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Global Spherically Symmetric Solutions for a Coupled Compressible Navier–Stokes/Allen–Cahn System

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Abstract In this paper, we consider the global spherically symmetric solutions for the initial boundary value problem of a coupled compressible Navier–Stokes/Allen–Cahn system which describes the motion of two-phase viscous compressible fluids. We prove the existence and uniqueness of global classical solution, weak solution and strong solution under the assumption of spherically symmetry condition for initial data ρ_0 without vacuum state.

Keywords Diffuse interface model, global solutions, Navier–Stokes equations, Allen–Cahn equation, spherically symmetry

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1 Introduction

In this paper, we are interested in a diffuse interface model for two-phase flows of viscous compressible fluids. As we know, in classical models, the interface between fluids is usually assumed to be separated by a sharp interface. But when the topological transitions such as droplet formation, coalescence or break-up of droplets occur, the sharp interface is replaced by a narrow transition layer and the fluids may mix undergoing phase transitions. Therefore, a phase field variable χ is introduced and a mixing energy can be defined as χ and its spatial gradient in this model. Moreover, this model is a strong coupling system between the Navier–Stokes equations and an Allen–Cahn equation by the Cauchy stress tensor. In this case, topological transitions can be described in a natural way. The effect of phase transition can also be described by different modified convective Cahn–Hilliard or other types of equations [3, 18, 27].

In this paper, we shall consider the following coupled Navier–Stokes/Allen–Cahn system

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proposed by Blesgen [5], which had been investigated in [17] as follows

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x(\mathbb{T}), \\ \partial_t(\rho\chi) + \mathrm{div}_x(\rho\chi\mathbf{u}) = -\mu(\rho, \chi, \Delta\chi), \\ \rho\mu = -\delta\Delta\chi + \rho\dfrac{\partial f(\rho,\chi)}{\partial\chi}, \end{cases}$$

where $\rho, \mathbf{u}$ and χ denote the density function,the velocity filed, and the phase variable, respectively. The Cauchy stress tensor

$$\mathbb{T} = \mathbb{S} - \delta\Big(\nabla_x\chi \otimes \nabla_x\chi - \frac{|\nabla_x\chi|^2}{2}\mathbb{I}\Big) - p(\rho,\chi)\mathbb{I},$$

$$\mathbb{S} = \nu(\chi)\Big(\nabla_x\mathbf{u} + \nabla_x^T\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big) + \eta(\chi)\mathrm{div}_x\mathbf{u}\mathbb{I},$$

$p = \rho^2\frac{\partial f(\rho,\chi)}{\partial\rho}$ is the thermodynamic pressure, μ represents the chemical potential and $f(\rho,\chi)$ is the potential energy density.

If we take the density ρ be a positive constant, then the system above reduces to an incompressible one. The diffuse interface models for two-phase flows of incompressible fluids have been extensively studied. For the coupled Navier–Stokes/Cahn–Hilliard system, Boyer [7] studied the existence of global weak and strong solutions in 2D, the existence of unique strong solution in 3D and the stability of the stationary solutions. Tachim-Medjo [31] proved the existence and the uniqueness of a coupled Cahn–Hilliard/Navier–Stokes model in a two dimensional bounded domain. There are also some results of the well-posedness, the asymptotic behavior, the attractor in [1, 19, 21, 22] and the reference therein. For the coupled Navier–Stokes/Allen–Cahn system with $\frac{\partial f}{\partial\chi} = \frac{1}{\delta}(\chi^3 - \chi)$, where δ is a positive constant and $\sqrt{\delta}$ represents the thickness of the interface, Zhao et al. [39] investigated the vanishing viscosity limit and proved that the solutions of the Navier–Stokes/Allen–Cahn system converge to that of the Euler/Allen–Cahn system in a proper small time interval. Kotschote [23] derived the more general model, and proved the existence and uniqueness of local strong solutions on a problem with a mixed boundary condition in bounded domain. Chen et al. [8] studied the global strong solutions for the initial boundary value problem with the heat conductivity proportional to a positive power of the temperature. Li and Huang [25] studied the existence and uniqueness of local strong solutions in 3D case. Babak et al. [4] provided a unified and comparative description of the most prominent phase field based two-phase flow models and presented the numerical results of the application of Galerkin-based isogeometric analysis to incompressible Navier–Stokes/Cahn–Hilliard equations in velocity-pressure-phase field-chemical potential formulation. Zhang [37] established a regularity criterion for the 3D incompressible density-dependent system. Gal and Grasseli [19] showed the existence of the trajectory attractor for both incompressible Navier–Stokes/Allen–Cahn and Navier–Stokes/Cahn–Hilliard systems. Moreover, for numerical simulations, such as jet pinching-off and drop formation, we refer the readers to [6, 26, 32, 36].

As far as we know, there are less theoretical available results for the compressible models. For the Navier–Stokes/Cahn–Hilliard system, Abels and Feireisl [2] derived the existence of weak solutions without any restriction on the size of the initial data. Kotschote and Zacher

[24] established the local existence of unique strong solution. Deugoué et al. [14] proved the existence and uniqueness of strong solutions of the stochastic 3D globally modified Navier–Stokes/Cahn–Hilliard model, and discussed the relation of the stochastic 3D globally modified Navier–Stokes/Cahn–Hilliard equations to the stochastic 3D Navier–Stokes/Cahn–Hilliard equations by proving a convergence theorem. For the isentropic Navier–Stokes/Allen–Cahn system, Chen et al. [10] investigated the global existence and uniqueness of strong and classical solutions of 1D initial boundary value problem. Feireisl et al. [17] proved the existence of weak solutions in 3D, where the density ρ is a measurable function. A different compressible Navier–Stokes/Allen–Cahn system arising from the biological material change in the process differentiation had been studied in [33]. In [30], Song et al. obtained the existence of the time-periodic solution to the system by using an approach of parabolic regularization and combining with the topology degree theory. Chen et al. [11] obtained the 1D global well-posedness with vacuum and constant viscosity. Ding et al. [13] derived the existence and uniqueness of global strong solutions with free boundary condition. Very recently, Chen and Zhu [12] proved the existence and uniqueness of global classical solutions and obtained a blow-up criterion for strong solutions with some assumption conditions. For the non-isentropic compressible Navier–Stokes/Allen–Cahn system with constant viscosity, Chen et al. [8, 9] studied global strong solutions of initial-boundary value problem and Cauchy problem. And Yan et al. [35] established global strong solutions for initial-boundary value problem with phase variable dependent viscosity in 1D. Thanks to the ideas of the investigation on the liquid crystals in Ding et al. [16], Ding et al. [15] established the global solutions for a coupled compressible Navier–Stokes/Allen–Cahn system in 1D. Recently, Zhang [38] established the regularity of solutions in $H^i([0,1])$ $(i = 2, 4)$ to the 1D isentropic compressible Navier–Stokes-Allen–Cahn system for initial data ρ_0 without vacuum states.

Motivated by Ding et al. [15] and Zhang [38], we shall establish the existence and uniqueness of global classical solution, the existence of weak solution and the existence of unique strong solution with spherically symmetric data for the 3D compressible Navier–Stokes/Allen–Cahn system.

For the compressible Navier–Stokes equations, the pressure p is usually chosen as $p = A\rho^\gamma$ with $A > 0$ and $\gamma > 1$, and for Allen–Cahn equation, double-well structural potential is often considered. Thus we take the specific free energy f as follows

$$f(\rho, \chi) = \frac{A\rho^{\gamma-1}}{\gamma - 1} + \frac{1}{\delta}\left(\frac{\chi^4}{4} - \frac{\chi^2}{2}\right),$$

where $\delta > 0$ and $\sqrt{\delta}$ denotes the thickness of the interface. Now we assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain and consider the following boundary conditions $\mathbf{u}|_{\partial\Omega} = 0, \quad \nabla\chi \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal outward vector at the boundary $\partial\Omega$ and the initial conditions $(\rho, \mathbf{u}, \chi)|_{t=0} = (\rho_0, \mathbf{u}_0, \chi_0)$.

In this paper, we study the spherically symmetric solutions of the system with the initial boundary conditions above. Now we construct the corresponding system for radial solutions. Let $r = |x|$ and take in the Euler coordinates

$$\rho(x, t) = \rho(r, t), \quad \mathbf{u}(x, t) = u(r, t)\frac{x}{r}, \quad \chi(x, t) = \chi(r, t).$$

Then the Navier–Stokes/Allen–Cahn system in $3D$ becomes

$$\begin{cases} \rho_t + (\rho u)_r + \dfrac{2}{r}\rho u = 0, \\ (\rho u)_t + (\rho u^2)_r + \dfrac{2}{r}\rho u^2 + A(\rho^\gamma)_r = \left(\dfrac{4}{3}\nu + \eta\right)\left(u_r + \dfrac{2}{r}u\right)_r - \dfrac{1}{2}\delta(\chi_r^2)_r - \dfrac{2}{r}\chi_r^2, \\ (\rho\chi)_t + (\rho\chi u)_r + \dfrac{2}{r}\rho\chi u = -\mu, \\ \rho\mu = -\left(\chi_{rr} + \dfrac{2}{r}\chi_r\right) + \rho(\chi^3 - \chi) \end{cases} \tag{1.1}$$

for $(r, t) \in (a, b) \times (0, +\infty)$ with $0 < a < b < +\infty$, where $\rho > 0, u, \chi$ denote the total density, the mean velocity of the fluid mixture, the concentration difference of the two component, respectively. Accordingly, the initial conditions become

$$(\rho, u, \chi)|_{t=0} = (\rho_0, u_0, \chi_0), \quad r \in (a, b) \tag{1.2}$$

and the usual no-slip boundary condition for the viscous fluid and natural condition for the concentration become

$$(u, \chi_r)|_{r=a,b} = (0, 0), \quad t \geq 0. \tag{1.3}$$

In this paper, we shall use the following notation.

(1) Let $I = (a, b)$, $\partial I = \{a, b\}$, $Q_T = I \times [0, T]$ for $T > 0$.

(2) For $p \geq 1$, $L^p = L^p(I)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, $W^{k,p} = W^{k,p}(I)$ denotes the Sobolev space with the norm $\|\cdot\|_{W^{k,p}}$, and $H^k = W^{k,2}$.

(3) For any points $P_1(r_1, t_1), P_2(r_2, t_2) \in Q_T$, we define the parabolic distance between them as

$$d(P_1, P_2) = (|r_1 - r_2|^2 + |t_1 - t_2|)^{\frac{1}{2}}.$$

(4) If $\varphi(x, t)$ is a function on Q_T, for $0 < \alpha < 1$, we define

$$[\varphi]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} = \sup_{P_1,P_2 \in Q_T, P_1 \neq P_2} \left\{ \frac{|\varphi(P_1) - \varphi(P_2)|}{d^\alpha(P_1, P_2)} \right\},$$

which is a semi-norm, and denote by $C^{\alpha,\frac{\alpha}{2}}(Q_T)$ the set of all function on Q_T such that $[\varphi]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} < +\infty$, endowed with norm

$$\|\varphi\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} = \|\varphi\|_{C^0(Q_T)} + [\varphi]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)},$$

where $\|\varphi\|_{C^0(Q_T)}$ is the maximum norm of $\varphi(x, t)$ on Q_T.

In the system (1.1), the density ρ is coupled in the Allen–Cahn equations $(1.1)_{3,4}$, which leads to more strong coupling than the corresponding incompressible one. More precisely, not only the velocity u enters in the equations for χ as a coefficient of lower order term χ_r, but also ρ appears in the coefficient of highest order term χ_t or χ_{rr}. Moreover, the concentration χ enters in Navier–Stokes equations $(1.1)_{1,2}$ for ρ and u with high order χ_{rr} which describes the capillary forces due to surface tension. Therefore, it will be essential to get good regularity estimates of χ. But it is not enough only basing on the regularities given by the energy estimates due to the strong coupling. So we must deal with the bounds and the regularities of ρ firstly. Song and Wang [29] have proved the local existence of unique classical solution by applying the contraction mapping theorem. Therefore, we shall extend it to the global one in terms of some a priori estimates. And by using weakly convergent method, we show the existence of weak solutions and the existence of unique strong solution.

Our first result is about classical solution.

Theorem 1.1 *Assume that $\rho_0 \in C^{1,\alpha}(I)$ satisfies $0 < c_0^{-1} \le \rho_0 \le c_0$ for some constants $\alpha \in (0,1)$ and $c_0 > 0$, $u_0, \chi_0 \in C^{2,\alpha}(I)$ with $u_0(0) = u_0(1) = 0$. Then the initial boundary value problem (1.1)–(1.3) admits a unique global classical solution (ρ, u, χ) satisfying that, for any $T > 0$, there exists a constant $c = c(c_0, T)$ such that*

$$(\rho_t, \rho_r) \in C^{\alpha,\frac{\alpha}{2}}(Q_T), \quad 0 < c^{-1} \le \rho \le c \text{ on } Q_T, \quad \textit{and} \quad (u, \chi) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T).$$

The following two theorems are concerned with the weak and strong solutions.

