Acta Mathematica Sinica, English Series
1999, Oct., Vol.15, No.4, p. 569–574

Acta Mathematica Sinica, English Series

Linearized Oscillations in Nonlinear Delay Difference Equations

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Abstract Consider the nonlinear delay difference equation

$$x_{n+1} - x_n + \sum_{j=1}^{m} p_j f_j(x_{n-k_j}) = 0.$$

We establish a linearized oscillation result of this equation, which is the extension of the result in the paper [1].

Keywords Delay difference equation, Oscillatory, Nonoscillatory
1991MR Subject Classification 34C10, 34C15, 39A10

1 Main Results

Consider the following equation

$$x_{n+1} - x_n + \sum_{j=1}^{m} p_j f_j(x_{n-k_j}) = 0, \quad n \geq n_0, \tag{1}$$

and its linearized equation

$$x_{n+1} - x_n + \sum_{j=1}^{m} p_j x_{n-k_j} = 0, \quad n \geq n_0, \tag{2}$$

Received September 17, 1997, Revised February 23, 1998, Accepted June 12, 1998
Supported by the National Natural Science Foundation of China

where $n_0 \in \{0, 1, 2, \ldots\}$. A linearized oscillation result of Eq. (1) has been established by Jurang Yan et al. [1], provided: for all $j = 1, 2, \ldots, m$

$$p_j \in (0, +\infty) \qquad k_j \in \{0, 1, 2, \ldots\}, \tag{3}$$

$$f_j \in C(R, R), \qquad u f_j(u) > 0 \quad (u \neq 0), \tag{4}$$

$$\lim_{u \to 0} \frac{f_j(u)}{u} = 1. \tag{5}$$

For further research, we find only few functions satisfy Condition (4), so the application of this theorem is limited. For this reason, condition (4) is transformed into

$$f_j \in C(R, R), \quad \exists \alpha > 0, \quad u f_j(u) > 0 (u \neq 0), \quad u \in (-\alpha, \alpha). \tag{4$'$}$$

This work is motivated by [3]. We obtain the following result:

Theorem *If conditions* (3), (4)$'$ *and* (5) *are satisfied, then Eq.* (1) *oscillates if and only if Eq.* (2) *oscillates.*

2 Proof of the Theorem

Before proving the theorem, we introduce the following lemmas.

Lemma 1 [2] *Assume that* (3) *is satisfied. Then every solution of Eq.* (2) *oscillates if and only if*

$$x_{n+1} - x_n + \sum_{j=1}^{m} p_j x_{n-k_j} \leq 0 \tag{6}$$

has no eventually positive solution.

Lemma 2 [2] *Every solution of Eq.* (2) *oscillates if and only if its characteristic equation*

$$\lambda - 1 + \sum_{j=1}^{m} p_j \lambda^{-k_j} = 0 \tag{7}$$

has no positive roots.

Lemma 3 *Assume that* (3), (4)$'$ *and* (5) *are satisfied. If every solution of the following equation oscillates:*

$$x_{n+1} - x_n + (1 - \epsilon) \sum_{j=1}^{m} p_j x_{n-k_j} = 0, \qquad n \geq n_0, \tag{8}$$

then every solution of Eq. (1) *oscillates, where* $0 < \epsilon < 1$.

Proof Otherwise, there is a nonoscillatory solution x_n, and without loss of generality we assume that there is an eventually positive solution; we know there exists $n_1 \geq n_0$ such that $n \geq n_1$; we have $x_n, x_{n-k_j} > 0, (j = 1, \ldots, m)$. So $\Delta x_n < 0$ for $n \geq n_1$, that is, x_n is strictly decreasing.

Then there must be $\beta \geq 0$, such that $\lim_{n\to\infty} x_n = \beta$, we easily prove $\beta = 0$ by condition (4)$'$ and $p_j > 0$.

Since $\lim_{u\to 0} \frac{f_j(u)}{u} = 1$ $(j = 1, \ldots, m)$, there is $\delta > 0$ such that $u \in (0, \delta)$, we have

$$(1-\epsilon)u < f_j(u) < (1+\epsilon)u, \quad j = 1, 2, \ldots, m. \tag{9}$$

Since x_n is strictly decreasing on $\{n_1, n_1+1, \ldots\}$ and $\lim_{n\to\infty} x_n = 0$, there is $n_2 \geq n_1$ such that $n \geq n_2$, $x_n, x_{n-k_j} \in (0,\delta)(j = 1, \ldots, m)$ hold. Hence by the positivity of solution x_n of Eq. (1) and inequality (9), we have

$$x_{n+1} - x_n + (1-\epsilon)\sum_{j=1}^{m} p_j x_{n-k_j} \leq x_{n+1} - x_n + \sum_{j=1}^{m} p_j f_j(x_{n-k_j}) = 0.$$

By Lemma 1, we know this contradicts the fact that (8) oscillates.

