Acta Mathematica Sinica, English Series
April, 2002, Vol.18, No.2, pp. 209–224

Acta Mathematica Sinica, English Series

Two-Dimensional Graphs Moving by Mean Curvature Flow

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Abstract A surface Σ is a graph in $\mathbb{R}^4$ if there is a unit constant 2-form ω on $\mathbb{R}^4$ such that $\langle e_1 \wedge e_2, \omega \rangle \geq v_0 > 0$ where $\{e_1, e_2\}$ is an orthonormal frame on Σ. We prove that, if $v_0 \geq \frac{1}{\sqrt{2}}$ on the initial surface, then the mean curvature flow has a global solution and the scaled surfaces converge to a self-similar solution. A surface Σ is a graph in $M_1 \times M_2$ where M_1 and M_2 are Riemann surfaces, if $\langle e_1 \wedge e_2, \omega_1 \rangle \geq v_0 > 0$ where ω_1 is a Kähler form on M_1. We prove that, if M is a Kähler-Einstein surface with scalar curvature R, $v_0 \geq \frac{1}{\sqrt{2}}$ on the initial surface, then the mean curvature flow has a global solution and it sub-converges to a minimal surface, if, in addition, $R \geq 0$ it converges to a totally geodesic surface which is holomorphic.

Keywords Mean curvature flow, 2-dimensional graphs in $\mathbb{R}^4$, Self-similar solution
2000 MR Subject Classification 53C44, 53C21

1 Introduction

For the classical solution of the mean curvature flow of hypersurfaces, Huisken showed in [1] and [2] that if the initial hypersurface is compact and uniformly convex in a complete manifold with bounded geometry then it converges to a single point under the mean curvature flow in a finite time and the normalized flow (area is fixed) converges to a sphere of that area in infinite (rescaled) time. Ecker and Huisken [3] proved that, if the initial hypersurface is an entire graph, the mean curvature has a long time solution and the solution of the normalized equation

Received July 25, 2001, Accepted October 11, 2001
The research is supported in part by a Sloan fellowship and an NSERC grant for Chen, by a grant from NSF of China for Li, by a grant from NSF of USA for Tian.

converges to a self-similar solution as time goes to infinity. For the higher co-dimensional case, Altschuler [4] and Altschuler-Grayson [5] studied the curvature flow of curves in R^3, they showed (in [5]) that, if the initial curve is a ramp, then the flow has a long time solution which converges to a line at infinity.

We consider the motion of an immersed surface in a 4-dimensional manifold M, $\mathbf{F}_0 : \Sigma \to M$, moving by its mean curvature in M. That is, we consider a one-parameter family $\mathbf{F}_t = \mathbf{F}(\cdot, t)$ of surfaces with corresponding images $\Sigma_t = \mathbf{F}_t(\Sigma)$ such that

$$\begin{cases} \dfrac{d}{dt}\mathbf{F}(x,t) = \mathbf{H}(x,t), \\ \quad \mathbf{F}(x,0) = \mathbf{F}_0(x), \end{cases} \tag{1.1}$$

where $\mathbf{H}(x,t)$ is the mean curvature vector of Σ_t at $\mathbf{F}(x,t)$. The area element of the induced metric $g_{ij} = \langle \nabla_i \mathbf{F}, \nabla_j \mathbf{F} \rangle$ on Σ_t is $\det(g_{ij}) dxdy$. It is well known that

$$\frac{d}{dt}\det(g_{ij}) = -|\mathbf{H}|^2 \det(g_{ij}).$$

Logarithmic integration implies that $\mathbf{F}$ remains immersed as long as the solution of (1.1) exists.

Let Σ be a 2-dimensional oriented surface and let $\mathbf{F}_0 : \Sigma \to \mathbb{R}^4$ be an immersion, and denote $\Sigma_0 = \mathbf{F}_0(\Sigma)$. We say that Σ_0 is a *graph*, if there exists a unit constant 2-form ω in $\mathbb{R}^4$ such that

$$v = \langle e_1 \wedge e_2, \omega \rangle \geq v_0 > 0,$$

for some constant v_0, where $\{e_1, e_2\}$ is an orthonormal frame on Σ_0.

Let ω be a unit constant 2-form in $\mathbb{R}^4$ with respect to which Σ_0 is a graph. Let $v = \langle e_1 \wedge e_2, \omega \rangle$, where e_1, e_2 is a normal frame on Σ_t. Suppose that Σ_0 has bounded curvature. We prove in this paper that if $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$ for all x, then Equation (1.1) has a global solution $\mathbf{F}$. We then consider the rescaled surface $\widetilde{\Sigma}_s$ defined by

$$\widetilde{\mathbf{F}}(\cdot, s) = \frac{1}{\sqrt{2t+1}}\mathbf{F}(\cdot, t),$$

where $s = \frac{1}{2}\log(2t+1)$, $0 \leq s < \infty$. We prove that, if in addition to the above assumption on v_0,

$$|\mathbf{F}^{\perp}|^2 \leq C(1 + |\mathbf{F}|^2)^{1-\delta}$$

on the initial surface Σ_0 for some $C > 0$, $\delta > 0$, then the normalized flow $\widetilde{\Sigma}_s$ converges to a self-similar solution as $s \to \infty$.

Let $M = M_1 \times M_2$ be a Kähler-Einstein manifold, M_1 and M_2 be Riemann surfaces. Let ω_i be a unit Kähler form on M_i for $i = 1, 2$. Let Σ be a 2-dimensional oriented surface and let $\mathbf{F}_0 : \Sigma \to M$ be an immersion, and denote $\Sigma_0 = \mathbf{F}_0(\Sigma)$. In this case, we say that Σ_0 is a *graph* in M, if

$$v = \langle e_1 \wedge e_2, \omega_1 \rangle \geq v_0 > 0,$$

for some constant v_0, where $\{e_1, e_2\}$ is an orthonormal frame on Σ_0.

We also prove in this paper that, if $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$ for all x, then Equation (1.1) has a global solution $\mathbf{F}$ and it sub-converges to a minimal surface, if the scalar curvature of M is nonnegative it converges to a totally geodesic surface which is holomorphic.

Throughout this paper, summation is taken for all repeated indices.

2 Global Existence in $\mathbb{R}^4$

We assume that $\mathbf{F}(x,t)$ satisfies the mean curvature flow equation (1.1). Suppose that $\mathbf{H}$ is the mean curvature vector of the surface $\mathbf{F}(\Sigma,t)$ in M, $\mathbf{A}$ is the second fundamental form and denote the Riemannian metric on M by $\langle\cdot,\cdot\rangle$. In normal coordinates around a point in Σ, the induced metric on Σ_t from $\langle\cdot,\cdot\rangle$ is given by $g_{ij} = \langle \partial_i F, \partial_j F\rangle$ where ∂_i $(i = 1,2)$ are the partial derivatives with respect to the local coordinates. Let Δ and ∇ be the Laplace operator and the covariant derivative for the induced metric on Σ_t, respectively. We choose an orthonormal frame e_1, e_2, v_1, v_2 of M such that e_1, e_2 is a frame of $\Sigma_t = \mathbf{F}(\Sigma,t)$, and v_1, v_2 is a frame of the normal bundle over Σ_t. We can write:

$$\mathbf{A} = A^\alpha v_\alpha, \quad \mathbf{H} = -H^\alpha v_\alpha.$$

Let $A^\alpha = (h^\alpha_{ij})$, where (h^α_{ij}) is a matrix, the trace and the norm of the second fundamental form are

$$H^\alpha = g^{ij}h^\alpha_{ij} = h^\alpha_{ii}, \quad |\mathbf{A}|^2 = \sum_\alpha |A^\alpha|^2 = g^{ij}g^{kl}h^\alpha_{ik}h^\alpha_{jl} = h^\alpha_{ik}h^\alpha_{ik}.$$

The standard parabolic theory implies that (1.1) has a smooth solution for a short time. We state it in the following theorem:

Theorem 2.1 *Suppose that the initial surface Σ_0 has bounded curvature. There exists $T > 0$ such that* (1.1) *has a smooth solution in the time interval* $[0,T)$. *If* $\max_{\Sigma_t} |\mathbf{A}|^2$ *is bounded near* T*, the solution can be extended to* $[0, T+\epsilon)$ *for some* $\epsilon > 0$.

