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# Local Existence for Inhomogeneous Schrödinger Flow into Kähler Manifolds

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**Abstract** In this paper we show that there exists a unique local smooth solution for the Cauchy problem of the inhomogeneous Schrödinger flow for maps from a compact Riemannian manifold $M$ with $\dim(M) \leq 3$ into a compact Kähler manifold $(N, J)$ with nonpositive Riemannian sectional curvature

**Keywords** Inhomogeneous Schrödinger flow, Kähler manifold, Local existence
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## 1 Introduction

Let $\Omega$ be a smooth bounded domain in Euclidean space $R^m$ $(m = 1, 2, 3)$. Then the well-known inhomogeneous Heisenberg spin (IHS) chain system (also called inhomogeneous ferromagnetic spin chain system) is given by

$$\partial_t u(x,t) = \sigma(x)\{u(x,t) \times \Delta u(x,t)\} + \nabla\sigma(x) \cdot \{u(x,t) \times \nabla u(x,t)\}, \ x \in \Omega,$$

where $u(x,t) \in S^2 \subset R^3$, $\sigma(x)$ is a positive real function on $\Omega$, $\times$ denotes the cross product in $R^3$ and $\Delta$ is the Laplace operator on $R^m$ (see [1–5]).

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For a map $u : (M, g) \to (N, h)$ between Riemannian manifolds, we recall that, in local coordinates, the tension field can be written as

$$\tau^\alpha(u) = \Delta u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j},$$

where $\Delta$ is the Laplace-Beltrami operator on $M$ with respect to the metric $g$ and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols on the target manifold $(N, h)$. Thus, it is easy to see that the IHS chain system can be written as

$$\partial_t u(x,t) = \sigma(x)J(u(x,t))\tau(u(x,t)) + \nabla\sigma(x)\cdot J(u)\nabla u(x,t).$$

This shows that the IHS chain system is a nonlinear Schrödinger equation into $S^2$. Indeed, it can be viewed as an infinite dimensional Hamiltonian system with respect to the inhomogeneous energy functional (see [6–9]) given by $E_\sigma(u) = \int_\Omega |du|^2\sigma(x)dx$, where $|du|^2$ denotes the Hilbert-Schmidt norm of the tangent map $du : T\Omega \longrightarrow TN$.

For the homogeneous case (i.e., $\sigma \equiv 1$), Zhou, Guo and Tan [10] showed that for smooth initial data there exists a unique smooth solution for the Cauchy problem of the ferromagnetic spin chain system from $S^1$ into $S^2$. In [11, 12], Wang proved the existence of a global weak solution for the Cauchy problem of the ferromagnetic spin system from any closed manifold into $S^2$ with or without external magnetic field (see also [13, 14]).

In [6], Ding and Wang studied the Schrödinger flow for maps from a compact Riemannian manifold into a symplectic manifold. For a symplectic manifold $(N, J)$ with symplectic form $\omega$, where $J$ is an almost complex structure on $N$ such that $h(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric, the Schrödinger flow for maps from $(M, g)$ into $(N, J)$ is defined by the equation $\partial_t u = J(u)\tau(u)$. It can be viewed as an infinite dimensional Hamiltonian system. When $M$ is the unit circle and $(N, J)$ is a Kähler manifold, Ding and Wang [6] proved that the Schrödinger flow admits a unique local smooth solution. Furthermore, when $(N, J)$ is a compact Riemann surface with constant sectional curvature, they showed that the solution exists globally by exploiting a conservative law.

Chang, Shatah and Uhlenbeck [15] studied the Cauchy problem for the Schödinger flow from $R^m$ $(m = 1, 2)$ into a compact Riemann surface $N$. Using a generalized Hasimoto transformation, they showed that for $m = 1$ and smooth initial data, the Cauchy problem admits a unique global smooth solution. For $m = 2$, considering symmetric solutions, they proved the global existence and uniqueness under the small energy assumption. Terng and Uhlenbeck [16] studied the global existence for Schrödinger flow from $R^1$ into Grassmannians. Also, Ding [17] pointed out that the nonlinear Schrödinger equation with $K = -1$ is gauge equivalent to the the Schrödinger flow from $R^1$ into $H(-1)$.

In [18] and [19], we showed the global existence of Schrödinger flows on Hermitian locally symmetric manifolds. More precisely, the following result was obtained:

**Theorem** [19] *Let $(N, J, h)$ be a Hermitian locally symmetric manifold and let $M = S^1$ or $R^1$. Then, the Schrödinger flow from $M$ into $N$ obeys the following conservative law:*

$$\frac{d}{dt}\Big\{\int_M |\tau(u(t))|^2 dS - \frac{1}{4}\int_M R(u', Ju', u', Ju')\, dS\Big\} \equiv 0,$$

*where $R(\cdot,\cdot,\cdot,\cdot)$ denotes the curvature tensor of $N$. If $M = S^1$, then for smooth initial data, the Cauchy problem of the Schrödinger flow admits a unique global smooth solution. If $M = R^1$, a similar global existence result holds when the target manifold $N$ is compact.*

Moreover, in [18], the authors showed that the two-dimensional Schrödinger flow into a compact, nonpositively curved Kähler manifold is locally well-posed.

In spite of many developments of Schrödinger flow and nonlinear Schrödinger equation (see e.g. [20–26]), little is known about the inhomogeneous case. The inhomogeneous Schrödinger flow from a Riemannian manifold into a symplectic manifold $(N, J)$ is defined by

$$\partial_t u = \sigma(x)J(u)\tau(u) + \nabla\sigma(x)\cdot J(u)\,du,$$

where $\sigma$ is a positive real function on $M$. If $\{e_i\}$ is a local orthonormal frame on $M$, then

$$\nabla\sigma(x)\cdot J(u)\,du = \nabla_{e_i}\sigma J(u)\left(du(e_i)\right).$$

We note, for instance, that the anisotropic IHS chain system can be reformulated as the inhomogeneous Schrödinger flow into the Poincaré disk:

$$\frac{\partial z}{\partial t} = i\left\{\sigma(x)\left(\Delta z + \frac{2\bar{z}}{1-|z|^2}(\nabla z)^2\right) + \nabla\sigma(x)\cdot\nabla z\right\},$$

where $z(x,t) \in \{z \in C : |z| < 1\}$.

Recently, Wang and Wang [7] proved that there exists a unique global smooth solution for the Cauchy problem of the inhomogeneous Schrödinger flow from $S^1$ into a complete Kähler manifold with constant holomorphic sectional curvature. They exploited the symmetries of Kähler manifolds with constant holomorphic sectional curvature to derive certain a priori estimates. This is different from the approach of investigating the Schrödinger semigroup and employing the Strichartz inequality.

Motivated by [7] and [18], this paper establishes a local existence theory for the Cauchy problem of the inhomogeneous Schrödinger flow from an $n$-dimensional ($n \le 3$) compact Riemannian manifold into a compact Kähler manifold with nonpositive sectional curvature. Our main result is:

**Theorem 1** *Let $M$ be a closed Riemannian manifold with* $\dim(M) \le 3$ *and let $(N, J)$ be a Kähler manifold with nonpositive sectional curvature. Assume that $\sigma(x) \in C^4(M)$ and* $\min_{x\in M}|\sigma(x)| > 0$. *Then, given the initial map $u_0 \in H^{5,2}(M)$, the Cauchy problem of the inhomogeneous Schrödinger flow from $M$ into $(N, J)$ admits a unique local solution $u \in L^\infty([0,T), H^{5,2}(M))$.*

Facilitated by the preliminary results collected in Section 2, the proof of Theorem 1 will be given in Section 3.

**A Note on Notation** We shall use the symbol $C$ generically to denote certain scalar-valued terms in the estimates to be derived in the remainder of the paper. We will, however, normally specify the objects/quantities on which these terms depend by means of arguments to $C$. For example, the symbol $C(M,N)$ denotes a constant depending only on the manifolds $M$ and $N$, whereas the symbol $C(\|\tau(u)\|_{L^2}, E_\sigma(u))$ denotes a smooth scalar-valued function depending on the quantities $\|\tau(u)\|_{L^2}$ and $E_\sigma(u)$. Thus, the latter $C$ is not necessarily a constant but may vary with $u$. Also, unless otherwise specified, $C$ shall be assumed to depend on its arguments smoothly.