Theorem 1.2 *Assume that $\rho_0 \in H^1(I)$ satisfies $0 < c_0^{-1} \le \rho_0 \le c_0$ for some constant $c_0 > 0$ and $u_0 \in L^2(I)$, $\chi_0 \in H^1(I)$ with $u_0(0) = u_0(1) = 0$. Then the problem (1.1)–(1.3) admits a global weak solution (ρ, u, χ) satisfying that, for any $T > 0$, there exists a constant $c = c(c_0, T)$ such that*

$$\begin{aligned}
&\rho \in L^\infty(0,T;H^1(I)), \quad \rho_t \in L^2(0,T;L^2(I)), \quad 0 < c^{-1} \le \rho \le c, \\
&u \in L^\infty(0,T;L^2(I)) \cap L^2(0,T;H_0^1(I)), \\
&\chi \in L^\infty(0,T;H^1(I)) \cap L^2(0,T;H^2(I)), \quad \chi_t \in L^2(0,T;L^2(I)).
\end{aligned}$$

Theorem 1.3 *Assume that $\rho_0 \in H^1(I)$ satisfies $0 < c_0^{-1} \le \rho_0 \le c_0$ for some constant $c_0 > 0$ and $u_0 \in H_0^1(I)$, $\chi_0 \in H^2(I)$ with $u_0(0) = u_0(1) = 0$. Then the problem (1.1)–(1.3) admits a global strong solution (ρ, u, χ) satisfying that, for any $T > 0$, there exists a constant $c = c(c_0, T)$ such that*

$$\begin{aligned}
&\rho \in L^\infty(0,T;H^1(I)), \quad \rho_t \in L^\infty(0,T;L^2(I)), \quad 0 < c^{-1} \le \rho \le c, \\
&u \in L^\infty(0,T;H_0^1(I)) \cap L^2(0,T;H^2(I)), \quad u_t \in L^2(0,T;L^2(I)), \\
&\chi \in L^\infty(0,T;H^2(I)) \cap L^2(0,T;H^3(I)), \quad \chi_t \in L^\infty(0,T;L^2(I)) \cap L^2(0,T;H^1(I)).
\end{aligned}$$

Since the constants A, δ, ν don't play any key role in our analysis, we can assume that $A = \nu = \delta = 1$. Without loss of generality, we also set $\int_I r^2 \rho_0 dr = 1$. Moreover, we also assume $\eta = -\frac{1}{3}\nu$. Then we can also rewrite (1.1) as

$$\begin{cases}
(r^2\rho)_t + (r^2\rho u)_r = 0, \\
(r^2\rho u)_t + (r^2\rho u^2)_r + r^2(\rho^\gamma)_r = (r^2 u_r)_r - 2u - (r^2\chi_r^2)_r + \dfrac{1}{2}r^2(\chi_r^2)_r, \\
(r^2\rho\chi)_t + (r^2\rho\chi u)_r = -r^2\mu, \\
r^2\rho\mu = -(r^2\chi_r)_r + r^2\rho(\chi^3 - \chi).
\end{cases} \tag{1.4}$$

The rest of the paper is organized as follows. In Section 2, we shall establish some a priori estimates globally in time for the classical solution. Then basing on these a priori estimates, we shall prove our main theorems in Section 3.

2 A Priori Estimates

In this section, we shall establish some a priori estimates for solutions. The first estimate is about the energy equality.

Lemma 2.1 *For any $0 \le t < T$, we have the following equality*

$$\int_I \left(\frac{r^2\rho u^2}{2} + \frac{r^2\rho^\gamma}{\gamma-1} + \frac{r^2\chi_r^2}{2} + \frac{r^2\rho(\chi^2-1)^2}{4} \right) dr + \int_0^t \int_I (r^2u_r^2 + 2u^2 + r^2\mu^2) dr ds = E_0, \quad (2.1)$$

where

$$E_0 = \int_I \left(\frac{r^2\rho_0 u_0^2}{2} + \frac{\rho_0^\gamma}{\gamma-1} + \frac{r^2\chi_{0r}^2}{2} + \frac{r^2\rho_0(\chi_0^2-1)^2}{4} \right) ds, \quad \int_I r^2\rho dr = \int_I r^2\rho_0 dr = 1.$$

Proof Multiplying $(1.4)_2$ by u and then integrating the result over I yield

$$\int_I \{(r^2\rho u)_t u + (r^2\rho u^2)_r u + r^2(\rho^\gamma)_r u\} dr = \int_I \left\{ (r^2u_r)_r u - 2u^2 - (r^2\chi_r^2)_r u + \frac{1}{2} r^2(\chi_r^2)_r u \right\} dr. \quad (2.2)$$

We can rewrite $(1.4)_1$ as

$$(r^2\rho)_t + (r^2\rho u)_r = 0,$$

which, together with (2.2) and integration by parts, leads to

$$\frac{d}{dt}\int_I \frac{1}{2} r^2\rho u^2 dr + \frac{1}{\gamma-1}\frac{d}{dt}\int_I r^2\rho^\gamma dr + \int_I r^2u_r^2 dr + 2\int_I u^2 dr = -\int_I (r^2\chi_r)_r u\chi_r dr. \quad (2.3)$$

On the other hand, $(1.4)_3$ can be rewritten as

$$r^2\rho\chi_t + r^2\rho u\chi_r = -r^2\mu.$$

Multiplying this equation by μ, noting $(1.4)_4$ and integrating the result over I, we can get

$$\frac{d}{dt}\int_I \left\{ \frac{1}{2} r^2\chi_r^2 + r^2\rho\left(\frac{\chi^4}{4} - \frac{\chi^2}{2} \right) \right\} dr + \int_I r^2\mu^2 dr = \int_I (r^2\chi_r)_r u\chi_r dr. \quad (2.4)$$

Combining (2.3) with (2.4), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_I r^2\rho u^2 dr + \frac{1}{\gamma-1}\frac{d}{dt}\int_I r^2\rho^\gamma dr + \frac{1}{2}\frac{d}{dt}\int_I r^2\chi_r^2 dr + \frac{d}{dt}\int_I \left[r^2\rho\left(\frac{\chi^4}{4} - \frac{\chi^2}{2} \right) \right] dr \\ &\quad + \int_I r^2u_r^2 dr + 2\int_I u^2 dr + \int_I r^2\mu^2 dr = 0. \end{aligned}$$

Integrating the above equation over $(0,t)$, for any $t \in [0,T]$, we derive that (2.1) holds. The proof is complete. □

With the help of the energy equality (2.1), we can obtain an upper bound of the concentration difference χ.

Lemma 2.2 *For any $0 \le t < T$, we have the following estimate*

$$\|\chi\|_{L^\infty(I\times(0,T))} \le C(E_0). \quad (2.5)$$

Proof From (2.1) we see

$$\int_I r^2\rho(\chi^2-1)^2 dr \le E_0,$$

which implies

$$\begin{aligned} \int_I r^2\rho\chi^4 dr &\le 2\int_I r^2\rho\chi^2 dr - \int_I r^2\rho dr + c(E_0) \\ &\le \frac{1}{2}\int_I r^2\rho\chi^4 dr + 2\int_I r^2\rho dr - \int_I r^2\rho dr + c(E_0). \end{aligned}$$

Thus

$$\int_I r^2\rho\chi^4 dr \le c(E_0).$$

Furthermore,

$$\int_I r^2\rho\chi dr \le \frac{1}{4}\int_I r^2\rho\chi^4 dr + \frac{3}{4}\int_I r^2\rho dr \le c(E_0).$$

Noticing that $\int_I r^2\rho dr = 1$ and $\rho > 0$, for any $(r,t)\in I\times(0,T)$, we get

$$\begin{aligned}
|\chi(x,t)| &= \left|\chi(x,t)\int_I r^2\rho(r,t)dr\right| \\
&\le \left|\int_I r^2\rho(r,t)[\chi(x,t)-\chi(r,t)]dr\right| + \left|\int_I r^2\rho(r,t)\chi(r,t)dr\right| \\
&\le \left|\int_I r^2\rho(r,t)\left(\int_r^x \chi_\xi(\xi,t)d\xi\right)dr\right| + c(E_0) \\
&\le \int_I r^2\rho(r,t)dr\int_I |\chi_\xi(\xi,t)|d\xi + c(E_0) \\
&\le \int_I r^2\rho(r,t)dr\left(\frac{1}{2}\int_I \xi^2\chi_\xi^2 d\xi + \frac{1}{2}\int_I \frac{1}{\xi^2}d\xi\right) + c(E_0) \\
&\le c(E_0).
\end{aligned}$$

The proof is complete. □

Next we prove the upper bound of the density ρ.

Lemma 2.3 *For any $0 \le t < T$, there exists a constant $C_1 = C(c_0, E_0, T) > 0$, such that*

$$\|\rho\|_{L^\infty(I\times(0,T))} \le C_1. \tag{2.6}$$

Proof For any $t\in[0,T]$, we set

$$\begin{aligned}
w(r,t) = &\int_0^t\left(u_r + \frac{2}{r}u - \frac{1}{2}\chi_r^2 - \rho^\gamma - \rho u^2\right)ds \\
&- \int_0^t\int_a^r\left(\frac{2}{\xi}\chi_\xi^2 + \frac{2}{\xi}\rho u^2\right)d\xi ds + \int_a^r \rho_0 u_0 d\xi.
\end{aligned}$$

Then we have

$$w_t = u_r + \frac{2}{r}u - \frac{1}{2}\chi_r^2 - \rho^\gamma - \rho u^2 - \int_a^r\left(\frac{2}{\xi}\chi_\xi^2 + \frac{2}{\xi}\rho u^2\right)d\xi, \quad w_r = \rho u.$$

It follows from (2.1) that

$$\begin{aligned}
&\int_I(|w|+|w_r|)dr \le \int_0^t\int_a^b\left(|u_r| + \left|\frac{2}{r}u\right| + \frac{1}{2}\chi_r^2 + \rho^\gamma + \rho u^2\right)drds \\
&\quad + (b-a)\int_0^t\int_a^b\left(\frac{2}{r}\chi_r^2 + \frac{2}{r}\rho u^2\right)drds + (b-a)\int_a^b|\rho_0u_0|dr + \int_a^b|\rho u|dr \\
&\le c\int_0^t\int_a^b\left(\frac{1}{2}r^2u_r^2 + \frac{c}{r^2} + \frac{u^2}{2} + \frac{1}{2a^2}r^2\chi_r^2 + \rho^\gamma + \frac{1}{a^2}r^2\rho u^2 + \frac{2}{a^3}r^2\chi_r^2 + \frac{2}{a^3}r^2\rho u^2\right)drds \\
&\quad + \frac{c}{a^2}\int_a^b(r^2\rho_0 + r^2\rho_0u_0^2 + r^2\rho + r^2\rho u^2)dr \\
&\le C(E_0)(1+t).
\end{aligned}$$

Hence

$$\|w\|_{L^\infty(Q_T)} \le c\int_I (|w| + |w_r|)dr \le C(E_0)(1+t).$$

Since $\rho > 0$, it suffices to prove $\rho(y,s) \le c$. For any $(y,s) \in I \times (0,t)$, let $r(y,t)$ solve the following problem

$$\begin{cases} \dfrac{dr(y,t)}{dt} = u(r(y,t),t), & 0 \le t < s. \\ r(y,s) = y, & a \le y \le b \end{cases}$$

Denote $f = \exp(w)$. Then we have

$$\begin{aligned}\frac{d}{dt}(\rho f)(r((y,t),t)) &= (\rho_t + \rho_r u)f + \rho f(w_t + uw_r)\\ &= (\rho_t + \rho_r u + \rho w_t + \rho u w_r)f\\ &= \Big[\rho_t + \rho_r u + \rho u_r + \frac{2}{r}\rho u - \frac{1}{2}\rho\chi_r^2 - \rho^{\gamma+1} - \rho^2 u^2\\ &\quad - \rho\int_a^r \Big(\frac{2}{\xi}\chi_\xi^2 + \frac{2}{\xi}\rho u^2\Big)d\xi + \rho^2 u^2\Big]f\\ &= \Big(-\frac{1}{2}\rho\chi_r^2 - \rho^{\gamma+1} - \rho\int_a^r \Big(\frac{2}{\xi}\chi_\xi^2 + \frac{2}{\xi}\rho u^2\Big)d\xi\Big)f\\ &\le 0.\end{aligned}$$

Therefore, for any $s \in (0,T)$, integrating the above inequality with respect to $t \in (0,s)$, we see

$$\begin{aligned}\rho(y,s)f(y,s) &= \rho(r(y,s),s)f(r(y,s),s) \le \rho(r(y,0),0)f(r(y,0),0)\\ &\le c_0\exp(E_0).\end{aligned}$$

Furthermore,

$$\begin{aligned}\rho(y,s) &\le C(c_0,E_0)f^{-1}(y,s)\\ &= C(c_0,E_0)\exp\{-w(y,s)\}\\ &\le C(c_0,E_0)\exp\{\|w\|_{L^\infty(Q_T)}\}\\ &\le C(c_0,E_0)\exp\{C(E_0)(1+t)\}.\end{aligned}$$

The proof is complete. □

Lemma 2.4 *For any* $0 \le t < T$, *we have*

$$\int_0^t\int_I \chi_{rr}^2 drds \le C_1. \tag{2.7}$$

Proof The equation $(1.4)_4$ implies that

$$r^2\chi_{rr} = -\rho\mu r^2 - 2r\chi_r + r^2\rho\chi^3 - r^2\rho\chi.$$

Combining (2.1), (2.5) and (2.6), we can get

$$\begin{aligned}\int_0^t\int_I r^4\chi_{rr}^2 drds &\le c\int_0^t\int_I (r^4\rho^2\mu^2 + r^2\chi_r^2 + r^4\rho^2\chi^6 + r^4\rho^2\chi^2)drds\\ &\le b^2\|\rho\|^2_{L^\infty(I\times(0,T))}\int_0^t\int_I r^2\mu^2 drds + \int_0^t\int_I r^2\chi_r^2 drds\end{aligned}$$

$$
\begin{aligned}
&+ b^2\|\rho\|_{L^\infty(I\times(0,T))}\|\chi\|^2_{L^\infty(I\times(0,T))}\int_0^t\int_I r^2\rho\chi^4 drds\\
&+ b^2\|\rho\|_{L^\infty(I\times(0,T))}\int_0^t\int_I r^2\rho\chi^2 drds\\
\le{}& C_1,
\end{aligned}
$$

which implies

$$
\int_0^t\int_I \chi_{rr}^2 drds \le C_1.
$$

The proof is complete. □

Now we deal with the lower bound of ρ and establish the following lemma.

