighted weak Hardy space. The Sharpness of our results is also discussed in this theorem. The boundedness of other generalizations of Calderón–Zygmund operators is given in Theorems 8 and 9.

2 Preliminary Results

Let $1 < p < \infty$. Following [2,3], a weight $\omega \geq 0$ is a Muckenhoupt $A_p(\mathbb{R}^n)$ weight if

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq c,$$

where c is a constant independent of the cube Q, and where, and in what follows, all the cubes have their sides parallel to the axes. The class $A_1(\mathbb{R}^n)$ is defined by letting $p \to 1$, namely,

$$\frac{1}{|Q|}\int_Q \omega(x)\,dx \leq c\,\mathrm{essinf}_{x\in Q}\,\omega(x),$$

where c is independent of Q. The smallest value of c is called the $A_p(\mathbb{R}^n)$-constant of ω. We also define $A_\infty(\mathbb{R}^n) = \bigcup_{p\geq 1} A_p(\mathbb{R}^n)$ and for $\omega \in A_\infty(\mathbb{R}^n)$, we set

$$q_\omega = \inf\{q \geq 1 :\ \omega \in A_q(\mathbb{R}^n)\}$$

and call q_ω the critical index of ω (see [2]).

Let $0 < p < \infty$ and let ω be a locally integrable non-negative function. We denote the weighted space $L^p(\mathbb{R}^n, \omega(x)dx)$ by $L^p_\omega(\mathbb{R}^n)$ and set

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\Big)^{1/p}.$$

We also denote the weak $L^p_\omega(\mathbb{R}^n)$ by $WL^p_\omega(\mathbb{R}^n)$ and set

$$\|f\|_{WL^p_\omega(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda[\omega(\{x\in\mathbb{R}^n :\ |f(x)| > \lambda\})]^{1/p},$$

where, and in what follows, $\omega(E) = \int_E \omega(x)\,dx$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the class of Schwartz functions and let $\mathcal{S}'(\mathbb{R}^n)$ be its dual space. We now introduce the weighted Hardy space.

Definition 1 *Let $\omega \in A_\infty(\mathbb{R}^n)$ and $p \in (0,1]$. The weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ is defined by*

$$H^p_\omega(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) :\ \phi^*(f)(x) \equiv \sup_{t>0} |\phi_t * f(x)| \in L^p_\omega(\mathbb{R}^n)\},$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a fixed function with $\int_{\mathbb{R}^n}\phi(x)\,dx \neq 0$ and $\phi_t(y) = t^{-n}\phi(y/t)$ for any $t > 0$. Moreover, we define $\|f\|_{H^p_\omega(\mathbb{R}^n)} = \|\phi^*(f)\|_{L^p_\omega(\mathbb{R}^n)}$.

It is well known that Definition 1 does not depend on the choice of ϕ (see [2] and [13]).

In what follows, if $\omega(x) \equiv 1$, we will denote $L^p_\omega(\mathbb{R}^n)$, $WL^p_\omega(\mathbb{R}^n)$ and $H^p_\omega(\mathbb{R}^n)$ simply by $L^p(\mathbb{R}^n)$, $WL^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$.

Definition 2 *Let $p \in (0,1]$ and $\omega \in A_\infty(\mathbb{R}^n)$. A p-atom with respect to ω is a function a supported in a cube Q such that*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \omega(Q)^{-1/p} \tag{2.1}$$

$\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for every multi-index α with $|\alpha| \le [n(q_\omega/p-1)]$, where, and in what follows, $[s]$ denotes the greatest integer less than or equal to s. (2.2)

The following lemma is Proposition 1.5 in [2] (see also [3]).

Lemma 1 *Let $\omega \in A_\infty(\mathbb{R}^n)$ and $p \in (0,1]$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in H^p_\omega(\mathbb{R}^n)$ if and only if f can be written as a series*

$$f = \sum_j \lambda_j a_j \tag{2.3}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where each a_j is a p-atom with respect to ω and the coefficients λ_j satisfy

$$\sum_j |\lambda_j|^p < \infty. \tag{2.4}$$

Moreover, the infimum of the sums in (2.4) *over all decompositions* (2.3) *is equivalent to* $\|f\|^p_{H^p_\omega(\mathbb{R}^n)}$.

It is well known that the Lebesgue space $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ and the Hardy space $H^p(\mathbb{R}^n)$ for all $p \in (0,1]$ are both special cases of homogeneous Triebel–Lizorkin spaces (see [14], p. 244). However, this is not true for the weak Lebesgue space and the weak Hardy space.

We now turn to the weighted weak Hardy space, which is a good substitute for the weighted Hardy spaces in the study of the boundedness of operators. Also, the weak Lebesgue space and the weak Hardy space arise naturally as intermediate spaces of the real method of interpolation between the $H^p(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$ spaces (see [7,8,10,12,14]).

Let $p \in (0,1]$ and $\omega \in A_\infty(\mathbb{R}^n)$. Define

$$\mathcal{A}_{p,\omega} = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha| \le [n(q_\omega/p-1)]+1} \sup_{x\in\mathbb{R}^n} (1+|x|)^{[n(q_\omega/p-1)]+n+1} |D^\alpha \phi(x)| < \infty \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}\cup\{0\})^n$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. Moreover, we set

$$\|\phi\|_{\mathcal{A}_{p,\omega}} \equiv \sup_{x\in\mathbb{R}^n} \sup_{|\alpha| \le [n(q_\omega/p-1)]+1} (1+|x|)^{[n(q_\omega/p-1)]+n+1} |D^\alpha \phi(x)|.$$

For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define

$$G_{p,\omega} f(x) = \sup_{\phi\in\mathcal{A}_{p,\omega},\, \|\phi\|_{\mathcal{A}_{p,\omega}} \le 1} \sup_{|x-y|<t} |(f * \phi_t)(y)|.$$

$G_{p,\omega} f$ is usually called the grand maximal function of f (see [5], p. 90).

Definition 3 *Let $\omega \in A_\infty(\mathbb{R}^n)$ and $p \in (0,1]$. Then the weighted weak Hardy space $WH^p_\omega(\mathbb{R}^n)$ is defined by*

$$WH^p_\omega(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n):\ G_{p,\omega}f \in WL^p_\omega(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{WH^p_\omega(\mathbb{R}^n)} = \|G_{p,\omega}f\|_{WL^p_\omega(\mathbb{R}^n)}$.

If $\omega(x) \equiv 1$, the weak Hardy space $WH^1(\mathbb{R}^n) \equiv WH^1_\omega(\mathbb{R}^n)$ was introduced by Fefferman and Soria in [7]. $WH^p(\mathbb{R}^n)$ first appeared in [8] (see also [10]). In [9], Zhang introduced the space $WH^p_\omega(\mathbb{R}^n)$ for $\omega \in A_1(\mathbb{R}^n)$ and established its atomic decomposition. We now generalize Zhang's result to weight $\omega \in A_\infty(\mathbb{R}^n)$.

Lemma 2 *Let $p \in (0,1]$ and $\omega \in A_\infty(\mathbb{R}^n)$. For $f \in WH^p_\omega(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k=-\infty}^\infty$ of bounded measurable functions such that*

$$f = \sum_{k=-\infty}^\infty f_k \ \text{in}\ \mathcal{S}'(\mathbb{R}^n). \tag{2.5}$$

Each f_k can be further decomposed into $f_k(x) = \sum_i b_{ki}(x)$, where the sequence $\{b_{ki}\}_i$ satisfies

$$\text{supp}\ b_{ki} \subset Q_{ki}, \quad \text{and}\ \ Q_{ki}\ \text{is a cube} \tag{2.6}$$

$$\sum_i \omega(Q_{ki}) \le c_1 2^{-kp}; \quad \sum_i \chi_{Q_{ki}}(x) \le c_1, \tag{2.6_1}$$

χ_E being the characteristic function of the set E, c_1 a constant and $c_1 \le c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}$;

$$\|b_{ki}\|_{L^\infty(\mathbb{R}^n)} \le c2^k \ \text{and} \int_{\mathbb{R}^n} b_{ki}(x)x^\alpha\, dx = 0,\ \text{for}\ |\alpha| \le [n(q_\omega/p - 1)]. \tag{2.6_2}$$

Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ has a decomposition satisfying (2.5) and (2.6), then $f \in WH^p_\omega(\mathbb{R}^n)$ and $\|f\|^p_{WH^p_\omega(\mathbb{R}^n)} \le cc_1$, where c is a constant.

In what follows, c always denotes a constant which is independent of the main parameters, but may vary from line to line.

To prove the lemma, we need the following weak-type summable principle (see [7] and [8]).

Lemma 3 *Let (X,μ) be any measurable space and $p \in (0,1)$. Let $\{f_k\}_k$ be a sequence of measurable functions such that for any $\lambda > 0$ and all $k \in \mathbb{Z}$, we have*

$$\mu(\{x \in X:\ |f_k(x)| > \lambda\}) \le \lambda^{-p}.$$

If $\sum_k |c_k|^p < \infty$, then $\sum_k c_k f_k(x)$ is absolutely convergent almost everywhere and

$$\mu\left(\left\{x \in X:\ \left|\sum_k c_k f_k(x)\right| > \lambda\right\}\right) \le \frac{2-p}{1-p}\left(\sum_k |c_k|^p\right)\lambda^{-p}.$$

Proof of Lemma 2 The proof is motivated by the atomic decomposition for $H^p(\mathbb{R}^n)$ (see [5]). Let $f \in WH^p_\omega(\mathbb{R}^n)$. Following the same argument as in the proof of Theorem 4.1 in [9], we obtain an atomic decomposition of f satisfying (2.5) and (2.6).