## 2 Preliminaries

Let $\pi : E \longrightarrow M$ be a Riemannian vector bundle over a Riemannian manifold $M$ and denote its tensor product with the exterior $p$ bundle by

$$\Lambda^p T^*M \otimes E \longrightarrow M, \quad p = 1, 2, \ldots, \dim(M).$$

Denote the set of smooth sections of $\Lambda^p T^*M \otimes E$ by $\Gamma(\Lambda^p T^*M \otimes E)$. The metrics on $T^*M$ and $E$ induce a metric on $\Lambda^p T^*M \otimes E$: for any $s_1, s_2 \in \Gamma(\Lambda^p T^*M \otimes E)$,

$$\langle s_1, s_2 \rangle = \sum_{i_1 < i_2 < \cdots < i_p} \langle s_1(e_{i_1}, \ldots, e_{i_p}), s_2(e_{i_1}, \ldots, e_{i_p}) \rangle,$$

where $\{e_i\}$ is an orthonormal local frame of $TM$. This further induces an inner product on $\Gamma(\Lambda^p T^*M \otimes E)$:

$$(s_1, s_2) = \int_M \langle s_1, s_2 \rangle(x)\, dM = \int_M \langle s_1, s_2 \rangle(x) * 1.$$

The space $L^2(M, \Lambda^p T^*M \otimes E)$ is the completion of $\Gamma(\Lambda^p T^*M \otimes E)$ with respect to the above inner product.

Let $\nabla$ be the covariant differential induced by the metric on $E$, then the Sobolev space $H^{k,r}(M,E)$ is defined as the completion of the smooth sections of $E$ with respect to the norm

$$\|s\|_{k,r} = \|s\|_{H^{k,r}(M,E)} = \Big( \sum_{i=0}^{k} \int_M |\nabla^i s|^r \, dM \Big)^{\frac{1}{r}},$$

where $|\nabla^i s| = \langle \underbrace{\nabla \cdots \nabla}_{i \text{ times}} s, \underbrace{\nabla \cdots \nabla}_{i \text{ times}} s \rangle^{\frac{1}{2}}$. The Sobolev spaces $H^{k,r}(M, \Lambda^p T^*M \otimes E)$ are defined analogously (see Appendix in [27] and [28] for details). These Sobolev spaces are the so-called *bundle-valued Sobolev spaces*. We recall two standard results when $M$ is a compact Riemannian manifold, namely, the Rellich theorem and the Sobolev embedding theorem.

On the other hand, it is well-known that a compact Riemannian manifold $N$ can be isometrically embedded into a Euclidean space $R^d$ for some positive integer $d > \dim(N)$. Hence, one can define the Sobolev spaces:

$$W^{k,p}(M,N) \equiv \{g : g \in W^{k,p}(M, R^d), g(x) \in N \text{ a.e. } x \in M\}.$$

These spaces are called *Sobolev spaces of maps* (*functions*).

We note that for any $f : M \longrightarrow N$, $df$ can be regarded as a 1-form with values in $f^*TN$, i.e., $df \in \Gamma(T^*M \otimes f^*TN)$. The energy density of $f$ is defined by $e(f) = \frac{1}{2}|df|^2$, and the energy functional is given by

$$E(f) = \frac{1}{2}\int_M |df|^2\, dM.$$

Let $\{e_i\}$ be a local orthonormal frame on $M$ and $\{e_i^*\}$ its dual frame. Let $\{\bar{e}_\alpha\}$ be a local orthonormal frame on $N$. Then, with respect to the above frames, $df = f_i^\alpha e_i^* \otimes \bar{e}_\alpha$, and $\tau(f) = \nabla_i f_i$ where $f_i = f_i^\alpha \bar{e}_\alpha = f_* e_i$.

In [18], we showed the following propositions:

**Proposition 2.1** [18] *Let $M$ be a closed Riemannian manifold and $N$ a complete Riemannian manifold with nonpositive sectional curvature. For a smooth map $u : M \to N$, there exists a constant $C(M)$ such that*

$$\int_M |\nabla du|^2\, dM \le \int_M |\tau(u)|^2\, dM + C(M)E(u).$$

*In particular, if $M$ has nonnegative Ricci curvature, then*

$$\int_M |\nabla du|^2\, dM \le \int_M |\tau(u)|^2\, dM.$$

In the following two propositions, $N$ is regarded as isometrically embedded in the Euclidean space $R^d$. For convenience, we denote $\|\cdot\|_{C^0(M,R^d)}$ and $\|\cdot\|_{W^{k,p}(M,R^d)}$ by $\|\cdot\|_{C^0}$ and $\|\cdot\|_{W^{k,p}}$ respectively. First, we prove the 3-dimensional analog of Proposition 2.3 in [18]:

**Proposition 2.2** *Let $M$ be a closed Riemannian manifold with $\dim(M) \le 3$ and $N$ a compact Riemannian manifold with nonpositive sectional curvature. For a smooth map $u : M \to N$,*

$$\|du\|_{W^{2,2}} \le C(M, N, E(u), \|\nabla\tau(u)\|_{L^2}).$$

*Proof* We need to consider only the case $\dim(M) = 3$. The proof is in fact similar to that of Proposition 2.3 in [18] and makes use of the bootstrap technique in regularity considerations in elliptic theory.

**Proposition 2.3** [18] *With the same assumptions as in Proposition 2.2, for $k \ge 1$, there exist constants $C(M, N)$ such that*

$$\|du\|_{W^{2k+1,2}} \le C(M,N)\|\Delta^k\tau(u)\|_{L^2} + C_{2k+1}(\|\nabla\Delta^{k-1}\tau(u)\|_{L^2}, \ldots, \|\nabla\tau(u)\|_{L^2}, E(u)),$$

*and*

$$\|du\|_{W^{2k+2,2}} \le C(M,N)\|\nabla\Delta^k\tau(u)\|_{L^2} + C_{2k+2}(\|\Delta^k\tau(u)\|_{L^2}, \ldots, \|\nabla\tau(u)\|_{L^2}, E(u)).$$

**Proposition 2.4** [18] *Let $E \to M$ be a Riemannian vector bundle over a closed $m$-dimensional Riemannian manifold. For $s \in \Gamma(E)$, there exists a constant $C(M)$, which does not depend on $E$, such that*

$$\|\nabla s\|_{L^p} \le C(M)\|\nabla s\|_{H^{1,r}}^a \|s\|_{L^q}^{1-a}, \tag{2.1}$$

*where* $\frac{1}{p} = \frac{1}{m} + a\left(\frac{1}{r} - \frac{2}{m}\right) + (1-a)\frac{1}{q}$, *for all* $a$ *in the interval* $\frac{1}{2} \le a \le 1$ *for which* $p$ *is nonnegative.*

**Proposition 2.5** *Let* $M$ *be a closed Riemannian manifold and* $N$ *a compact Riemannian manifold with nonpositive sectional curvature. Assume that* $\sigma(x)$ *is a smooth function on* $M$ *with* $\min_{x\in M}|\sigma(x)| > 0$. *Then, for a smooth map* $u : M \to N$, *there exists a constant* $C(\sigma)$ *such that*

$$\|\nabla\tau(u)\|_{L^2}^2 \le C(\sigma)\{\|\nabla\tau_\sigma(u)\|_{L^2} + \|\tau_\sigma(u)\|_{L^2}^2 + E_\sigma(u)\},$$

$$\|\Delta\tau(u)\|_{L^2}^2 \le C(\sigma)\|\Delta\tau_\sigma(u)\|_{L^2} + C(\|\nabla\tau_\sigma(u)\|_{L^2}^2, E_\sigma(u)),$$

*and*

$$\|\nabla\Delta\tau(u)\|_{L^2}^2 \le C(\sigma)\|\nabla\Delta\tau_\sigma(u)\|_{L^2} + C(\|\Delta\tau_\sigma(u)\|_{L^2}^2, E_\sigma(u)),$$

*where* $\tau_\sigma(u) \equiv \sigma(x)\tau(u) + \nabla\sigma(x)\cdot du$.

