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Moving Symplectic Curves in Kähler-Einstein Surfaces

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Abstract We derive a parabolic equation for the Kähler angle of a real surface evolving under the mean curvature flow in a Kähler-Einstein surface and show that a symplectic curve remains symplectic with the flow.

Keywords Mean curvature flow, Kähler angle, Symplectic curve
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1 Minimal Surfaces

Let M be a compact Kähler-Einstein surface and Σ an orientable compact surface without boundary which is immersed in M. It is well known that one can define the so-called Kähler angle α at each point of Σ, which carries fundamental geometric information of Σ in M. In particular, Σ is a holomorphic curve if $\alpha \equiv 0$, it is a Lagrangian curve if $\alpha \equiv \frac{\pi}{2}$ and is a symplectic curve if $0 \le \alpha < \frac{\pi}{2}$. When Σ is a minimal surface in a Kähler-Einstein surface, this angle α satisfies a nonlinear elliptic equation (cf. [1]). There are many consequences of this nonlinear equation. For instance, by the maximum principle, we can easily show that if Σ is a minimal and symplectic surface in a Kähler-Einstein surface with positive scalar curvature, then Σ must be holomorphic. Moreover, in [1] Wolfson proved

Theorem 1.1 *Let M be a Kähler-Einstein surface with scalar curvature R and Σ a compact totally real branched minimal surface without boundary which is smoothly immersed in M. Then*

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(1) Σ is Lagrangian when $R<0$, (2) Σ is holomorphic with respect to some complex structure of M when M is K3.

In [2], the authors obtain the following:

Theorem 1.2 *Let Σ be an admissible minimal surface in a Kähler-Einstein surface with non-positive scalar curvature. If $c_1(\Sigma)=\chi(T\Sigma)+\chi(N\Sigma)$ then Σ is either symplectic or Lagrangian.*

In this paper, we study the behavior of the Kähler angle of immersed surfaces under the mean curvature flow, and derive a few corollaries. In particular, we shall show that a symplectic curve will remain symplectic with the evolution.

2 Moving Surfaces by the Mean Curvature Flow

Consider a smooth family of immersions $F:\Sigma\times[0,T)\to M$. We would like to describe the Kähler angle associated with each point of $\Sigma_t=F(\Sigma,t)$. Let ω_1 and ω_2 be $(1,0)$-forms on M such that the Kähler metric of M is

$$ds_M^2=\omega_1\overline{\omega}_1+\omega_2\overline{\omega}_2. \tag{2.1}$$

The induced metric $ds^2_{\Sigma_t}$ on Σ_t is then

$$\phi\overline{\phi}=\omega_1\overline{\omega}_1+\omega_2\overline{\omega}_2. \tag{2.2}$$

Notice that (2.1) determines ω_1 and ω_2 up to unitary transformations. As in [3], we can choose ω_1 and ω_2 such that

$$\omega_1=s_1\phi,\quad \omega_2=s_2\overline{\phi}, \tag{2.3}$$

where s_1 and s_2 are complex-valued functions on Σ_t. Substituting (2.3) into (2.2) yields

$$|s_1|^2+|s_2|^2=1. \tag{2.4}$$

By unitary transformations, we may further assume that when restricted to Σ_t

$$\omega_1=\cos\frac{\alpha}{2}\phi,\quad \omega_2=\sin\frac{\alpha}{2}\overline{\phi}, \tag{2.5}$$

where α is a continuous function on Σ_t with values between 0 and π. A point p on Σ_t is called *complex* if $\alpha(p)=0$, *anti-complex* if $\alpha(p)=\pi$ and *Lagrangian* if $\alpha(p)=\frac{\pi}{2}$. The Kähler form of M restricted on Σ_t is then

$$\omega_M=\frac{\sqrt{-1}}{2}(\omega_1\wedge\overline{\omega}_1+\omega_2\wedge\overline{\omega}_2)|_{\Sigma_t}=\frac{\sqrt{-1}}{2}\cos\alpha\,\phi\wedge\overline{\phi}. \tag{2.6}$$

Note that $\frac{\sqrt{-1}}{2}\phi\wedge\overline{\phi}$ is the area element of Σ_t.