However, in general $\max_{\Sigma_t} |\mathbf{A}|^2$ becomes unbounded as $t \to T$. In this section, we will give a condition to guarantee the global existence of the mean curvature flow (1.1).

In this and the following section, we consider the case where $M = \mathbb{R}^4$.

Let $H(\mathbf{X}, \mathbf{X}_0, t)$ be the backward heat kernel on $\mathbb{R}^4$. Define

$$\rho(\mathbf{X},t) = 4\pi(t_0 - t)H(\mathbf{X},\mathbf{X}_0,t) = \frac{1}{4\pi(t_0-t)}\exp\left(-\frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)}\right),$$

for $t < t_0$. We prove a monotonicity inequality, which was essentially proved by Huisken [6] (also see [3]).

Proposition 2.2 *Suppose that* $\mathbf{F}$ *satisfies Equation* (1.1)*, and that $f(x,t)$ is a smooth function defined $\Sigma \times \mathbb{R}^+$. We have*

$$\frac{\partial}{\partial t}\int_{\Sigma_t} f\rho(\mathbf{F},t)d\mu_t = \int_{\Sigma_t}\left(\frac{df}{dt} - \Delta f\right)\rho(\mathbf{F},t)d\mu_t - \int_{\Sigma_t} f\rho(\mathbf{F},t)\left|\mathbf{H} + \frac{(\mathbf{F}-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2 d\mu_t. \quad (2.1)$$

Proof It is clear that

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Sigma_t} f\rho(\mathbf{F},t)d\mu_t &= \int_{\Sigma_t}\Big(\frac{\partial}{\partial t}f\Big)\rho(\mathbf{F},t)d\mu_t + \int_{\Sigma_t} f\frac{\partial}{\partial t}\rho(\mathbf{F},t)d\mu_t - \int_{\Sigma_t} f\rho(\mathbf{F},t)|\mathbf{H}|^2d\mu_t\\ &= \int_{\Sigma_t}\Big(\Big(\frac{\partial}{\partial t}-\Delta\Big)f\Big)\rho(\mathbf{F},t)d\mu_t + \int_{\Sigma_t} f\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho(\mathbf{F},t)d\mu_t\\ &\quad -\int_{\Sigma_t}\rho(\mathbf{F},t)|\mathbf{H}|^2d\mu_t.\end{aligned}$$

A straight forward computation leads to

$$\frac{\partial}{\partial t}\rho(\mathbf{X},t) = \Big(\frac{1}{t_0-t} - \frac{1}{2(t_0-t)}\langle\mathbf{H},\mathbf{X}-\mathbf{X}_0\rangle - \frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)^2}\Big)\rho(\mathbf{X},t)$$

and

$$\nabla\exp\Big(-\frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)}\Big) = -\exp\Big(-\frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)}\Big)\frac{\langle\mathbf{X}-\mathbf{X}_0,\nabla\mathbf{X}\rangle}{2(t_0-t)}$$

and

$$\begin{aligned}\Delta\exp\Big(-\frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)}\Big) = \exp\Big(-\frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)}\Big)\Big(&\frac{|\langle\mathbf{X}-\mathbf{X}_0,\nabla\mathbf{X}\rangle|^2}{4(t_0-t)^2}\\ &-\frac{\langle\mathbf{X}-\mathbf{X}_0,\Delta\mathbf{X}\rangle}{2(t_0-t)} - \frac{|\nabla\mathbf{X}|^2}{2(t_0-t)}\Big).\end{aligned}$$

Note that in the induced metric on Σ_t, $|\nabla\mathbf{F}|^2 = 2$ and $\Delta\mathbf{F} = \mathbf{H}$, so we have

$$\Big(\frac{\partial}{\partial t}+\Delta\Big)\rho(\mathbf{F},t) = -\Big(\frac{\langle\mathbf{F}-\mathbf{X}_0,\mathbf{H}\rangle}{(t_0-t)} + \frac{|(\mathbf{F}-\mathbf{X}_0)^{\perp}|^2}{4(t_0-t)^2}\Big)\rho(\mathbf{F},t). \tag{2.2}$$

Then the proposition follows.

Using Proposition 2.2, one can show the following maximum principle as Ecker-Huisken did for Corollary 1.1 in [3]:

Proposition 2.3 *Suppose that $f(x,t)$ is a smooth function defined by $\Sigma\times\mathbb{R}^+$, which satisfies the inequality*

$$\frac{\partial f}{\partial t} - \Delta f \le \mathbf{a}\cdot\nabla f,$$

for some vector field $\mathbf{a}$ on Σ_t. If $\mathbf{a}_0 = \sup_{\Sigma\times[0,t_1]}|\mathbf{a}| < \infty$ for some $t_1 > 0$, then

$$\sup_{\Sigma_t} f \le \sup_{\Sigma_0} f,$$

for all $t\in[0,t_1]$.

Note that the function f does not need to be non-negative.

Let ω be a unit constant 2-form in $\mathbb{R}^4$. As before, we set $v = \langle e_1\wedge e_2,\omega\rangle$.

Lemma 2.4 *We have*

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)v = &\sum_{\alpha=1,2}\big((h_{11}^{\alpha})^2 + 2(h_{12}^{\alpha})^2 + (h_{22}^{\alpha})^2\big)v\\ &-\big(2h_{11}^1h_{12}^2 - 2h_{11}^2h_{12}^1 + 2h_{21}^1h_{22}^2 - 2h_{21}^2h_{22}^1\big)\langle v_1\wedge v_2,\omega\rangle.\end{aligned}$$

Proof We first calculate $\frac{\partial}{\partial t}e_i$. We have

$$\begin{aligned}\frac{\partial}{\partial t}e_i &= \left\langle \frac{\partial}{\partial t}e_i, e_j \right\rangle e_j + \left\langle \frac{\partial}{\partial t}e_i, v_\alpha \right\rangle v_\alpha \\ &= \left\langle \frac{\partial}{\partial t}e_i, e_j \right\rangle e_j - \left\langle e_i, \frac{\partial}{\partial t}v_\alpha \right\rangle v_\alpha \\ &= \left\langle \frac{\partial}{\partial t}e_i, e_j \right\rangle e_j - \nabla_i H^\alpha v_\alpha - H^\gamma C^\alpha_{i\gamma} v_\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}v &= \left\langle \frac{\partial}{\partial t}e_1 \wedge e_2, \omega \right\rangle + \left\langle e_1 \wedge \frac{\partial}{\partial t}e_2, \omega \right\rangle \\ &= -\left(\nabla_1 H^\alpha + H^\gamma C^\alpha_{1\gamma}\right)\langle v_\alpha \wedge e_2, \omega\rangle \\ &\quad -\left(\nabla_2 H^\alpha + H^\gamma C^\alpha_{2\gamma}\right)\langle e_1 \wedge v_\alpha, \omega\rangle.\end{aligned}$$