*Proof* It is easy to see that there exists a constant depending only on $\sigma$ such that

$$\|\tau(u)\|_{L^2}^2 \le C(\sigma)\{\|\tau_\sigma(u)\|_{L^2}^2 + E_\sigma(u)\}. \tag{2.2}$$

Since $\min_{x\in M}|\sigma(x)| > 0$, by a direct calculation, we have

$$\begin{aligned}
|\nabla\tau(u)|^2 \le\ & |\nabla(\sigma^{-1}(\sigma\tau(u) + \nabla\sigma\cdot du))|^2 + |\nabla(\sigma^{-1}\nabla\sigma\cdot du)|^2 \\
\le\ & |\nabla(\sigma^{-1}\tau_\sigma(u))|^2 + |\nabla(\sigma^{-1}\nabla\sigma\cdot du)|^2 \\
\le\ & |\sigma^{-1}\nabla\tau_\sigma(u)|^2 + |\sigma^{-2}\langle\nabla\sigma,\tau_\sigma(u)\rangle|^2 + C(\sigma)(|du|^2 + |\nabla du|^2) \\
\le\ & C(\sigma)\{|\nabla\tau_\sigma(u)|^2 + |\tau_\sigma(u)|^2 + |du|^2 + |\nabla du|^2\}.
\end{aligned} \tag{2.3}$$

Thus, it follows from Proposition 2.1, (2.2) and (2.3) that

$$\begin{aligned}
\|\nabla\tau(u)\|_{L^2}^2 \le\ & C(\sigma)\{\|\nabla\tau_\sigma(u)\|_{L^2}^2 + \|\tau_\sigma(u)\|_{L^2}^2 + E(u) + \|\nabla du\|_{L^2}^2\} \\
\le\ & C(\sigma)\{\|\nabla\tau_\sigma(u)\|_{L^2}^2 + \|\tau_\sigma(u)\|_{L^2}^2 + E(u) + \|\tau(u)\|_{L^2}^2\} \\
\le\ & C(\sigma)\{\|\nabla\tau_\sigma(u)\|_{L^2}^2 + \|\tau_\sigma(u)\|_{L^2}^2 + E_\sigma(u)\},
\end{aligned}$$

where the last inequality follows because $E(u) \le C(\sigma)E_\sigma(u)$.

The second inequality follows by direct calculation and Proposition 2.2. The last inequality then follows from Proposition 2.3.

## 3 Proof of Theorem 1

In this section we will establish the local existence result for the Cauchy problem of the inhomogeneous Schrödinger flow from $M$ into a Kähler manifold $(N, J)$ defined by

$$\begin{cases} \partial_t u(t) = & J(u)\{\sigma(x)\tau(u(t)) + \nabla\sigma(x)\cdot du(t)\}, \\ u(x,0) = & u_0(x), \end{cases} \tag{3.1}$$

where $\sigma$ is a positive real function on $M$ and $u_0 : M \longrightarrow N$. To prove the local existence result, we shall consider the following approximating equations parameterized by $\varepsilon \in (0, 1)$:

$$\begin{cases} \partial_t u(x,t) = & \varepsilon\tau_\sigma(u(x,t)) + J(u)\tau_\sigma(u(x,t)), \\ u(x,0) = & u_0(x). \end{cases} \tag{3.2}$$

If $\min_{x\in M}\sigma(x) > 0$, it is easy to see that this is a second-order uniformly parabolic system. Hence, by the classical theory (see [29]), given any smooth map $u_0$, there is a unique local smooth solution to (3.2). Also, obviously,

$$\frac{d}{dt}\int_M |du|^2\sigma(x)\,dM \le 0,$$

and this leads to the energy inequality [7] $E_\sigma(u(t)) \le E_\sigma(u_0)$. Furthermore,

$$[\min_{x\in M}\sigma(x)]E(u) \le E_\sigma(u) \le [\max_{x\in M}\sigma(x)]E(u).$$

These energy estimates will be used extensively in the subsequent arguments.

We also need the following estimates on the Sobolev norms of the tension field $\tau(u)$:

**Lemma 3.1** *Let $M$ be a closed Riemannian manifold with $\dim(M) \le 3$ and $N$ a compact Kähler manifold with nonpositive sectional curvature. Let $\sigma(x)$ be a smooth function defined on $M$ such that $\min_{x\in M}|\sigma(x)| > 0$. If $u$ is a smooth solution of the Cauchy problem (3.2), then there exists $T > 0$, which is independent on $0 < \varepsilon < 1$, such that the following inequality holds uniformly for $0 < \varepsilon < 1$:*

$$\sup_{t\in[0,T)}\int_M |\nabla\tau(u)|^2\,dM \le C(T, u_0).$$

*($C(T, u_0)$ also depends on the $C^2$-norm of $\sigma$).*

*Proof* We will only consider the case $\dim(M) = 3$. Without loss of generality, we may assume that $M$ is the flat torus $T^m \equiv R^m/Z^m$ and $\min_{x\in M}\sigma(x) > 0$. For ease of notation, denote $\frac{\partial u}{\partial t}$ by $\dot{u}$, and $u_*e_i$ by $u_i$. As $\nabla J \equiv 0$, by direct calculation,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_M |\dot{u}|^2\,dM = & \int_M \langle \dot{u}, \nabla_t\dot{u}\rangle\,dM \\ = & \int_M \langle \dot{u}, \nabla_t(\varepsilon\tau_\sigma(u) + J(u)\tau_\sigma(u))\rangle\,dM \\ = & \int_M \varepsilon\langle \dot{u}, (\sigma(x)\nabla_t\nabla_j u_j + \nabla_l\sigma(x)\cdot\nabla_t u_l)\rangle\,dM \\ & + \int_M \langle \dot{u}, J(u)(\sigma(x)\nabla_t\nabla_j u_j + \nabla_l\sigma(x)\cdot\nabla_t u_l)\rangle\,dM \\ = & \int_M \varepsilon\langle \dot{u}, \sigma(x)\nabla_j\nabla_j\dot{u} + \nabla_l\sigma(x)\cdot\nabla_l\dot{u}\rangle\,dM \\ & + \int_M \langle \dot{u}, J(u)(\sigma(x)\nabla_j\nabla_j\dot{u} + \nabla_l\sigma(x)\cdot\nabla_t u_l)\rangle\,dM \end{aligned}$$

$$\begin{aligned}
&\quad +\int_M \varepsilon\langle \dot{u}, \sigma(x)R(u_j,\dot{u})u_j\rangle\, dM \\
&\quad +\int_M \langle \dot{u}, J(u)(\sigma(x)R(u_j,\dot{u})u_j)\rangle\, dM \\
&= -\varepsilon\int_M \langle \nabla_j\dot{u}, \nabla_j\dot{u}\rangle\sigma(x)\, dM + \int_M \varepsilon\langle \dot{u}, \sigma(x)R(u_j,\dot{u})u_j\rangle\, dM \\
&\quad +\int_M \langle \dot{u}, J(u)(\sigma(x)R(u_j,\dot{u})u_j)\rangle\, dM,
\end{aligned} \tag{3.3}$$

where $R(\cdot,\cdot)$ denotes the Riemannian curvature operator of $N$ and

$$R(u_j,\dot{u})u_j = \nabla_t\nabla_j u_j - \nabla_j\nabla_t u_j = \nabla_t\nabla_j u_j - \nabla_j\nabla_j\dot{u}.$$

Now, applying Hölder's inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_M |\dot{u}|^2\, dM + \varepsilon\int_M \langle \nabla_j\dot{u}, \nabla_j\dot{u}\rangle\, dM &\le C(\sigma,N)\int_M |du|^2|\dot{u}|^2\, dM \\
&\le C(\sigma,N)\|du\|^2_{C^0}\int_M |\dot{u}|^2\, dM.
\end{aligned} \tag{3.4}$$

Note that by differentiating both sides of (3.1), we obtain $\nabla\dot{u} = \varepsilon\nabla\tau_\sigma(u) + J(u)\nabla\tau_\sigma(u)$. Hence, as $\langle \nabla\tau_\sigma(u), J(u)\nabla\tau_\sigma(u)\rangle \equiv 0$,

$$|\nabla\dot{u}|^2 = (1+\varepsilon^2)|\nabla\tau_\sigma(u)|^2. \tag{3.5}$$

Similarly,

$$|\dot{u}|^2 = (1+\varepsilon^2)|\tau_\sigma(u)|^2. \tag{3.6}$$

Now, using Proposition 2.5 and the monotonicity of inhomogeneous energy, one can prove that

$$\|\nabla\tau(u)\|_{L^2} \le C(\sigma)\{\|\nabla\dot{u}\|_{L^2} + \|\dot{u}\|_{L^2} + \sqrt{E_\sigma(u_0)}\}.$$