To get the underlying Riemannian structure into the picture, we set

$$\psi_1=\overline{s}_1\omega_1+s_2\overline{\omega}_2=\theta_1+\sqrt{-1}\theta_2, \tag{2.7}$$

$$\psi_2 = \overline{s}_2\omega_1 - s_1\overline{\omega}_2 = \theta_4 + \sqrt{-1}\theta_3. \tag{2.8}$$

Then $\theta_1, \ldots, \theta_4$ form an orthonormal coframe of M. They define a positive orientation. One can see this as follows:

$$\psi_1 \wedge \overline{\psi}_1 \wedge \psi_2 \wedge \overline{\psi}_2 = (s_1\overline{s}_1 - s_2\overline{s}_2)^2\omega_1 \wedge \overline{\omega}_1 \wedge \omega_2 \wedge \overline{\omega}_2 = (s_1\overline{s}_1 - s_2\overline{s}_2)^2 dV_M, \tag{2.9}$$

where dV_M is the volume form of M. On the other hand, in terms of θ_i, we have

$$\psi_1 \wedge \overline{\psi}_1 \wedge \psi_2 \wedge \overline{\psi}_2 = (-2\sqrt{-1}\theta_1 \wedge \theta_2) \wedge (-2\sqrt{-1}\theta_4 \wedge \theta_3) = 4\,\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4. \tag{2.10}$$

Let e_i be the dual of θ_i in the tangent bundle TM along Σ_t. Therefore at each point $p \in \Sigma_t$, e_i form an orthonormal basis of T_pM. Moreover, along Σ_t we have

$$\overline{s}_2\,\omega_1 - s_1\overline{\omega}_2 = 0, \tag{2.11}$$

which means that θ_1, θ_2 form an orthonormal basis of $T_p\Sigma_t$ and θ_3, θ_4 form an orthonormal basis of the normal plane $N_p\Sigma_t$. We therefore obtain a canonical basis e_3, e_4 of the normal bundle at each point of Σ_t.

Now we consider a smooth family of mappings $F : \Sigma \times [0, T) \to M$, $0 < T < \infty$, which satisfies the mean curvature flow equation

$$\frac{\partial F}{\partial t} = H, \tag{2.12}$$

where $H = H^1 e_3 + H^2 e_4$ is the mean curvature of Σ_t in M.

We first consider how the pull-back 2-form $F^*\omega$ varies with time t where ω is the Kähler form of M. Let $v^1, ..., v^4$ be the local coordinates of M near Σ_t which are determined by $e_1, ..., e_4$. Hereinafter we use the summation convention for repeated indices and when no confusion with the Kähler angle arises we also use α for the index. Write the Kähler form as

$$\omega = \omega_{\alpha\beta} dv^\alpha \wedge dv^\beta. \tag{2.13}$$

Note that $\omega_{\alpha\beta}$ is antisymmetric

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \tag{2.14}$$

and ω being closed implies

$$\omega_{\alpha\beta,\gamma} + \omega_{\gamma\alpha,\beta} + \omega_{\beta\gamma,\alpha} = 0. \tag{2.15}$$

Lemma 2.1 *Let F be a solution of the mean curvature flow equation* (2.12). *If x, y are local coordinates on Σ, then*

$$\frac{\partial}{\partial t} F^*\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = F^* d\omega(H, \cdot)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right). \tag{2.16}$$

Proof Differentiating

$$F^*\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \omega_{\alpha\beta}\frac{\partial F^\alpha}{\partial x}\frac{\partial F^\beta}{\partial y} \tag{2.17}$$

yields

$$\frac{\partial}{\partial t}F^*\omega\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\omega_{\alpha\beta,\gamma}H^\gamma\frac{\partial F^\alpha}{\partial x}\frac{\partial F^\beta}{\partial y}+\omega_{\alpha\beta}\left(\frac{\partial H^\alpha}{\partial x}\frac{\partial F^\beta}{\partial y}+\frac{\partial H^\beta}{\partial y}\frac{\partial F^\alpha}{\partial x}\right). \tag{2.18}$$

On the other hand, the interior product of H and ω defines a 1-form

$$\omega(H,\cdot)=\omega_{\alpha\beta}H^\alpha dv^\beta. \tag{2.19}$$

It follows that

$$d\omega(H,\cdot)=\omega_{\alpha\beta,\gamma}H^\alpha dv^\gamma\wedge dv^\beta+\omega_{\alpha\beta}\frac{\partial H^\alpha}{\partial v^\gamma}dv^\gamma\wedge dv^\beta. \tag{2.20}$$