Conversely, let $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfy (2.5) and (2.6). For any given $\lambda > 0$, we choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \le \lambda < 2^{k_0+1}$. Now write

$$f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k \equiv F_1 + F_2.$$

Then $G_{p,\omega} f(x) \le G_{p,\omega} F_1(x) + G_{p,\omega} F_2(x)$. Since

$$G_{p,\omega} F_1(x) \le \sum_{k=-\infty}^{k_0} G_{p,\omega} f_k(x) \le c \sum_{k=-\infty}^{k_0} 2^k \le c_0 \lambda,$$

we have

$$\begin{aligned} \{x \in \mathbb{R}^n :\ G_{p,\omega} f(x) > (c_0+1)\lambda\} \quad &\subset \{x \in \mathbb{R}^n :\ G_{p,\omega} F_1(x) > c_0 \lambda\} \\ &\quad \bigcup \{x \in \mathbb{R}^n :\ G_{p,\omega} F_2(x) > \lambda\} \\ &= \{x \in \mathbb{R}^n :\ G_{p,\omega} F_2(x) > \lambda\}. \end{aligned}$$

Set $A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i Q_{ki}^*$, where Q_{ki}^* is the cube with the same center as Q_{ki} and side length $2\sqrt{n}+1$ times the side length of Q_{ki}. Noting that $\omega \in A_{q_\omega+\epsilon}(\mathbb{R}^n)$ for any $\epsilon > 0$ (if $q_\omega = 1$, then ϵ can be 0), we have

$$\begin{aligned} \omega(A_{k_0}) \quad &\le \sum_{k=k_0+1}^{\infty} \sum_i \omega(Q_{ki}^*) \le c \sum_{k=k_0+1}^{\infty} \sum_i \omega(Q_{ki}) \left(\frac{|Q_{ki}^*|}{|Q_{ki}|} \right)^{q_\omega+\epsilon} \\ &\le c \sum_{k=k_0+1}^{\infty} \sum_i \omega(Q_{ki}) \le c c_1 \sum_{k=k_0+1}^{\infty} 2^{-kp} \le c c_1 \lambda^{-p}. \end{aligned}$$

We now estimate $\omega\{x \notin A_{k_0} :\ G_{p,\omega} F_2(x) > \lambda\}$. We first have

$$G_{p,\omega} F_2(x) \le \sum_{k=k_0+1}^{\infty} \sum_i G_{p,\omega} b_{ki}(x).$$

Choose any $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\phi\|_{\mathcal{A}_{p,\omega}} \le 1$. Let x_{ki} be the center of Q_{ki} and $P_x(y)$ be the $[n(q_\omega/p-1)]$-order Taylor expansion of $\phi\Big(\frac{x-y}{t}\Big)$ in y with respect to $y = x_{ki}$. Then, if $x \notin A_{k_0}$, we have

$$\begin{aligned} |b_{ki} * \phi_t(x)| \quad &= \frac{1}{t^n} \Big| \int_{\mathbb{R}^n} b_{ki}(y) \Big[\phi\Big(\frac{x-y}{t}\Big) - P_x(y) \Big]\, dy \Big| \\ &\le \frac{c 2^k}{t^n} \sum_{|\beta| = [n(q_\omega/p-1)]+1} \int_{Q_{ki}} \Big| (D^\beta \phi) \Big[\frac{x - x_{ki} + \theta(x_{ki} - y)}{t} \Big] \Big| \\ &\qquad \times \Big| \frac{y - x_{ki}}{t} \Big|^{[n(q_\omega/p-1)]+1} dy \\ &\le c 2^k \left(\frac{l(Q_{ki})}{|x - x_{ki}|} \right)^{[n(q_\omega/p-1)]+n+1}, \end{aligned}$$

where $\theta \in (0,1)$ depends on x, x_{ki} and y, $l(Q_{ki})$ is the side length of Q_{ki} and c is independent of k, i and ϕ. Thus for $x \notin A_{k_0}$, we have

$$G_{p,\omega} b_{ki}(x) \le c 2^k \left(\frac{l(Q_{ki})}{|x - x_{ki}|} \right)^{[n(q_\omega/p-1)]+n+1}.$$

Now choose $q \in \left(\frac{nq_\omega}{[n(q_\omega/p-1)]+n+1},\ p\right)$ and $0<\epsilon<q([n(q_\omega/p-1)]+n+1)/n-q_\omega$. Then

$$\begin{aligned}\omega\left(\left\{x\notin Q^*_{ki}:\right.\right.&\left.\left.\left(\frac{l(Q_{ki})}{|x-x_{ki}|}\right)^{[n(q_\omega/p-1)]+n+1}>\lambda\right\}\right)\\ &\le\frac{1}{\lambda^q}\int_{\mathbb{R}^n\setminus Q^*_{ki}}\left(\frac{l(Q_{ki})}{|x-x_{ki}|}\right)^{q([n(q_\omega/p-1)]+n+1)}\omega(x)\,dx\\ &\le\frac{c}{\lambda^q}\sum_{j=0}^{\infty}\int_{\sqrt{n}l(Q_{ki})2^j\le|x-x_{ki}|<\sqrt{n}l(Q_{ki})2^{j+1}}\left(\frac{l(Q_{ki})}{|x-x_{ki}|}\right)^{q([n(q_\omega/p-1)]+n+1)}\\ &\qquad\times\omega(x)\,dx\\ &\le\frac{c}{\lambda^q}\sum_{j=0}^{\infty}\frac{\omega(B(x_{ki},\sqrt{n}l(Q_{ki})2^{j+1}))}{2^{jq([n(q_\omega/p-1)]+n+1)}}\\ &\le\frac{c\omega(Q_{ki})}{\lambda^q}\sum_{j=0}^{\infty}2^{j\{nq_\omega+n\epsilon-q([n(q_\omega/p-1)]+n+1)\}}\\ &\le\frac{c_2\omega(Q_{ki})}{\lambda^q},\end{aligned}$$

where c_2 is a constant independent of $k,\ i$ and λ, and $B(x,r)\equiv\{y\in\mathbb{R}^n:\ |y-x|<r\}$. Thus, by Lemma 3, we have

$$\omega(\{x\notin A_{k_0}:\ G_{p,\omega}F_2(x)>\lambda\})\le\frac{2-q}{1-q}\frac{c}{\lambda^q}\sum_{k=k_0+1}^{\infty}\sum_i 2^{kq}\omega(Q_{ki})\le\frac{cc_1}{\lambda^q}\sum_{k=k_0+1}^{\infty}2^{k(q-p)}\le\frac{cc_1}{\lambda^p}.$$

This completes the proof of Lemma 2.

3 θ-Type Calderón–Zygmund Operators

Following Yabuta [6], we generalize the Calderón–Zygmund operator to its θ-type.

Definition 4 *Let θ be a non-negative non-decreasing function on $\mathbb{R}^+\equiv(0,\infty)$ with $\int_0^1\theta(t)t^{-1}\,dt<\infty$. A measurable function K on $\mathbb{R}^n\times\mathbb{R}^n\setminus\{(x,x):\ x\in\mathbb{R}^n\}$ is said to be a θ-type kernel if it satisfies*

(i) $|K(x,y)|\le c/|x-y|^n$ *for* $x\ne y$,

(ii) $|K(x,y)-K(x+h,y)|+|K(y,x)-K(y,x+h)|\le c\theta(|h|/|x-y|)/|x-y|^n$ *for* $|h|<|x-y|/2$.

Definition 5 *Let T be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$. We say T is a θ-type Calderón–Zygmund operator if*

(i) *T can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into weak $L^2(\mathbb{R}^n)$,*

(ii) *There is a θ-type kernel $K(x,y)$ such that $Tf(x)=\int_{\text{supp } f}K(x,y)f(y)\,dy$ for all $f\in C_c^\infty(\mathbb{R}^n)$ and for all $x\notin$ supp f, where $C_c^\infty(\mathbb{R}^n)$ is the space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.*

The following theorem is motivated by the result in ([4], p. 49) and is of interest in itself.

Theorem 1 *Let θ be a non-negative non-decreasing function on $\mathbb{R}^+$ with $\int_0^1 \theta(t)t^{-1}\,dt < \infty$. Let T be a linear operator associated with a θ-type kernel. Then the following conditions are equivalent:*

$\int_Q |Ta(x)|\,dx \le c\|a\|_{L^\infty(\mathbb{R}^n)}|Q|$ *for* $a \in L^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp} a \subset Q$, *a cube in* $\mathbb{R}^n$; (3.1)

T is a bounded map from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$; (3.2)

T is a bounded map from $L_c^\infty(\mathbb{R}^n) \equiv \{f \in L^\infty(\mathbb{R}^n) : \operatorname{supp} f \text{ is compact }\}$ *into* $\mathrm{BMO}(\mathbb{R}^n)$; (3.3)

T is bounded from $L^q(\mathbb{R}^n)$ into $WL^q(\mathbb{R}^n)$ for some $q \in (1,\infty)$; (3.4)

T is bounded on $L^q(\mathbb{R}^n)$ for some $q \in (1,\infty)$; (3.5)

T is bounded on $L^2(\mathbb{R}^n)$. (3.6)

To prove Theorem 1, we need the following interpolation of sublinear operators in ([4], p. 43).