Therefore, it follows from Proposition 2.2 that

$$\begin{aligned}
\|du\|_{C^0} &\le C(M,N,E(u_0),\|\nabla\tau(u)\|_{L^2}) \\
&\le C(\sigma,M,N,E(u_0),\|\dot{u}\|_{L^2},\|\nabla\dot{u}\|_{L^2}) \\
&\le C(\sigma,M,N,E_\sigma(u_0),\|\dot{u}\|_{H^{1,2}(\sigma)}),
\end{aligned} \tag{3.7}$$

where

$$\|\dot{u}\|_{H^{1,2}(\sigma)} \equiv \|\dot{u}\|_{L^2} + \left\{\int_M |\nabla\dot{u}|^2\sigma(x)\, dM\right\}^{\frac{1}{2}}.$$

Substituting (3.7) into (3.4), we get

$$\frac{d}{dt}\int_M |\dot{u}|^2\, dM \le C(\sigma,M,N,E_\sigma(u_0),\|\dot{u}\|_{H^{1,2}(\sigma)}).$$

It follows from (3.6) and the last inequality that

$$\frac{d}{dt}\int_M |\tau_\sigma(u)|^2\, dM \le F_1^\sigma(\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)}), \tag{3.8}$$

where $F_1^\sigma(\cdot)$ is a smooth positive function which depends on $\sigma$, $E_\sigma(u_0)$ and the geometry of $M$ and $N$, but not on $\varepsilon$.

On the other hand, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_M|\nabla\dot u|^2\sigma(x)\,dM &= \int_M\langle\nabla_i\dot u,\nabla_t\nabla_i\dot u\rangle\sigma(x)\,dM\\
&= \int_M\langle\nabla_i\dot u,\nabla_t\nabla_i(\varepsilon\tau_\sigma(u)+J(u)\tau_\sigma(u))\rangle\sigma(x)\,dM\\
&= \int_M\{\varepsilon\langle\nabla_i\dot u,\nabla_t\nabla_i(\sigma(x)\nabla_ju_j+\nabla_l\sigma(x)\cdot u_l)\rangle\\
&\quad+\langle\nabla_i\dot u,J(u)\nabla_t\nabla_i(\sigma(x)\nabla_ju_j+\nabla_l\sigma(x)\cdot u_l)\rangle\}\sigma(x)\,dM\\
&= I_1+I_2,
\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}
I_1 = &\int_M\varepsilon\langle\nabla_i\dot u,\nabla_t\nabla_i(\sigma(x)\nabla_ju_j+\nabla_l\sigma(x)\cdot\nabla_lu)\rangle\sigma(x)\,dM\\
= &\int_M\varepsilon\{\langle\nabla_i\dot u,\sigma(x)\nabla_t\nabla_i\nabla_ju_j\rangle+\langle\nabla_i\sigma(x)\cdot\nabla_i\dot u,\nabla_t\nabla_ju_j\rangle\}\sigma(x)\,dM\\
&+\int_M\varepsilon\{\langle\nabla_i\dot u,\nabla_l\sigma(x)\cdot\nabla_t\nabla_iu_l\rangle+\langle\nabla_i\dot u,\nabla_i\nabla_l\sigma(x)\cdot\nabla_tu_l\rangle\}\sigma(x)\,dM
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
I_2 = &\int_M\langle\nabla_i\dot u,J(u)\nabla_t\nabla_i(\sigma(x)\nabla_ju_j+\nabla_l\sigma\cdot\nabla_lu)\rangle\sigma(x)\,dM\\
= &\int_M\{\langle\nabla_i\dot u,J(u)\sigma(x)\nabla_t\nabla_i\nabla_ju_j\rangle+\langle\nabla_i\sigma(x)\cdot\nabla_i\dot u,J(u)\nabla_t\nabla_ju_j\rangle\}\sigma(x)\,dM\\
&+\int_M\{\langle\nabla_i\dot u,J(u)\nabla_l\sigma(x)\cdot\nabla_t\nabla_iu_l\rangle+\langle\nabla_i\dot u,J(u)\nabla_i\nabla_l\sigma(x)\cdot\nabla_tu_l\rangle\}\sigma(x)\,dM\\
\equiv\ & A_1+A_2+A_3+A_4.
\end{aligned}\tag{3.11}$$

We compute the quantities $A_1-A_4$ below.

By the commutative relation of covariant differential, we have

$$\begin{aligned}
\nabla_t\nabla_i\nabla_ju_j = &\ \nabla_i\nabla_t\nabla_ju_j+R(u_i,\dot u)\nabla_ju_j\\
= &\ \nabla_i\nabla_j\nabla_tu_j+\nabla_i(R(u_j,\dot u)u_j)+R(u_i,\dot u)\nabla_ju_j\\
= &\ \nabla_i\nabla_j\nabla_j\dot u+R(u_i,\dot u)\nabla_ju_j+R(\nabla_iu_j,\dot u)u_j\\
&+R(u_j,\nabla_i\dot u)u_j+R(u_j,\dot u)\nabla_iu_j+(\nabla_iR)(u_j,\dot u)u_j.
\end{aligned}\tag{3.12}$$

Also,

$$\int_M\langle\nabla_i\dot u,J(u)\sigma(x)\nabla_i\nabla_j\nabla_j\dot u\rangle\sigma(x)\,dM=-2\int_M\langle\nabla_i\sigma(x)\cdot\nabla_i\dot u,J(u)\nabla_j\nabla_j\dot u\rangle\sigma(x)\,dM.\tag{3.13}$$

Using (3.12) and (3.13), we can deduce that

$$\begin{aligned}
A_1 = &\int_M\langle\nabla_i\dot u,J(u)\sigma(x)\nabla_t\nabla_i\nabla_ju_j\rangle\sigma(x)\,dM\\
\le &\ -2\int_M\langle\nabla_i\sigma(x)\cdot\nabla_i\dot u,J(u)\nabla_j\nabla_j\dot u\rangle\sigma(x)\,dM
\end{aligned}$$

$$+C(M,N,\sigma)\int_M |\nabla\dot{u}|\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |du|^3|\dot{u}|\}\,dM. \tag{3.14}$$

Similarly,

$$\begin{aligned} A_2 = & \int_M \langle \nabla_i\sigma(x)\cdot\nabla_i\dot{u}, J(u)\nabla_t\nabla_j u_j\rangle\}\sigma(x)\,dM \\ \le & \int_M \langle \nabla_i\sigma(x)\cdot\nabla_i\dot{u}, J(u)\nabla_j\nabla_j\dot{u}\rangle\sigma(x)\,dM \\ & +C(N,\sigma)\int_M |\nabla\dot{u}||\dot{u}||du|^2\,dM. \end{aligned} \tag{3.15}$$

To compute $A_3$, note that, by integrating by parts,

$$\begin{aligned} A_3 = & \int_M \langle \nabla_i\dot{u}, J(u)\nabla_l\sigma(x)\cdot\nabla_t\nabla_i u_l\rangle\sigma(x)\,dM \\ = & -\int_M \langle \dot{u}, J(u)\nabla_i\nabla_l\sigma(x)\cdot\nabla_t\nabla_i u_l\rangle\,dM \\ & -\int_M \langle \dot{u}, J(u)\nabla_l\sigma(x)\cdot\nabla_i\nabla_t\nabla_i u_l\rangle\sigma(x)\,dM \\ & -\int_M \langle \dot{u}, J(u)\nabla_l\sigma(x)\cdot\nabla_t\nabla_i u_l\rangle\nabla_i\sigma(x)\,dM. \end{aligned} \tag{3.16}$$