Lemma 4 *Assume that* (3), (4)$'$ *and* (5) *are satisfied. If every solution of the following equation does not oscillate:*

$$x_{n+1} - x_n + (1+\epsilon)\sum_{j=1}^{m} p_j x_{n-k_j} = 0 \qquad n \geq n_0, \tag{10}$$

then every solution of Eq. (1) *does not oscillate, where* $0 < \epsilon < 1$.

Proof By (5), we know there is $\delta > 0$ such that $u \in (0, \delta)$, so we have

$$(1-\epsilon)u \leq f_j(u) \leq (1+\epsilon)u \quad j = 1, 2, \ldots, m. \tag{11}$$

By Lemma 2, we know that the characteristic equation of Eq. (10)

$$\lambda - 1 + (1+\epsilon)\sum_{j=1}^{m} p_j \lambda^{-k_j} = 0$$

has a positive root η, that is, η satisfies

$$\eta - 1 + (1+\epsilon)\sum_{j=1}^{m} p_j \eta^{-k_j} = 0. \tag{12}$$

Obviously, $0 < \eta < 1$. Let $k = \min_{1\leq j\leq m}\{k_j\}$. We denote by X the Banach space of real bounded continuous functions on $\{n_0 - k, n_0 - k + 1, \ldots\}$ with supremum norm. Suppose $S \subseteq X$ be made up of x_n with the following properties:

(i) $n \geq n_0$, x_n is nonincreasing and $x_n \equiv x_0\eta^{(n-n_0)}$ if $n \in \{n_0 - k, n_0 - k + 1, \ldots, n_0\}$,

(ii) $n \geq n_0$, $x_0\eta^{(n-n_0)} \leq x_n \leq x_0\eta^{-n}$,

(iii) $n \geq n_0$, $x_{n-k_j} \leq x_n\eta^{-k_j}$, $j = 1, \ldots, m$.

where $x_0 = x_{n_0}$ and satisfies $0 < x_0 < \delta\eta^k$.

Define operator F on S :

$$(Fx)(n) = \begin{cases} x_0\eta^{(n-n_0)} & n_0 - k \leq n \leq n_0, \\ x_0 \prod_{s=n_0}^{n-1} \left[1 - \sum_{j=1}^{m} p_j \dfrac{f_j(x_{s-k_j})}{x_s}\right], & n > n_0. \end{cases}$$

In the following, we will prove $FS \subseteq S$. Obviously $(Fx)(n)$ is nonincreasing and $(Fx)(n) \leq x_0\eta^{-k}$. By (11), (12) and the definition of F, we have

$$\begin{aligned}(Fx)(n) &= x_0 \prod_{s=n_0}^{n-1} \left[1 - \sum_{j=1}^{m} p_j \frac{f_j(x_{s-k_j})}{x_s}\right] \\ &\geq x_0 \prod_{s=n_0}^{n-1} \left[1 - (1+\epsilon) \sum_{j=1}^{m} p_j \frac{x_{s-k_j}}{x_s}\right] \\ &\geq x_0 \prod_{s=n_0}^{n-1} \left[1 - (1+\epsilon) \sum_{j=1}^{m} p_j \eta^{-k_j}\right] \\ &= x_0 \prod_{s=n_0}^{n-1} \eta = x_0 \eta^{(n-n_0)},\end{aligned}$$

and for each $i = 1, 2, \ldots, m$ and $n \geq n_0$, we have

$$\begin{aligned}\frac{(Fx)(n-k_i)}{(Fx)(n)} &= \frac{1}{\prod_{s=n-k_i}^{n-1}[1 - \sum_{j=1}^{m} p_j \frac{f_j(x_{s-k_j})}{x_s}]} \\ &\leq \frac{1}{\prod_{s=n-k_i}^{n-1}[1 - (1+\epsilon)\sum_{j=1}^{m} p_j \frac{x_{s-k_j}}{x_s}]} \\ &\leq \frac{1}{\prod_{s=n-k_i}^{n-1}[1 - (1+\epsilon)\sum_{j=1}^{m} p_j \eta^{-k_j}]} \\ &= \frac{1}{\prod_{s=n-k_i}^{n-1} \eta} = \eta^{-k_i};\end{aligned}$$

so, $FS \subseteq S$. Obviously S is nonempty (since $x_0\eta^{n-n_0} \in S$), close, convex, and we easily know the operator F is continuous. In order to prove FS is relative compact in X, we only prove $\triangle(Fx)(n)$ is uniformly bounded. Since

$$\triangle(Fx)(n) = -\sum_{j=1}^{m} p_j \frac{f_j(x_{n-k_j})}{x_n}(Fx)(n),$$

so

$$\begin{aligned}|\triangle(Fx)(n)| &= \sum_{j=1}^{m} p_j \frac{f_j(x_{n-k_j})}{x_n}(Fx)(n) \\ &\leq (1+\epsilon)\sum_{j=1}^{m} p_j \frac{x_{n-k_j}}{x_n} x_0 \eta^{-k} \\ &\leq (1+\epsilon)\sum_{j=1}^{m} p_j \eta^{-k_j} x_0 \eta^{-k} \\ &= (1-\eta)x_0\eta^{-k} = x_0(1-\eta)\eta^{-k}.\end{aligned}$$

Taking the above into account, we have proved that the operator F satisfies all the conditions of the Shauder fixed-point theorem. Hence F has fixed-point x in S such that $Fx = x$, and obviously x is eventually a positive solution of Eq. (1) with $x_{n_0} = x_0$.