Recall that $\nabla_i e_j = -h^\alpha_{ij} v_\alpha$. We have

$$\nabla_1 v = -\left(h^\alpha_{11}\langle v_\alpha \wedge e_2, \omega\rangle + h^\alpha_{12}\langle e_1 \wedge v_\alpha, \omega\rangle\right),$$

$$\nabla_2 v = -\left(h^\alpha_{21}\langle v_\alpha \wedge e_2, \omega\rangle + h^\alpha_{22}\langle e_1 \wedge v_\alpha, \omega\rangle\right).$$

Then,

$$\begin{aligned}\nabla_1^2 v &= -\nabla_1 h^\alpha_{11}\langle v_\alpha \wedge e_2, \omega\rangle - h^\alpha_{11}\langle \nabla_1 v_\alpha \wedge e_2, \omega\rangle + h^\alpha_{11}h^\beta_{12}\langle v_\alpha \wedge v_\beta, \omega\rangle \\ &\quad - \nabla_1 h^\alpha_{12}\langle e_1 \wedge v_\alpha, \omega\rangle - h^\alpha_{12}\langle e_1 \wedge \nabla_1 v_\alpha, \omega\rangle + h^\alpha_{11}h^\beta_{12}\langle v_\alpha \wedge v_\beta, \omega\rangle \\ &= -\nabla_1 h^\alpha_{11}\langle v_\alpha \wedge e_2, \omega\rangle - (h^\alpha_{11})^2\langle e_1 \wedge e_2, \omega\rangle - h^\gamma_{11}C^\alpha_{1\gamma}\langle v_\alpha \wedge e_2, \omega\rangle \\ &\quad - \nabla_1 h^\alpha_{12}\langle e_1 \wedge v_\alpha, \omega\rangle - (h^\alpha_{12})^2\langle e_1 \wedge e_2, \omega\rangle - h^\gamma_{12}C^\alpha_{1\gamma}\langle e_1 \wedge v_\alpha, \omega\rangle \\ &\quad + 2h^\alpha_{11}h^\beta_{12}\langle v_\alpha \wedge v_\beta, \omega\rangle\end{aligned}$$

and

$$\begin{aligned}\nabla_2^2 v &= -\nabla_2 h^\alpha_{21}\langle v_\alpha \wedge e_2, \omega\rangle - (h^\alpha_{21})^2\langle e_1 \wedge e_2, \omega\rangle - h^\gamma_{21}C^\alpha_{2\gamma}\langle v_\alpha \wedge e_2, \omega\rangle \\ &\quad - \nabla_2 h^\alpha_{22}\langle e_1 \wedge v_\alpha, \omega\rangle - (h^\alpha_{22})^2\langle e_1 \wedge e_2, \omega\rangle - h^\gamma_{22}C^\alpha_{2\gamma}\langle e_1 \wedge v_\alpha, \omega\rangle \\ &\quad + 2h^\alpha_{21}h^\beta_{22}\langle v_\alpha \wedge v_\beta, \omega\rangle.\end{aligned}$$

Noticing that

$$\nabla_1 h^\alpha_{21} = \nabla_2 h^\alpha_{11} + h^\gamma_{11}C^\alpha_{2\gamma} - h^\gamma_{21}C^\alpha_{1\gamma}, \quad \nabla_2 h^\alpha_{12} = \nabla_1 h^\alpha_{22} + h^\gamma_{22}C^\alpha_{1\gamma} - h^\gamma_{12}C^\alpha_{2\gamma},$$

we therefore obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) v &= \sum_{\alpha=1,2}\left((h^\alpha_{11})^2 + 2(h^\alpha_{12})^2 + (h^\alpha_{22})^2\right)\langle e_1 \wedge e_2, \omega\rangle \\ &\quad - \left(2h^1_{11}h^2_{12} - 2h^2_{11}h^1_{12} + 2h^1_{21}h^2_{22} - 2h^2_{21}h^1_{22}\right)\langle v_1 \wedge v_2, \omega\rangle.\end{aligned}$$

This proves the lemma.

Then by Proposition 2.3, we can show the following theorem:

Proposition 2.5 *Let ω be a unit constant 2-form on $\mathbb{R}^4$. If $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$ for all x, then $v(x,t) \geq v_0$ for all $t > 0$ and x.*

Proof It is clear that $\langle e_1 \wedge e_2, \omega\rangle^2 + \langle v_1 \wedge v_2, \omega\rangle^2 \leq 1$. By Lemma 2.4, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) v \geq |\mathbf{A}|^2 \left(v - \sqrt{1-v^2}\right) = |\mathbf{A}|^2 \frac{2v^2 - 1}{v + \sqrt{1-v^2}}.$$

Assume that t_1 is the first point where

$$\inf_{\Sigma_{t_1}} v = v_1, \quad \frac{1}{\sqrt{2}} < v_1 < v_0. \tag{2.3}$$

It is clear that $t_1 > 0$. Hence we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) v \geq 0,$$

in $\Sigma \times [0, t_1]$. Applying Proposition 2.3 to $-v$, we conclude that $v \geq v_0$ in $\Sigma \times [0, t_1]$, which contradicts (2.3).

Theorem 2.6 *Let ω be a unit constant 2-form in $\mathbb{R}^4$ with respect to which Σ_0 is a graph. Suppose that the curvature on Σ_0 is bounded. If $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$ for all $x \in \Sigma_0$, then Equation (1.1) has a global solution $\mathbf{F}$.*

Proof It suffices to show that $\max_{\Sigma_t} |\mathbf{A}|$ is bounded for all $t > 0$. For this purpose, we consider the functions $u_1 = \langle e_1 \wedge e_2, \omega + *\omega\rangle$ and $u_2 = \langle e_1 \wedge e_2, \omega - *\omega\rangle$. By Lemma 2.4, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) u_1 = & \sum_{\alpha=1,2} \left((h^\alpha_{11})^2 + 2(h^\alpha_{12})^2 + (h^\alpha_{22})^2\right) u_1 \\ & - \left(2h^1_{11}h^2_{12} - 2h^2_{11}h^1_{12} + 2h^1_{21}h^2_{22} - 2h^2_{21}h^1_{22}\right) u_1,\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) u_2 = & \sum_{\alpha=1,2} \left((h^\alpha_{11})^2 + 2(h^\alpha_{12})^2 + (h^\alpha_{22})^2\right) u_2 \\ & + \left(2h^1_{11}h^2_{12} - 2h^2_{11}h^1_{12} + 2h^1_{21}h^2_{22} - 2h^2_{21}h^1_{22}\right) u_2.\end{aligned}$$

Applying Proposition 2.5 and the minimum principle, we get

$$u_i(x,t) \geq u_i(x,0) \geq v_0 - \frac{1}{\sqrt{2}} > 0, \quad i = 1, 2,$$

because

$$u_i = \langle e_1 \wedge e_2, \omega\rangle + (-1)^{i+1}\langle v_1 \wedge v_2, \omega\rangle.$$