Then, using the commutative relation of covariant differential

$$\begin{aligned} \nabla_i\nabla_t\nabla_i u_l = & \ \nabla_i\nabla_i\nabla_l\dot{u} + \nabla_i(R(u_i,\dot{u})u_l) \\ = & \ \nabla_i\nabla_i\nabla_l\dot{u} + R(\nabla_i u_i,\dot{u})u_l + R(u_i,\nabla_i\dot{u})u_l + R(u_i,\dot{u})\nabla_i u_l + (\nabla_i R)(u_i,\dot{u})u_l \\ = & \ \nabla_i(\nabla_l\nabla_i\dot{u} + R(u_l,u_i)\dot{u}) + R(\nabla_i u_i,\dot{u})u_l + R(u_i,\nabla_i\dot{u})u_l \\ & +R(u_i,\dot{u})\nabla_i u_l + (\nabla_i R)(u_i,\dot{u})u_l \\ = & \ \nabla_l\nabla_i\nabla_i\dot{u} + 2R(u_l,u_i)\nabla_i\dot{u} + R(\nabla_i u_l,u_i)\dot{u} + R(u_l,\nabla_i u_i)\dot{u} + (\nabla_i R)(u_l,u_i)\dot{u} \\ & +R(\nabla_i u_i,\dot{u})u_l + R(u_i,\nabla_i\dot{u})u_l + R(u_i,\dot{u})\nabla_i u_l + (\nabla_i R)(u_i,\dot{u})u_l. \end{aligned}$$

Hence, the second term on the right-hand side of (3.16) becomes

$$\begin{aligned} & \int_M \langle \dot{u}, J(u)\nabla_l\sigma(x)\cdot\nabla_i\nabla_t\nabla_i u_l\rangle\sigma(x)\,dM \\ & = \int_M \langle \dot{u}, J(u)\nabla_l\sigma(x)\cdot\nabla_l\nabla_i\nabla_i\dot{u}\rangle\sigma(x)\,dM + \int_M \langle \dot{u}, P(\sigma)(du,\nabla du,\dot{u},\nabla\dot{u})\rangle\sigma(x)\,dM \\ & = -\int_M \langle \nabla_l\sigma(x)\cdot\nabla_l\dot{u}, J(u)\nabla_i\nabla_i\dot{u}\rangle\}\sigma(x)\,dM - \int_M \langle \dot{u}, J(u)\Delta\sigma(x)\nabla_i\nabla_i\dot{u}\rangle\}\sigma(x)\,dM \\ & \quad +\int_M \langle \dot{u}, P(\sigma)(du,\nabla du,\dot{u},\nabla\dot{u})\rangle\sigma(x)\,dM \\ & = \int_M \langle \dot{u}, P_1(\sigma)(du,\nabla du,\dot{u},\nabla\dot{u})\rangle\sigma(x)\,dM - \int_M \langle \nabla_l\sigma(x)\cdot\nabla_l\dot{u}, J(u)\nabla_i\nabla_i\dot{u}\rangle\}\sigma(x)\,dM, \end{aligned}$$

where $P_1(\sigma)(du,\nabla du,\dot{u},\nabla\dot{u})$ is a multilinear functional with coefficients dependent on $\sigma$ and satisfies

$$|P_1(\sigma)(du,\nabla du,\dot{u},\nabla\dot{u})| \le C(M,N,\sigma)\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |\nabla\dot{u}| + |\dot{u}||du|^3\}.$$

The other terms on the right-hand side of (3.16) are handled similarly. Finally, direct calculation yields

$$
\begin{aligned}
A_3 = & \int_M \langle \nabla_i \dot{u}, J(u)\nabla_l \sigma(x)\cdot \nabla_t \nabla_i u_l\rangle \sigma(x)\, dM \\
= & \int_M \langle \nabla_i \dot{u}, J(u)\nabla_i\nabla_l \sigma(x)\cdot \nabla_l \dot{u}\rangle \sigma(x)\, dM \\
& + \int_M \langle \nabla_l \sigma(x)\cdot \nabla_l \dot{u}, J(u)\nabla_i\nabla_i \dot{u}\rangle \sigma(x)\, dM \\
& + \int_M \langle \nabla_i \dot{u}, J(u)\nabla_l \sigma(x)\cdot \nabla_l \dot{u}\rangle \nabla_i\sigma(x)\, dM \\
& + \int_M \langle \dot{u}, P_2(\sigma)(du, \nabla du, \dot{u}, \nabla\dot{u})\rangle\, dM,
\end{aligned} \tag{3.17}
$$

where $P_2(\sigma)(du, \nabla du, \dot{u}, \nabla \dot{u})$ is a multilinear functional with coefficients dependent on $\sigma$ and satisfies

$$
\begin{aligned}
|P_2(\sigma)(du, \nabla du, \dot{u}, \nabla\dot{u})| \quad & \leq C(M,N,\sigma)\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| \\
& + |\dot{u}||du|^2 + |du|^3|\dot{u}| + |\nabla\dot{u}|\}.
\end{aligned} \tag{3.18}
$$

Thus,

$$
\begin{aligned}
A_3 \leq & \int_M \langle \nabla_l \sigma(x)\cdot \nabla_l \dot{u}, J(u)\nabla_i\nabla_i \dot{u}\rangle \sigma(x)\, dM + C(\sigma)\int_M |\nabla\dot{u}|^2\, dM \\
& + C(M,N,\sigma)\int_M |\dot{u}|\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |\dot{u}||du|^2 + |du|^3|\dot{u}| + |\nabla\dot{u}|\}\, dM.
\end{aligned} \tag{3.19}
$$

Finally, it is obvious that

$$
A_4 \leq C(\sigma)\int_M |\nabla\dot{u}|^2\, dM. \tag{3.20}
$$

Substituting (3.13), (3.15), (3.19) and (3.20) into (3.11), we obtain that

$$
\begin{aligned}
I_2 = \quad & A_1 + A_2 + A_3 + A_4 \\
\leq \quad & C(\sigma)\int_M |\nabla\dot{u}|^2\, dM \\
& + C(M,N,\sigma)\int_M |\dot{u}|\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |\dot{u}||du|^2 + |du|^3|\dot{u}| + |\nabla\dot{u}|\}\, dM \\
& + C(M,N,\sigma)\int_M |\nabla\dot{u}|\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |du|^3|\dot{u}| + |\dot{u}||du|^2\}\, dM.
\end{aligned} \tag{3.21}
$$

By the same argument as done in the estimates for $I_2$, we also infer that, for $0 < \varepsilon < 1$,

$$
\begin{aligned}
I_1 \leq \quad & -\varepsilon\int_M \langle \Delta\dot{u}, \Delta\dot{u}\rangle \sigma^2(x)\, dM + C(\sigma)\int_M |\nabla\dot{u}|^2\, dM \\
& + C(M,N,\sigma)\int_M |\dot{u}|\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |\dot{u}||du|^2 + |du|^3|\dot{u}| + |\nabla\dot{u}|\}\, dM \\
& + C(M,N,\sigma)\int_M |\nabla\dot{u}|\{|\dot{u}||du||\nabla du| + |du|^2|\nabla\dot{u}| + |\dot{u}||du|^2 + |du|^3|\dot{u}|\}\, dM.
\end{aligned} \tag{3.22}
$$

Combining (3.21) and (3.22), we obtain from (3.9) that, for $0 < \varepsilon < 1$,

$$\begin{aligned}
&\frac{d}{dt}\int_M |\nabla \dot{u}|^2 \sigma(x)\, dM + 2\varepsilon \int_M \langle \Delta \dot{u}, \Delta \dot{u}\rangle \sigma^2(x)\, dM \\
&\le C(\sigma)\left\{ \int_M |\nabla \dot{u}|^2\, dM + \int_M |\dot{u}|^2 |du\|\nabla du|\, dM \right.\\
&\quad \left. + \int_M \dot{u}(|du|^2|\nabla \dot{u}| + |\dot{u}||du|^2 + |du|^3|\dot{u}| + |\nabla \dot{u}|)\, dM \right\}\\
&\quad + C(\sigma)\int_M |\nabla \dot{u}|\{|\dot{u}\|du\|\nabla du| + |du|^2|\nabla \dot{u}| + |du|^3|\dot{u}| + |\dot{u}||du|^2\}\, dM.
\end{aligned}$$