Lemma 2.5 *The inequalities*

$$
\int_I \rho_r^2 dr \le C_2, \tag{2.8}
$$

and

$$
\left\|\frac{1}{\rho}\right\|_{L^\infty(I\times(0,T))} \le C_2, \tag{2.9}
$$

hold, where the constant $C_2 = C(c_0, E_0, \|\rho_0\|_{H^1}, T) > 0$.

Proof From $(1.1)_1$, we can obtain

$$
\frac{1}{2}\frac{d}{dt}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr = \int_I \left(\frac{1}{r^2\rho}\right)_r u_{rr}dr.
$$

Multiplying $(1.4)_2$ by $(\frac{1}{r^2\rho})_r$ and integrating the result over I, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + \gamma\int_I \frac{\rho_r^2}{r^2\rho^{3-\gamma}}dr\\
&= -2\gamma\int_I \frac{\rho_r}{r^3\rho^{2-\gamma}}dr + \int_I\left[\frac{2}{r^2}u\left(\frac{1}{r^2\rho}\right)_r - \frac{2}{r}u_r\left(\frac{1}{r^2\rho}\right)_r\right]dr\\
&\quad + \int_I\left[(\rho u)_t\left(\frac{1}{r^2\rho}\right)_r + (\rho u^2)_r\left(\frac{1}{r^2\rho}\right)_r + \frac{2}{r}\rho u^2\left(\frac{1}{r^2\rho}\right)_r\right]dr\\
&\quad + \int_I\left[\frac{1}{2}(\chi_r^2)_r\left(\frac{1}{r^2\rho}\right)_r + \frac{2}{r}\chi_r^2\left(\frac{1}{r^2\rho}\right)_r\right]dr\\
&= \sum_{i=1}^4 J_i.
\end{aligned} \tag{2.10}
$$

Since $\int_I r^2\rho dr = \int_I r^2\rho_0 dr = 1$, by the mean value theorem, there exists $\zeta(t) \in I$ such that $\zeta(t)^2\rho(\zeta(t),t)(b-a) = \int_I r^2\rho dr = 1$. Hence we have

$$
\begin{aligned}
\frac{1}{r^2\rho(r,t)} &= \frac{1}{r^2\rho(r,t)} - \frac{1}{\zeta(t)^2\rho(\zeta(t),t)} + (b-a)\\
&= \int_{\zeta(t)}^r \left(\frac{1}{\xi^2\rho(\xi,t)}\right)_\xi d\xi + (b-a)\\
&= -\int_{\zeta(t)}^r \frac{(\xi^2\rho(\xi,t))_\xi}{(\xi^2\rho(\xi,t))^2}d\xi + (b-a)\\
&\le \frac{1}{2}\int_a^b (b-a)r^2\rho\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + \frac{1}{2}\int_a^b \frac{1}{r^2\rho}\frac{1}{b-a}dr + (b-a)
\end{aligned}
$$

$$\leq \frac{1}{2}\int_a^b (b-a)r^2\rho\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + \frac{1}{2}\left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))} + (b-a)$$

which implies that

$$\left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))} \leq (b-a)\int_a^b r^2\rho\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + 2(b-a). \tag{2.11}$$

Now we can turn to estimate each term in the right-hand side of (2.10). For J_1, we have

$$\begin{aligned}
J_1 &= -2\gamma\int_I \frac{\rho_r}{r^3\rho^{2-\gamma}}dr \\
&= 2\gamma\int_I \frac{1}{r\rho^{-\gamma}}\left(\frac{1}{r^2\rho}\right)_r dr - 4\gamma\int_I \frac{1}{r^4\rho^{1-\gamma}}dr \\
&\leq \gamma^2\int_I \frac{1}{r^4\rho^{1-2\gamma}}dr + \frac{1}{2}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr - 4\gamma\int_I \frac{1}{r^4\rho^{1-2\gamma}}dr \\
&\leq \frac{1}{2}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + c\int_I \frac{1}{r^4\rho^{1-2\gamma}}dr.
\end{aligned} \tag{2.12}$$

Using the Cauchy inequality and (2.11), we also have

$$\begin{aligned}
J_2 &= \int_I \left[\frac{2}{r}u\left(\frac{1}{r^2\rho}\right)_r - \frac{2}{r}u_r\left(\frac{1}{r^2\rho}\right)_r\right]dr \\
&\leq c\int_I u^2\frac{1}{r^2\rho}dr + \frac{1}{2}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + c\int_I \frac{r^2u_r^2}{r^2\rho}dr + \frac{1}{2}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr \\
&\leq c\left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))}\int_I u^2 dr + c\left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))}\int_I r^2u_r^2 dr + \int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr \\
&\leq c\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr\left(1 + \int_I u^2 dr + \int_I r^2u_r^2 dr\right).
\end{aligned} \tag{2.13}$$

Noting the fact $|\ln\rho| \leq \rho + \frac{1}{\rho}$, using Lemma 2.3 and the Cauchy inequality, we can calculate

$$\begin{aligned}
J_3 &= \int_I \left[(\rho u)_t\left(\frac{1}{r^2\rho}\right)_r + (\rho u^2)_r\left(\frac{1}{r^2\rho}\right)_r + \frac{2}{r}\rho u^2\left(\frac{1}{r^2\rho}\right)_r\right]dr \\
&= \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + \int_I (\rho u)_r\left(\frac{1}{r^2\rho}\right)_t dr + \int_I \left[(\rho u^2)_r\left(\frac{1}{r^2\rho}\right)_r + \frac{2}{r}\rho u^2\left(\frac{1}{r^2\rho}\right)_r\right]dr \\
&= \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + \int_I \left[\frac{u_r^2}{r^2} - \frac{2uu_r}{r^3} - \frac{4u^2}{r^4} - \frac{2\rho_r u^2}{r^3\rho}\right]dr \\
&\leq \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + c\int_I (u^2+u_r^2)dr - \int_I 2u^2\frac{1}{r^3}(\ln\rho)_r dr \\
&= \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + c\int_I (u^2+u_r^2)dr + \int_I 2uu_r\frac{1}{r^3}\ln\rho dr - \int_I 6u^2\frac{1}{r^4}\ln\rho dr \\
&\leq \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + c\int_I (u^2+u_r^2)dr + \int_I \left|2uu_r\frac{1}{r^3}\left(\rho+\frac{1}{\rho}\right)\right|dr + \int_I |6u^2\frac{1}{r^4}\left(\rho+\frac{1}{\rho}\right)|dr \\
&\leq \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + c\int_I (u^2+u_r^2)dr + c\int_I \rho^2u^2 dr + c\left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))}\int_I u^2 dr \\
&\leq \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + c\int_I (u^2+u_r^2)dr + c\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr\int_I u^2 dr.
\end{aligned} \tag{2.14}$$

Using the Sobolev embedding theorem, we have

$$\begin{aligned}
J_4 &= \int_I \left[\frac{1}{2}\left(\chi_r^2\right)_r \left(\frac{1}{r^2\rho}\right)_r + \frac{2}{r}\chi_r^2\left(\frac{1}{r^2\rho}\right)_r\right] dr \\
&\le \frac{1}{2}\int_I r^2\rho\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 \chi_r^2 dr + \frac{1}{2}\int_I \frac{1}{r^2\rho}\chi_{rr}^2 dr + \int_I \frac{2}{r^2}\chi_r^2 \frac{1}{r^2\rho} dr - \int_I \frac{4}{r}\chi_r\chi_{rr}\frac{1}{r^2\rho} dr \\
&\le \frac{1}{2}\|\chi_r\|_{L^\infty(I)}^2 \int_I r^2\rho\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + c\left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))}\int_I (\chi_{rr}^2 + \chi_r^2) dr \\
&\le c\int_I r^2\rho\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr \int_I (\chi_{rr}^2+\chi_r^2) dr. \qquad (2.15)
\end{aligned}$$

Inserting (2.12)–(2.15) into (2.10), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + \gamma\int_I \frac{r\rho_r^2}{r^3\rho^{3-\gamma}} dr \\
&\quad\le \frac{d}{dt}\int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr + c\int_I \rho^{\gamma-1} dr + \int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr \int_I (u^2+u_r^2+\chi_{rr}^2+\chi_r^2) dr \\
&\quad\quad + \int_I (u^2+u_r^2) dr + \int_I r^2\rho u^2 dr.
\end{aligned}$$

Integrating the above inequality over $(0,t)$, and combining (2.1) and (2.6) yield

$$\begin{aligned}
&\frac{1}{2}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + \gamma\int_0^t\int_I \frac{r\rho_r^2}{r^3\rho^{3-\gamma}} drds \\
&\le \frac{1}{2}\int_I (r^2\rho_0)\left|\left(\frac{1}{r^2\rho_0}\right)_r\right|^2 dr + \int_I \rho u\left(\frac{1}{r^2\rho}\right)_r dr - \int_I \rho_0 u_0\left(\frac{1}{r^2\rho_0}\right)_r dr + c\int_0^t\int_I \rho^{\gamma-1} drds \\
&\quad + \int_0^t \left[\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr \int_I (u^2+u_r^2+\chi_{rr}^2+\chi_r^2) dr\right] ds + \int_0^t\int_I (u^2+u_r^2+r^2\rho u^2) drds \\
&\le c\left\|\frac{1}{\rho_0}\right\|_{L^\infty(I)}^3 \|\rho_0\|_{H^1(I)} + \frac{1}{4}\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + c\int_I r^2\rho u^2 dr \\
&\quad + c\left\|\frac{1}{\rho_0}\right\|_{L^\infty(I)}^2 \|\rho_0\|_{H^1(I)} + \frac{1}{4}\int_I (r^2\rho_0)\left|\left(\frac{1}{r^2\rho_0}\right)_r\right|^2 dr + c\|\rho\|_{L^\infty(I\times(0,T))}^{\gamma-1} \\
&\quad + \int_0^t \left\{\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr\left[\int_I (u^2+u_r^2+\chi_{rr}^2) dr + c(E_0)\right]\right\} ds + cE_0 \\
&\le c\int_0^t \left\{\left[\int_I (u^2+u_r^2+\chi_{rr}^2) dr + E_0\right]\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr\right\} ds + C(C_1, \|\rho_0\|_{H^1(I)}).
\end{aligned}$$

By the Gronwall inequality and (2.1), (2.6), (2.7), we obtain

$$\begin{aligned}
&\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + \gamma\int_0^t\int_I \frac{r\rho_r^2}{r^3\rho^{3-\gamma}} drds \\
&\le C(C_1, \|\rho_0\|_{H^1(I)})\int_0^t \left\{\left[\int_I (u^2+u_r^2+\chi_{rr}^2) dr + E_0\right]\right. \\
&\quad \left.\times \exp\left[c\int_s^t \left(\int_I (u^2+u_r^2+\chi_{rr}^2) dr + E_0\right) d\xi\right]\right\} ds + C(C_1, \|\rho_0\|_{H^1(I)}) \\
&\le C(C_1, \|\rho_0\|_{H^1(I)}). \qquad (2.16)
\end{aligned}$$

Notice that

$$\int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr = \int_I \frac{4r^2\rho^2 + r^6\rho_r^2 + 4r^3\rho\rho_r}{r^6\rho^3} dr$$
$$\geq \frac{1}{b^6\|\rho\|_{L^\infty(I\times(0,T))}} \int_I (4r^2\rho^2 + r^6\rho_r^2 + 4r^3\rho\rho_r) dr,$$

which implies

$$\int_I \rho_r^2 dr \leq c\|\rho\|_{L^\infty(I\times(0,T))} \int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + c\int_I (\rho^2 + \rho\rho_r)$$
$$\leq c\|\rho\|_{L^\infty(I\times(0,T))} \int_I (r^2\rho)\left|\left(\frac{1}{r^2\rho}\right)_r\right|^2 dr + c\|\rho\|^2_{L^\infty(I\times(0,T))} + \varepsilon \int_I \rho_r^2 dr.$$

Taking ε small enough, we have

$$\int_I \rho_r^2 dr \leq C_2.$$

Moreover, by virtue of (2.11) and (2.16), we can get

$$\frac{1}{b^2}\|\rho\|_{L^\infty(I\times(0,T))} \leq \left\|\frac{1}{r^2\rho}\right\|_{L^\infty(I\times(0,T))} \leq C_2.$$

The proof is complete. □

Now we turn to estimates for χ.