Setting $u = u_1 \cdot u_2$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 2|\mathbf{A}|^2 u - 2\nabla u_1 \cdot \nabla u_2 = 2|\mathbf{A}|^2 u - 2\frac{\nabla u_1}{u_1} \cdot \nabla u + 2\frac{|\nabla u_1|^2 u}{u_1^2}.$$

Let $\phi = \frac{|\mathbf{A}|^2}{u}$. By Proposition 2.6 in [7], we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\phi &= \frac{1}{u}\left(\frac{\partial}{\partial t} - \Delta\right)|\mathbf{A}|^2 - \frac{|\mathbf{A}|^2}{u^2}\left(\frac{\partial}{\partial t} - \Delta\right)u + 2\nabla|\mathbf{A}|^2 \cdot \frac{\nabla u}{u^2} - 2|\mathbf{A}|^2\frac{|\nabla u|^2}{u^3} \\ &\leq \frac{-2|\nabla|\mathbf{A}||^2}{u} + 2\nabla\phi \cdot \frac{\nabla u}{u} + 2\phi\frac{\nabla u_1}{u_1} \cdot \frac{\nabla u}{u} - 2\phi\frac{|\nabla u_1|^2}{u_1^2} \\ &\leq 2\nabla\phi \cdot \frac{\nabla u}{u} - \nabla\phi \cdot \frac{\nabla u}{u} + 2\phi\frac{\nabla u_1}{u_1} \cdot \frac{\nabla u}{u} - 2\phi\frac{|\nabla u_1|^2}{u_1^2} - \frac{|\nabla\phi|^2}{2\phi} - \frac{\phi|\nabla u|^2}{2u^2} \\ &\leq \nabla\phi \cdot \frac{\nabla u}{u}. \end{aligned} \tag{2.4}$$

By Proposition 2.3, we have $\max_{\Sigma_t}\phi \leq \max_{\Sigma_0}\phi$. Therefore $|\mathbf{A}|$ is uniformly bounded for all t, and this implies the desired result.

3 Asymptotic Behavior

In the following theorem, we give an estimation of the second fundamental form:

Theorem 3.1 *Let ω be a unit constant 2-form in $\mathbb{R}^4$. Suppose that the curvature on Σ_0 is bounded. If $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$, then $\max_{\Sigma_t} t|\mathbf{A}|^2 \leq C$, where $C > 0$ depends on Σ_0.*

Proof We set $\phi = \frac{|\mathbf{A}|^2}{u}$, where $u = u_1 \cdot u_2$ is as defined in the proof of Theorem 2.6. By (2.4), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(t\phi + \frac{1}{u}\right) \leq t\left(\nabla\phi \cdot \frac{\nabla u}{u} + \phi - 2\frac{|\nabla u|^2}{u^3} - \frac{2|\mathbf{A}|^2}{u^2} + 2\frac{\nabla u_1}{u_1} \cdot \frac{\nabla u}{u^2} - 2\frac{|\nabla_1|^2}{u_1^2 u}\right.$$

It follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(t\phi + \frac{1}{u}\right) \leq \frac{\nabla u}{u} \cdot \nabla\left(t\phi + \frac{1}{u}\right).$$

On the other hand, Theorem 2.6 asserts that at any finite time $t_1 > 0$ there exists a positive constant C which may depend on t_1 such that $|\mathbf{A}(x,t_1)|^2 \leq C$ for all x. Moreover, we have seen in the proof of Lemma 2.4 that $|\nabla u|^2 \leq 2|\mathbf{A}|^2$, and from Theorem 2.5, we can see that $u \geq v_0 - \frac{1}{\sqrt{2}} > 0$. Therefore, at any finite time $t_1 > 0$, we have $\sup_x |\frac{\nabla u}{u}|(x,t_1) < \infty$; and then, we conclude the proof of the theorem in view of Proposition 2.3.

The theorem implies that, if the mean curvature flow converges at infinity, it converges to a plane. However, it may move out to infinity. As in [3], we consider the rescaled surface $\widetilde{\Sigma}_s$ defined by

$$\widetilde{\mathbf{F}}(\cdot,s) = \frac{1}{\sqrt{2t+1}}\mathbf{F}(\cdot,t),$$

where $s = \frac{1}{2}\log(2t+1)$, $0 \leq s < \infty$. The normalized equation then becomes

$$\frac{\partial}{\partial s}\widetilde{\mathbf{F}} = \widetilde{\mathbf{H}} - \widetilde{\mathbf{F}}. \tag{3.1}$$

It is clear that

$$\widetilde{v}(x,s) = \langle \widetilde{e}_1 \wedge \widetilde{e}_2, \omega \rangle = v(x,t), \quad |\widetilde{\mathbf{A}}|^2(x,s) = (2t+1)|\mathbf{A}|^2(x,t) \le C,$$

and it follows that

$$\begin{aligned}\left|\widetilde{\mathbf{F}}(x,s) - \frac{1}{\sqrt{2t+1}}\mathbf{F}(x,0)\right| &= \frac{1}{\sqrt{2t+1}}|\mathbf{F}(x,t) - \mathbf{F}(x,0)| \le \frac{1}{\sqrt{2t+1}}\int_0^t \left|\frac{\partial \mathbf{F}}{\partial t}\right| \\ &\le \frac{1}{\sqrt{2t+1}}\int_0^t |\mathbf{H}| \le \frac{1}{\sqrt{2t+1}}\int_0^t |\mathbf{A}| \le C.\end{aligned}$$

So, $\widetilde{\mathbf{F}}$ converges at infinity. In the rest of this section, we will study what equation the limiting surface satisfies.

Theorem 3.2 *Let ω be a unit constant 2-form in $\mathbb{R}^4$. Suppose that the curvature on Σ_0 is bounded. Assume that on the initial surface $v(x,0) \ge v_0 > \frac{1}{\sqrt{2}}$, and for some $C > 0$, $\delta > 0$, $|\mathbf{F}^\perp|^2 \le C(1+|\mathbf{F}|^2)^{1-\delta}$. Then the rescaled surface $\widetilde{\Sigma}_s$ converges to a limiting surface $\widetilde{\Sigma}_\infty$ as $s \to \infty$, and $\widetilde{\Sigma}_\infty$ satisfies the equation*

$$\mathbf{F}_\infty^\perp = -\mathbf{H}_\infty.$$

We begin with some computations. Note that $\mathbf{F}^\perp = \langle \mathbf{F}, v_\alpha \rangle v_\alpha$, where the summation is taken over α.