By Hölder's inequality, it follows from the last inequality that

$$\begin{aligned}
&\frac{d}{dt}\int_M |\nabla \dot{u}|^2 \sigma(x)\, dM + 2\varepsilon \int_M \langle \Delta \dot{u}, \Delta \dot{u}\rangle \sigma^2(x)\, dM \\
&\le C(\sigma)\left\{ \int_M |\nabla \dot{u}|^2\, dM + \|du\|_{C^0}\int_M |\dot{u}|^2 |\nabla du|\, dM \right\}\\
&\quad + C(\sigma)\int_M \|du\|_{C^0}\{|\nabla \dot{u}||\dot{u}||\nabla du| + \|du\|_{C^0}(|\nabla \dot{u}|^2 + (1 + \|du\|_{C^0})|\dot{u}||\nabla \dot{u}|)\}\, dM \\
&\quad + C(\sigma, \|du\|^2_{C^0})\int_M (|\nabla \dot{u}||\dot{u}| + |\dot{u}|^2(1 + \|du\|_{C^0}))\, dM \\
&\le C(\sigma, \|du\|_{C^0})\{\|\nabla \dot{u}\|^2_{L^2} + \|\dot{u}\|_{L^2} + \|\dot{u}\|^2_{L^4} + \|\nabla du\|_{L^4}\}\\
&\quad + C(\|du\|_{C^0}, \|\dot{u}\|_{L^2}, E_\sigma(u_0)).
\end{aligned} \tag{3.23}$$

Here we have used the following inequality derived from Proposition 2.1:

$$\begin{aligned}
\|\nabla du\|^2_{L^2} \le\ & C\{\|\tau(u)\|^2_{L^2} + E_\sigma(u_0)\}\\
\le\ & C(\sigma)\{\|\tau_\sigma(u)\|^2_{L^2} + E_\sigma(u_0)\}\\
\le\ & C(\sigma)\{\|\dot{u}\|^2_{L^2} + E_\sigma(u_0)\}.
\end{aligned} \tag{3.24}$$

By Propositions 2.2, 2.4 and 2.5,

$$\begin{aligned}
\|\nabla du\|_{L^4} \le\ & C(M,N)\{\|du\|_{W^{1,4}} + \|du\|^2_{C^0}\}\\
\le\ & C(M,N)\{\|du\|_{W^{1,4}} + \|du\|^2_{C^0}\}\\
\le\ & C(M,N)\{\|du\|^{\frac{3}{4}}_{W^{2,2}}\|du\|^{\frac{1}{4}}_{L^2} + \|du\|^2_{C^0}\}\\
\le\ & C(M, N, E_\sigma(u), \|\nabla \tau(u)\|_{L^2})\\
\le\ & C(\sigma, E_\sigma(u_0), \|\dot{u}\|_{L^2}, \|\nabla \dot{u}\|_{L^2})\\
\le\ & C(\sigma, E_\sigma(u_0), \|\dot{u}\|_{H^{1,2}(\sigma)}),
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
\|\dot{u}\|_{L^4} \le\ & C(M,N)\|\dot{u}\|_{W^{1,2}}\\
\le\ & C(\sigma, E_\sigma(u_0), \|\dot{u}\|_{L^2}, \|\nabla \dot{u}\|_{L^2})\\
\le\ & C(\sigma, E_\sigma(u_0), \|\dot{u}\|_{H^{1,2}(\sigma)}).
\end{aligned} \tag{3.26}$$

Plugging the last two inequalities into (3.23), we obtain

$$\frac{d}{dt}\int_M |\nabla \dot{u}|^2 \sigma(x)\, dM \le C(\sigma, E_\sigma(u_0), \|\dot{u}\|_{H^{1,2}(\sigma)}). \tag{3.27}$$

Consequently, by (3.5), it follows that

$$\frac{d}{dt}\int_M |\nabla \tau_\sigma(u)|^2 \sigma(x)\, dM \le C(\sigma, M, N, E_\sigma(u_0))(\|\tau(u)\|_{H^{1,2}(\sigma)}), \tag{3.28}$$

where $F_2^\sigma(\cdot)$ is a smooth positive function which depends on $\sigma$, $E_\sigma(u_0)$, $M$ and $N$, but not on $\varepsilon$. Thus, combining (3.8) and (3.28), we have

$$\begin{aligned}\frac{d}{dt}\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)} \le \quad & F_1^\sigma(\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)}) + F_2^\sigma(\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)}) \\ \equiv \quad & F^\sigma(\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)}),\end{aligned} \tag{3.29}$$

where $F^\sigma(\,\cdot\,)$ does not depend on $\varepsilon$.

Now consider the initial value problem for the ordinary differential equation

$$\begin{cases} \dfrac{dq(x)}{dt} = F^\sigma(q(x)), \\ q(0) = \|\tau_\sigma(u_0)\|_{H^{1,2}(\sigma)}. \end{cases}$$

It is easy to see that this problem has a local smooth solution. Hence, by the comparison principle of ordinary differential equations, there exists a positive real number $T(u_0)$, which does not depend on $\varepsilon$, such that on $[0, T(u_0))$

$$\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)} \le C(M, N, T(u_0)).$$

Combining this with the inequality

$$\int_M |\nabla \tau(u)|^2\, dM \le C(\sigma)\{\|\tau_\sigma(u)\|_{H^{1,2}(\sigma)} + \sqrt{E_\sigma(u_0)}\},$$

we obtain

$$\sup_{t\in[0,T)} \int_M |\nabla \tau(u)|^2\, dM \le C(T, u_0),$$

where $C(T, u_0)$ depends on the $C^2$-norm of $\sigma$, but does not depend on $\varepsilon$. This completes the proof of Lemma 3.1.

**Remark 1** From the above proof, we see that $T = T(E_\sigma(u_0))$ depends only on $E_\sigma(u_0)$, $\sigma(x)$, and the geometry of $M$ and $N$.

**Remark 2** It is easy to see that the following estimates hold uniformly for $0 < \varepsilon < 1$:

$$\sup_{t\in[0,T)} \|\dot{u}(t)\|_{L^2}^2 \le C(T, u_0) \text{ and } \sup_{t\in[0,T)} \|\nabla \dot{u}(t)\|_{L^2}^2 \le C(T, u_0).$$

**Lemma 3.2** *Under the same assumptions as in Lemma* 3.1*, there exists a positive $T$, which is independent of* $0 < \varepsilon < 1$*, such that the following hold uniformly for* $0 < \varepsilon < 1$*:*

$$\sup_{t\in[0,T)} \int_M |\Delta \tau(u)|^2\, dM \le C(T, u_0),$$

*and*

$$\sup_{t\in[0,T)}\int_M |\nabla\Delta\tau(u)|^2\,dM \le C(T,u_0).$$

*Here, $C(T,u_0)$ depends the $C^4$-norm of $\sigma$, but does not depend on $\varepsilon$.*

*Proof* As $\nabla J \equiv 0$, by a direct calculation we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_M |\Delta\dot u|^2\sigma^2\,dM = \;& \int_M \langle\Delta\dot u, \nabla_t\Delta\dot u\rangle\sigma^2\,dM\\ =\;& \varepsilon\int_M \langle\Delta\dot u, \nabla_t\nabla_i\nabla_i(\sigma\nabla_j u_j + \nabla_j\sigma\cdot u_j)\rangle\sigma^2\,dM\\ &+\int_M \langle\Delta\dot u, J(u)\nabla_t\nabla_i\nabla_i(\sigma\nabla_j u_j + \nabla_j\sigma\cdot u_j)\rangle\sigma^2\,dM\\ \equiv\;& \mathcal{I}_1+\mathcal{I}_2.\end{aligned}\tag{3.30}$$

We write

$$\begin{aligned}\mathcal{I}_1 =\;& \varepsilon\int_M\langle\Delta\dot u, \sigma\nabla_t\nabla_i\nabla_i\tau(u)\rangle\sigma^2\,dM + 2\varepsilon\int_M\langle\Delta\dot u, \nabla_i\sigma\cdot\nabla_t\nabla_i\tau(u)\rangle\sigma^2\,dM\\ &+\varepsilon\int_M\langle\Delta\dot u, \nabla_i\nabla_i\sigma\cdot\nabla_t\tau(u)\rangle\sigma^2\,dM + \varepsilon\int_M\langle\Delta\dot u, \nabla_j\sigma\cdot\nabla_t\nabla_i\nabla_i u_j\rangle\sigma^2\,dM\\ &+2\varepsilon\int_M\langle\Delta\dot u, \nabla_i\nabla_j\sigma\cdot\nabla_t\nabla_i u_j\rangle\sigma^2\,dM + \varepsilon\int_M\langle\Delta\dot u, \nabla_i\nabla_i\nabla_j\sigma\cdot\nabla_t u_j\rangle\sigma^2\,dM\\ \equiv\;& \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{A}_5+\mathcal{A}_6.\end{aligned}\tag{3.31}$$

and

$$\begin{aligned}\mathcal{I}_2 =\;& \int_M\langle\Delta\dot u, \sigma J(u)\nabla_t\nabla_i\nabla_i\tau(u)\rangle\sigma^2\,dM\\ &+2\int_M\langle\Delta\dot u, J(u)\nabla_i\sigma\cdot\nabla_t\nabla_i\tau(u)\rangle\sigma^2\,dM\\ &+\int_M\langle\Delta\dot u, J(u)\nabla_i\nabla_i\sigma\cdot\nabla_t\tau(u)\rangle\sigma^2\,dM\\ &+\int_M\langle\Delta\dot u, J(u)\nabla_j\sigma\cdot\nabla_t\nabla_i\nabla_i u_j\rangle\sigma^2\,dM\\ &+2\int_M\langle\Delta\dot u, J(u)\nabla_i\nabla_j\sigma\cdot\nabla_t\nabla_i u_j\rangle\sigma^2\,dM\\ &+\int_M\langle\Delta\dot u, J(u)\nabla_i\nabla_i\nabla_j\sigma\cdot\nabla_t u_j\rangle\sigma^2\,dM\\ \equiv\;& \mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4+\mathcal{B}_5+\mathcal{B}_6.\end{aligned}\tag{3.32}$$