Lemma 4 *Let T be a sublinear operator which is bounded from $L_c^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ and from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Then T is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$.*

Proof of Theorem 1 The equivalence of (3.1), (3.2) and (3.3) is a simple consequence of the theorem in ([4], p. 49) and (2.3), (2.4) and (2.5) in ([6], p. 21). Now, suppose (3.1) holds; then (3.2) and (3.3) also hold. Therefore, by Lemma 4, we know that T is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1,\infty)$. Thus, (3.6), (3.5) and (3.4) all hold. Obviously, (3.6) implies (3.5) and (3.5) implies (3.4). Now we need to show that (3.4) implies (3.1). To do so, we first prove that if T satisfies (3.4), then T is bounded from $L^1(\mathbb{R}^n)$ into $WL^1(\mathbb{R}^n)$. Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Using the Calderón–Zygmund decomposition (see [3], p. 140; or [5], p. 17), we know that $f = g + b$, where $b = \sum_k b_k$ and there exists a sequence of non-overlapping cubes $\{Q_k\}_k$, so that

$|g(x)| \le c\lambda$, for a.e. $x \in \mathbb{R}^n$. (3.7)

Each b_k is supported in Q_k,

$$\int_{\mathbb{R}^n} |b_k(x)|\,dx \le c\lambda|Q_k|, \text{ and } \int_{\mathbb{R}^n} b_k(x)\,dx = 0; \tag{3.8}$$

$$\sum_k |Q_k| \le c\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}. \tag{3.9}$$

Thus,

$$\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \subset \{x \in \mathbb{R}^n : |Tg(x)| > \lambda/2\} \cup \{x \in \mathbb{R}^n : |Tb(x)| > \lambda/2\} \equiv I_\lambda \cup II_\lambda.$$

Since T is bounded from $L^q(\mathbb{R}^n)$ into $WL^q(\mathbb{R}^n)$ for some $q \in (1,\infty)$, we have

$$|I_\lambda| \le c2^q\lambda^{-q}\|g\|^q_{L^q(\mathbb{R}^n)} \le c\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)},$$

where in the last inequality we use (3.7), (3.9) and the fact that if $x \notin Q_k$, then $g(x) = f(x)$. This gives our desired estimate for $|I_\lambda|$.

Now set $E_\lambda = \cup_k Q_k^*$, where Q_k^* denotes the cube with the same centre as Q_k and side length $2\sqrt{n}+1$ times the side length of Q_k. It is clear that

$$\begin{aligned} |II_\lambda| &= |II_\lambda \cap E_\lambda| + |II_\lambda \setminus E_\lambda| \\ &\le |E_\lambda| + 2\lambda^{-1} \int_{\mathbb{R}^n \setminus E_\lambda} |Tb(x)|\, dx \\ &\le c\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)} + 2\lambda^{-1} \int_{\mathbb{R}^n \setminus E_\lambda} |Tb(x)|\, dx. \end{aligned}$$

Suppose that x_k is the centre of Q_k and $l(Q_k)$ is the side length of Q_k. Since $\int_{Q_k} b_k(y)\, dy = 0$ we have

$$\begin{aligned} F_\lambda &\equiv \int_{\mathbb{R}^n \setminus E_\lambda} |Tb(x)|\, dx \\ &\le \int_{\mathbb{R}^n \setminus E_\lambda} \Big| \sum_k \int_{Q_k} K(x,y) b_k(y)\, dy \Big|\, dx \\ &\le \sum_k \int_{\mathbb{R}^n \setminus E_\lambda} \int_{Q_k} |K(x,y) - K(x,x_k)| |b_k(y)|\, dy\, dx \\ &\le c \sum_k \int_{Q_k} |b_k(y)| \Big\{ \int_{\mathbb{R}^n \setminus E_\lambda} \theta\Big(\frac{|y-x_k|}{|x-x_k|} \Big) \frac{1}{|x-x_k|^n}\, dx \Big\}\, dy \\ &\le c \Big\{ \sum_{l=0}^{\infty} \theta(2^{-l}) \Big\} \sum_k \int_{Q_k} |b_k(y)|\, dy \\ &\le c \Big\{ \int_0^1 \frac{\theta(t)}{t}\, dt \Big\} \lambda \sum_k |Q_k| \\ &\le c \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

It follows that $|II_\lambda| \le c\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}$. It now follows from our estimates of $|I_\lambda|$ and $|II_\lambda|$ that T is bounded from $L^1(\mathbb{R}^n)$ into $WL^1(\mathbb{R}^n)$. By our hypothesis (3.4) and the Marcinkiewicz interpolation theorem, we conclude that T is bounded on $L^p(\mathbb{R}^n)$ for any $p \in (1,q)$. Therefore, for a as in (3.1), we have

$$\int_Q |Ta(x)|\, dx \le c\|Ta\|_{L^p(\mathbb{R}^n)} |Q|^{1-1/p} \le c\|a\|_{L^p(\mathbb{R}^n)} |Q|^{1-1/p} \le c\|a\|_{L^\infty(\mathbb{R}^n)} |Q|.$$

Thus (3.1) holds and the proof of Theorem 1 is complete.

The following theorem follows from Theorem 1 and ([6], Theorem 2.4).

Theorem 2 *Let θ be a non-negative non-decreasing function on $\mathbb{R}^+$ with $\int_0^1 \theta(t) t^{-1}\, dt < \infty$. Let T be a linear operator associated with a θ-type kernel and let T satisfy any one of* (3.1)–(3.6) *in Theorem* 1. *Then*

For any weight $\omega \in A_p(\mathbb{R}^n)$ with $p \in (1,\infty)$,

$$\|Tf\|_{L^p_\omega(\mathbb{R}^n)} \le c\|f\|_{L^p_\omega(\mathbb{R}^n)}, \tag{3.10}$$

where c depends only on n, p and the $A_p(\mathbb{R}^n)$-constant of ω.

For any weight $\omega \in A_1(\mathbb{R}^n)$,

$$\|Tf\|_{WL^1_\omega(\mathbb{R}^n)} \le c\|f\|_{L^1_\omega(\mathbb{R}^n)}, \tag{3.11}$$

where c depends only on n and the $A_1(\mathbb{R}^n)$-constant of ω.

(3.11) *and* (3.12) *also hold for the truncated maximal operator T_* of T, where*

$$T_*f(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} K(x,y)f(y)\,dy \right|. \tag{3.12}$$

We do not know whether all θ-type Calderón–Zygmund operators are bounded on $H^p_\omega(\mathbb{R}^n)$ or $WH^p_\omega(\mathbb{R}^n)$. However, we shall prove in Theorems 3 and 4 that the answer is affirmative for the following special case of θ-type operators.

Definition 6 *Let $\delta \in (0,1]$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$ is said to be a δ-type Calderón–Zygmund operator if T satisfies any one of* (3.1)–(3.6) *and there is a kernel $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and all $x \notin \operatorname{supp} f$, we have*

$$Tf(x) = \int_{\operatorname{supp} f} K(x,y)f(y)\,dy, \tag{3.13}$$

where K satisfies

$$|K(x,y)| \le c/|x-y|^n \text{ for } x \ne y, \tag{3.13$_1$}$$

$$|K(x,y) - K(x,y+h)| \le c|h|^\delta/|x-y|^{n+\delta}, \text{ if } |h| < |x-y|/2, \tag{3.13$_2$}$$

$$|K(y,x) - K(y+h,x)| \le c|h|^\delta/|x-y|^{n+\delta}, \text{ if } |h| < |x-y|/2. \tag{3.13$_3$}$$

As a consequence of Theorem 1 and ([2], Theorem 2.8) we have the following result.

Lemma 5 *Let $\delta \in (0,1]$ and let T be a δ-type Calderón–Zygmund operator. We have:*

If $p \in (1,\infty)$ and $\omega \in A_p(\mathbb{R}^n)$, then T can be extended to a bounded operator on $L^p_\omega(\mathbb{R}^n)$; (3.14)

If $p \in (n/(n+\delta), 1]$ and $\omega \in A_q(\mathbb{R}^n)$ with $q \in [1,(1+\delta/n)p)$, then T can be extended to a bounded operator from $H^p_\omega(\mathbb{R}^n)$ into $L^p_\omega(\mathbb{R}^n)$. (3.15)

In all cases, the operator norm of T depends only on the constants for the kernel and the weight.