Lemma 3.3 *We have*

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\langle \mathbf{F}, v_\alpha\rangle v_\alpha|^2 = 2|\mathbf{A}|^2|\langle \mathbf{F}, v_\alpha\rangle v_\alpha|^2 - 4H^\alpha\langle \mathbf{F}, v_\alpha\rangle - 2|\widetilde{\nabla}(\langle \mathbf{F}, v_\alpha\rangle v_\alpha)|^2,$$

and

$$\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)|\widetilde{\mathbf{H}} + \langle \widetilde{\mathbf{F}}, \widetilde{v}_\alpha\rangle\widetilde{v}_\alpha|^2 \le 2\left(|\widetilde{\mathbf{A}}|^2 - 1\right)\left(|\widetilde{\mathbf{H}} + \langle \widetilde{\mathbf{F}}, \widetilde{v}_\alpha\rangle\widetilde{v}_\alpha|^2 - 2|\widetilde{\widetilde{\nabla}}(\widetilde{\mathbf{H}} + \langle \widetilde{\mathbf{F}}, \widetilde{v}_\alpha\rangle\widetilde{v}_\alpha)|^2\right),$$

where $\widetilde{\Delta}$ is the Laplace operator on $\widetilde{\Sigma}_s$, $\widetilde{\widetilde{\nabla}}$ is the covariant differentiation on $\mathrm{Hom}(T\widetilde{\Sigma}_s \times T\widetilde{\Sigma}_s, \mathrm{Nor}\widetilde{\Sigma}_s)$ determined by the covariant differentiation on $T\widetilde{\Sigma}_s$ and D on the normal bundle, D is the normal connection for the immersion $\widetilde{\Sigma}_s \subset \mathbb{R}^4$.

Proof By Lemma 2.2 in [7], we have

$$\frac{\partial}{\partial t}\langle \mathbf{F}, v_\alpha\rangle = -H^\alpha + \left\langle \mathbf{F}, \frac{\partial v_\alpha}{\partial t}\right\rangle = -H^\alpha + \langle \mathbf{F}, \nabla H^\alpha\rangle + H^\gamma C^\alpha_{i\gamma}\langle \mathbf{F}, e_i\rangle + b^\beta_\alpha\langle \mathbf{F}, v_\beta\rangle.$$

It is clear that

$$\nabla_i\langle \mathbf{F}, v_\alpha\rangle = \langle \mathbf{F}, h^\alpha_{ij}e_j\rangle + \langle \mathbf{F}, C^\beta_{i\alpha}v_\beta\rangle.$$

So,

$$\begin{aligned}\Delta\langle \mathbf{F}, v_\alpha\rangle &= H^\alpha - h^\alpha_{ij}h^\beta_{ij}\langle \mathbf{F}, v_\beta\rangle + C^\beta_{i\alpha}C^\gamma_{i\beta}\langle \mathbf{F}, v_\gamma\rangle \\ &\quad + C^\beta_{i\alpha}h^\beta_{ij}\langle \mathbf{F}, e_j\rangle + \nabla_i C^\beta_{i\alpha}\langle \mathbf{F}, v_\beta\rangle + \langle \mathbf{F}, \nabla_i h^\alpha_{ij}e_j\rangle.\end{aligned}$$

Noting that

$$\nabla_i h^\alpha_{jj} = \nabla_j h^\alpha_{ij} + h^\beta_{ij} C^\alpha_{j\beta} - H^\beta C^\alpha_{i\beta},$$

we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\langle \mathbf{F}, v_\alpha\rangle &= -2H^\alpha + h^\alpha_{ij} h^\beta_{ij}\langle \mathbf{F}, v_\beta\rangle - \nabla_i C^\beta_{i\alpha}\langle \mathbf{F}, v_\beta\rangle \\ &\quad - 2C^\beta_{i\alpha} h^\beta_{ij}\langle \mathbf{F}, e_j\rangle - C^\beta_{i\alpha} C^\gamma_{i\beta}\langle \mathbf{F}, v_\gamma\rangle + b^\beta_\alpha \langle \mathbf{F}, v_\beta\rangle.\end{aligned}$$

It is clear that

$$\begin{aligned}|\widetilde{\nabla}(\langle \mathbf{F}, v_\alpha\rangle v_\alpha)|^2 &= \nabla_i \langle \mathbf{F}, v_\beta\rangle \cdot \nabla_i\langle \mathbf{F}, v_\beta\rangle + 2C^\beta_{i\alpha}\nabla_i\langle \mathbf{F}, v_\beta\rangle\langle \mathbf{F}, v_\alpha\rangle \\ &\quad + C^\beta_{i\alpha} C^\beta_{i\gamma}\langle \mathbf{F}, v_\alpha\rangle\langle \mathbf{F}, v_\gamma\rangle \\ &= \nabla_i \langle \mathbf{F}, v_\beta\rangle \cdot \nabla_i\langle \mathbf{F}, v_\beta\rangle + 2h^\beta_{ij} C^\beta_{i\alpha}\langle \mathbf{F}, e_j\rangle\langle \mathbf{F}, v_\alpha\rangle \\ &\quad - C^\beta_{i\alpha} C^\beta_{i\gamma}\langle \mathbf{F}, v_\alpha\rangle\langle \mathbf{F}, v_\gamma\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)|\langle \mathbf{F}, v_\alpha\rangle v_\alpha|^2 &= 2\langle \mathbf{F}, v_\alpha\rangle\left(\frac{\partial}{\partial t} - \Delta\right)\langle \mathbf{F}, v_\alpha\rangle - 2\nabla_l\langle \mathbf{F}, v_\alpha\rangle\nabla_l\langle \mathbf{F}, v_\alpha\rangle \\ &= 2|\mathbf{A}|^2|\langle \mathbf{F}, v_\alpha\rangle v_\alpha|^2 - 4H^\alpha\langle \mathbf{F}, v_\alpha\rangle - 2|\widetilde{\nabla}(\langle \mathbf{F}, v_\alpha\rangle v_\alpha)|^2.\end{aligned}$$

This proves the identity in the lemma.

Using Equation (3.1), we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\langle \widetilde{\mathbf{F}}, \widetilde{v}_\alpha\rangle &= -2\widetilde{H}^\alpha + \widetilde{h}^\alpha_{ij}\widetilde{h}^\beta_{ij}\langle \widetilde{\mathbf{F}}, \widetilde{v}_\beta\rangle - \widetilde{\nabla}_i \widetilde{C}^\beta_{i\alpha}\langle \widetilde{\mathbf{F}}, \widetilde{v}_\beta\rangle \\ &\quad - 2\widetilde{C}^\beta_{i\alpha}\widetilde{h}^\beta_{ij}\langle \widetilde{\mathbf{F}}, \widetilde{e}_j\rangle - \widetilde{C}^\beta_{i\alpha}\widetilde{C}^\gamma_{i\beta}\langle \widetilde{\mathbf{F}}, \widetilde{v}_\gamma\rangle + \widetilde{b}^\beta_\alpha\langle \widetilde{\mathbf{F}}, \widetilde{v}_\beta\rangle - \langle \widetilde{\mathbf{F}}, \widetilde{v}_\alpha\rangle.\end{aligned}$$

By an argument similar to the one used in the proof of Proposition 2.6 in [7], we can obtain:

$$\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\widetilde{H}^\alpha = -\widetilde{H}^\gamma\widetilde{h}^\gamma_{il}\widetilde{h}^\alpha_{il} + \widetilde{H}^\gamma\widetilde{C}^\beta_{i\gamma}\widetilde{C}^\alpha_{i\beta} + \widetilde{H}^\gamma\widetilde{\nabla}_i\widetilde{C}^\alpha_{i\gamma} + 2\widetilde{\nabla}_i\widetilde{H}^\beta\widetilde{C}^\alpha_{i\beta} - \widetilde{H}^\beta\widetilde{b}^\alpha_\beta + \widetilde{H}^\alpha.$$

The inequality in the lemma then follows from a straightforward computation (See the proof of Proposition 2.6 in [7]).