Lemma 2.6 *For any $0 \leq t < T$, we have*

$$\int_I \chi_{rr}^2 dr + \int_0^t \int_I (\chi_{rt}^2 + \chi_{rrr}^2) dr ds \leq C_3, \tag{2.17}$$

where the constant $C_3 = C(c_0, E_0, \|\rho_0\|_{H^1(I)}, \|\chi_0\|_{H^2(I)}, T) > 0$.

Proof We rewrite the equations $(1.4)_3$ and $(1.4)_4$ as

$$r^2\rho^2\chi_t + r^2\rho^2\chi_r u = 2r\chi_r + r^2\chi_{rr} - r^2\rho(\chi^3 - \chi). \tag{2.18}$$

Differentiating (2.18) with respect to r, multiplying the result by χ_{rt} and integrating over I yield

$$\frac{1}{2}\frac{d}{dt}\int_I r^2\chi_{rr}^2 dr + \int_I r^2\rho^2\chi_{rt}^2 dr$$
$$= \int_I (2\chi_r\chi_{rt} + 4r\chi_{rr}\chi_{rt}) dr - \int_I (2r\rho^2\chi_r\chi_{rt} + 2r^2\rho\rho_r\chi_t\chi_{rt} + 2r\rho^2 u\chi_r\chi_{rt}) dr$$
$$- \int_I (2r^2\rho\rho_r u\chi_r\chi_{rt} + r^2\rho^2 u\chi_{rr}\chi_{rt} + r^2\rho^2 u_r\chi_r\chi_{rt} + 2r\rho(\chi^3 - \chi)\chi_{rt}) dr$$
$$- \int_I (r^2\rho_r(\chi^3 - \chi)\chi_{rt} + r^2\rho(3\chi^2 - 1)\chi_r\chi_{rt}) dr$$
$$\leq \frac{1}{2}\int_I r^2\rho^2\chi_{rt}^2 dr + c\int_I \left(\frac{\chi_r^2}{r^2\rho^2} + \frac{\chi_{rr}^2}{\rho^2} + \rho^2\chi_t^2 + r^2\rho_r^2\chi_t^2 + \rho^2 u^2\chi_r^2 + r^2\rho_r^2 u^2\chi_r^2\right) dr$$
$$+ c\int_I (r^2\rho^2 u^2\chi_{rr}^2 + r^2\rho^2 u_r^2\chi_r^2) dr + c\int_I (\chi^3 - \chi)^2 dr + c\int_I \frac{r^2\rho_r^2(\chi^3 - \chi)^2}{\rho^2} dr$$
$$+ c\int_I r^2(3\chi^2 - 1)^2\chi_r^2 dr.$$

Noting that the following Sobolev embedding inequalities

$$\|u\|_{L^\infty(I)} \le c\|u_r\|_{L^2(I)}, \quad \|\chi_r\|_{L^\infty(I)} \le c\|\chi_{rr}\|_{L^2(I)},$$
$$\|\chi_t\|^2_{L^\infty(I)} \le \int_I (\chi_t^2 + 2|\chi_t\chi_{tr}|)dr \le c\int_I \chi_t^2 dr + \varepsilon \int_I \chi_{tr}^2 dr,$$

we can derive

$$\begin{aligned}
&\frac{d}{dt}\int_I r^2\chi_{rr}^2 dr + \int_I r^2\rho^2\chi_{rt}^2 dr \\
&\le c\left\|\frac{1}{\rho}\right\|^2_{L^\infty}\int_I \chi_r^2 dr + c\left\|\frac{1}{\rho}\right\|^2_{L^\infty}\int_I r^2\chi_{rr}^2 dr + c\|\rho\|^2_{L^\infty}\int_I \chi_t^2 dr + c\|\chi_t\|^2_{L^\infty}\int_I \rho_r^2 dr \\
&\quad + \|\rho\|^2_{L^\infty}\|u\|^2_{L^\infty}\int_I \chi_r^2 dr + \|u\|^2_{L^\infty}\|\chi_r\|^2_{L^\infty}\int_I \rho_r^2 dr + \|\rho\|^2_{L^\infty}\|u\|^2_{L^\infty}\int_I r^2\chi_{rr}^2 dr \\
&\quad + \|\rho\|^2_{L^\infty}\|\chi_r\|^2_{L^\infty}\int_I u_r^2 dr \\
&\quad + c(\|\chi\|^6_{L^\infty} + \|\chi\|^4_{L^\infty} + \|\chi\|^2_{L^\infty})\left(1 + \left\|\frac{1}{\rho}\right\|_{L^\infty}\int_I \rho_r^2 dr + \int_I \chi_r^2 dr\right).
\end{aligned} \tag{2.19}$$

Moreover, we also know that for any function φ

$$\int_I \varphi^2 dr \le \frac{1}{a^2}\int_I r^2\varphi^2 dr \le c\int_I r^2\varphi^2 dr.$$

In addition, from (2.18), we can get

$$\chi_t = -\chi_r u + \frac{2\chi_r}{r\rho^2} + \frac{\chi_{rr}}{\rho^2} - \frac{\chi^3 - \chi}{\rho}.$$

Thus we have

$$\begin{aligned}
\int_I \chi_t^2 dr &\le c\int_I \left(\chi_r^2 u^2 + \frac{\chi_r^2}{r^2\rho^4} + \frac{\chi_{rr}^2}{\rho^4} + \frac{\chi^6}{\rho^2} + \frac{\chi^4}{\rho^2} + \frac{\chi^2}{\rho^2}\right)dr \\
&\le c(\|u\|^2_{L^\infty(I)} + 1)\int_I \chi_r^2 dr + c\int_I \chi_{rr}^2 dr \\
&\le c\|u\|_{L^\infty(I)}\int_I \chi_r^2 dr + \left\|\frac{1}{\rho}\right\|^4_{L^\infty(Q_T)}\int_I \chi_r^2 dr + \left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\int_I \chi_{rr}^2 dr \\
&\quad + \left\|\frac{1}{\rho}\right\|^2_{L^\infty(Q_T)}(\|\chi\|^6_{L^\infty(Q_T)} + \|\chi\|^4_{L^\infty(Q_T)} + \|\chi\|^2_{L^\infty(Q_T)}) \\
&\le c\left(1 + \int_I u_r^2 dr\right)\int_I \chi_r^2 dr + c\int_I \chi_{rr}^2 dr + C_2.
\end{aligned}$$

Combining the above inequalities and (2.1), (2.5), (2.6), (2.8), from (2.19), we can get

$$\begin{aligned}
&\frac{d}{dt}\int_I r^2\chi_{rr}^2 dr + \int_I r^2\rho^2\chi_{rt}^2 dr \\
&\quad \le C_2\left(1 + \int_I r^2 u_r^2 dr\right)\int_I r^2\chi_{rr}^2 dr + C_2\left(1 + \int_I r^2 u_r^2 dr\right).
\end{aligned}$$

By the Gronwall inequality and (2.1), we have

$$\int_I r^2\chi_{rr}^2 dr + \int_0^t\int_I r^2\rho^2\chi_{rt}^2 drds \le C(C_2, \|\chi_0\|_{H^2(I)}),$$

which implies

$$\int_I \chi_{rr}^2 dr + \int_0^t \int_I \chi_{rt}^2 drds \le C(C_2, \|\chi_0\|_{H^2(I)}).$$

Differentiating (2.18) with respect to r, we can get

$$\begin{aligned}\chi_{rrr} = &\frac{2}{r}\rho^2\chi_t + 2\rho\rho_r\chi_t + \rho^2\chi_{rt} + \frac{2}{r}\rho^2\chi_r u + 2\rho\rho_r\chi_r u + \rho^2\chi_{rr}u \\ &+ \rho^2\chi_r u_r - \frac{2}{r}\chi_r - \frac{4}{r}\chi_{rr} + \frac{2\rho}{r}(\chi^3-\chi) + \rho_r(\chi^3-\chi) + \rho(3\chi^2-1)\chi_r.\end{aligned}$$

Then we can derive from Lemmas 2.1–2.5

$$\begin{aligned}\int_0^t\int_I \chi_{rrr}^2 drds \le\ & c\int_0^t\int_I \{\rho^4\chi_t^2 + \rho^2\rho_r^2\chi_t^2 + \rho^4\chi_{rt}^2 + \rho^4\chi_r^2u^2 + \rho^2\rho_r^2\chi_r^2u^2 \\ &+ \rho^4\chi_{rr}^2u^2 + \rho^4\chi_r^2u_r^2 + \chi_r^2 + \chi_{rr}^2 + \rho^2(\chi^3-\chi)^2 \\ &+ \rho_r^2(\chi^3-\chi)^2 + \rho^2(3\chi^2-1)^2\chi_r^2\} drds \\ \le\ & c\|\rho\|_{L^\infty(Q_T)}^4 \int_0^t\int_I \chi_t^2 drds + c\|\rho\|_{L^\infty(Q_T)}^2 \int_0^t \left\{\|\chi_t\|_{L^\infty(I)}^2 \int_I \rho_r^2 dr\right\} ds \\ &+ c\|\rho\|_{L^\infty(Q_T)}^4 \int_0^t\int_I \chi_{rt}^2 drds + c\|\rho\|_{L^\infty(Q_T)}^4 \int_0^t \left\{\|u\|_{L^\infty(I)}^2 \int_I \rho_r^2 dr\right\} ds \\ &+ c\|\rho\|_{L^\infty(Q_T)}^2 \int_0^t \left\{\|u\|_{L^\infty(I)}^2 \|\chi_r\|_{L^\infty(I)}^2 \int_I \rho_r^2 dr\right\} ds \\ &+ c\|\rho\|_{L^\infty(Q_T)}^4 \int_0^t \left\{\|u\|_{L^\infty(I)}^2 \int_I \chi_{rr}^2 dr\right\} ds \\ &+ c\|\rho\|_{L^\infty(Q_T)}^4 \int_0^t \left\{\|\chi_r\|_{L^\infty(I)}^2 \int_I u_r^2 dr\right\} ds \\ &+ c\|\rho\|_{L^\infty(Q_T)}^2 (\|\chi\|_{L^\infty(Q_T)}^6 + \|\chi\|_{L^\infty(Q_T)}^4 + \|\chi\|_{L^\infty(Q_T)}^2) \\ &+ (\|\chi\|_{L^\infty(Q_T)}^6 + \|\chi\|_{L^\infty(Q_T)}^4 + \|\chi\|_{L^\infty(Q_T)}^2) \int_0^t\int_I \rho_r^2 drds \\ &+ c\|\rho\|_{L^\infty(Q_T)}^2 (\|\chi\|_{L^\infty(Q_T)}^4 + \|\chi\|_{L^\infty(Q_T)}^3 + \|\chi\|_{L^\infty(Q_T)}^2) \int_0^t\int_I \chi_r^2 drds \\ \le\ & C(C_2, \|\chi_0\|_{H^2(I)}).\end{aligned}$$

The proof is complete. □

Now we turn to establish some a priori estimates on u.

Lemma 2.7 *For any $0 \le t < T$, we have*

$$\int_I (u_r^2 + u^2) dr + \int_0^t \int_I (\rho u_t^2 + u_{rr}^2) drds \le C_4, \tag{2.20}$$

where the constant $C_4 = C(c_0, E_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^1}, \|\chi_0\|_{H^2}, \gamma, T) > 0$.