Definition 7 *Let T be a Calderón–Zygmund operator of θ-type (see Definition 5) or of semi-(θ,p)-type (see Definition 8 in Section 4). We say $T^*1 = 0$ if $\int_{\mathbb{R}^n} Ta(x)\,dx = 0$ for all $a \in L^\infty(\mathbb{R}^n)$ with compact support and $\int_{\mathbb{R}^n} a(x)\,dx = 0$.*

Theorem 3 *Let $\delta \in (0,1]$ and let T be a δ-type Calderón–Zygmund operator. If $T^*1 = 0$, $n/(n+\delta) < p \le 1$ and $\omega \in A_q(\mathbb{R}^n)$ with $q \in [1, (1+\delta/n)p)$, then T is bounded on $H^p_\omega(\mathbb{R}^n)$.*

Proof Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx \ne 0$. By Lemma 1, we only need to show that for any p-atom a with respect to ω, $\|(Ta)^*\|_{L^p_\omega(\mathbb{R}^n)} \le c$ with c independent of a. Suppose $\operatorname{supp} a \subset B(x_0,r)$. Let $\omega \in A_q(\mathbb{R}^n)$ with $q \in [1,(1+\delta/n)p)$. We choose $p_0 \in (q,(1+\delta/n)p)$, and write

$$\|(Ta)^*\|^p_{L^p_\omega(\mathbb{R}^n)} = \int_{B(x_0,4r)} [(Ta)^*(x)]^p \omega(x)\,dx + \int_{\mathbb{R}^n \setminus B(x_0,4r)} [(Ta)^*(x)]^p \omega(x)\,dx \equiv L_1 + L_2.$$

By Lemma 5, we have

$$\begin{aligned} L_1 &\leq \left(\int_{B(x_0,4r)} [(Ta)^*(x)]^{p_0} \omega(x)\,dx \right)^{p/p_0} \omega(B(x_0,4r))^{1-p/p_0} \\ &\leq c \left(\int_{B(x_0,r)} |a(x)|^{p_0} \omega(x)\,dx \right)^{p/p_0} \omega(B(x_0,r))^{1-p/p_0} \\ &\leq c, \end{aligned}$$

where c is independent of a.

To estimate L_2, we first estimate $(Ta)^*(x)$ when $x \notin B(x_0,4r)$. In fact, we have

$$\begin{aligned} |Ta * \varphi_t(x)| &= \left| \int_{\mathbb{R}^n} Ta(y) \frac{1}{t^n} \varphi\Big(\frac{x-y}{t}\Big)\,dy \right| \\ &= \left| \int_{\mathbb{R}^n} Ta(y) \frac{1}{t^n} \left(\varphi\Big(\frac{x-y}{t}\Big) - \varphi\Big(\frac{x-x_0}{t}\Big) \right) dy \right| \\ &\leq \frac{1}{t^n} \int_{|y-x_0|<2r} |Ta(y)| \left| \varphi\Big(\frac{x-y}{t}\Big) - \varphi\Big(\frac{x-x_0}{t}\Big) \right| dy \\ &\quad + \frac{1}{t^n} \int_{2r \leq |y-x_0| < \frac{|x-x_0|}{2}} \cdots + \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \cdots \\ &\equiv E_1 + E_2 + E_3. \end{aligned}$$

By the mean value theorem and Hölder's inequality, we have

$$\begin{aligned} E_1 &\leq \frac{1}{t^{n+1}} \|Ta\|_{L^2(\mathbb{R}^n)} \left(\int_{|y-x_0|<2r} \left| \nabla\varphi\Big(\frac{x-x_0-\gamma(y-x_0)}{t}\Big) \right|^2 |y-x_0|^2\,dy \right)^{1/2} \\ &\leq c \frac{r^{n+1}}{|x-x_0|^{n+1} \omega(B(x_0,r))^{1/p}}, \end{aligned}$$

where $\gamma \in (0,1)$ depends on x, y and x_0, and $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Here, we use the inequalities

$$\frac{|x-x_0-\gamma(y-x_0)|^{n+1}}{t^{n+1}} \left| \nabla\varphi\Big(\frac{x-x_0-\gamma(y-x_0)}{t}\Big) \right| \leq c$$

and $|x-x_0-\gamma(y-x_0)| \geq |x-x_0| - |y-x_0| \geq |x-x_0|/2$. Using the same inequalities, we have

$$\begin{aligned} E_2 &= \frac{1}{t^{n+1}} \int_{2r \leq |y-x_0| < \frac{|x-x_0|}{2}} \left| \int_{B(x_0,r)} a(z)(K(y,z) - K(y,x_0))\,dz \right| \\ &\qquad \times \left| \nabla\varphi\Big(\frac{x-x_0-\gamma(y-x_0)}{t}\Big) \right| |y-x_0|\,dy \\ &\leq c \frac{r^{n+\delta}}{|x-x_0|^{n+1} \omega(B(x_0,r))^{1/p}} \int_{2r \leq |y-x_0| < \frac{|x-x_0|}{2}} \frac{1}{|y-x_0|^{n+\delta-1}}\,dy \\ &\leq \begin{cases} c \dfrac{r^{n+1}}{|x-x_0|^{n+1} \omega(B(x_0,r))^{1/p}} \ln\Big(\dfrac{|x-x_0|}{4r}\Big), & \text{if } \delta = 1, \\ c \dfrac{r^{n+\delta}}{|x-x_0|^{n+\delta} \omega(B(x_0,r))^{1/p}}, & \text{if } \delta \in (0,1). \end{cases} \end{aligned}$$

Now consider E_3. We have

$$\begin{aligned} E_3 \quad &\le \frac{1}{t^n}\int_{|y-x_0|\ge\frac{|x-x_0|}{2}}\left|\int_{B(x_0,r)} a(z)\,(K(y,z)-K(y,x_0))\;dz\right| \\ &\qquad\qquad \times\left(\left|\varphi\Big(\frac{x-y}{t}\Big)\right|+\left|\varphi\Big(\frac{x-x_0}{t}\Big)\right|\right)\,dy \\ &\le c\,\frac{r^{n+\delta}}{t^n\omega(B(x_0,r))^{1/p}}\int_{|y-x_0|\ge\frac{|x-x_0|}{2}}\frac{1}{|y-x_0|^{n+\delta}}\left(\left|\varphi\Big(\frac{x-y}{t}\Big)\right|+\left|\varphi\Big(\frac{x-x_0}{t}\Big)\right|\right)\,dy \\ &\le c\,\frac{r^{n+\delta}}{|x-x_0|^{n+\delta}\omega(B(x_0,r))^{1/p}}. \end{aligned}$$

Let $x\notin B(x_0,4r)$. If $\delta=1$ then

$$(Ta)^*(x)\le c_\epsilon\frac{r^{n+\epsilon}}{|x-x_0|^{n+\epsilon}\omega(B(x_0,r))^{1/p}} \tag{3.16}$$

holds for any $\epsilon\in(0,1)$. If $\delta\in(0,1)$, then

$$(Ta)^*(x)\le c_\delta\frac{r^{n+\delta}}{|x-x_0|^{n+\delta}\omega(B(x_0,r))^{1/p}}. \tag{3.17}$$

Note that $q<(1+\delta/n)p$. If $\delta=1$ we choose $\epsilon\in(0,1)$ such that $q<(1+\epsilon/n)p$; if $\delta\in(0,1)$ we choose $\epsilon=\delta$. Then, in all cases, we have $(n+\epsilon)p>nq$ and

$$\begin{aligned} L_2 \quad &\le c_\epsilon\int_{\mathbb{R}^n\setminus B(x_0,4r)}\frac{r^{(n+\epsilon)p}}{|x-x_0|^{(n+\epsilon)p}\omega(B(x_0,r))}\omega(x)\,dx \\ &= c_\epsilon\frac{r^{(n+\epsilon)p}}{\omega(B(x_0,r))}\sum_{l=0}^{\infty}\int_{2^{l+2}r\le|x-x_0|<2^{l+3}r}\frac{\omega(x)}{|x-x_0|^{(n+\epsilon)p}}\,dx \\ &\le c_\epsilon\sum_{l=0}^{\infty}\frac{1}{2^{l[(n+\epsilon)p-nq]}} \\ &\le c_\epsilon, \end{aligned}$$

where c_ϵ is independent of a.

This completes the proof of Theorem 3.

Theorem 4 *Let $\delta\in(0,1]$ and let T be a δ-type Calderón–Zygmund operator. If $T^*1=0$, $n/(n+\delta)<p\le1$ and $\omega\in A_q(\mathbb{R}^n)$ with $q\in[1,(1+\delta/n)p)$, then T is bounded on $WH^p_\omega(\mathbb{R}^n)$.*

Proof For any given $\lambda>0$, let $k_0\in\mathbb{Z}$ such that $2^{k_0}\le\lambda<2^{k_0+1}$. Let $\omega\in A_q(\mathbb{R}^n)$ and $f\in WH^p_\omega(\mathbb{R}^n)$. Then by Lemma 2, we write

$$f=\sum_{k=-\infty}^{\infty}f_k=\sum_{k=-\infty}^{\infty}\sum_i b_{k,i}=\sum_{k=-\infty}^{k_0}\sum_i\cdots+\sum_{k=k_0+1}^{\infty}\sum_i\cdots\equiv F_1+F_2,$$

where $b_{k,i}$'s are as in Lemma 2. We choose $p_0\in(q,(1+\delta/n)p)$. By Lemma 5, T is bounded on

$L^{p_0}_\omega(\mathbb{R}^n)$. Then,

$$\begin{aligned}\omega(\{x\in\mathbb{R}^n : G_{p,\omega}(TF_1)(x)>\lambda\}) &\le \frac{1}{\lambda^{p_0}}\|G_{p,\omega}(TF_1)\|^{p_0}_{L^{p_0}_\omega(\mathbb{R}^n)}\\ &\le \frac{c}{\lambda^{p_0}}\|F_1\|^{p_0}_{L^{p_0}_\omega(\mathbb{R}^n)}\\ &\le \frac{c}{\lambda^{p_0}}\left(\sum_{k=-\infty}^{k_0}2^k\left[\sum_i\omega(Q_{ki})\right]^{1/p_0}\right)^{p_0}\\ &\le \frac{c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}}{\lambda^p},\end{aligned}$$

which is our desired estimate.