Lemma 3.4 *Suppose that Σ_0 satisfies the hypotheses in Theorem* 3.2*. On $\widetilde{\Sigma}_s$, we have*

$$|\langle \widetilde{\mathbf{F}}, \widetilde{v}_\alpha\rangle\widetilde{v}_\alpha|^2 \le C(s)(1 + |\widetilde{\mathbf{F}}|^2)^{1-\delta},$$

where $C(s)$ depends on s.

Proof Because we need only to show the lemma for a constant $C(s)$ depending on s, it suffices to prove the inequality for $|\langle \mathbf{F}, v_\alpha\rangle v_\alpha|^2$. We set $\eta(x,t) = |\mathbf{F}|^2 + 4t + 1$. We can easily verify that

$(\frac{\partial}{\partial t}-\Delta)\eta=0$. Set $g=|\langle\mathbf{F},v_\alpha\rangle v_\alpha|$ and $\eta_\delta=\eta^{\delta-1}$. Applying Lemma 3.3 and using the fact that $|\widetilde{\mathbf{A}}|\le C$, we obtain:

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)g^2\eta_\delta &= \eta_\delta\left(\frac{\partial}{\partial t}-\Delta\right)g^2+g^2\left(\frac{\partial}{\partial t}-\Delta\right)\eta_\delta-4g\nabla g\cdot\nabla\eta_\delta\\ &\le C\left(g^2+1\right)\eta_\delta+g^2\eta_\delta\left(\frac{1}{\eta_\delta}\left(\frac{\partial}{\partial t}-\Delta\right)\eta_\delta+2\eta_\delta^{-2}|\nabla\eta_\delta|^2\right)\\ &\quad-2\nabla\left(g^2\eta_\delta\right)\cdot\nabla\log\eta_\delta\\ &\le C\left(g^2+1\right)\eta_\delta+g^2\eta_\delta\left((\delta-1)\delta\eta^{-2}|\nabla\eta|^2\right)-2\nabla\left(g^2\eta_\delta\right)\cdot\nabla\log\eta_\delta\\ &\le C\left(g^2\eta_\delta+1\right)-2\nabla\left(g^2\eta_\delta\right)\cdot\nabla\log\eta_\delta,\end{aligned}$$

where we assume that $0<\delta<1$ without loss of generality.

Let $f(x,t)=e^{-Ct}g^2\eta_\delta$. Then

$$\left(\frac{\partial}{\partial t}-\Delta\right)f(x,t)\le-2\nabla f(x,t)\cdot\nabla\log\eta_\delta+Ce^{-Ct}.$$

From Proposition 2.3, we have

$$f(x,t)-Ce^{-Ct}\le\sup_x(f(x,0)-C).$$

Therefore,

$$\begin{aligned}|\langle\mathbf{F},v_\alpha\rangle v_\alpha|^2=g^2&\le(C+\sup_x(f(x,0)-C)e^{Ct})\eta_\delta^{-1}\\ &=(C+\sup_x(f(x,0)-C)e^{Ct})(|\mathbf{F}|^2+4t+1)^{1-\delta}.\end{aligned}$$

This completes the proof.

Proof of Theorem 3.2

$$\varphi=\frac{1}{\widetilde{u}}|\widetilde{\mathbf{H}}+\langle\widetilde{\mathbf{F}},\widetilde{v}_\alpha\rangle\widetilde{v}_\alpha|^2,$$

where $\widetilde{u}=\langle\widetilde{e}_1\wedge\widetilde{e}_2,\omega+*\omega\rangle\cdot\langle\widetilde{e}_1\wedge\widetilde{e}_2,\omega-*\omega\rangle$. It is clear that

$$\left(\frac{\partial}{\partial s}-\widetilde{\Delta}\right)\varphi\le2\varphi+\widetilde{\nabla}\varphi\frac{\widetilde{\nabla}\widetilde{u}}{\widetilde{u}}.$$

For $0<\epsilon<\delta$, let $G(x,s)=(\eta^\alpha(\widetilde{\mathbf{F}}))^{\epsilon-1}e^{\beta s}$, where $\eta^\alpha(\widetilde{\mathbf{F}})=1+\alpha|\widetilde{\mathbf{F}}|^2$. We have

$$\left(\frac{\partial}{\partial s}-\widetilde{\Delta}\right)\eta^\alpha=-2\alpha(|\widetilde{\mathbf{F}}|^2+2),$$

and

$$\left(\frac{\partial}{\partial s}-\widetilde{\Delta}\right)G\le(\beta+2(1-\epsilon)(2\alpha+1))G.$$

It follows that

$$\left(\frac{\partial}{\partial s}-\widetilde{\Delta}\right)\varphi G\le2\varphi G-G\widetilde{\nabla}\varphi\frac{\widetilde{\nabla}\widetilde{u}}{\widetilde{u}}+\varphi\left(\frac{\partial}{\partial s}-\widetilde{\Delta}\right)G-2\widetilde{\nabla}\varphi\widetilde{\nabla}G.$$

Observing that

$$-G\widetilde{\nabla}\varphi\frac{\widetilde{\nabla}\widetilde{u}}{\widetilde{u}} - 2\widetilde{\nabla}\varphi\widetilde{\nabla}G = \widetilde{\nabla}(\varphi G)\left(\frac{\widetilde{\nabla}\widetilde{u}}{\widetilde{u}} + 2\frac{\widetilde{\nabla}G}{G}\right) + \varphi\left(-\frac{\widetilde{\nabla}\widetilde{u}}{\widetilde{u}} + 2\frac{\widetilde{\nabla}G}{G}\right)\widetilde{\nabla}G,$$

and $|\widetilde{\nabla}\widetilde{u}| \le C|\widetilde{\mathbf{A}}|$, and $|\frac{\widetilde{\nabla}G}{G}| \le \sqrt{\alpha}$, by choosing α and β sufficiently small, we have

$$\left(\frac{\partial}{\partial s} - \widetilde{\Delta}\right)\varphi G \le -\widetilde{\nabla}(\varphi G)\left(-\frac{\widetilde{\nabla}\widetilde{u}}{\widetilde{u}} + 2\frac{\widetilde{\nabla}G}{G}\right).$$

By Proposition 2.3, we have

$$\sup_{\widetilde{\Sigma}_s}\frac{\varphi}{(1+\alpha|\widetilde{\mathbf{F}}|^2)^{1-\epsilon}} \le e^{-\beta s}\sup_{\Sigma_0}\frac{\varphi}{(1+\alpha|\widetilde{\mathbf{F}}|^2)^{1-\epsilon}}.$$

Letting $s \to \infty$, we get $\mathbf{F}_\infty^\perp = -\mathbf{H}_\infty$. This proves the theorem.

4 Global Existence in $M_1 \times M_2$

Let $M = M_1 \times M_2$ be a Kähler-Einstein surface, M_1 and M_2 be Riemann surfaces. Let ω_i be a unit Kähler form on M_i for $i = 1, 2$. In this section we consider the global existence of the mean curvature flow (1.1) in M.

Let Σ be a 2-dimensional oriented surface and let $\mathbf{F}_0 : \Sigma \to M$ be an immersion, and denote $\Sigma_0 = \mathbf{F}_0(\Sigma)$. We say that Σ_0 is a *graph* in M, if $v = \langle e_1 \wedge e_2, \omega_1\rangle \ge v_0 > 0$ for some constant v_0, where $\{e_1, e_2\}$ is an orthonormal frame on Σ_0.