Then we have

$$\begin{aligned}F^*d\omega(H,\cdot)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)&=H^\alpha\frac{\partial F^\gamma}{\partial x}\frac{\partial F^\beta}{\partial y}(\omega_{\alpha\beta,\gamma}-\omega_{\alpha\gamma,\beta})+\omega_{\alpha\beta}\frac{\partial H^\alpha}{\partial x}\frac{\partial F^\beta}{\partial y}-\omega_{\alpha\beta}\frac{\partial H^\alpha}{\partial y}\frac{\partial F^\beta}{\partial x}\\&=H^\alpha\frac{\partial F^\gamma}{\partial x}\frac{\partial F^\beta}{\partial y}\omega_{\gamma\beta,\alpha}+\omega_{\alpha\beta}\frac{\partial H^\alpha}{\partial x}\frac{\partial F^\beta}{\partial y}+\omega_{\beta\alpha}\frac{\partial H^\alpha}{\partial y}\frac{\partial F^\beta}{\partial x}\end{aligned} \tag{2.21}$$

by using (2.14) and (2.15) in the last equality. Now comparing (2.18) and (2.21) proves the lemma.

Relative to ω_1,ω_2, a unitary connection $\omega_{\alpha\overline{\beta}}$ is uniquely determined by the equations

$$d\omega_\alpha=\omega_{\alpha\overline{1}}\wedge\omega_1+\omega_{\alpha\overline{2}}\wedge\omega_2, \tag{2.22}$$

$$\omega_{\alpha\overline{\beta}}=-\omega_{\overline{\beta}\alpha}. \tag{2.23}$$

Along the surface Σ_t, we have

$$\sin\frac{\alpha}{2}\omega_1-\cos\frac{\alpha}{2}\overline{\omega}_2=0. \tag{2.24}$$

Taking the exterior derivative of the above equation and using Cartan's equations leads to

$$\frac{1}{2}\left(d\alpha+\sin\alpha(\omega_{1\overline{1}}+\omega_{2\overline{2}})\right)=a\phi+b\overline{\phi}, \tag{2.25}$$

$$\omega_{1\overline{2}}=b\phi+c\overline{\phi} \tag{2.26}$$

for some complex-valued functions a,b,c on Σ_t. Let $\{\theta_{ij}\}$ be the connection forms in the coframe $\theta_1,...,\theta_4$. Then the second fundamental form of Σ_t is given by

$$\mathbb{I}_k=\alpha_k\theta_1^2+2\beta_k\theta_1\theta_2+\gamma_k\theta_2^2, \tag{2.27}$$

where

$$\theta_{k1}=\alpha_k\theta_1+\beta_k\theta_2, \tag{2.28}$$

$$\theta_{k2}=\beta_k\theta_1+\gamma_k\theta_2, \tag{2.29}$$

for $k=3,4$. It follows that the mean curvature normals are given by

$$H^1=\alpha_3+\gamma_3, \tag{2.30}$$

$$H^2=\alpha_4+\gamma_4. \tag{2.31}$$

Recall that $\overline{s}_2\,\omega_1 - s_1\overline{\omega}_2 = \theta_4 + \sqrt{-1}\theta_3$. Comparing their covariant derivatives along Σ_t, we see the following relations (cf. [4]):

$$2b + a + c = \alpha_4 + \sqrt{-1}\alpha_3, \tag{2.32}$$

$$2b - a - c = \gamma_4 + \sqrt{-1}\gamma_3, \tag{2.33}$$

$$\sqrt{-1}(a - c) = \beta_4 - \sqrt{-1}\beta_3. \tag{2.34}$$

Therefore, we have:

Lemma 2.2 *Let b be defined as in (2.25) and (2.26). Then*

$$H^2 + \sqrt{-1}H^1 = 4b. \tag{2.35}$$

In particular, $b = 0$ if and only if Σ_t is minimal.

To derive an evolution equation for α, we observe that α is real and $\omega_{1\overline{1}} + \omega_{2\overline{2}}$ is purely imaginary. Then by taking the summation of (2.25) with its complex conjugate, we have

$$d\alpha = a\phi + b\overline{\phi} + \overline{a}\overline{\phi} + \overline{b}\phi, \tag{2.36}$$