Now set $A_{k_0}=\bigcup_{k=k_0+1}^\infty\bigcup_i Q^*_{ki}$, where Q^*_{ki} is the cube with the same centre as Q_{ki} and side length $2\sqrt{n}+1$ times the side length of Q_{ki}. Noting that $\omega\in A_q(\mathbb{R}^n)$, we easily obtain

$$\omega(A_{k_0})\le\sum_{k=k_0+1}^\infty\sum_i\omega(Q^*_{ki})\le c\sum_{k=k_0+1}^\infty\sum_i\omega(Q_{ki})\le\frac{c}{\lambda^p}\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}.$$

To finish the proof, we still need to estimate $\omega(\{x\notin A_{k_0}:\ G_{p,\omega}(TF_2)(x)>\lambda\})$. Note that

$$G_{p,\omega}(TF_2)(x)\le\sum_{k=k_0+1}^\infty\sum_i G_{p,\omega}(Tb_{k,i})(x).$$

Choose q_1 such that $nq/(n+\delta)<q_1=n/(n+\delta-\epsilon)<p$ and let $\phi\in\mathcal{A}_{p,\omega}$. Let $y_{k,i}$ be the centre of Q_{ki}. Arguing as in the proofs of (3.16) and (3.17) we have, for $x\notin Q^*_{ki}$,

$$|Tb_{k,i}*\phi_t(x)|\le c\|\phi\|_{\mathcal{A}_{p,\omega}}\frac{2^k|Q_{ki}|^{q/q_1}}{|x-y_{ki}|^{nq/q_1}}.$$

Therefore, for $x\notin Q^*_{ki}$,

$$G_{p,\omega}(Tb_{k,i})(x)\le c\frac{2^k|Q_{ki}|^{q/q_1}}{|x-y_{ki}|^{nq/q_1}}. \tag{3.18}$$

We claim that for any $\lambda>0$,

$$\omega\left(\left\{x\notin A_{k_0}:\frac{|Q_{ki}|^{q/q_1}}{\omega(Q_{ki})^{1/q_1}|x-y_{ki}|^{nq/q_1}}>\lambda\right\}\right)\le\frac{c}{\lambda^{q_1}}, \tag{3.19}$$

where c is independent of k and i. Note that if $x\notin A_{k_0}$, then $|x-y_{ki}|>\sqrt{n}l_{ki}$, where l_{ki} is half the side length of Q_{ki}. Thus, for $\lambda\ge\omega(Q_{ki})^{-1/q_1}(\sqrt{n})^{-nq/q_1}$, we have

$$\omega\left(\left\{x\notin A_{k_0}:\frac{|Q_{ki}|^{q/q_1}}{\omega(Q_{ki})^{1/q_1}|x-y_{ki}|^{nq/q_1}}>\lambda\right\}\right)=\omega(\emptyset)\le\frac{c}{\lambda^{q_1}}.$$

For $\lambda<\omega(Q_{ki})^{-1/q_1}(\sqrt{n})^{-nq/q_1}$, choose $r=\left(\frac{|Q_{ki}|}{\lambda^{q_1/q}\omega(Q_{ki})^{1/q}}\right)^{1/n}$ so that $r\ge\sqrt{n}l_{ki}$. Then,

$$\omega\left(\left\{x\notin A_{k_0}:\frac{|Q_{ki}|^{q/q_1}}{\omega(Q_{ki})^{1/q_1}|x-y_{ki}|^{nq/q_1}}>\lambda\right\}\right)\le\omega(B(y_{ki},r))=\frac{c}{\lambda^{q_1}}.$$

Thus (3.19) holds. By (3.19) and Lemmas 3 and 5, we have

$$\omega(\{x\notin A_{k_0}:\ G_{p,\omega}(TF_2)(x)>\lambda\})\le\frac{c}{\lambda^{q_1}}\sum_{k=k_0+1}^\infty\sum_i 2^{kq_1}\omega(Q_{ki})\le\frac{c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}}{\lambda^p},$$

where c is independent of λ and f. This completes the proof of Theorem 4.

When $p \in [n/(n+1), 1)$ and $\delta = n(1/p - 1)$, we have the following theorem on the boundedness of δ-type Calderón–Zygmund operators.

Theorem 5 *Let $p \in [n/(n+1), 1)$ and $\delta = n(1/p - 1)$. If T is a δ-type Calderón–Zygmund operator, $T^*1 = 0$ and $\omega \in A_1(\mathbb{R}^n)$, then T is bounded from $H^p_\omega(\mathbb{R}^n)$ into $WH^p_\omega(\mathbb{R}^n)$.*

Proof By Lemma 1, it suffices to show that for any p-atom a with respect to ω and for any $\lambda > 0$, we have

$$\omega(\{x \in \mathbb{R}^n : \ G_{p,\omega}(Ta)(x) > \lambda\}) \le \frac{c}{\lambda^p}, \tag{3.20}$$

where c is independent of a.

Suppose supp $a \subset Q$ and let Q^* be defined as in the proof of Theorem 4. By (3.14) in Lemma 5, we have that T is bounded for any $p_0 \in (1, \infty)$ and $\omega \in A_{p_0}(\mathbb{R}^n)$. Therefore,

$$\begin{aligned}\omega(\{x \in Q^* : \ G_{p,\omega}(Ta)(x) > \lambda\}) &\le \frac{c}{\lambda^p} \int_{\{x \in Q^* : G_{p,\omega}(Ta)(x) > \lambda\}} [G_{p,\omega}(Ta)(x)]^p \omega(x)\, dx \\ &\le \frac{c}{\lambda^p} \|a\|^p_{L^{p_0}_\omega(\mathbb{R}^n)} \omega(Q^*)^{1-p/p_0} \le \frac{c}{\lambda^p},\end{aligned} \tag{3.21}$$

where c is independent of a.

On the other hand, note that $p = n/(n+1)$ implies that $\int_{\mathbb{R}^n} a(x) x^\alpha\, dx = 0$ for any $\alpha \in (\mathbb{N} \cup \{0\})^n$ and $|\alpha| = 1$. As in the proofs of (3.16), (3.17) and (3.18), we have

$$G_{p,\omega}(Ta)(x) \le c \frac{|Q|^{n/p}}{\omega(Q)^{1/p} |x - x_0|^{n/p}}$$

for $x \notin Q^*$, where x_0 is the centre of Q. Following the proof of (3.19), we obtain

$$\omega(\{x \notin Q^* : \ G_{p,\omega}(Ta)(x) > \lambda\}) \le \frac{c}{\lambda^p}. \tag{3.22}$$

Combining (3.21) and (3.22), we establish (3.20) and this completes the proof of Theorem 5.

4 Other Generalizations of Calderón–Zygmund Operators

We now further weaken our θ-type operators to the following semi-(θ, p_0)-type operators. We shall prove in Theorem 6 the boundedness of these weaker operators on certain $L^p_\omega(\mathbb{R}^n)$ spaces.

Definition 8 *Let $1 < p_0 < \infty$ and θ be a non-negative non-decreasing function on $\mathbb{R}^+$ with $\int_0^1 \theta(t) t^{-1}\, dt < \infty$. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ is said to be a semi-(θ, p_0)-type Calderón–Zygmund operator if T satisfies the following two properties:*

(1) *T can be extended to a bounded linear operator from $L^{p_0}(\mathbb{R}^n)$ into $WL^{p_0}(\mathbb{R}^n)$;*

(2) *There is a kernel $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and all $x \notin$ supp f, we have $Tf(x) = \int_{\text{supp } f} K(x, y) f(y)\, dy$, where K satisfies the following two conditions:*

$(2)_1$ *$|K(x, y)| \le c/|x - y|^n$ for all $x \ne y$;*

(2)$_2$ $|K(x,y)-K(x,z)|\le c\theta(|y-z|/|x-z|)/|x-z|^n$, *if* $|y-z|<|x-z|/2$.

Our next theorem complements Theorem 2.

Theorem 6 *Let θ be a non-negative non-decreasing function on $\mathbb{R}^+$ such that*

$$\int_0^1 \theta(t)/t\,dt<\infty$$

and $1<p_0<\infty$. Let T be a semi-(θ,p_0)-type Calderón–Zygmund operator. Then T is bounded on $L^p_\omega(\mathbb{R}^n)$ for $1<p<p_0$, where $\omega\in A_p(\mathbb{R}^n)$ satisfies

$$\sup_{2^{k-2}\le|x|\le 2^{k+1}}\omega(x)\le c_1\inf_{2^{k-2}\le|x|\le 2^{k+1}}\omega(x),\quad k\in\mathbb{Z}. \tag{4.1}$$

Proof As in the proof of Theorem 1, T is bounded from $L^1(\mathbb{R}^n)$ into $WL^1(\mathbb{R}^n)$. By hypothesis, T is bounded from $L^{p_0}(\mathbb{R}^n)$ into $WL^{p_0}(\mathbb{R}^n)$. It follows from the Marcinkiewicz interpolation theorem that T is bounded on $L^p(\mathbb{R}^n)$ for any $p\in(1,p_0)$. Now Theorem 6 is a simple consequence of Theorem 1 in [15].

Note that the power weight obviously satisfies restriction (4.1) above. Some other examples of weights satisfying (4.1) can be found in [15].