Proof It follows from $(1.4)_1$ and $(1.4)_2$ that

$$r^2\rho u_t + r^2\rho u u_r + r^2\gamma\rho^{\gamma-1}\rho_r = 2ru_r + r^2u_{rr} - 2u - 2r\chi_r^2 - r^2\chi_r\chi_{rr}. \tag{2.21}$$

Multiplying (2.21) by u_t, integrating the results over I, we have

$$\frac{1}{2}\frac{d}{dt}\int_I r^2u_r^2 dr + \frac{d}{dt}\int_I u^2 dr + \int_I r^2\rho u_t^2 dr$$

$$= -\int_I r^2\rho u u_r u_t dr - \int_I r^2\gamma\rho^{\gamma-1}\rho_r u_t dr - \int_I 2r\chi_r^2 u_t dr - \int_I r^2\chi_r\chi_{rr}u_t dr$$
$$= \frac{1}{2}\int_I r^2\rho u_t^2 dr + c\int_I r^2\rho u^2 u_r^2 dr + c\gamma^2\int_I r^2\rho^{2r-3}\rho_r^2 dr + c\int_I \frac{\chi_r^4}{\rho}dr + c\int_I \frac{r^2\chi_r^2\chi_{rr}^2}{\rho}dr.$$

Then we have

$$\begin{aligned}
&\frac{d}{dt}\int_I r^2u_r^2dr + \frac{d}{dt}\int_I 2u^2dr + \int_I r^2\rho u_t^2 dr\\
&\le c\int_I r^2\rho u^2u_r^2dr + c\gamma^2\int_I r^2\rho^{2r-3}\rho_r^2dr + c\int_I\frac{\chi_r^4}{\rho}dr + c\int_I\frac{r^2\chi_r^2\chi_{rr}^2}{\rho}dr\\
&\le c\|\rho\|^2_{L^\infty(Q_T)}\|u\|^2_{L^\infty(I)}\int_I u_r^2dr + c\gamma^2\|\rho\|^{2\gamma-3}_{L^\infty(Q_T)}\int_I\rho_r^2dr\\
&\quad + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\|\chi_r^2\|^2_{L^\infty(I)}\int_I\chi_r^2dr + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\|\chi_r^2\|^2_{L^\infty(I)}\int_I\chi_{rr}^2dr\\
&\le c\|\rho\|^2_{L^\infty(Q_T)}\int_I r^2u_r^2dr\cdot\int_I r^2u_r^2dr + c\gamma^2\|\rho\|^{2\gamma-3}_{L^\infty(Q_T)}\int_I\rho_r^2dr\\
&\quad + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\int_I\chi_{rr}^2dr\cdot\int_I\chi_r^2dr + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\int_I\chi_{rr}^2dr\cdot\int_I\chi_{rr}^2dr\\
&\le C_1\left(\int_I r^2u_r^2dr\right)^2 + \gamma^2C_3,
\end{aligned}$$

where we have used (2.1), (2.6), (2.8), (2.9), (2.17) and the Sobolev embedding inequalities in the last step. By the Gronwall inequality and (2.1), we have

$$\int_I r^2u_r^2dr + \int_I 2u^2dr + \int_0^t\int_I r^2\rho u_t^2drds \le C(C_3, \|u_0\|_{H^1(I)}, \gamma),$$

which implies

$$\int_I u_r^2dr + \int_I 2u^2dr \le C(C_3, \|u_0\|_{H^1(I)}, \gamma).$$

The equation (2.21) implies

$$u_{rr} = \rho u_t + \rho u u_r + \gamma\rho^{\gamma-1}\rho_r - \frac{2u_r}{r} + \frac{2u}{r^2} + \frac{2\chi_r^2}{r} + \chi_r\chi_{rr}.$$

Then we can obtain from Lemmas 2.1–2.6,

$$\begin{aligned}
\int_0^t\int_I u_{rr}^2drds &\le c\int_0^t\int_I\left\{\rho^2u_t^2 + \rho^2u^2u_r^2 + \gamma^2\rho^{2\gamma-2}\rho_r^2 + \frac{4u_r^2}{r^2} + \frac{4u^2}{r^4} + \frac{4\chi_r^4}{r^2} + \chi_r^2\chi_{rr}^2\right\}drds\\
&\le c\|\rho\|_{L^\infty(Q_T)}\int_0^t\int_I\rho u_t^2drds + c\|\rho\|^2_{L^\infty(Q_T)}\int_0^t\left\{\|u\|_{L^\infty(I)}\int_I u_r^2dr\right\}ds\\
&\quad + c\gamma^2\|\rho\|^{2\gamma-2}_{L^\infty(Q_T)}\int_0^t\int_I\rho_r^2drds + c\int_0^t\int_I(u_r^2+u^2)drds\\
&\quad + c\int_0^t\left\{\|\chi_r\|_{L^\infty(I)}\int_I\chi_r^2dr\right\}ds + c\int_0^t\left\{\|\chi_r\|_{L^\infty(I)}\int_I\chi_{rr}^2dr\right\}ds\\
&\le C(C_3, \|u_0\|_{H^1(I)}, \gamma).
\end{aligned}$$

The proof is complete. □

Lemma 2.8 *For any $0 \le t < T$, we have*

$$\int_I (\rho u_t^2 + u_{rr}^2)dr + \int_0^t \int_I u_{rt}^2 drds \le C_5, \tag{2.22}$$

where the constant $C_5 = C(c_0, E_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|\chi_0\|_{H^2}, \gamma, T) > 0$.

Proof Differentiating (2.21) with respect to t, multiplying the results by u_t, then integrating over I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I r^2\rho u_t^2 dr + \int_I r^2 u_{rt}^2 dr \\
&= -\frac{1}{2}\int_I (r^2\rho)_t u_t^2 dr - \int_I \{(r^2\rho)_t u u_r u_t + r^2\rho u_t^2 u_r + r^2\rho u u_{rt} u_t + r^2(\rho^\gamma)_{rt} u_t \\
&\quad + 2u_t^2 + 4r\chi_t\chi_{rt}u_t + (r^2\chi_r\chi_{rr})_t u_t\}dr \\
&= -\int_I \{r^2\rho u u_t u_{rt} + r^2\rho u u_r^2 u_t + r^2\rho u^2 u_{rr} u_t + r^2\rho u^2 u_r u_{rt} + r^2\rho u_t^2 u_r + r^2\rho u u_{rt} u_t\}dr \\
&\quad - \gamma\int_I \{2r\rho^{\gamma-1}\rho_r u u_t + 2r\rho^{\gamma-1}u_r u_t + 4\rho^\gamma u u_t + r^2\rho^{\gamma-1}\rho_r u u_{rt} + r^2\rho^\gamma u_r u_{rt} + 2r\rho^\gamma u u_{rt}\}dr \\
&\quad - \int_I \{2u_t^2 + 4r\chi_r\chi_{rt}u_t + r^2\chi_{rt}\chi_{rr}u_t + r^2\chi_r\chi_{rrt}u_t\}dr \\
&= \sum_{i=1}^8 K_i.
\end{aligned} \tag{2.23}$$

Now we estimate each term on the right-hand side of (2.23). Using the equation $(1.4)_1$, the integration by parts, the Young inequality, the embedding theorem and Lemmas 2.3 and 2.7, we can obtain

$$\begin{aligned}
K_1 = -\int_I r^2\rho u u_t u_{rt} dr &\le \varepsilon\int_I r^2 u_{rt}^2 dr + c\int_I r^2\rho^2 u^2 u_t^2 dr \\
&\le \varepsilon\int_I r^2 u_{rt}^2 dr + c\|\rho\|_{L^\infty(Q_T)}\|u\|_{L^\infty(I)}^2 \int_I r^2\rho u_t^2 dr \\
&\le \varepsilon\int_I r^2 u_{rt}^2 dr + c\|\rho\|_{L^\infty(Q_T)}\int_I u_r^2 dr \int_I r^2\rho u_t^2 dr \\
&\le \varepsilon\int_I r^2 u_{rt}^2 dr + c\int_I r^2\rho u_t^2 dr.
\end{aligned}$$

Noting that the equation $(1.4)_1$ and using the Young inequality, the embedding theorem and Lemmas 2.3 and 2.7, we can also obtain

$$\begin{aligned}
K_2 &= -\int_I r^2\rho u u_r^2 u_t dr - \int_I r^2\rho u^2 u_{rr} u_t dr - \int_I r^2\rho u^2 u_r u_{rt} dr \\
&\le c\int_I r^2\rho u^2 u_t^2 dr + c\int_I r^2\rho u_r^4 dr + c\int_I r^2\rho u^2 u_{rr}^2 dr + \varepsilon\int_I r^2 u_{rt}^2 dr + c\int_I r^2\rho^2 u^4 u_r^2 dr \\
&\le c\|u\|_{L^\infty(I)}^2\int_I r^2\rho u_t^2 dr + c\|\rho\|_{L^\infty(Q_T)}\|u_r\|_{L^\infty(I)}^2\int_I u_r^2 dr \\
&\quad + c\|\rho\|_{L^\infty(Q_T)}\|u\|_{L^\infty(I)}^2\int_I u_{rr}^2 dr + \varepsilon\int_I r^2 u_{rt}^2 dr + c\|\rho\|_{L^\infty(Q_T)}^2\|u\|_{L^\infty(I)}^4\int_I u_r^2 dr \\
&\le c\int_I u_r^2 dr\int_I r^2\rho u_t^2 dr + c\|\rho\|_{L^\infty(Q_T)}\int_I (u_r^2 + u_{rr}^2)dr\int_I u_r^2 dr
\end{aligned}$$

$$+\varepsilon\int_I r^2u_{rt}^2dr+c\|\rho\|_{L^\infty(Q_T)}^2\left\{\int_I u_r^2dr\right\}^3$$
$$\le c+c\int_I r^2\rho u_t^2dr+c\int_I u_{rr}^2dr+\varepsilon\int_I r^2u_{rt}^2dr.$$

Using the Young inequality and the Cauchy inequality, we can obtain

$$\begin{aligned}K_3&=-\int_I r^2\rho u_t^2u_rdr\le c\|u_r\|_{L^\infty(I)}\int_I r^2\rho u_t^2dr\\&\le c\left\{\int_I(u_r^2+u_{rr}^2)dr\right\}^{\frac{1}{2}}\int_I r^2\rho u_t^2dr\\&\le c\left\{\int_I u_{rr}^2dr+1\right\}\int_I r^2\rho u_t^2dr.\end{aligned}$$

Similarly, we have

$$\begin{aligned}K_4&=-\int_I r^2\rho uu_{rt}u_tdr\le\varepsilon\int_I r^2u_{rt}^2dr+c\int_I r^2\rho^2u^2u_t^2dr\\&\le\varepsilon\int_I r^2u_{rt}^2dr+c\|\rho\|_{L^\infty(Q_T)}\|u\|_{L^\infty(I)}^2\int_I r^2\rho u_t^2dr\\&\le\varepsilon\int_I r^2u_{rt}^2dr+c\int_I r^2\rho u_t^2dr.\end{aligned}$$

Using the equation $(1.4)_1$, the Young inequality, the embedding theorem and Lemmas 2.3 and 2.7, we easily obtain

$$\begin{aligned}K_5&=-\gamma\int_I 2r\rho^{\gamma-1}\rho_ruu_tdr-\gamma\int_I 2r\rho^\gamma u_ru_tdr-\gamma\int_I 4\rho^\gamma\rho_ruu_tdr\\&\quad-\gamma\int_I r^2\rho^{\gamma-1}\rho_ruu_{rt}dr-\gamma\int_I r^2\rho^\gamma u_ru_{rt}dr-\gamma\int_I r\rho^\gamma uu_{rt}dr\\&\le c\int_I r^2\rho u_t^2dr+c\int_I\rho^{2\gamma-3}\rho_r^2u^2dr+c\int_I\rho^{2\gamma-1}u_r^2dr+c\int_I\frac{1}{r^2}\rho^{2\gamma-1}u^2dr\\&\quad+\varepsilon\int_I r^2u_{rt}^2dr+c\gamma\int_I r^2\rho^{2\gamma-2}\rho_r^2u^2dr+c\int_I r^2\rho^{2\gamma}u_r^2dr+c\int_I\rho^{2\gamma}u^2dr\\&\le c\gamma\int_I r^2\rho u_t^2dr+c\gamma\|\rho\|_{L^\infty(Q_T)}^{2\gamma-3}\|u\|_{L^\infty(I)}^2\int_I\rho_r^2dr\\&\le c\int_I r^2\rho u_t^2dr+c\|\rho\|_{L^\infty(Q_T)}^{2\gamma-3}\int_I u_r^2dr\int_I\rho_r^2dr+c\gamma\|\rho\|_{L^\infty(Q_T)}^{2\gamma-1}\int_I u_r^2dr\\&\quad+c\|\rho\|_{L^\infty(Q_T)}^{2\gamma-1}\|u\|_{L^\infty(I)}^2\|\rho_r\|^2+\varepsilon\int_I r^2u_{rt}^2dr+c\gamma\|\rho\|_{L^\infty(Q_T)}^{2\gamma-2}\|u\|_{L^\infty(I)}^2\int_I\rho_r^2dr\\&\quad+c\|\rho\|_{L^\infty(Q_T)}^{2\gamma}\|u\|_{L^\infty(I)}^2\\&\le c\int_I r^2\rho u_t^2dr+\varepsilon\int_I r^2u_{rt}^2dr+c.\end{aligned}$$

By virtue of the Young inequality, we have

$$K_6=-\int_I u_t^2dr\le c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\int_I r^2\rho u_t^2dr.$$

We also deduce from the Cauchy inequality that

$$K_7=-\int_I 4r\chi_r\chi_{rt}u_tdr\le c\int_I r^2\rho u_t^2dr+c\int_I\frac{\chi_r^2\chi_{rt}^2}{\rho}dr$$

$$\le c\int_I r^2\rho u_t^2 dr + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)} \|\chi_r\|_{L^\infty(I)}^2 \int_I \chi_{rt}^2 dr$$
$$\le c\int_I r^2\rho u_t^2 dr + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)} \int_I \chi_{rr}^2 dr \int_I \chi_{rt}^2 dr.$$