and by taking the $(0,1)$-part of both sides of the above equation, we have

$$\overline{\partial}\alpha = (\overline{a} + b)\overline{\phi}. \tag{2.37}$$

Then it follows from (2.37) and (2.25) that

$$\begin{aligned}
\partial\overline{\partial}\alpha &= d\overline{\partial}\alpha \\
&= d(\overline{a}\overline{\phi} + \overline{b}\phi) + d(b\overline{\phi} - \overline{b}\phi) \\
&= \frac{1}{2}d(d\alpha - \sin\alpha(\omega_{1\overline{1}} + \omega_{2\overline{2}})) + d(b\overline{\phi} - \overline{b}\phi) \\
&= -\frac{1}{2}\cos\alpha\, d\alpha \wedge (\omega_{1\overline{1}} + \omega_{2\overline{2}}) - \frac{\sqrt{-1}}{2}\sin\alpha\,\mathrm{Ric} + d(b\overline{\phi} - \overline{b}\phi),
\end{aligned} \tag{2.38}$$

where Ric denotes the Ricci curvature of M pulled back to Σ_t by the immersion F (cf. [1]). Using (2.25) again, we have

$$\begin{aligned}
-\frac{\cos\alpha}{2}d\alpha \wedge (\omega_{1\overline{1}} + \omega_{2\overline{2}}) &= -\frac{\cos\alpha}{\sin\alpha}d\alpha \wedge \left(a\phi + b\overline{\phi}\right) \\
&= -\frac{\cos\alpha}{\sin\alpha}\left(\overline{a}\overline{\phi} + \overline{b}\phi\right) \wedge \left(a\phi + b\overline{\phi}\right) \\
&= -\frac{\cos\alpha}{\sin\alpha}\left(\overline{\partial}\alpha - b\overline{\phi} + \overline{b}\phi\right) \wedge \left(\partial\alpha - \overline{b}\phi + b\overline{\phi}\right) \\
&= \frac{\cos\alpha}{\sin\alpha}\left(\partial\alpha \wedge \overline{\partial}\alpha - d\alpha \wedge (b\overline{\phi} - \overline{b}\phi)\right).
\end{aligned} \tag{2.39}$$

Therefore, we obtain

$$\partial\overline{\partial}\alpha = \frac{\cos\alpha}{\sin\alpha}\left(\partial\alpha \wedge \overline{\partial}\alpha - d\alpha \wedge (b\overline{\phi} - \overline{b}\phi)\right) - \frac{\sqrt{-1}}{2}\sin\alpha\mathrm{Ric} + d(b\overline{\phi} - \overline{b}\phi). \tag{2.40}$$

Lemma 2.3 *If* $\sin\alpha \neq 0$*, then*

$$b\overline{\phi} - \overline{b}\phi = \frac{\sqrt{-1}}{2\sin\alpha} F^*\omega(H,\cdot). \tag{2.41}$$

Proof From (2.5), (2.6), (2.7) and (2.8), we can write

$$F^*\omega = \frac{\sqrt{-1}}{2}\cos\alpha\left(\psi_1 \wedge \overline{\psi}_1 - \psi_2 \wedge \overline{\psi}_2\right) + \frac{\sqrt{-1}}{2}\sin\alpha\left(\psi_1 \wedge \overline{\psi}_2 + \psi_2 \wedge \overline{\psi}_1\right). \tag{2.42}$$

Then it follows that

$$F^*\omega(H,\cdot) = \sin\alpha\ (\theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4)(H) = \sin\alpha\ \left(H^1\theta_1 - H^2\theta_2\right). \tag{2.43}$$

But from Lemma 2.2, we have

$$b\overline{\phi} - \overline{b}\phi = \frac{\sqrt{-1}}{2}\left(H^1\theta_1 - H^2\theta_2\right). \tag{2.44}$$

Now (2.43) and (2.44) imply the lemma.

It is known (cf. [5]) that the area element $\frac{\sqrt{-1}}{2}\phi \wedge \overline{\phi}$ of Σ_t in the induced metric evolves with the mean curvature flow by the formula

$$\frac{\partial}{\partial t}\left(\phi \wedge \overline{\phi}\right) = -|H|^2 \phi \wedge \overline{\phi}. \tag{2.45}$$

By differentiating $F^*\omega = \frac{\sqrt{-1}}{2}\cos\alpha\,\phi \wedge \overline{\phi}$ in t, we conclude from (2.45) that

$$\frac{\partial}{\partial t}F^*\omega = \frac{\sqrt{-1}}{2}\frac{\partial\cos\alpha}{\partial t}\phi \wedge \overline{\phi} - \frac{\sqrt{-1}}{2}\cos\alpha|H|^2\phi \wedge \overline{\phi}. \tag{2.46}$$