In what follows, we say a semi-(θ,p_0)-type Calderón–Zygmund operator is a semi-(δ,p_0)-type Calderón–Zygmund operator if $\theta=t^\delta$. The following theorem shows that Theorems 3, 4 and 5 can be improved for weights satisfying (4.1).

Theorem 7 *Let $\delta\in(0,1]$ and $p_0\in(1,\infty)$ and let T be a semi-(δ,p_0)-type Calderón–Zygmund operator.*

(1) *If $T^*1=0$ and $n/(n+\delta)<p\le 1$, then T is bounded on $H^p_\omega(\mathbb{R}^n)$ and $WH^p_\omega(\mathbb{R}^n)$ separately, where $\omega\in A_q(\mathbb{R}^n)$ satisfies (4.1) with $1\le q<\min\{p_0,(1+\delta/n)p\}$,*

(2) *If $n/(n+1)\le p<1$, $\delta=n(1/p-1)$ and $T^*1=0$, then T is bounded from $H^p_\omega(\mathbb{R}^n)$ into $WH^p_\omega(\mathbb{R}^n)$, where $\omega\in A_1(\mathbb{R}^n)$ satisfies (4.1),*

(3) *For $n/(n+1)\le p<1$, there exists a semi-(δ,p_0)-type Calderón–Zygmund operator satisfying $T^*1=0$ and T is not bounded on $H^p(\mathbb{R}^n)$, where $\delta=n(1/p-1)$.*

Proof With Theorem 6 replacing Lemma 5 in the proofs of Theorems 3, 4 and 5 we can prove (1) and (2). We omit the details.

We now prove (3) for $n=1$. The other cases are similar. For $k=0,1,2,\ldots$, let I_k^+ be the interval $[k,(2k+1)/2)$ and I_k^- be the interval $[(2k+1)/2,k+1)$. Note that $(2k+1)/2$ is the midpoint of the interval $[k,k+1]$. Now we define the kernel $K(x,y)$ on $\mathbb{R}\times\mathbb{R}$ as follows:

$$K(x,y)=\begin{cases} y^{1/p-1}/([x]+1)^{1/p}, & \text{if } 0\le[y]\le[x],\ x\in I_k^+ \text{ for some } k\in\mathbb{N},\\ -y^{1/p-1}/([x]+1)^{1/p}, & \text{if } 0\le[y]\le[x],\ x\in I_k^- \text{ for some } k\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

Define T on $L^2(\mathbb{R})$ by

$$Tf(x)\equiv \text{p.v.}\int_{\mathbb{R}}K(x,y)f(y)\,dy.$$

We first claim that $T^*1 = 0$. Let a be a p-atom. Then $Ta(x) = 0$ if $x \le 0$. Note that for each $x > 0$ with $x \in [k, (k+1)/2]$ for some $k \in \mathbb{N} \cup \{0\}$, there exists $\tilde{x} = x + 1/2$ such that $Ta(x) = -Ta(\tilde{x})$. Thus $\int_{\mathbb{R}} Ta(x)\, dx = 0$.

We next show that T is bounded from $L^{p_0}(\mathbb{R})$ into $WL^{p_0}(\mathbb{R})$. Let $f \in L^{p_0}(\mathbb{R})$ with compact support J. Then for $x \in \mathbb{R}$ we have

$$|Tf(x)| = \Big|\int_J K(x,y)f(y)\, dy\Big| \le \|K(x,\cdot)\chi_{J\cap\mathbb{R}^+}\|_{L^{p_0'}(\mathbb{R})}\|f\|_{L^{p_0}(\mathbb{R})} \le \frac{c\|f\|_{L^{p_0}(G)}}{|x|^{1/p_0}},$$

where $1/p_0 + 1/p_0' = 1$. Hence T is bounded from $L^{p_0}(\mathbb{R})$ into $WL^{p_0}(\mathbb{R})$.

Note that $K(x,y)$ obviously satisfies $(2)_1$ of Definition 8. To see that $K(x,y)$ satisfies $(2)_2$ of Definition 8, we first assume that $0 \le z < y$. Let $0 < [y] \le [x]$ be such that $2|y-z| < |x-z|$. Then we have $0 \le z < y < x$. Hence

$$|K(x,y) - K(x,z)| \le \Big|\frac{|y|^{1/p-1}}{([x]+1)^{1/p}} - \frac{|z|^{1/p-1}}{([x]+1)^{1/p}}\Big| \le \frac{|y|^{1/p-1} - |z|^{1/p-1}}{([x]+1)^{1/p}} = c\frac{|y-z|^{1/p-1}}{|x-z|^{1/p}},$$

where the last inequality follows from $0 \le z < x$ which implies that $x - z < [x]+1$. Hence $(2)_2$ of Definition 8 is also valid. For $z < 0 \le y$ and $0 \le [y] \le [x]$, we have

$$|K(x,y) - K(x,z)| \le \Big|\frac{|y|^{1/p-1}}{([x]+1)^{1/p}}\Big| \le \frac{|y-z|^{1/p-1}}{|x-z|^{1/p}}$$

for $|x-z| > 2|y-z|$. Note that the last inequality above follows because $z < 0$ and $0 \le x, y$ imply that $|x-z| = x + |z|$ and $|y-z| = y + |z|$. Hence $|x-z| > 2|y-z|$ implies that $x + |z| > 2(y + |z|)$, i.e. $x > 2y + |z|$. Thus $2([x]+1) > x + |z| = |x-z|$. Hence $(2)_2$ of Definition 8 is also valid.

Finally, we give an example to show that T is not bounded on $H^p(\mathbb{R})$. Let $a = \chi_{[1,3]} - 2\chi_{[1,2]}$. Then a is a p-atom. For $3 < x$ we have

$$\begin{aligned}
|Ta(x)| &= \Big|\int_{[1,3]} \frac{|y|^{1/p-1}}{|[x]+1|^{1/p}}\, dy - \int_{[1,2]} \frac{2|y|^{1/p-1}}{|[x]+1|^{1/p}}\, dy\Big| \\
&\ge \frac{1}{2|x|^{1/p}}\left(\Big|\int_{[1,3]} |y|^{1/p-1}\, dy - \int_{[1,2]} 2|y|^{1/p-1}\, dy\Big|\right) \\
&= \frac{c}{|x|^{1/p}},
\end{aligned}$$

where c is a constant independent of x. Thus, we have $(Ta)^*(x) \ge c/|x|^{1/p}$ if $3 < x$. Consequently $Ta \notin H^p(\mathbb{R})$. Hence the theorem is established.

In the next theorem we consider the boundedness of another generalization of Calderón–Zygmund operators.

Theorem 8 *Let θ be a non-negative non-decreasing function on $\mathbb{R}^+$ and let K be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that*

(a) $|K(x,y)| \le c/|x-y|^n$ *for all* $x \ne y$,

(b) $|K(x,y) - K(x,z)| \le c\theta(|y-z|/|x-z|)/|x-z|^n$, *if* $|y-z| < |x-z|/2$.

Let $\bar{p} \in [1,\infty)$ and let $\omega \in A_{\bar{p}}(\mathbb{R}^n)$. We then have:

(i) *Let T be bounded from $L^{p_0}_\omega(\mathbb{R}^n)$ into $WL^{p_0}_\omega(\mathbb{R}^n)$ with $p_0 \geq \bar{p}$ and $p_0 \neq 1$ such that*

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} K(x,y) f(y)\, dy.$$

If $\int_0^1 \frac{\theta(t)^p |\ln t|}{t^{(\bar{p}-p)n+1}}\, dt < \infty$, then T is bounded from $WH^p_\omega(\mathbb{R}^n)$ to $WL^p_\omega(\mathbb{R}^n)$ for $0 < p \leq 1$.

(ii) *Let T be bounded on $L^{p_0}_\omega(\mathbb{R}^n)$ with $p_0 \geq \bar{p}$ and $p_0 \neq 1$ such that*

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} K(x,y) f(y)\, dy.$$

If $T^*1 = 0$ and $\int_0^1 \frac{\theta(t) |\ln t|^{2/p+\epsilon}}{t^{(\bar{p}-p)n/p+1}}\, dt < \infty$ for some $\epsilon > 0$, then T is bounded on $WH^p_\omega(\mathbb{R}^n)$ for $n/(n+1) < p \leq 1$ and $\bar{p} \in [1, (1+1/n)p)$.

Proof (i) For any given $\lambda > 0$, we choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \lambda < 2^{k_0+1}$. Let $f \in WH^p_\omega(\mathbb{R}^n)$. We write $f \equiv F_1 + F_2$ as in the proof of Theorem 4.

