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A Minimizing Property of Lagrangian Solutions

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Abstract In this paper, we prove that the Lagrangian solutions to the three-body problem minimize the action function.

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It is known that the Keplerian orbits minimize the Lagrangian action of the two-body problem (see [1, 2 and 3]), and in this short paper, we will show that the Lagrangian solutions [4] to the three-body problem also minimize the action functional. Note the fact that circular Lagrangian solutions minimize the action functional on the zero mean loop space has already been known (see [5, 6]).

For a given choice of the masses $(m_1, m_2, m_3) \in \mathbb{R}^3_+$, the configuration space of the three-body problem in $\mathbb{C}$ is given by

$$F = \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3; \sum_{i=1}^{3} m_i x_i = 0, \text{ and } x_i \neq x_j, i \neq j \right\}.$$

The loop space $\mathfrak{M}$ we will deal with is the $W^{1,2}$ completion of the following smooth loop space:

$$\begin{aligned} \mathfrak{C} \;\; = \;\; & \{q(t) = (q_1(t), q_2(t), q_3(t)) \in C^\infty(\mathbb{R}/T\mathbb{Z}, F); \\ & \deg(q_2 - q_1) \neq 0, \deg(q_3 - q_2) \neq 0, \text{ and } \deg(q_1 - q_3) \neq 0\}, \end{aligned}$$

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and the Lagrangian action is given by

$$f(q) = \frac{1}{2}\int_0^T \sum_{i=1}^{3} m_i|\dot{q}_i|^2 dt + \int_0^T \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|} dt.$$

The solution to the three-body problem is a critical point of the Lagrangian f on $\mathfrak{M}$. The main result in this paper is the following theorem:

Theorem 1 *The minimal regular solutions to the three-body problem in $\mathfrak{M}$ are precisely the Lagrangian elliptical solutions.*

We start with a brief review of Keplerian orbits, Gordon's result [2] and Lagrangian solutions.

The Keplerian orbits are the regular periodic solutions to the equation

$$\ddot{x}(t) = -\frac{x(t)}{|x(t)|^3}, \text{ where } x(t) \in C^\infty(\mathbb{R}, \mathbb{C}).$$

It is easy to verify that the energy $E = \frac{1}{2}|\dot{x}|^2 - |x|^{-1}$ and the angular momentum $G = x \times \dot{x}$ are constants of the motion (see [1] or [3]). In polar coordinates, the Kepler orbits can be written as $x(t) = r(t)\exp(\sqrt{-1}\alpha(t))$, and $r(t)$ and $\alpha(t)$ satisfy the equation

$$r(t) = \frac{G^2}{1 + \sqrt{1 + 2EG^2}\cos(\alpha(t) - \beta)}, \text{ and } G = r^2(t)\dot{\alpha}(t).$$

The curve is an ellipse with eccentricity $\sqrt{1 + 2EG^2}$, semimajor axis $(-2E)^{-1}$, and the period of the close orbit is $T = 2\pi(-2E)^{-3/2}$.

The Keplerian orbits are solutions to the two-body problem. Let $\mathfrak{N}$ be the $W^{1,2}$ completion of the loop space

$$\{x(t) \in C^\infty(\mathbb{R}/T\mathbb{Z}, \mathbb{C}), x(t) \neq 0 \text{ and } \deg x \neq 0\};$$

and the action functional is defined by

$$f_1(x) = \int_0^T \left(\frac{|\dot{x}(t)|^2}{2} + \frac{1}{|x(t)|}\right) dt.$$

The critical point of the functional f_1 in $\mathfrak{N}$ is called the solution to the two-body problem.

Gordon [2] proved the following theorem:

Theorem 2 *The minimal regular solutions to the two-body problem in $\mathfrak{N}$ are precisely the Keplerian orbits, and the minimum of the action functional f_1 equals $A = (3\pi)(T/2\pi)^{1/3}$.*

For the three-body problem, there are well-known solutions which were discovered by Lagrange [4] in 1772. For three masses $(m_1, m_2, m_3) \in \mathbb{R}^3_+$ $(m_1 + m_2 + m_3 = 1)$, put the masses at the vertices of an equilateral triangle $\{t_1, t_2, t_3\} \subset \mathbb{C}$ of side 1 such that $\sum_i m_i t_i = 0$, and pick a Keplerian orbit $x(t)$; then $q(t) = x(t)(t_1, t_2, t_3)$ is called a Lagrangian solution. To see that the Lagrangian solution $q(t)$ is a solution to the three-body problem, we have the following lemma:

Lemma 3 *Lagrangian solutions to the three-body problem minimize the action functional f in $\mathfrak{M}$.*