So far, we have only used the fact that M is Kähler. When M is a Kähler-Einstein surface with scalar curvature R, the pulled back Ricci 2-form by the immersion F is

$$\mathrm{Ric} = \frac{\sqrt{-1}}{2}R\cos\alpha\phi \wedge \overline{\phi}. \tag{2.47}$$

Now we go back to equation (2.40). It follows from Lemma 2.3 that

$$\begin{aligned}
\partial\overline{\partial}\alpha &= \frac{\cos\alpha}{\sin\alpha}\left(\partial\alpha \wedge \overline{\partial}\alpha - d\alpha \wedge \frac{\sqrt{-1}}{2\sin\alpha}F^*\omega(H,\cdot)\right) \\
&\quad + \frac{R}{4}\sin\alpha\cos\alpha\phi \wedge \overline{\phi} + \frac{\sqrt{-1}}{2\sin\alpha}dF^*\omega(H,\cdot) \\
&= \frac{\cos\alpha}{4\sin\alpha}|\nabla\alpha|^2\phi \wedge \overline{\phi} + \frac{R}{4}\sin\alpha\cos\alpha\phi \wedge \overline{\phi} \\
&\quad - \frac{\sqrt{-1}\cos\alpha}{\sin^2\alpha}d\alpha \wedge F^*\omega(H,\cdot) + \frac{\sqrt{-1}}{2\sin\alpha}dF^*\omega(H,\cdot) \\
&= \frac{\cos\alpha}{4\sin\alpha}|\nabla\alpha|^2\phi \wedge \overline{\phi} + \frac{R}{4}\sin\alpha\cos\alpha\phi \wedge \overline{\phi} - \frac{\sqrt{-1}\cos\alpha}{\sin^2\alpha}d\alpha \wedge F^*\omega(H,\cdot) \\
&\quad - \frac{1}{4\sin\alpha}\frac{\partial\cos\alpha}{\partial t}\phi \wedge \overline{\phi} + \frac{\cos\alpha}{4\sin\alpha}|H|^2\phi \wedge \overline{\phi}.
\end{aligned} \tag{2.48}$$

We used Lemma 2.1 and (2.46) in the last step. On the other hand, we have

$$\Delta \cos\alpha = -\sin\alpha\Delta\alpha - \cos\alpha|\nabla\alpha|^2 \tag{2.49}$$

and

$$\Delta\alpha\phi\wedge\overline{\phi} = 4\partial\overline{\partial}\alpha. \tag{2.50}$$

Hence, we obtain:

Theorem 2.4 *If $F : \Sigma\times[0,T)\to M$ is a family of immersions which satisfies the mean curvature flow equation, then the Kähler angle α of the immersed surfaces satisfies*

$$\begin{aligned}\frac{\partial\cos\alpha}{\partial t} &= \Delta\cos\alpha + 2\cos\alpha|\nabla\alpha|^2 + R\sin^2\alpha\cos\alpha + |H|^2\cos\alpha \\ &\quad + \frac{4\sqrt{-1}\cos\alpha}{\sin^2\alpha} d\cos\alpha\wedge F^*\omega(H,\cdot)(\phi^*\wedge\overline{\phi}^*),\end{aligned} \tag{2.51}$$

where $\phi^*, \overline{\phi}^*$ stand for the dual vectors of the 1-forms $\phi, \overline{\phi}$, provided $\alpha\neq 0,\pi$.

Theorem 2.4 and the minimum principle for parabolic equations lead to interesting applications. Recall that an immersed surface Σ_t in M is called *symplectic* if its Kähler angle satisfies $0\le\alpha<\pi/2$ at every point.

Theorem 2.5 *Let M be a Kähler-Einstein surface and let Σ be a compact orientable surface without boundary. Suppose that $F:\Sigma\times[0,T)\to M$ is a family of immersions which satisfies the mean curvature flow equations. If the initial surface is symplectic, then it will remain symplectic for any $t\in[0,T)$.*

Proof First we notice that if α is identically equal to zero on the initial surface, then the initial surface is holomorphic hence automatically minimal. So the mean curvature flow keeps every point still. In this case, the theorem is trivially true. Second, by assumption the initial surface is symplectic hence there are no anti-complex points on it, i.e. $\alpha\neq\pi$. In particular, α is smooth in a neighborhood of the points where $\cos\alpha$ attains its minimum and the minimum principle is applicable, provided α is not identically zero. Denote the scalar curvature of the Kähler-Einstein metric on M by R. Let $\overline{x}(t)$ be a point on Σ_t such that

$$\cos\alpha(\overline{x}(t)) = \min_{x(t)\in\Sigma_t}\cos\alpha(x(t)). \tag{2.52}$$