Let $\Sigma_t = \mathbf{F}_0(\Sigma, t)$. Let

$$J_1 = |h_{11}^2 + h_{12}^1|^2 + |h_{21}^2 + h_{22}^1|^2 + |h_{12}^2 - h_{11}^1|^2 + |h_{22}^2 - h_{21}^1|^2,$$

and

$$J_2 = |h_{11}^2 - h_{12}^1|^2 + |h_{21}^2 - h_{22}^1|^2 + |h_{12}^2 + h_{11}^1|^2 + |h_{22}^2 + h_{21}^1|^2.$$

Note that $J_1 + J_2 = 2|\mathbf{A}|^2$. We set $u_1 = \langle e_1 \wedge e_2, \omega_1 + \omega_2\rangle$ and $u_2 = \langle e_1 \wedge e_2, \omega_1 - \omega_2\rangle$.

Proposition 4.1 [8] *Assume that $M = M_1 \times M_2$ is a Kähler-Einstein surface with constant scalar curvature R, M_1 and M_2 are Riemann surfaces. Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right)u_1 = J_1 u_1 + R(1 - u_1^2)u_1,$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)u_2 = J_2 u_2 + R(1 - u_2^2)u_2.$$

The first identity was proved in [7] (Proposition 3.2). Instead of considering the orientation $\{e_1, e_2, v_1, v_2\}$, we consider the orientation $\{e_1, e_2, v_1, -v_2\}$, and using Proposition 3.2 in [7], we obtain the second identity.

As a consequence, we have:

Proposition 4.2 *If $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$ for all $x \in \Sigma_0$, then $u_i(x,t) \geq e^{-Ct}(v_0 - \frac{1}{\sqrt{2}})$ for all $t > 0$, $x \in \Sigma$, and $i = 1, 2$, where $C = \max\{-R, 0\}$, R is the scalar curvature of M.*

Proof By Proposition 4.1, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) e^{Ct} u_i = e^{Ct}(J_i + C + R(1 - u_i^2))u_i, \quad i = 1, 2.$$

Note that $(J_i + C + R(1 - u_i^2)) \geq 0$ for $i = 1, 2$; applying the minimum principle, we conclude that

$$e^{Ct} u_i(x,t) \geq \min_{x \in \Sigma} u_i(x,0) > 0, \quad i = 1, 2.$$

This proves the proposition.

Theorem 4.3 *Assume that $M = M_1 \times M_2$ is a Kähler-Einstein surface with constant scalar curvature R, M_1 and M_2 are Riemann surfaces, Σ_0 is a graph. If $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$ for all $x \in \Sigma_0$, then Equation* (1.1) *has a global solution* $\mathbf{F}$*. If $R \geq 0$, $|\mathbf{A}|$ is uniformly bounded for all t.*

Proof Set $u = u_1 \cdot u_2$, and by Proposition 4.2, $u > 0$. By Proposition 4.1, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) u &\geq 2|\mathbf{A}|^2 u - 2\nabla u_1 \cdot \nabla u_2 - Cu \\ &\geq 2|\mathbf{A}|^2 u - 2\frac{\nabla u_1}{u_1} \cdot \nabla u + 2\frac{|\nabla u_1|^2 u}{u_1^2} - Cu,\end{aligned}$$

where $C = \max\{-R, 0\}$, R is the scalar curvature of M. Let $\phi = \frac{|\mathbf{A}|^2}{u}$. By Proposition 2.6 in [7], we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\phi &= \frac{1}{u}\left(\frac{\partial}{\partial t} - \Delta\right)|\mathbf{A}|^2 - \frac{|\mathbf{A}|^2}{u^2}\left(\frac{\partial}{\partial t} - \Delta\right)u + 2\nabla|\mathbf{A}|^2 \cdot \frac{\nabla u}{u^2} - 2|\mathbf{A}|^2\frac{|\nabla u|^2}{u^3} \\ &\leq \frac{-2|\nabla|\mathbf{A}||^2}{u} + 2\nabla\phi \cdot \frac{\nabla u}{u} + 2\phi\frac{\nabla u_1}{u_1} \cdot \frac{\nabla u}{u} - 2\phi\frac{|\nabla u_1|^2}{u_1^2} + C\phi \\ &\leq 2\nabla\phi \cdot \frac{\nabla u}{u} - \nabla\phi \cdot \frac{\nabla u}{u} + 2\phi\frac{\nabla u_1}{u_1} \cdot \frac{\nabla u}{u} \\ &\quad - 2\phi\frac{|\nabla u_1|^2}{u_1^2} - \frac{|\nabla\phi|^2}{2\phi} - \frac{\phi|\nabla u|^2}{2u^2} + C\phi \\ &\leq \nabla\phi \cdot \frac{\nabla u}{u} + C\phi.\end{aligned}$$

By the maximum principle, we have $\max_{\Sigma_t} \phi \leq e^{Ct} \max_{\Sigma_0} \phi$. So $|\mathbf{A}|$ is bounded for all finite time t, and Equation (1.1) has a global solution $\mathbf{F}$. If $C = 0$, $|\mathbf{A}|$ is uniformly bounded for all t.

5 Convergence at Infinity

In this section, we consider the convergence of the mean curvature flow. We do not require the ambient space M has a product structure in Theorem 5.2.

Theorem 5.1 *Assume that $M = M_1 \times M_2$ is a Kähler-Einstein surface with nonnegative scalar curvature R, M_1 and M_2 are Riemann surfaces, Σ_0 is a graph. If $v(x, 0) \geq v_0 > \frac{1}{\sqrt{2}}$, then the global solution $\mathbf{F}(\cdot, t)$ of Equation (1.1) converges to $\mathbf{F}_\infty$ in C^2 as $t \to \infty$ and $\Sigma_\infty = \mathbf{F}_\infty(\Sigma)$ is totally geodesic.*

Proof By Theorem 4.3, we know that $|\mathbf{A}| \leq C$ for all $t > 0$ and $x \in \Sigma$. It follows that $\mathbf{F}(\cdot, t)$ converges to $\mathbf{F}_\infty$ in C^2 as $t \to \infty$. Since

$$\frac{\partial}{\partial t}\int_{\Sigma_t} d\mu_t = -\int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t,$$

we have

$$\mu_t(\Sigma_t) \leq \mu_0(\Sigma_0) \text{ and } \int_0^\infty \int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t dt \leq \mu_0(\Sigma_0).$$

By Proposition 4.1, we have

$$\frac{\partial}{\partial t}\int_{\Sigma_t} u_1 d\mu_t \geq \int_{\Sigma_t} u_1 J_1 d\mu_t - \int_{\Sigma_t} u_1 |\mathbf{H}|^2 d\mu_t,$$

and

$$\frac{\partial}{\partial t}\int_{\Sigma_t} u_2 d\mu_t \geq \int_{\Sigma_t} u_2 J_2 d\mu_t - \int_{\Sigma_t} u_2 |\mathbf{H}|^2 d\mu_t.$$

Integration in t implies that

$$2\mu_0(\Sigma_0) + 2\int_0^\infty \int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t dt \geq \min_{i,x} u_i(x, 0) \int_0^\infty \int_{\Sigma_t} 2|\mathbf{A}|^2 d\mu_t dt,$$

and then

$$\int_0^\infty \int_{\Sigma_t} |\mathbf{A}|^2 d\mu_t dt < \infty.$$

So, there is a sequence $t_i \to \infty$, such that

$$\int_{\Sigma_{t_i}} |\mathbf{A}|^2 d\mu_{t_i} \to 0 \text{ as } i \to \infty.$$

It follows that $|\mathbf{A}_\infty| = 0$, that is, Σ_∞ is totally geodesic. This proves the theorem.