By Lemma 2 and the fact that T is bounded from $L^{p_0}_\omega(\mathbb{R}^n)$ into $WL^{p_0}_\omega(\mathbb{R}^n)$, we easily obtain

$$\omega(\{x \in \mathbb{R}^n :\ |TF_1(x)| > \lambda\}) \leq \frac{c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}}{\lambda^p}. \tag{4.2}$$

Now let us estimate $TF_2(x)$. Let $B_{k_0} = \bigcup_{k=k_0+1}^\infty \bigcup_i \widetilde{Q}_{ki}$, where $\widetilde{Q}_{ki}$ is the cube with the same centre as Q_{ki} and side length $(3/2)^{(k-k_0)p/(n\bar{p})}\sqrt{n}$ times the side length of Q_{ki}. Since $\omega \in A_{\bar{p}}(\mathbb{R}^n)$, Lemma 2 implies that

$$\begin{aligned} \omega(B_{k_0}) &\leq \sum_{k=k_0+1}^\infty \sum_i \omega(\widetilde{Q}_{ki}) \leq c \sum_{k=k_0+1}^\infty \sum_i \omega(Q_{ki}) \left(\frac{|\widetilde{Q}_{ki}|}{|Q_{ki}|}\right)^{\bar{p}} \\ &\leq c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)} \sum_{k=k_0+1}^\infty 2^{-kp}\left(\frac{3}{2}\right)^{(k-k_0)p} \leq c\lambda^{-p}\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}. \end{aligned}$$

On the other hand, let l_{ki} be half the side length of Q_{ki} and let y_{ki} be the centre of Q_{ki}. Then

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \widetilde{Q}_{ki}} |Tb_{k,i}(x)|^p \omega(x)\, dx \\ &\quad = \int_{\mathbb{R}^n \setminus \widetilde{Q}_{ki}} \left| \int_{Q_{ki}} [K(x,y) - K(x,y_{ki})] b_{k,i}(y)\, dy \right|^p \omega(x)\, dx \\ &\quad \leq c2^{kp}|Q_{ki}|^p \int_{\mathbb{R}^n \setminus \widetilde{Q}_{ki}} \theta\left(\frac{\sqrt{n}l_{ki}}{|x-y_{ki}|}\right)^p \frac{1}{|x-y_{ki}|^{np}}\omega(x)\, dx \\ &\quad \leq c2^{kp}|Q_{ki}|^p \times \sum_{j=1}^\infty \int_{2^j(\frac{3}{2})^{(k-k_0)p/(n\bar{p})}\sqrt{n}l_{ki} \leq |x-y_{ki}| < 2^{j+1}(\frac{3}{2})^{(k-k_0)p/(n\bar{p})}\sqrt{n}l_{ki}} \theta\left(\frac{\sqrt{n}l_{ki}}{|x-y_{ki}|}\right)^p \\ &\qquad\qquad \times \frac{1}{|x-y_{ki}|^{np}}\omega(x)\, dx \\ &\quad \leq c2^{kp} \sum_{j=1}^\infty \theta(2^{-j}(2/3)^{(k-k_0)p/(n\bar{p})})^p 2^{-jnp}(2/3)^{(k-k_0)p^2/\bar{p}}\omega(Q_{kij}) \\ &\quad \leq c2^{kp} \sum_{j=1}^\infty \theta(2^{-j}(2/3)^{(k-k_0)p/(n\bar{p})})^p 2^{-jnp}(2/3)^{(k-k_0)p^2/\bar{p}}\omega(Q_{ki})\left(\frac{|Q_{kij}|}{|Q_{ki}|}\right)^{\bar{p}} \\ &\quad \leq c2^{kp}\omega(Q_{ki}) \sum_{j=1}^\infty \theta(2^{-j}(2/3)^{(k-k_0)p/(n\bar{p})})^p 2^{jn(\bar{p}-p)}(2/3)^{(k-k_0)(p-\bar{p})p/\bar{p}}, \end{aligned}$$

where Q_{kij} denotes the cube with side length $2^{j+2}(3/2)^{(k-k_0)p/(n\bar{p})}\sqrt{n}l_{ki}$ and centre y_{ki}. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_{k_0}} |TF_2(x)|^p\omega(x)\,dx \\
&\quad\le \sum_{k=k_0+1}^{\infty}\sum_i \int_{\mathbb{R}^n\setminus \widetilde{Q}_{ki}} |Tb_{k,i}(x)|^p\omega(x)\,dx \\
&\quad\le c\sum_{k=k_0+1}^{\infty} 2^{kp}\sum_i \omega(Q_{ki})\sum_{j=1}^{\infty}\theta(2^{-j}(2/3)^{(k-k_0)p/(n\bar{p})})^p 2^{jn(\bar{p}-p)}(2/3)^{(k-k_0)(p-\bar{p})p/\bar{p}} \\
&\quad\le c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}\sum_{k=k_0+1}^{\infty}\sum_{j=1}^{\infty}\theta(2^{-j}(2/3)^{(k-k_0)p/(n\bar{p})})^p 2^{jn(\bar{p}-p)}(2/3)^{(k-k_0)(p-\bar{p})p/\bar{p}} \\
&\quad\le c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}\sum_{k=k_0+1}^{\infty}(2/3)^{(k-k_0)(p-\bar{p})p/\bar{p}}\sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{1-j}}\frac{\theta(t(2/3)^{(k-k_0)p/(n\bar{p})})^p}{t^{n(\bar{p}-p)+1}}\,dt \\
&\quad\le c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}\int_0^1\frac{\theta(t)^p|\ln t|}{t^{n(\bar{p}-p)+1}}\,dt \\
&\quad\le c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}.
\end{aligned}$$

Hence we have

$$\omega(\{x\in\mathbb{R}^n:\ |TF_2(x)|>\lambda\})\le\frac{c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}}{\lambda^p}. \tag{4.3}$$

Thus (i) follows from (4.2) and (4.3).

(ii) Routine arguments as in (i) above show that

$$\omega(\{x\in\mathbb{R}^n:\ G_{p,\omega}(TF_1)(x)>\lambda\})\le\frac{c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}}{\lambda^p}$$

and $\omega(B_{k_0})\le\dfrac{c\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}}{\lambda^p}$.

Let us now estimate $\int_{\mathbb{R}^n\setminus B_{k_0}}[G_{p,\omega}(TF_2)(x)]^p\omega(x)\,dx$. We assume for the time being that for $x\notin\widetilde{Q}_{ki}$, we have

$$G_{p,\omega}(Tb_{k,i})(x)\le c2^k|Q_{ki}|^{\bar{p}/p}|x-y_{ki}|^{-n\bar{p}/p}|\ln(|x-y_{ki}||Q_{ki}|^{-1/n})|^{-2/p-\epsilon}, \tag{4.4}$$

where c is independent of i and k. We then have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_{k_0}}[G_{p,\omega}(TF_2)(x)]^p\omega(x)\,dx \\
&\quad\le\sum_{k=k_0+1}^{\infty}\sum_i\int_{\mathbb{R}^n\setminus\widetilde{Q}_{ki}}[G_{p,\omega}(Tb_{k,i})(x)]^p\omega(x)\,dx \\
&\quad\le c\sum_{k=k_0+1}^{\infty}2^{kp}\sum_i|Q_{ki}|^{\bar{p}} \\
&\qquad\times\sum_{j=1}^{\infty}\int_{2^j(\frac{3}{2})^{(k-k_0)p/(n\bar{p})}\sqrt{n}l_{ki}\le|x-y_{ki}|<2^{j+1}(\frac{3}{2})^{(k-k_0)p/(n\bar{p})}\sqrt{n}l_{ki}}\frac{1}{|x-y_{ki}|^{n\bar{p}}} \\
&\qquad\quad\times|\ln(|x-y_{ki}||Q_{ki}|^{-1/n})|^{-2-\epsilon p}\omega(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le c \sum_{k=k_0+1}^{\infty} 2^{kp}(2/3)^{(k-k_0)p} \sum_i \sum_{j=1}^{\infty} 2^{-j\bar{p}n} \left(\frac{k-k_0}{n}+j\right)^{-2-\epsilon p} \omega(Q_{kij}) \\
&\le c \sum_{k=k_0+1}^{\infty} 2^{kp} \sum_i \omega(Q_{ki}) \sum_{j=1}^{\infty} \left(\frac{k-k_0}{n}+j\right)^{-2-\epsilon p} \\
&\le c\|f\|_{WH^p_\omega(\mathbb{R}^n)}^p \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left(\frac{k}{n}+j\right)^{-2-\epsilon p} \\
&\le c\|f\|_{WH^p_\omega(\mathbb{R}^n)}^p.
\end{aligned}
$$

Thus (ii) is proved once we establish (4.4). We now prove (4.4). Let $x \notin \widetilde{Q}_{ki}$ and choose any $\varphi \in \mathcal{A}_{p,\omega}$. As in the proofs of (3.16) and (3.17), we have

$$
\begin{aligned}
|Tb_{k,i} * \varphi_t(x)| &= \left| \int_{\mathbb{R}^n} Tb_{k,i}(y) \frac{1}{t^n} \varphi\left(\frac{x-y}{t}\right) dy \right| \\
&= \left| \int_{\mathbb{R}^n} Tb_{k,i}(y) \frac{1}{t^n} \left\{ \varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-y_{ki}}{t}\right) \right\} dy \right| \\
&\le \frac{1}{t^n} \int_{|y-y_{ki}|<2l_{ki}} |Tb_{k,i}(y)| \left| \varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-y_{ki}}{t}\right) \right| dy \\
&\qquad + \frac{1}{t^n} \int_{2l_{ki} \le |y-y_{ki}| < |x-y_{ki}|/2} \cdots + \frac{1}{t^n} \int_{|y-y_{ki}| \ge |x-y_{ki}|/2} \cdots \\
&\equiv Q_1 + Q_2 + Q_3.
\end{aligned}
$$