Now, we compute the terms in $\mathcal{I}_1$. Note that

$$\begin{aligned}\nabla_t\nabla_i\nabla_i\tau(u) &= \nabla_i\nabla_t\nabla_i\tau(u) + R(u_i,\dot u)\nabla_i\tau(u)\\ &= \nabla_i\nabla_i\nabla_t\tau(u) + \nabla_i(R(u_i,\dot u)\tau(u)) + R(u_i,\dot u)\nabla_i\tau(u)\\ &= \nabla_i\nabla_i\nabla_j\nabla_j\dot u + \nabla_i\nabla_i(R(u_j,\dot u)u_j) + \nabla_i(R(u_i,\dot u)\tau(u))\\ &\quad +R(u_i,\dot u)\nabla_i\tau(u).\end{aligned}\tag{3.33}$$

Also, we have

$$\begin{aligned}\nabla_i(R(u_i,\dot u)\tau(u)) =\;& R(\tau(u),\dot u)\tau(u) + R(u_i,\nabla_i\dot u)\tau(u)\\ &+R(u_i,\dot u)\nabla_i\tau(u) + (\nabla_i R)(u_i,\dot u)\tau(u),\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}\nabla_i\nabla_i(R(u_j,\dot{u})u_j) =& \nabla_i\{(\nabla_i R)(u_j,\dot{u})u_j + R(\nabla_i u_j,\dot{u})u_j + R(u_j,\nabla_i\dot{u})u_j + R(u_j,\dot{u})\nabla_i u_j\}\\ =& R(\Delta u_j,\dot{u})u_j + R(\nabla_i u_j,\nabla_i\dot{u})u_j + R(\nabla_i u_j,\dot{u})\nabla_i u_j\\ &+ R(\nabla_i u_j,\nabla_i\dot{u})u_j + R(u_j,\Delta\dot{u})u_j + R(u_j,\nabla_i\dot{u})\nabla_i u_j\\ &+ R(\nabla_i u_j,\dot{u})\nabla_i u_j + R(u_j,\nabla_i\dot{u})\nabla_i u_j + R(u_j,\dot{u})\Delta u_j + (\nabla_i\nabla_i R)(u_j,\dot{u})u_j\\ &+ 2(\nabla_i R)(\nabla_i u_j,\dot{u})u_j + 2(\nabla_i R)(u_j,\nabla_i\dot{u})u_j + 2(\nabla_i R)(u_j,\dot{u})\nabla_i u_j. \quad (3.35)\end{aligned}$$

Here $\Delta\dot{u} = \nabla_i\nabla_i\dot{u}$ and $\Delta u_j = \nabla_i\nabla_i u_j$. Substituting (3.33)–(3.35) into the first term $\mathcal{A}_1$ of the right-hand side of (3.31) and using the estimate $\|du\|_{C^0} \le C(T,u_0)$, we have

$$\begin{aligned}\mathcal{A}_1 \le& -\varepsilon\int_M |\nabla\Delta\dot{u}|^2\sigma^3\,dM - 3\int_M \langle\Delta\dot{u},\nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM\\ &+ C(M,N,T,u_0)\left\{\int_M |\Delta\dot{u}|\{|\tau(u)|^2|\dot{u}| + |\nabla\dot{u}||\tau(u)| + |\dot{u}||\nabla\dot{u}|\}\,dM\right.\\ &\left.+\int_M |\Delta\dot{u}|\{|\nabla^2 du||\dot{u}| + |\Delta\dot{u}| + |\nabla\dot{u}||\nabla du| + |\nabla du|^2|\dot{u}| + |\nabla du||\dot{u}| + |\nabla\dot{u}|\}\,dM\right\}. \quad (3.36)\end{aligned}$$

Thus, by using Hölder's inequality on the integral terms on the right-hand side of (3.36), it follows that

$$\begin{aligned}&C(M,N,T,u_0)\left\{\int_M |\Delta\dot{u}|\{|\tau(u)|^2|\dot{u}| + |\nabla\dot{u}||\tau(u)| + |\dot{u}||\nabla\dot{u}|\}\,dM\right.\\ &\quad\left.+\int_M |\Delta\dot{u}|\{|\nabla^2 du||\dot{u}| + |\Delta\dot{u}| + |\nabla\dot{u}||\nabla du| + |\nabla du|^2|\dot{u}| + |\nabla du||\dot{u}| + |\nabla\dot{u}|\}\,dM\right\}\\ &\le C(M,N,T,u_0)\{\|\Delta\dot{u}\|_{L^2}^2\|\tau(u)\|_{L^6}\|\dot{u}\|_{L^6}^2 + \|\tau(u)\|_{C^0}\|\nabla\dot{u}\|_{L^2}\|\Delta\dot{u}\|_{L^2}\\ &\quad+ \|\dot{u}\|_{C^0}\|\nabla^2 du\|_{L^2}\|\Delta\dot{u}\|_{L^2} + \|\Delta\dot{u}\|_{L^2}^2 + \|\Delta\dot{u}\|_{L^2}\|\nabla du\|_{C^0}\|\nabla\tau(u)\|_{L^2}\\ &\quad+ \|\Delta\dot{u}\|_{L^2}\|\nabla du\|_{L^6}^2\|\dot{u}\|_{L^2} + \|\nabla du\|_{L^2}\|\dot{u}\|_{C^0} + \|\nabla\dot{u}\|_{L^2}\}. \quad (3.37)\end{aligned}$$

By the Sobolev imbedding theorem, we have

$$\|\tau(u)\|_{L^6} \le C(M,N)\|\tau(u)\|_{H^{1,2}}, \quad \|\dot{u}\|_{L^6} \le C(M,N)\|\dot{u}\|_{H^{1,2}}. \quad (3.38)$$

Further, the Sobolev imbedding theorem and Proposition 2.3 imply that

$$\begin{aligned}\|\tau(u)\|_{C^0} \le& C\{\|\Delta\tau(u)\|_{L^2} + C(M,N)(\|\nabla\tau(u)\|_{L^2}, E(u_0))\}\\ \le& C\{\|\Delta\tau_\sigma(u)\|_{L^2} + C(M,N)(\|\nabla\tau_\sigma(u)\|_{L^2}, E_\sigma(u_0))\}, \quad (3.39)\end{aligned}$$

and

$$\begin{aligned}\|\nabla du\|_{C^0} \le& \quad C(M,N)\{\|\Delta\tau(u)\|_{L^2} + C(\|\nabla\tau(u)\|_{L^2}, E(u_0))\}\\ \le& \quad C(M,N)\{\|\Delta\tau_\sigma(u)\|_{L^2} + C(\|\nabla\tau_\sigma(u)\|_{L^2}, E_\sigma(u_0))\}.\end{aligned} \quad (3.40)$$