Theorem 5.2 *Let M be a Kähler-Einstein surface. Suppose that the smooth solution of the mean curvature flow (1.1) exists on $[0, \infty)$. Then there is a sequence of $t_i \to \infty$ such that Σ_{t_i} converges to a minimal surface possibly with finitely many singularities. Outside the singularity set of the minimal surface, the convergence is in C^2. If the scalar curvature of M is non-negative, the minimal surface is a holomorphic curve.*

Proof By the Gauss equation

$$R_{1212} = K_{1212} + (h_{11}^\alpha h_{22}^\alpha - h_{12}^\alpha h_{12}^\alpha),$$

we get

$$(H^\alpha - h^\alpha_{22})h^\alpha_{22} - (h^\alpha_{12})^2 = -K_{1212} + R_{1212},$$

and

$$(H^\alpha - h^\alpha_{11})h^\alpha_{11} - (h^\alpha_{12})^2 = -K_{1212} + R_{1212}.$$

Adding the last two identities, one gets

$$|\mathbf{A}|^2 = |\mathbf{H}|^2 - 2R_{1212} + 2K_{1212}.$$

We therefore have

$$\int_{\Sigma_t} |\mathbf{A}|^2 d\mu_t \le \int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t + C\mu_t(\Sigma_t) + 4g - 4,$$

where g is the genus of the initial surface Σ_0. Because Σ_t is a continuous deformation of Σ_0, so its genus is also g. Since

$$\frac{\partial}{\partial t}\int_{\Sigma_t} d\mu_t = -\int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t,$$

we have

$$\mu_t(\Sigma_t) \le \mu_0(\Sigma_0) \text{ and } \int_0^\infty \int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t dt \le \mu_0(\Sigma_0).$$

So,

$$\int_{\Sigma_t} |\mathbf{A}|^2 d\mu_t \le \int_{\Sigma_t} |\mathbf{H}|^2 d\mu_t + C,$$

and there is a sequence $t_i \to \infty$, such that

$$\int_{\Sigma_{t_i}} |\mathbf{H}|^2 d\mu_{t_i} \to 0 \text{ as } i \to \infty. \tag{5.1}$$

It follows that

$$\int_{\Sigma_{t_i}} |\mathbf{A}|^2 d\mu_{t_i} \le C. \tag{5.2}$$

Suppose that Σ_{t_i} blows up around a point $p \in M$. We have

$$\lambda_i^2 = \max_{\Sigma_{t_i} \cap \overline{B}^M_r(p)} |\mathbf{A}|^2 \to \infty.$$

Assume that $\lambda_i = |\mathbf{A}(x_i)|$ and that $\mathbf{F}(x_i, t_i) \to p$ as $i \to \infty$. Considering the blow-up sequence

$$\mathbf{F}_i = \lambda_i(\mathbf{F}(x + x_i, t_i) - \mathbf{F}(x_i, t_i)),$$

we can see that $\mathbf{F}_i \to \mathbf{F}_\infty$ as $i \to \infty$ and $\mathbf{F}_\infty$ is a minimal surface in $\mathbb{R}^4$ with $|\mathbf{A}| \le |\mathbf{A}(0)| = 1$.

Lemma 5.3 *There is an absolute constant ϵ_0 such that for all the minimal surfaces Σ in $\mathbb{R}^4$ with $|\mathbf{A}| \le |\mathbf{A}(0)| = 1$, we have*

$$\int_{B^4_1(0)\cap\Sigma} |\mathbf{A}|^2 d\mu \ge \epsilon_0.$$

Proof Otherwise, there are Σ^k with $|\mathbf{A}^k| \leq |\mathbf{A}^k(0)| = 1$, $\mathbf{H}^k = 0$, and

$$\int_{B_1^4(0)\cap\Sigma} |\mathbf{A}^k|^2 d\mu \to 0.$$

By Proposition 2.6 in [7], we have

$$\Delta|\mathbf{A}^k|^2 = -2|\widetilde{\nabla}\mathbf{A}^k|^2 + 2|\mathbf{A}^k|^4 \leq 2|\mathbf{A}^k|^2.$$

By the mean value inequality, we obtain

$$1 = |\mathbf{A}^k(0)| \leq C\int_{B_1^2(0)} |\mathbf{A}^k|^2 \leq C\int_{B_1^4(0)\cap\Sigma} |\mathbf{A}^k|^2 \to 0.$$

This proves the lemma.

By Lemma 5.3, we have

$$\epsilon_0 \leq \int_{B_1^4(0)\cap\Sigma_i} |\mathbf{A}^i|^2 d\mu_i = \int_{B^4_{\lambda_i^{-2}}(0)\cap\Sigma_{t_i}} |\mathbf{A}|^2 d\mu_{t_i}.$$

By (5.2), one can see that the blow-up set is at most a finite set of points. We can see from (5.1) that Σ_∞ is a minimal surface.

By Proposition 3.2 in [7], we have

$$\frac{\partial}{\partial t}\int_{\Sigma_t} \cos\alpha d\mu_t = \int_{\Sigma_t} \cos\alpha|\nabla^M J_t|^2 d\mu_t + \int_{\Sigma_t} R\sin^2\alpha\cos\alpha d\mu_t - \int_{\Sigma_t} \cos\alpha|\mathbf{H}|^2 d\mu_t,$$

where

$$|\nabla^M J_t|^2 = |h_{11}^2 + h_{12}^1|^2 + |h_{21}^2 + h_{22}^1|^2 + |h_{12}^2 - h_{11}^1|^2 + |h_{22}^2 - h_{21}^1|^2$$

for the second fundamental form h_{ij}^α of Σ_t in M. However,

$$\int_{\Sigma_t} \cos\alpha d\mu_t = \int_{\Sigma_t} \omega$$

is constant under the continuous deformation in t since ω is closed. Therefore, if $R \geq 0$ we have

$$\int_{\Sigma_t} \cos\alpha d\mu_t \leq \int_{\Sigma_t} \cos\alpha|\mathbf{H}|^2 d\mu_t.$$

By (5.1), we then obtain

$$\int_{\Sigma_{t_i}} |\nabla^M J_{t_i}|^2 d\mu_{t_i} \to 0 \text{ as } i \to \infty.$$

So, Σ_∞ is a holomorphic curve. This proves the theorem.

If a minimal surface is a graph, we know that it is smooth ([9], Theorem 7.2, see also [10], Theorem 4.2). So, we have the following corollary:

Corollary 5.4 *Assume that $M = M_1 \times M_2$ is a Kähler-Einstein surface, M_1 and M_2 are Riemann surfaces, Σ_0 is a graph. If $v(x,0) \geq v_0 > \frac{1}{\sqrt{2}}$, then the global solution $\mathbf{F}(\cdot,t)$ of the* *equation* (1.1) *sub-converges to* $\mathbf{F}_\infty$ *in* C^2 *as* $t \to \infty$*, possibly outside a finite set of points, and* $\Sigma_\infty = \mathbf{F}_\infty(\Sigma)$ *is a smooth minimal surface.*

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