Lemma 5 *Assume that Eq. (7) has no positive roots. Then there is $0 < \epsilon_0 < 1$ such that $|\epsilon| < \epsilon_0$, the equation*

$$\lambda - 1 + (1-\epsilon)\sum_{j=1}^{m} p_j \lambda^{-k_j} = 0 \tag{13}$$

has no positive roots.

Proof Let $F(\lambda) = \lambda - 1 + \sum_{j=i}^{m} p_j \lambda^{-k_j}$.

Since (7) does not have positive roots, it is easily seen that $F(\lambda) > 0 (\lambda > 0)$ and $h = \min_{\lambda>0} F(\lambda)$ exists and is positive. Supposing $F(\lambda_0) = h$, then

$$\lambda - 1 + \sum_{j=i}^{m} p_j \lambda^{-k_j} \geq F(\lambda_0) = h, \quad \lambda > 0.$$

Let the function

$$G(\epsilon, \lambda) = \lambda - 1 + (1-\epsilon)\sum_{j=1}^{m} p_j \lambda^{-k_j} \quad ((\epsilon, \lambda) \in (-1,1) \times (0, +\infty)).$$

Then

$$G'_\lambda(\epsilon, \lambda) = 1 - (1-\epsilon)\sum_{j=1}^{m} p_j k_j \lambda^{(-k_j-1)}.$$

Obviously $G'_\lambda(0, \lambda_0) = 0$. We consider equation $G'_\lambda(\epsilon, \lambda) = 0$ in a small enough neighborhood of the point $(0, \lambda_0)$, and easily check that function $G'_\lambda(\epsilon, \lambda)$ satisfies the conditions of implicit function theorem in the neighborhood of $(0, \lambda)$. So in the neighborhood of $(0, \lambda_0)$ there uniquely exists a continuous function $\lambda = \lambda(\epsilon)$, defined in the neighborhood $O(0, \theta)$ of point $\epsilon = 0$, and satisfying $G'_\lambda(\epsilon, \lambda(\epsilon)) \equiv 0$ and $\lambda_0 = \lambda(0)$, which implies that the uniquely minimum point $\lambda(\epsilon)$ of function $\lambda - 1 + (1-\epsilon)\sum_{j=1}^{m} p_j \lambda^{-k_j}$ is continuous about ϵ. So $\lim_{\epsilon \to 0} \lambda(\epsilon) = \lambda_0$, and

$$\lim_{\epsilon \to 0} \left[\lambda(\epsilon) - 1 + (1-\epsilon)\sum_{j=1}^{m} p_j \lambda(\epsilon)^{-k_j}\right] = \lambda_0 - 1 + \sum_{j=1}^{m} p_j \lambda_0^{-k_j} = h.$$

Therefore, there is $0 < \epsilon_0 < 1$ such that $|\epsilon| < \epsilon_0$; we have

$$\lambda(\epsilon) - 1 + (1-\epsilon)\sum_{j=1}^{m} p_j \lambda(\epsilon)^{-k_j} \geq h/2,$$

that is, when $|\epsilon| < \epsilon_0$, equation $\lambda - 1 + (1-\epsilon)\sum_{j=1}^{m} p_j \lambda^{-k_j} = 0$ has no positive roots.

Using a similar method for Lemma 5, we have

Lemma 6 *Assume Eq. (7) has positive roots. Then there is $0 < \epsilon_0 < 1$ such that $|\epsilon| < \epsilon_0$, the equation*

$$\lambda - 1 + (1+\epsilon)\sum_{j=1}^{m} p_j \lambda^{-k_j} = 0$$

has positive roots.

Proof the theorem If Eq. (2) oscillates, then by Lemma 2, we know Eq. (7) has no positive roots. By Lemma 5, there is $\epsilon_0 > 0$ such that $|\epsilon| < \epsilon_0$, the characteristic equation of Eq. (8)

has no positive roots. Because of Lemma 2, we know Eq. (8) oscillates, then Eq. (1) oscillates by Lemma 3.

If Eq. (2) does not oscillate, by Lemma 2, Eq. (7) has positive roots, by Lemma 6, there is $\epsilon_0 > 0$ such that $|\epsilon| < \epsilon_0$, Eq. (14) has positive roots. Because of Lemma 2, Eq. (10) does not oscillate, then Eq. (1) does not oscillate by Lemma 4.

For example Consider delay difference equations

$$x_{n+1} - x_n + \sin x_{n-k} = 0 \quad (k \geq 1)$$

and

$$x_{n+1} - x_n + x_{n-k} = 0 \quad (k \geq 1).$$

Obviously all coefficients satisfy conditions (3), (4) and (5), then we know they are equivalent on properties of oscillation by the theorem. But we can not draw this conclusion by the theorem in [1].

Acknowledgment The author would like to thank the referee for many valuable suggestions that improved the original manuscript.

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