By the mean value theorem and Hölder's inequality, we have

$$
\begin{aligned}
Q_1 &\le \frac{1}{t^{n+1}} \|Tb_{k,i}\|_{L^{p_0}(\mathbb{R}^n)} \left(\int_{Q_{ki}} \left| \nabla\varphi\left(\frac{x-y_{ki}-\gamma(y-y_{ki})}{t}\right) \right|^{p_0'} |y-y_{ki}|^{p_0'} dy \right)^{1/p_0'} \\
&\le c2^k |Q_{ki}|^{1/p_0} \frac{l_{ki}|Q_{ki}|^{1/p_0'}}{|x-y_{ki}|^{n+1}} \le c2^k \frac{|Q_{ki}|^{1+1/n}}{|x-y_{ki}|^{n+1}} \\
&\le c2^k |Q_{ki}|^{\bar{p}/p} |x-y_{ki}|^{-n\bar{p}/p} |\ln(|x-y_{ki}||Q_{ki}|^{-1/n})|^{-2/p-\epsilon},
\end{aligned}
$$

since $\bar{p} < (1+1/n)p$. Here $\gamma \in (0,1)$, $1/p_0 + 1/p_0' = 1$ and we have used the fact that

$$
\sup_{x \in \mathbb{R}^n} |x|^{n+1} |\nabla\varphi(x)| < \infty
$$

and $|x-y_{ki}-\gamma(y-y_{ki})| > |x-y_{ki}|/2$. Using the same fact, we have

$$
\begin{aligned}
Q_2 &\le \frac{1}{t^{n+1}} \int_{2l_{ki} \le |y-y_{ki}| < |x-y_{ki}|/2} \left(\int_{Q_{ki}} |b_{ki}(z)||K(y,z)-K(y,y_{ki})|\, dz \right) \\
&\qquad \times \left| \nabla\varphi\left(\frac{x-y_{ki}-\gamma(y-y_{ki})}{t}\right) \right| |y-y_{ki}|\, dy \\
&\le \frac{c2^k}{|x-y_{ki}|^{n+1}} \int_{2l_{ki} \le |y-y_{ki}| < |x-y_{ki}|/2} \int_{Q_{ki}} \theta\left(\frac{|z-y_{ki}|}{|y-y_{ki}|}\right) \frac{1}{|y-y_{ki}|^{n-1}}\, dz\, dy \\
&\le \frac{c2^k}{|x-y_{ki}|^{n+1}} \int_{Q_{ki}} \left(\int_{\frac{2|z-y_{ki}|}{|x-y_{ki}|}}^{\frac{|z-y_{ki}|}{2l_{ki}}} \frac{\theta(r)}{r^2}\, dr \right) |z-y_{ki}|\, dz \\
&= \frac{c2^k}{|x-y_{ki}|^{n+1}} \int_{Q_{ki}} \left(\int_{\frac{2|z-y_{ki}|}{|x-y_{ki}|}}^{\frac{|z-y_{ki}|}{2l_{ki}}} \frac{\theta(r)|\ln r|^{2/p+\epsilon}}{r^{n(\bar{p}-p)/p+1}} \frac{|\ln r|^{-2/p-\epsilon}}{r^{1-n(\bar{p}-p)/p}}\, dr \right) |z-y_{ki}|\, dz.
\end{aligned}
$$

Note that if r is sufficiently small, then $\frac{|\ln r|^{-2/p-\epsilon}}{r^{1-n(\bar{p}-p)/p}}$ is non-increasing on r, and we have

$$\begin{aligned} Q_2 &\leq \frac{c2^k}{|x-y_{ki}|^{n\bar{p}/p}} \int_{Q_{ki}} \left| \ln\left(\frac{2|z-y_{ki}|}{|x-y_{ki}|}\right)\right|^{-2/p-\epsilon} |z-y_{ki}|^{n\bar{p}/p-n}\, dz \\ &\leq c2^k|Q_{ki}|^{\bar{p}/p}|x-y_{ki}|^{-n\bar{p}/p}|\ln(|x-y_{ki}||Q_{ki}|^{-1/n})|^{-2/p-\epsilon}. \end{aligned}$$

For Q_3, we have

$$\begin{aligned} Q_3 &\leq \frac{1}{t^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \left(\int_{Q_{ki}} |b_{k,i}(z)||K(y,z)-K(y,y_{ki})|\,dz\right) \\ &\qquad \times \left\{\left|\varphi\left(\frac{x-y}{t}\right)\right| + \left|\varphi\left(\frac{x-y_{ki}}{t}\right)\right|\right\} dy \\ &\leq \frac{c2^k}{t^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \left\{\int_{Q_{ki}} \theta\left(\frac{|z-y_{ki}|}{|y-y_{ki}|}\right) dz\right\} \frac{1}{|y-y_{ki}|^n}\left|\varphi\left(\frac{x-y}{t}\right)\right| dy \\ &\qquad + \frac{c2^k}{t^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \left\{\int_{Q_{ki}} \theta\left(\frac{|z-y_{ki}|}{|y-y_{ki}|}\right) dz\right\} \frac{1}{|y-y_{ki}|^n}\left|\varphi\left(\frac{x-y_{ki}}{t}\right)\right| dy \\ &\equiv Q_{31}+Q_{32}. \end{aligned}$$

For Q_{31}, we have

$$\begin{aligned} Q_{31} &\leq \frac{c2^k}{t^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \left\{\int_{Q_{ki}} \theta\left(\frac{2|z-y_{ki}|}{|x-y_{ki}|}\right) dz\right\} \frac{1}{|y-y_{ki}|^n}\left|\varphi\left(\frac{x-y}{t}\right)\right| dy \\ &\leq \frac{c2^k}{|x-y_{ki}|^n}\int_0^{l_{ki}} \theta\left(\frac{2r}{|x-y_{ki}|}\right) r^{n-1}\,dr \\ &= c2^k \int_0^{\frac{2l_{ki}}{|x-y_{ki}|}} \theta(r) r^{n-1}\,dr \\ &= c2^k \int_0^{\frac{2l_{ki}}{|x-y_{ki}|}} \frac{\theta(r)|\ln r|^{2/p+\epsilon}}{r^{(\bar{p}-p)n/p+1}} r^{n\bar{p}/p}|\ln r|^{-2/p-\epsilon}\,dr \\ &\leq c2^k|Q_{ki}|^{\bar{p}/p}|x-y_{ki}|^{-n\bar{p}/p}|\ln(|x-y_{ki}||Q_{ki}|^{-1/n})|^{-2/p-\epsilon}, \end{aligned}$$

where we use the fact that $r^{n\bar{p}/p}|\ln r|^{-2/p-\epsilon}$ is increasing on r when r is small. Using this fact for Q_{32}, we have

$$\begin{aligned} Q_{32} &\leq \frac{c2^k}{|x-y_{ki}|^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \left\{\int_{Q_{ki}} \theta\left(\frac{|z-y_{ki}|}{|y-y_{ki}|}\right) dz\right\} \frac{1}{|y-y_{ki}|^n}\,dy \\ &\leq \frac{c2^k}{|x-y_{ki}|^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \int_0^{\frac{l_{ki}}{|y-y_{ki}|}} \frac{\theta(r)|\ln r|^{2/p+\epsilon}}{r^{(\bar{p}-p)n/p+1}} r^{n\bar{p}/p}|\ln r|^{-2/p-\epsilon}\,dr\,dy \\ &\leq \frac{c2^k}{|x-y_{ki}|^n}\int_{|y-y_{ki}|\geq|x-y_{ki}|/2} \frac{|Q_{ki}|^{\bar{p}/p}}{|y-y_{ki}|^{n\bar{p}/p}}\left|\ln\left(\frac{|y-y_{ki}|}{|Q_{ki}|^{1/n}}\right)\right|^{-2/p-\epsilon} dy \\ &\leq \frac{c2^k}{|x-y_{ki}|^n}\left|\ln\left(\frac{|x-y_{ki}|}{|Q_{ki}|^{1/n}}\right)\right|^{-2/p-\epsilon} \int_{|w|\geq|x-y_{ki}|/2} \frac{|Q_{ki}|^{\bar{p}/p}}{|w|^{n\bar{p}/p}}\,dw \\ &\leq c2^k|Q_{ki}|^{\bar{p}/p}|x-y_{ki}|^{-n\bar{p}/p}|\ln(|x-y_{ki}||Q_{ki}|^{-1/n})|^{-2/p-\epsilon}. \end{aligned}$$

Thus (4.4) is established and this completes the proof of (ii).

Combining Theorem 6 with Theorem 8, we easily obtain the following boundedness theorem for semi-(θ, p_0)-type Calderón–Zygmund operators.

Theorem 9 *Let $p_0 \in (1, \infty)$ and θ be a non-negative non-decreasing function on $\mathbb{R}^+$. Let T be a semi-(θ, p_0)-type Calderón–Zygmund operator. Let $\bar{p} \in [1, p_0)$ and let $\omega \in A_{\bar{p}}(\mathbb{R}^n)$ satisfy* (4.1). *Then we have:*

(i) *If $\int_0^1 \frac{\theta(t)^p |\ln t|}{t^{(\bar{p}-p)n+1}}\, dt < \infty$, then T is bounded from $WH^p_\omega(\mathbb{R}^n)$ to $WL^p_\omega(\mathbb{R}^n)$ for $0 < p \le 1$;*

(ii) *If $T^*1 = 0$ and $\int_0^1 \frac{\theta(t) |\ln t|^2/p+\epsilon}{t^{(\bar{p}-p)n/p+1}}\, dt < \infty$ for some $\epsilon > 0$, then T is bounded on $WH^p_\omega(\mathbb{R}^n)$ for $n/(n+1) < p \le 1$ and $\bar{p} \in [1, (1+1/n)p)$.*

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