Also, by Proposition 2.3, it is not difficult to find that $\|du\|_{H^{2,2}} \le \|du\|_{W^{2,2}} + C(\|\tau(u)\|_{L^2}, E(u_0))$. (see Lemma 4.3 in [18]) Using these estimates, the Kato inequality, Propositions 2.4

and 2.5, we obtain that

$$\begin{aligned}
\|\nabla du\|_{L^6} \le \quad & C(M,N)\{\|\nabla du\|_{L^2} + \|\nabla|\nabla du|\|_{L^2}\}^{\frac{2}{3}}\|\nabla du\|_{L^2}^{\frac{1}{3}} \\
\le \quad & C(M,N)\{\|\nabla du\|_{L^2} + \|\nabla^2 du\|_{L^2}\}^{\frac{2}{3}}\|\nabla du\|_{L^2}^{\frac{1}{3}} \\
\le \quad & C(M,N)\|du\|_{H^{2,2}}^{\frac{2}{3}}\|\nabla du\|_{L^2}^{\frac{1}{3}} \\
\le \quad & C(m,N,\|\nabla\tau(u)\|_{L^2}^2, E_\sigma(u_0))\|\nabla du\|_{L^2}^{\frac{1}{3}} \\
\le \quad & C(T,u_0).
\end{aligned} \tag{3.41}$$

Subsituting (3.37)–(3.41) into (3.36), we obtain

$$\begin{aligned}
\mathcal{A}_1 \le \quad & -\varepsilon\int_M |\nabla\Delta\dot{u}|^2\sigma^2\,dM - 3\varepsilon\int_M \langle\Delta\dot{u}, \nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM \\
& + C_1(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}.
\end{aligned} \tag{3.42}$$

Note that, by invoking the commutation relation of the covariant differentials,

$$\int_M \langle\nabla_i\nabla_j\dot{u}, \nabla_i\nabla_j\dot{u}\rangle\,dM = -\int_M \langle\nabla_i\nabla_i\nabla_j\dot{u}, \nabla_j\dot{u}\rangle\,dM \le \int_M \langle\Delta\dot{u},\Delta\dot{u}\rangle\,dM + C(E_\sigma(u), \|\dot{u}\|_{H^{1,2}}).$$

Using this inequality and similar arguments, we obtain

$$\mathcal{A}_2 \le 2\varepsilon\int_M \langle\Delta\dot{u}, \nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM + C_2(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}, \tag{3.43}$$

$$\mathcal{A}_4 \le \varepsilon\int_M \langle\Delta\dot{u}, \nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM + C_3(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}, \tag{3.44}$$

and

$$\mathcal{A}_3 + \mathcal{A}_5 + \mathcal{A}_6 \le C_4(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}. \tag{3.45}$$

It follows from (3.42)–(3.45), and the fact that $\min|\sigma(x)| > 0$, that

$$\mathcal{I}_1 \le -\varepsilon\int_M |\nabla\Delta\dot{u}|^2\sigma^2\,dM + C(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\sigma^2\,dM + 1\Big\}, \tag{3.46}$$

where $C(T,u_0)$ does not depend on $0<\varepsilon<1$.

Next, we turn to $\mathcal{I}_2$. By similar arguments, we have

$$\mathcal{B}_1 \le -3\int_M \langle\Delta\dot{u}, J(u)\nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM + C_5(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}, \tag{3.47}$$

$$\mathcal{B}_2 \le 2\int_M \langle\Delta\dot{u}, J(u)\nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM + C_6(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}, \tag{3.48}$$

$$\mathcal{B}_4 \le \int_M \langle\Delta\dot{u}, J(u)\nabla_i\sigma\cdot\nabla_i\Delta\dot{u}\rangle\sigma^2\,dM + C_7(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}, \tag{3.49}$$

and

$$\mathcal{B}_3 + \mathcal{B}_5 + \mathcal{B}_6 \le C_8(T,u_0)\Big\{\int_M |\Delta\dot{u}|^2\,dM + 1\Big\}. \tag{3.50}$$

It follows from (3.47)–(3.50) that

$$\mathcal{I}_2 \leq C(T, u_0)\Big\{ \int_M |\Delta \dot{u}|^2 \sigma^2 \, dM + 1 \Big\}, \tag{3.51}$$

where $C(T, u_0)$ does not depend on $0 < \varepsilon < 1$.

Finally, combining (3.30), (3.46) and (3.51), we obtain

$$\frac{d}{dt} \int_M |\Delta \dot{u}|^2 \sigma^2 \, dM \leq C(T, u_0)\Big\{ \int_M |\Delta \dot{u}|^2 \sigma^2 \, dM + 1 \Big\}. \tag{3.52}$$

It follows immediately by Gronwall's inequality that $\int_M |\Delta \dot{u}|^2 \, dM \leq C(T, u_0) \leq \infty$, $t \in [0, T)$, and hence by Proposition 2.5 and Lemma 3.1 $\int_M |\Delta \tau(u)|^2 \, dM \leq C(T, u_0) \leq \infty$, $t \in [0, T)$.

To derive the second estimate in the lemma, we consider the quantity $\frac{d}{dt} \int_M |\nabla \Delta \dot{u}|^2 \sigma^3 \, dM$. By the same arguments as above, we can dispose of the terms containing the sixth-order derivatives and employ the estimates established above for the remaining terms. Omitting the details, we eventually obtain

$$\frac{d}{dt} \int_M |\nabla \Delta \dot{u}|^2 \sigma^3 \, dM \leq C(M, N, \sigma, u_0) \left\{ \int_M |\nabla \Delta \dot{u}|^2 \sigma^3 \, dM + 1 \right\}.$$

This implies that $\int_M |\nabla \Delta \dot{u}|^2 \, dM \leq C(T, u_0)$. Hence, it follows that $\int_M |\nabla \Delta \tau(u)|^2 \, dM \leq C(T, u_0)$, and the proof of Lemma 3.2 is complete.

*Proof of Theorem* 1 If the initial map $u_0$ is smooth, from Lemmas 3.1 and 3.2, we know that, when $0 < \varepsilon < 1$, there exists a positive constant $C(T, u_0) \leq \infty$ such that

$$\sup_{t \in [0,T)} \int_M \{ |\tau(u^\varepsilon)|^2 + |\nabla \tau(u^\varepsilon)|^2 + |\Delta \tau(u^\varepsilon)|^2 + |\nabla \Delta \tau(u^\varepsilon)|^2 \} \, dM \leq C(T, u_0), \ t \in [0, T),$$

uniformly for all $0 < \varepsilon < 1$. Therefore, by Proposition 2.3, we are able to select a sequence $\{\varepsilon_i\}$, $\varepsilon_i \searrow 0$, such that $u^{\varepsilon_i} \to u$ [weak*] in $L^\infty([0, T), W^{5,2}(M))$. Obviously, $u$ is a solution of (3.1).

Now, assume that $u_0 \in H^{5,2}(M)$. By the well-known approximation theorem of Sobolev maps, there is a sequence $\{u_{0k}\}$ in $C^\infty(M, N)$ such that $u_{0k} \longrightarrow u_0$ in $H^{5,2}(M)$ as $k \to \infty$. By Lemmas 3.1 and 3.2, there is a positive $T'(E_\sigma(u_0)) > 0$, which does not depend on $k$, such that the Cauchy problem (3.2) with initial map $u_{0k}$ is well-posed on $M \times [0, T']$. The corresponding solution to the Cauchy problem is denoted by $u_k^\varepsilon$. First, let $\varepsilon_i \searrow 0$, then, upon extracting a subsequence and reindexing if necessary, $u_k^\varepsilon$ converges to $u_k \in L^\infty([0, T'], H^{5,2}(M))$. Moreover, $u_k$ satisfies (3.1) in the classical sense, and $\sup_{t \in [0,T']} \|u_k\|_{H^{5,2}} \leq C(T', E_\sigma(u_0))$. Finally, by sending $k$ to infinity, we obtain a classical solution to (3.1) with initial map $u_0$. The uniqueness has been addressed in [7].

**Final Remark** More generally, we may consider the Cauchy problem

$$\begin{cases} u_t = J(u)\{f(x,t)\tau(u) + \nabla f(x,t) \cdot du\}, \\ u(\cdot, 0) = u_0, \end{cases}$$

where $u(x, t)$ is a map from $M \times [0, T)$ into a Kähler manifold $(N, J)$. We would like to call the above nonautonomous, inhomogeneous Schrödinger equation the nonautonomous Schrödinger flow (NSF). When $N$ is compact and has nonpositive sectional curvature, under certain technical assumptions on $f(x,t)$, it is not difficult to establish the local existence theory for the Cauchy problem of NSF by using arguments presented in this paper for initial maps $u_0$ belonging to appropriate Sobolev spaces.

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