By virtue of integrating by parts and the Young inequality, we can obtain

$$K_8 = -\int_I (r^2\chi_{rt}\chi_{rr}u_t + r^2\chi_r\chi_{rrt}u_t)dr = \int_I 2r\chi_{rt}\chi_r u_t dr + \int_I r^2\chi_{rt}\chi_r u_{rt} dr$$
$$\le c\int_I r^2\rho u_t^2 dr + c\int_I \frac{\chi_{rt}^2\chi_r^2}{\rho} dr + \varepsilon\int_I r^2 u_{rt}^2 dr + c\int_I r^2\chi_{rt}^2\chi_r^2 dr$$
$$\le \varepsilon\int_I r^2 u_{rt}^2 dr + c\int_I r^2\rho u_t^2 dr + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)} \|\chi_r\|_{L^\infty(I)}^2 \int_I \chi_{rt}^2 dr + c\|\chi_r\|_{L^\infty(I)}^2 \int_I \chi_{rt}^2 dr$$
$$\le \varepsilon\int_I r^2 u_{rt}^2 dr + c\int_I r^2\rho u_t^2 dr + c\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)} \int_I \chi_{rr}^2 dr \int_I \chi_{rt}^2 dr + c\int_I \chi_{rr}^2 dr \int_I \chi_{rt}^2 dr.$$

Then combining above results and (2.6), (2.8), (2.9), (2.17), (2.20), we can get

$$\frac{d}{dt}\int_I r^2\rho u_t^2 dr + \int_I r^2 u_{rt}^2 dr$$
$$\le C_4\left(1 + \int_I u_{rr}^2 dr\right)\int_I r^2\rho u_t^2 dr + C_4\left\{1 + \int_I (u_{rr}^2 + \chi_{rt}^2) dr\right\}.$$

Applying the Gronwall inequality and (2.17), (2.20), we get

$$\int_I r^2\rho u_t^2 dr + \int_0^t\int_I r^2 u_{rt}^2 dr ds \le C(C_4, \|u_0\|_{H^2(I)}),$$

which implies

$$\int_I \rho u_t^2 dr + \int_0^t\int_I u_{rt}^2 dr ds \le C(C_4, \|u_0\|_{H^2(I)}).$$

Combining above inequalities and equation (2.21), we get

$$\int_I u_{rr}^2 dr \le c\int_I \left\{\rho^2 u_t^2 + \rho^2 u^2 u_r^2 + \gamma^2\rho^{2\gamma-2}\rho_r^2 + \frac{u_r^2}{r^2} + \frac{u^2}{r^4} + \frac{\chi_r^2}{r^4} + \chi_r^2\chi_{rr}^2\right\} dr$$
$$\le c\int_I \rho^2 u_t^2 dr + c\|\rho\|_{L^\infty(Q_T)}^2\|u\|_{L^\infty(I)}^2 \int_I u_r^2 dr + c\|\rho\|_{L^\infty(Q_T)}^{2\gamma-2}\int_I \rho_r^2 dr$$
$$+ c\int_I u_r^2 dr + c\int_I u^2 dr + c\int_I \chi_r^2 dr + c\|\chi_r\|_{L^\infty(I)}^2 \int_I \chi_{rr}^2 dr$$
$$\le C(C_4, \|u_0\|_{H^2(I)}).$$

The proof is complete. □

3 Proofs of Main Theorems

In this section, we shall show Theorems 1.1–1.3. Our main idea is to extend the local classical solution to the global one, based on some priori estimates in Section 2 and the following well-known lemma from [28].

Lemma 3.1 *Let X, E, Y are Banach spaces. Assume that $X \subset E \subset Y$ and $X \hookrightarrow\hookrightarrow E$. Then the following embedding are compact*:

i) $$\left\{\psi : \psi \in L^q(0,T;X), \frac{\partial\psi}{\partial t} \in L^1(0,T;Y)\right\} \hookrightarrow\hookrightarrow L^q(0,T;E), \quad \text{if } 1 \le q \le +\infty,$$

$$\text{ii)}\quad \left\{\psi : \psi \in L^\infty(0,T;X), \frac{\partial \psi}{\partial t} \in L^r(0,T;Y)\right\} \hookrightarrow\hookrightarrow C(0,T;E), \quad \text{if } 1 < r \le +\infty.$$

Now we first show Theorem 1.1.

Proof of Theorem 1.1 Suppose that it was false. Then, for problem (1.1)–(1.3), there exists a maximal time internal $0 < T_* < +\infty$ such that there is a unique classical solution $(\rho, u, \chi) : I \times [0, T_*) \to \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, but at least one of the following properties fails:

$$\begin{aligned}
&\text{(i)}\quad (\rho_t, \rho_r) \in C^{\alpha, \frac{\alpha}{2}}(Q_{T_*}),\\
&\text{(ii)}\quad 0 < c^{-1} \le \rho \le c < +\infty, \quad (r,t) \in Q_{T_*},\\
&\text{(iii)}\quad (u, \chi) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_{T_*}).
\end{aligned}$$

For Lemmas 2.3 and 2.5, we know that (ii) holds. Hence either (i) or (iii) fails. From Lemmas 2.1–2.8 and equations (1.1) (or equations (1.4)), we can prove

$$\begin{aligned}
&\max\{\|\rho\|_{C^{\frac{1}{2},\frac{1}{4}}(Q_{T_*})}, \|\chi\|_{C^{1,\frac{1}{2}}(Q_{T_*})}, \|u\|_{C^{1,\frac{1}{2}}(Q_{T_*})}\}\\
&\le C(c_0, E_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|\chi_0\|_{H^2}, T_*) < +\infty.
\end{aligned} \tag{3.1}$$

In fact, for any $(r_1, t), (r_2, t) \in Q_{T_*}$, by Lemma 2.5, we have

$$|\rho(r_1,t) - \rho(r_2,t)| = \left|\int_{r_2}^{r_1} \rho_r(r,t)dr\right| \le \left(\int_a^b \rho_r^2 dr\right)^{\frac{1}{2}} |r_1 - r_2|^{\frac{1}{2}} \le C(T)|r_1 - r_2|^{\frac{1}{2}}. \tag{3.2}$$

For any $(r, t_1), (r, t_2) \in Q_{T_*}$, we consider the case of $r \in [a, \frac{a+b}{2}]$. Suppose that $\Delta t = t_2 - t_1 \ge 0$ satisfies $(\Delta t)^{\frac{1}{2}} \le \frac{b-a}{2}$. Integrating the equation $(1.4)_1$ over $(r, r + (\Delta t)^{\frac{1}{2}}) \times (t_1, t_2)$, and using Lemmas 2.3, 2.5, 2.7 and the embedding theorem, we obtain

$$\begin{aligned}
&\int_r^{r+(\Delta t)^{\frac{1}{2}}} (\xi^2 \rho(\xi, t_2) - \xi^2 \rho(\xi, t_1)) d\xi\\
&= -\int_{t_1}^{t_2}\int_r^{r+(\Delta t)^{\frac{1}{2}}} (\xi^2 \rho_\xi u(\xi,t) + \xi^2 \rho u_\xi(\xi,t) + 2\xi\rho u(\xi,t)) d\xi dt\\
&\le \int_{t_1}^{t_2}\left(\int_r^{r+(\Delta t)^{\frac{1}{2}}} \xi^4 \rho_\xi^2 u^2 d\xi\right)^{\frac{1}{2}} (\Delta t)^{\frac{1}{4}} dt + \int_{t_1}^{t_2}\left(\int_r^{r+(\Delta t)^{\frac{1}{2}}} \xi^4 \rho^2 u_\xi^2 d\xi\right)^{\frac{1}{2}} (\Delta t)^{\frac{1}{4}} dt\\
&\quad + 2\int_{t_1}^{t_2}\left(\int_r^{r+(\Delta t)^{\frac{1}{2}}} \xi^2 \rho^2 u^2 d\xi\right)^{\frac{1}{2}} (\Delta t)^{\frac{1}{4}} dt\\
&\le c\int_{t_1}^{t_2} \|u\|_{L^\infty(I)}\left(\int_I \rho_\xi^2 d\xi\right)^{\frac{1}{2}} (\Delta t)^{\frac{1}{4}} dt + c\|\rho\|_{L^\infty Q_T}\int_{t_1}^{t_2}\left(\int_I u_\xi^2 d\xi\right)^{\frac{1}{2}} (\Delta t)^{\frac{1}{4}} dt\\
&\quad + c\|\rho\|_{L^\infty Q_T}\int_{t_1}^{t_2}\left(\int_I u^2 d\xi\right)^{\frac{1}{2}} (\Delta t)^{\frac{1}{4}} dt\\
&\le C(T)(\Delta t)^{\frac{5}{4}}.
\end{aligned}$$

Noticing $\rho \in L^\infty(0, T_*; H^1(I))$ and $\rho_t \in L^\infty(0, T_*; L^2(I))$, by Lemma 3.1, we have $\rho \in C([0,T]; L^2(I))$. By the integral mean value theorem, we know that there exists a point $r^* \in [r, r + (\Delta t)^{\frac{1}{2}}] \in [a, b]$ such that

$$\int_r^{r+(\Delta t)^{\frac{1}{2}}} (\xi^2 \rho(\xi, t_2) - \xi^2 \rho(\xi, t_1)) d\xi$$

$$
\begin{aligned}
&= (\rho(r^*,t_2)-\rho(r^*,t_1))\int_r^{r+(\Delta t)^{\frac12}}\xi^2 d\xi\\
&= (\rho(r^*,t_2)-\rho(r^*,t_1))\Big(r^2(\Delta t)^{\frac12}+r\Delta t+\frac13(\Delta t)^{\frac32}\Big)\\
&\ge a^2(\rho(r^*,t_2)-\rho(r^*,t_1))(\Delta t)^{\frac12}.
\end{aligned}
$$

Thus we arrive at

$$
|\rho(r^*,t_2)-\rho(r^*,t_1)|\le C(T)|t_2-t_1|^{\frac34}.
$$

Combining the above inequality with (3.2) gives

$$
\begin{aligned}
|\rho(r,t_1)-\rho(r,t_2)| &\le |\rho(r,t_1)-\rho(r^*,t_1)|+|\rho(r^*,t_1)-\rho(r^*,t_2)|+|\rho(r^*,t_2)-\rho(r,t_2)|\\
&\le C(T)|r-r^*|^{\frac12}+C(T)|t_1-t_2|^{\frac34}\\
&\le C(T)|t_1-t_2|^{\frac14}.
\end{aligned}
\tag{3.3}
$$

Similarly, for the case of $r\in[\frac{a+b}{2},b]$, integrating the equation $(1.4)_1$ over $(r-(\Delta t)^{\frac12},r)\times(t_1,t_2)$, we can also get the above inequality. Therefore, from (3.2) and (3.3), for any $(r_1,t_1),(r_2,t_2)\in Q_{T_*}$, it holds that

$$
\begin{aligned}
|\rho(r_1,t_1)-\rho(r_2,t_2)| &\le |\rho(r_1,t_1)-\rho(r_2,t_1)|+|\rho(r_2,t_1)-\rho(r_2,t_2)|\\
&\le C(T)(|r_1-r_2|^{\frac12}+|t_1-t_2|^{\frac14}),
\end{aligned}
$$

which implies that $\|\rho\|_{C^{\frac12,\frac14}(Q_{T_*})}\le C(T)<\infty$. Next we prove that $\|\chi\|_{C^{1,\frac12}(Q_{T_*})}<\infty$. Using the mean value theorem and Sobolev embedding inequalities, we have

$$
\begin{aligned}
|\chi(r_1,t)-\chi(r_2,t)| &= \Big|\int_{r_2}^{r_1}\chi_r(r,t)dr\Big|\\
&\le \|\chi_r\|_{L^\infty(I)}|r_1-r_2|\\
&\le |r_1-r_2|\int_I\chi_{rr}^2dr\\
&\le C(T)|r_1-r_2|.
\end{aligned}
$$

Notice that

$$
\chi_t=-\chi_r u+\frac{\chi_{rr}}{\rho^2}+\frac{2\chi_r}{r\rho^2}-\frac{\chi^2}{\rho}+\frac{\chi}{\rho}.
\tag{3.4}
$$