Let t_0 be any time such that $\cos\alpha(\overline{x}(t_0))>0$. By assumption, the initial surface Σ_0 satisfies this requirement. For any $\epsilon>0$, we have

$$\begin{aligned}\frac{\partial e^{\epsilon(t-t_0)}\cos\alpha}{\partial t} &= e^{\epsilon(t-t_0)}\Bigg(\Delta\cos\alpha + 2\cos\alpha|\nabla\alpha|^2 + R\sin^2\alpha\cos\alpha + |H|^2\cos\alpha \\ &\quad + \frac{4\sqrt{-1}\cos\alpha}{\sin^2\alpha} d\cos\alpha\wedge F^*\omega(H,\cdot)(\phi^*\wedge\overline{\phi}^*)\Bigg) + \epsilon e^{\epsilon(t-t_0)}\cos\alpha.\end{aligned} \tag{2.53}$$

In particular, if $R\ge 0$ the minimum principle asserts that

$$\frac{\partial e^{\epsilon(t-t_0)}\cos\alpha}{\partial t}\Big|_{\overline{x}(t_0)} \ge \epsilon\cos\alpha(\overline{x}(t_0)) > 0. \tag{2.54}$$

This implies that

$$\frac{\partial \cos\alpha}{\partial t}\Big|_{\overline{x}(t_0)} > -\epsilon \cos\alpha(\overline{x}(t_0)). \tag{2.55}$$

The left-hand side of (2.55) is independent of ϵ. So by letting ϵ go to zero we see that the t-derivative of $\cos\alpha$ is strictly positive in some neighborhood $U \times [t_0, t_0+\delta)$ where U is some open neighborhood of $\overline{x}(t_0)$ in Σ_{t_0} and δ is some positive number. For any $t_1 > t_0$ which is sufficiently close to t_0, there exists some point $\overline{x}(t_1) \in U \times [t_0, t_0+\delta)$. $\overline{x}(t_1)$ is evolved from some point y on Σ_{t_0}. Therefore

$$\cos\alpha(\overline{x}(t_1)) > \cos\alpha(y) \geq \cos\alpha(\overline{x}(t_0)) > 0. \tag{2.56}$$

Consequently, the minimum of $\cos\alpha$ is strictly increasing on $[0, T)$. If $R < 0$, we consider

$$\begin{aligned}\frac{\partial e^{-R(t-t_0)}\cos\alpha}{\partial t} &= e^{-R(t-t_0)}\Big(\Delta\cos\alpha + 2\cos\alpha|\nabla\alpha|^2 + R\sin^2\alpha\cos\alpha \\ &\quad + |H|^2\cos\alpha + \frac{4\sqrt{-1}\cos\alpha}{\sin^2\alpha} d\cos\alpha \wedge F^*\omega(H,\cdot)(\phi^* \wedge \overline{\phi}^*)\Big) \\ &\quad - Re^{-R(t-t_0)}\cos\alpha.\end{aligned} \tag{2.57}$$

At $\overline{x}(t_0)$, the minimum principle implies

$$\frac{\partial e^{-R(t-t_0)}\cos\alpha}{\partial t}\Big|_{\overline{x}(t_0)} \geq -R\cos^3\alpha(\overline{x}(t_0)) > 0. \tag{2.58}$$

Continuity then implies that in a neighborhood $U' \times [t_0, t_0+\delta')$ of $\overline{x}(t_0)$

$$\frac{\partial\cos\alpha}{\partial t} - R\cos\alpha > 0, \tag{2.59}$$

$$\cos\alpha > 0. \tag{2.60}$$

Again by continuity, we may assume that $\cos\alpha$ attains its minimum value at some $\overline{x}(t)$ in U' for all $t \in [t_0, t_0+\delta')$. Let $y(t)$ be the point in $U' \times t_0$ which evolves to $\overline{x}(t)$. It follows that

$$\begin{aligned}\cos\alpha(x(t)) &\geq \cos\alpha(\overline{x}(t)) \\ &> \cos\alpha(y(t))e^{R(t-t_0)} \\ &\geq \cos\alpha(\overline{x}(t_0))e^{R(t-t_0)},\end{aligned} \tag{2.61}$$

and therefore $\cos\alpha$ remains strictly positive for all t in $[t_0, t_0+\delta')$, hence in $[t_0, T)$.

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