Proof Since $\sum m_i q_i = 0$, then $\sum m_i \dot{q}_i = 0$ and then

$$\begin{aligned}\sum_{i<j} m_i m_j |\dot{q}_i - \dot{q}_j|^2 &= \frac{1}{2}\sum_{i\neq j} m_i m_j |\dot{q}_i - \dot{q}_j|^2 \\ &= \frac{1}{2}\sum_{i\neq j} m_i m_j (|\dot{q}_i|^2 + |\dot{q}_j|^2 - 2\langle \dot{q}_i, \dot{q}_j\rangle) \\ &= \sum_{i=1}^{3} m_i |\dot{q}_i|^2 \sum_{j\neq i} m_j - \sum_{i=1}^{3}\sum_{j\neq i} m_i m_j \langle \dot{q}_i, \dot{q}_j\rangle \\ &= \sum_{i=1}^{3} m_i |\dot{q}_i|^2 \sum_{j} m_j - \left\langle \sum_i m_i \dot{q}_i, \sum_j m_j \dot{q}_j \right\rangle \\ &= M \sum_i m_i |\dot{q}_i|^2 = \sum_i m_i |\dot{q}_i|^2,\end{aligned}$$

where $M = \sum m_i = 1$ is the total mass. Thus

$$f(q) = \sum_{i<j} m_i m_j \int_0^T \left(\frac{|\dot{q}_i - \dot{q}_j|^2}{2} + \frac{1}{|q_i - q_j|} \right) dt.$$

For each $i < j$, by Theorem 2, we have that

$$\int_0^T \left(\frac{|\dot{q}_i - \dot{q}_j|^2}{2} + \frac{1}{|q_i - q_j|} \right) dt \geq 3\pi \left(\frac{T}{2\pi} \right)^{1/3},$$

and the integral attains its minimum at Keplerian orbit $q_i(t) - q_j(t) = x(t)(t_i - t_j)$. This shows that the action functional f attains its minimum $3\pi(T/2\pi)^{1/3}\sum_{i<j} m_i m_j$ at the Lagrangian solution $q(t) = x(t)(t_1, t_2, t_3)$, and thus we complete the proof of the lemma.

Now we are in a position to prove Theorem 1. To prove Theorem 1, we need to verify that any regular minimal solution to the three-body problem is a Lagrangian solution.

Proof of Theorem 1 Suppose $q(t) = (q_1(t), q_2(t), q_3(t))$ is a regular minimum solution to the three-body problem. By Theorem 2 and the proof of Lemma 3, $x_1(t) = q_2(t) - q_3(t), x_2(t) = q_3(t) - q_1(t)$ and $x_3(t) = q_1(t) - q_2(t)$ are Kepler orbits of period T. So

$$\ddot{x}_i(t) = -\frac{x_i(t)}{|x_i(t)|^3}.$$

It is clear that

$$x_1(t) + x_2(t) + x_3(t) = 0.$$

This implies that $\ddot{x}_1(t) + \ddot{x}_2(t) + \ddot{x}_3(t) = 0$, i.e.

$$\frac{x_1(t)}{|x_1(t)|^3} + \frac{x_2(t)}{|x_2(t)|^3} + \frac{x_3(t)}{|x_3(t)|^3} = 0.$$

If, for some $t_0 \in \mathbb{R}/T\mathbb{Z}$, the three points $q_1(t_0), q_2(t_0)$ and $q_3(t_0)$ are not the vertices of an equilateral triangle, it must be collinear, and then collinear for all $t \in \mathbb{R}/T\mathbb{Z}$.

To see this, we assume that $|x_2(t_0)| \neq |x_3(t_0)|$ and the real part of $x_1(t_0) = 0$. Then

$$\Re e(x_2(t_0)) + \Re e(x_3(t_0)) = 0, \text{ and } \frac{\Re e(x_2(t_0))}{|x_2(t_0)|^3} + \frac{\Re e(x_3(t_0))}{|x_3(t_0)|^3} = 0,$$

and so $\Re e(x_2(t_0)) = \Re e(x_3(t_0)) = 0$.

In fact $q_1(t), q_2(t)$ and $q_3(t)$ cannot be collinear, since we know that the three Kepler orbits $x_1(t), x_2(t)$ and $x_3(t)$ have the same semimajor axes.

So now we know that $q(t)$ is of the following form:

$$q(t) = r(t)(q_1(0), q_2(0), q_3(0)), \text{ deg} r \neq 0,$$

where $(q_1(0), q_2(0), q_3(0))$ are the vertices of an equilateral triangle and $\sum_i m_i q_i(0) = 0$. The action functional f must also attain its minimum of f at $q(t)$ on the space

$$\left\{ r(t)(q_1(0), q_2(0), q_3(0)), \deg r \neq 0, \sum_i m_i q_i(0) = 0 \right\}.$$

Restricting f on this space, then we have

$$f(r(t)) = \int_0^T \left(\sum_i m_i |q_i(0)|^2 \frac{|\dot{r}(t)|^2}{2} + \sum_{i<j} \frac{m_i m_j}{|q_i(0) - q_j(0)|} \frac{1}{|r(t)|} \right) dt.$$

So $r(t)$ is a Kepler orbit and then $q(t)$ is a Lagrangian solution. This completes the proof of the theorem.

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