For any $(r,t_1),(r,t_2)\in Q_{T_*}$, we consider the case of $r\in[a,\frac{a+b}{2}]$. Suppose that $\Delta t=t_2-t_1\ge0$ satisfies $\Delta t\le\frac{b-a}{2}$. Integrating the above equation (3.4) over $(r,r+\Delta t)\times(t_1,t_2)$ yields

$$
\begin{aligned}
&\int_r^{r+\Delta t}(\chi(\xi,t_1)-\chi(\xi,t_2))d\xi\\
&\quad=\int_{t_1}^{t_2}\int_r^{r+\Delta t}\Big(-\chi_r u+\frac{\chi_{rr}}{\rho^2}+\frac{2\chi_r}{r\rho^2}-\frac{\chi^2}{\rho}+\frac{\chi}{\rho}\Big)(\xi,t)d\xi dt\\
&\quad\le\int_{t_1}^{t_2}\Big(\|u\|^2_{L^\infty(I)}\int_I\chi_r^2dr\Big)^{\frac12}(\Delta t)^{\frac12}dt+\Big\|\frac1\rho\Big\|^2_{L^\infty(Q_T)}\int_{t_1}^{t_2}\Big(\int_I\chi_{rr}^2dr\Big)^{\frac12}(\Delta t)^{\frac12}dt\\
&\qquad+2\Big\|\frac1\rho\Big\|^2_{L^\infty(Q_T)}\int_{t_1}^{t_2}\Big(\int_I\chi_r^2dr\Big)^{\frac12}(\Delta t)^{\frac12}dt+\Big\|\frac1\rho\Big\|_{L^\infty(Q_T)}\|\chi\|^2_{L^\infty(Q_T)}(\Delta t)^{\frac32}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\frac{1}{\rho}\right\|_{L^\infty(Q_T)}\|\chi\|_{L^\infty(Q_T)}(\Delta t)^{\frac{3}{2}}\\
&\le C(T)(\Delta t)^{\frac{3}{2}}.
\end{aligned}
$$

For the left-hand side of the above inequality, by the integral mean value theorem, there exists a point $r^*\in[r,r+\Delta t]\subset[a,b]$ such that

$$
\int_r^{r+\Delta t}(\chi(\xi,t_1)-\chi(\xi,t_2))d\xi=(\chi(r^*,t_1)-\chi(r^*,t_2))\Delta t.
$$

Then we get

$$
|\chi(r^*,t_1)-\chi(r^*,t_2)|\le(\Delta t)^{\frac{1}{2}},
$$

which leads to

$$
\begin{aligned}
&|\chi(r,t_1)-\chi(r,t_2)|\\
&\quad\le|\chi(r,t_1)-\chi(r^*,t_1)|+|\chi(r^*,t_1)-\chi(r^*,t_2)|+|\chi(r^*,t_1)-\chi(r,t_2)|\\
&\quad\le C(T)|r-r^*|+C(T)|t_1-t_2|^{\frac{1}{2}}\\
&\quad\le C(T)|t_1-t_2|^{\frac{1}{2}}.
\end{aligned}
$$

So we can get

$$
|\chi(r_1,t_1)-\chi(r_2,t_2)|\le C(T)(|r_1-r_2|+|t_1-t_2|^{\frac{1}{2}})\le\infty,
$$

which implies

$$
\|\chi\|_{C^{1,\frac{1}{2}}(Q_{T_*})}<\infty.
$$

By virtue of the same method above, we can easily get

$$
\|u\|_{C^{1,\frac{1}{2}}(Q_{T_*})}<\infty
$$

From (2.18), we can get

$$
\chi_t=-\chi_r u+\frac{\chi_{rr}}{\rho^2}+\frac{2\chi_r}{r\rho^2}-\frac{\chi^2}{\rho}+\frac{\chi}{\rho},
$$

then we can derive following result by the Schauder theory and (3.1)

$$
\|\chi\|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(Q_{T_*})}<\infty.
$$

Set $G(r,t)=-\frac{1}{2}(\chi_r^2)_r-\frac{2}{r}\chi_r^2$. Then $\|G\|_{C^{\frac{1}{2},\frac{1}{4}}(Q_{T_*})}<\infty$. It is convenient to switch now to the Lagrangian coordinate relative to matter flow $(t,r)\to(\tau,y)$. With the transformation rules: $\partial_\tau=\partial_t+u\partial_r$ and $\partial_y=\frac{1}{r^2\rho}\partial_r$, then $(1.1)_1$ and $(1.1)_2$ become

$$
\begin{cases}
\rho_\tau+r^2\rho^2u_y+\dfrac{2}{r}\rho u=0,\\
\dfrac{u_\tau}{r^2}=\dfrac{4u_y}{r}+r^2(\rho u_y)_y-\dfrac{2u}{r^4\rho}-(\rho^\gamma)_y+\dfrac{G}{r^2\rho}.
\end{cases}
\tag{3.5}
$$

Moreover, Lemmas 2.3, 2.5, 2.7 and 2.8 in the Lagrangian coordinate give

$$
0<C(T)^{-1}\le\rho\le C(T)<\infty,\tag{3.6}
$$

$$
\int_I\rho_y^2dy\le C(T)<\infty,\tag{3.7}
$$

$$
\int_I(u_{yy}^2+u_y^2+u^2)dy\le C(T)<\infty.\tag{3.8}
$$

Since $[(\ln\rho)_y]_\tau = \frac{4u_y}{r} + \frac{2u}{r^4\rho} - r^2(\rho u_y)_y$ and $\frac{u_\tau}{r^2} = (\frac{u}{r^2})_\tau - \frac{2u^2}{r^3}$, combining (3.5), we can get

$$\left(\frac{u}{r^2}\right)_\tau + [(\ln\rho)_y]_\tau = \frac{2u^2}{r^3} - (\rho^\gamma)_y + \frac{G}{r^2\rho} = \frac{2u}{r^3} - \gamma\rho^\gamma(\ln\rho)_y + \frac{G}{r^2\rho}.$$

Namely,

$$\begin{aligned}&\frac{d}{ds}\left\{\left[\frac{u}{r^2} + (\ln\rho)_y\right]\mathrm{e}^{-\gamma\int_s^\tau \rho^\gamma(y,\xi)d\xi}\right\}\\ &\quad= \left[\frac{2u}{r^3} - \gamma\rho(\ln\rho)_y + \frac{G}{r^2\rho}\right]\mathrm{e}^{-\gamma\int_s^\tau \rho^\gamma(y,\xi)d\xi} + \left[\frac{u}{r^2} + (\ln\rho)_y\right]\gamma\rho^\gamma\mathrm{e}^{-\gamma\int_s^\tau \rho^\gamma(y,\xi)d\xi}\\ &\quad= \left[\frac{u}{r^2}\gamma\rho^\gamma + \frac{2u}{r^3} + \frac{G}{r^2\rho}\right]\mathrm{e}^{-\gamma\int_s^\tau \rho^\gamma(y,\xi)d\xi}.\end{aligned}$$

Integrating the above equation over $(0,\tau)$ with respect to s, we have

$$\begin{aligned}&\frac{u}{r^2} + (\ln\rho)_y\\ &\quad= \left(\frac{u_0}{r_0^2} + (\ln\rho_0)_y\right)\mathrm{e}^{-\gamma\int_0^\tau \rho^\gamma(y,\xi)d\xi} + \int_0^\tau \left[\frac{u}{r^2}\gamma\rho^\gamma + \frac{2u}{r^3} + \frac{G}{r^2\rho}\right]\mathrm{e}^{-\gamma\int_s^\tau \rho^\gamma(y,\xi)d\xi}ds.\end{aligned}$$

So we have

$$\|\rho_y\|_{C^{\frac12,\frac14}(Q_{T_*})} < \infty. \tag{3.9}$$

Similarly to the argument in [29], we can apply the Schauder theory to $(3.5)_2$ and get

$$\|u\|_{C^{2+\frac12,1+\frac14}(Q_{T_*})} < \infty.$$

In particular,

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} + \|u_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} < \infty.$$

Applying these estimates to $(3.5)_1$, we get $\|\rho\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} < \infty$, which, together with (3.1), using the Schauder theory to $(3.5)_2$, gives

$$\|\chi\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})} < \infty.$$

Thus we have $\|G\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})} < \infty$. Using the same method once again and applying the Schauder theory to $(3.5)_2$, we have

$$\max\{\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T_*})}, \|\rho_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}, \|\rho_\tau\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T_*})}\} < \infty.$$

This contradicts the choice of T_*. Hence $T_* = \infty$. The proof of Theorem 1.1 is complete.

By virtue of Lemmas 2.1–2.5, we can show that there exist global weak solutions to the problem (1.1)–(1.3) under the assumptions $\rho_0 \in H^1(I)$ satisfying $0 < c_0^{-1} \le \rho_0 \le c_0$ and $u_0 \in L^2(I),\ \chi_0 \in H^1(I)$.

Proof of Theorem 1.2 First, by the standard mollification, we may assume that for any $\alpha \in (0,1)$, there exists a sequence of initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \chi_0^\varepsilon) \in C^{1,\alpha}(I)\times C^{2,\alpha}(I)\times C^{2,\alpha}(I)$ such that

(i) $0 < c_0^{-1} \le \rho_0^\varepsilon \le c_0 < +\infty$ on I,

(ii) $\lim_{\varepsilon\to 0}(\|\rho_0^\varepsilon - \rho_0\|_{H^1} + \|u_0^\varepsilon - u_0\| + \|\chi_0^\varepsilon - \chi_0\|_{H^1}) = 0$.

Let $(\rho^\varepsilon, u^\varepsilon, \chi^\varepsilon)$ be the unique global classical solution of the problem (1.1) with the initial conditions $(\rho_0^\varepsilon, u_0^\varepsilon, \chi_0^\varepsilon)$ and the boundary conditions $(u^\varepsilon, \chi_r^\epsilon)|_{r=a,b} = (0,0)$ for all $t \ge 0$. It follows from Lemmas 2.1–2.5 that, for any $0 < T < +\infty$, the following estimates hold

$$C(T)^{-1} \le \rho^\varepsilon \le C(T), \quad \text{in } I\times[0,T],$$

$$\|\rho^\varepsilon\|_{L^\infty(0,T;H^1(I))} + \|\rho^\varepsilon_t\|_{L^2(0,T;L^2(I))} \le C(T),$$
$$\|u^\varepsilon\|_{L^\infty(0,T;L^2(I))} + \|u^\varepsilon\|_{L^2(0,T;H^1_0(I))} \le C(T),$$
$$\|\chi^\varepsilon\|_{L^\infty(0,T;H^1(I))} + \|\chi^\varepsilon\|_{L^2(0,T;H^2(I))} + \|\chi^\varepsilon_t\|_{L^2(0,T;L^2(I))} \le C(T).$$

After taking possible subsequences, taking $\varepsilon \to 0$ and using Lemma 3.1, we have

$$(\rho^\varepsilon, \rho^\varepsilon_r) \rightharpoonup (\rho, \rho_r) \quad \text{weak * in } L^\infty(0,T;L^2(I)), \tag{3.10}$$
$$\rho^\varepsilon_t \rightharpoonup \rho_t \quad \text{weakly in } L^2(0,T;L^2(I)), \tag{3.11}$$
$$\rho^\varepsilon \to \rho \quad \text{strongly in } C(Q_T), \tag{3.12}$$
$$u^\varepsilon \rightharpoonup u \quad \text{weak * in } L^\infty(0,T;L^2(I)) \text{ weakly in } L^2(0,T;H^1_0(I)), \tag{3.13}$$
$$(\chi^\varepsilon, \chi^\varepsilon_r, \chi^\varepsilon_{rr}) \rightharpoonup (\chi, \chi_r, \chi_{rr}) \quad \text{weakly in } L^2(0,T;L^2(I)), \tag{3.14}$$
$$(\chi^\varepsilon, \chi^\varepsilon_r) \rightharpoonup (\chi, \chi_r) \quad \text{weak * in } L^\infty(0,T;L^2(I)), \tag{3.15}$$
$$\chi^\varepsilon_t \rightharpoonup \chi_t \quad \text{weakly in } L^2(0,T;L^2(I)), \tag{3.16}$$
$$\chi^\varepsilon \rightharpoonup \chi \quad \text{weakly in } C(Q_T) \cap L^2(0,T;C^1(I)). \tag{3.17}$$

It is easy to see that (3.11), (3.12) and (3.13) imply

$$\rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_r + \rho^\varepsilon u^\varepsilon \frac{2}{r} \to \rho_t + (\rho u)_r + \rho u \frac{2}{r}, \quad \text{in } \mathcal{D}'(Q_T). \tag{3.18}$$

Since

$$(\rho^\varepsilon u^\varepsilon)_t = u^\varepsilon_{rr} - \frac{2}{r^2}u^\varepsilon + \frac{2}{r}u^\varepsilon_r - \frac{1}{2}((\chi^\varepsilon_r)^2)_r - \frac{2}{r}(\chi^\varepsilon_r)^2 - (\rho^\varepsilon(u^\varepsilon)^2)_r - \frac{2}{r}\rho^\varepsilon(u^\varepsilon)^2$$
$$- ((\rho^\varepsilon)^\gamma)_r \in L^2(0,T;H^1(I)),$$

and $\rho^\varepsilon u^\varepsilon \rightharpoonup \rho u$ weak * in $L^\infty(0,T;L^2(I))$, by Lemma 3.1, we have

$$\rho^\varepsilon u^\varepsilon \to \rho u, \quad \text{strongly in } C(0,T;H^{-1}(I)), \tag{3.19}$$

which, together with (3.13), implies that

$$\rho^\varepsilon (u^\varepsilon)^2 \to \rho u^2 \quad \text{in } \mathcal{D}'(Q_T). \tag{3.20}$$

From (3.16), we see that $\chi^\varepsilon_{rt} \in L^2(0,T;H^{-1}(I))$. Combining this with (3.15), we have

$$\chi^\varepsilon_r \to \chi_r \quad \text{in } C(0,T;H^{-1}(I)),$$

which, together with (3.15), we get

$$(\chi^\varepsilon_r)^2 \to (\chi_r)^2 \quad \text{in } \mathcal{D}'(Q_T). \tag{3.21}$$

Hence, by (3.10), (3.12), (3.13) and (3.19)–(3.21), we arrive at

$$(\rho u_t) + (\rho u^2)_r + \frac{2}{r}\rho u^2 + (\rho^\gamma)_r = \left(u_r + \frac{2}{r}u\right)_r - \frac{1}{2}(\chi^2_r)_r - \frac{2}{r}\chi^2_r. \tag{3.22}$$

Similarly, we can obtain

$$(\rho^\varepsilon)^2\chi^\varepsilon_t \to \rho^2\chi_t, \quad (\rho^\varepsilon)^2\chi^\varepsilon_r u^\varepsilon \to \rho^2\chi_r u, \quad (\rho^\varepsilon)^2\chi^\varepsilon u^\varepsilon \to \rho^2\chi u.$$

Thus,

$$\rho^2\chi_t + \rho^2\chi_r u + \frac{2}{r}\rho^2\chi u = \chi_{rr} + \frac{2}{r}\chi_r - \rho(\chi^3 - \chi) \quad \text{in } \mathcal{D}'(Q_T). \tag{3.23}$$

From (3.18), (3.22) and (3.23), we conclude that the proof of Theorem 1.2 is complete.

In terms of Lemmas 2.1–2.7, we can derive the existence and uniqueness of strong solutions to the problem (1.1)–(1.3) under the assumptions $\rho_0 \in H^1(I)$ satisfying $0 < c_0^{-1} \le \rho_0 \le c_0$ and $u_0 \in H_0^1(I)$, $\chi_0 \in H^2(I)$.

Proof of Theorem 1.3 Since the initial data $\chi_0 \in H^2(I)$ and $u_0 \in H_0^1(I)$, we may assume that

$$\lim_{\varepsilon\to 0}(\|\rho_0^\varepsilon - \rho_0\|_{H^1(I)} + \|u_0^\varepsilon - u_0\|_{H^1(I)} + \|\chi_0^\varepsilon - \chi_0\|_{H^2(I)}) = 0.$$

Lemmas 2.6 and 2.7 imply

$$\sup_{0\le t\le T}\int_I(|u_r^\varepsilon|^2 + |\chi_{rr}^\varepsilon|^2 + |u^\varepsilon|^2 + |\chi_t^\varepsilon|^2)dr + \int_0^T\int_I(|u_{rr}^\varepsilon|^2 + |u_t^\varepsilon|^2 + |\chi_{rrr}^\varepsilon|^2 + |\chi_{rt}^\varepsilon|^2)drds \le C(T).$$

By the proof of Theorem 1.2 and the weak lower semi-continuity of the norm, we can easily derive that

$$\begin{aligned} &u \in L^\infty(0,T;H_0^1(I)) \cap L^2(0,T;H^2(I)), \quad u_t \in L^2(0,T;L^2(I)),\\ &\chi \in L^\infty(0,T;H^2(I)) \cap L^2(0,T;H^3(I)), \quad \chi_t \in L^\infty(0,T;L^2(I)) \cap L^2(0,T;H^1(I)). \end{aligned}$$

From $(1.1)_1$ we can also get $\rho_t \in L^\infty(0,T;L^2(I))$. Thus we complete the existence result.

Next, we prove the uniqueness. Let (ρ_i, u_i, χ_i) $(i = 1, 2)$ be two solutions to the problem (1.1)–(1.3) obtained as above. Denote $\tilde\rho = \rho_1 - \rho_2$, $\tilde u = u_1 - u_2$ and $\tilde\chi = \chi_1 - \chi_2$. Then

$$\begin{cases} \tilde\rho_t = -\tilde\rho_r u_2 - \rho_{1r}\tilde u - \tilde\rho u_{2r} - \rho_1\tilde u_r - \dfrac{2}{r}\tilde\rho u_2 - \dfrac{2}{r}\rho_1\tilde u,\\ \rho_1\tilde u_t - \tilde u_{rr} = -\tilde\rho u_{2t} - \rho_1 u_1\tilde u_r - \rho_1\tilde u u_{2r} - \tilde\rho u_2 u_{2r} - [(\rho_1^\gamma)_r - (\rho_2^\gamma)_r] - \dfrac{1}{r^2}\tilde u\\ \qquad + \dfrac{2}{r}\tilde u_r - \tilde\chi_r\chi_{1rr} - \chi_{2r}\tilde\chi_{rr} - \dfrac{2}{r}\tilde\chi_r(\chi_{1r} + \chi_{2r}),\\ \rho_1^2\tilde\chi_t - \tilde\chi_{rr} = -\tilde\rho(\rho_1+\rho_2)\chi_{2t} - \rho_1^2\tilde\chi_r u_1 - \rho_1^2\chi_{2r}\tilde u - \tilde\rho(\rho_1+\rho_2)\chi_{2r}u_2 + \dfrac{2}{r}\tilde\chi_r\\ \qquad - \rho_1\tilde\chi(\chi_1^2 + \chi_1\chi_2 + \chi_2^2) - \tilde\rho\chi_2^3 + \rho_1\tilde\chi + \tilde\rho\chi_2 \end{cases} \tag{3.24}$$

for $(r,t) \times (0,+\infty)$, subject to the initial boundary value conditions

$$(\tilde\rho, \tilde u, \tilde\chi)\,|_{t=0} = 0 \quad \text{in } [a,b], \quad (\tilde u, \tilde\chi_r)\,|_{\partial I} = 0, \quad \text{for } t > 0.$$

Multiplying $(3.24)_1$ by $\tilde\rho$, integrating the result over I gives

$$\begin{aligned} \frac12\frac{d}{dt}\int_I\tilde\rho^2dr &= -\int_I\left(\tilde\rho_r\tilde\rho u_2 + \tilde\rho\rho_{1r}\tilde u + \tilde\rho^2u_{2r} + \tilde\rho\rho_1\tilde u_r + \frac{2}{r}\tilde\rho^2u_2 + \frac{2}{r}\tilde\rho\rho_1\tilde u\right)dr\\ &= \frac12\int_I\tilde\rho^2u_{2r}dr - \int_I\left(\tilde\rho\rho_{1r}\tilde u + \tilde\rho^2u_{2r} + \tilde\rho\rho_1\tilde u_r + \frac{2}{r}\tilde\rho^2u_2 + \frac{2}{r}\tilde\rho\rho_1\tilde u\right)dr\\ &\le \frac12\|u_{2r}\|_{L^\infty(I)}\int_I\tilde\rho^2dr + \|\tilde u\|_{L^\infty(I)}\|\tilde\rho\|_{L^2(I)}\|\rho_{1r}\|_{L^2(I)} + \|u_{2r}\|_{L^\infty(I)}\int_I\tilde\rho^2dr\\ &\quad + \|\rho_1\|_{L^\infty(Q_T)}\|\tilde u_r\|_{L^2(I)}\|\tilde\rho\|_{L^2(I)} + c\|u_2\|_{L^\infty(I)}\|\tilde\rho\|_{L^2(I)}\\ &\quad + c\|\tilde u\|_{L^\infty(I)}\|\rho_1\|_{L^\infty(Q_T)}\|\tilde\rho\|_{L^2(I)}. \end{aligned}$$

Since $\tilde u(0,t) = 0$, we have $\tilde u(r,t) = \int_0^r\tilde u_\xi(\xi,t)d\xi$ for $(r,t) \in Q_T$ and hence

$$\|\tilde u\|_{L^\infty(I)} \le \|\tilde u_r\|_{L^2(I)}, \quad t \in [0,T]. \tag{3.25}$$

It follows from (3.25) and the regularities of (ρ_i, u_i)that

$$\frac{d}{dt}\int_I \tilde\rho^2 dr \le c\Big(\|u_2\|_{H^2(I)} + \int_I \rho_{1r}^2 dr + 1\Big)\int_I \tilde\rho^2 dr + \int_I \tilde u_r^2 dr. \tag{3.26}$$

Multiplying $(3.24)_2$ by $\tilde u$, integrating the result over I yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I \rho_1\tilde u^2 dr + \int_I \tilde u_r^2 dr &= \frac{1}{2}\int_I \rho_{1t}\tilde u^2 dr - \int_I \rho_1 u_1\tilde u_r\tilde u dr - \int_I \big(\rho_1\tilde u^2 u_{2r} + \tilde\rho u_2 u_{2r}\tilde u + \tilde\rho u_{2t}\tilde u\big)\,dr\\ &\quad + \int_I [(\rho_1^\gamma) - (\rho_2^\gamma)]\tilde u_r dr - \int_I \frac{2}{r^2}\tilde u^2 dr + \int_I \frac{2}{r}\tilde u\tilde u_r dr - \int_I \tilde\chi_r\chi_{1rr}\tilde u dr\\ &\quad + \int_I (\chi_{2rr}\tilde\chi_r\tilde u + \chi_{2r}\tilde\chi_r\tilde u_r)\,dr - \int_I \left(\frac{2}{r}\tilde\chi_r(\chi_{1r}+\chi_{2r})\tilde u\right) dr.\end{aligned}$$

Since $\rho_{1t} = -(\rho_1 u_1)_r - \frac{2}{r}\rho_1 u_1$, then we have

$$\frac{1}{2}\int_I \rho_{1t}\tilde u^2 dr - \int_I \rho_1 u_1 \tilde u_r\tilde u dr = -\frac{1}{2}\int_I \frac{2}{r}\rho_1 u_1\tilde u^2 dr.$$

Moreover, $\int_I |\varphi| dr = \int_I \frac{\rho_i}{\rho_i}|\varphi| dr = \|\frac{1}{\rho_i}\|_{L^\infty(Q_T)}\int_I |\rho_i\varphi| dr$ $(i = 1, 2)$. By using the above results, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I \rho_1\tilde u^2 dr + \int_I \tilde u_r^2 dr\\ &\quad = c\|u_1\|_{L^\infty(I)}\int_I \rho_1\tilde u^2 dr + \|u_{2r}\|_{L^\infty(I)}\int_I \rho_1\tilde u^2 dr + \|u_2\|_{L^\infty(I)}\|u_{2r}\|_{L^\infty(I)}\|\tilde\rho\|_{L^2(I)}\|\sqrt{\rho_1}\tilde u\|_{L^2(I)}\\ &\qquad + \|\tilde u\|_{L^\infty(I)}\|u_{2t}\|_{L^2(I)}\|\tilde\rho\|_{L^2(I)} + c\|\tilde\rho\|_{L^2(I)}\|\tilde u_r\|_{L^2(I)} + c\int_I \rho_1\tilde u^2 dr + c\|\tilde u_r\|_{L^2(I)}\|\sqrt{\rho_1}\tilde u\|_{L^2(I)}\\ &\qquad + \|\tilde u\|_{L^\infty(I)}\|\chi_{1rr}\|_{L^2(I)}\|\tilde\chi_r\|_{L^2(I)} + \|\tilde u\|_{L^\infty(I)}\|\chi_{2rr}\|_{L^2(I)}\|\tilde\chi_r\|_{L^2(I)}\\ &\qquad + \|\chi_{2r}\|_{L^\infty(I)}\|\tilde\chi_r\|_{L^2(I)}\|\tilde u_r\|_{L^2(I)} + c\|\chi_{1r}\|_{L^\infty(I)}\|\tilde\chi_r\|_{L^2(I)}\|\sqrt{\rho_1}\tilde u\|_{L^2(I)}\\ &\qquad + c\|\chi_{2r}\|_{L^\infty(I)}\|\tilde\chi_r\|_{L^2(I)}\|\sqrt{\rho_1}\tilde u\|_{L^2(I)}\\ &\quad \le \frac{1}{2}\int_I \tilde u_r^2 dr + c\left(\int_I \chi_{1rr}^2 dr + \int_I \chi_{2rr}^2 dr\right)\int_I \tilde\chi_r^2 dr + c\left(\int_I u_{2t}^2 dr + 1\right)\int_I \tilde\rho^2 dr\\ &\qquad + c\left(\int_I u_{1r}^2 dr + \int_I u_{2r}^2 dr + \int_I u_{2rr}^2 dr + 1\right)\int_I \rho_1\tilde u^2 dr,\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dt}\int_I \rho_1\tilde u^2 dr + \int_I \tilde u_r^2 dr &\le c\int_I \tilde\chi_r^2 dr + c\left(\int_I u_{2t}^2 dr + 1\right)\int_I \tilde\rho^2 dr\\ &\quad + c\left(\int_I u_{1r}^2 dr + \int_I u_{2r}^2 dr + \int_I u_{2rr}^2 dr + 1\right)\int_I \rho_1\tilde u^2 dr.\end{aligned} \tag{3.27}$$

Multiplying $(3.24)_3$ by $\tilde\chi$, integrating over I, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I \rho_1^2\tilde\chi^2 dr + \int_I \tilde\chi_r^2 dr\\ &\quad = \int_I \rho_1\rho_{1t}\tilde\chi^2 dr - \int_I \tilde\rho(\rho_1+\rho_2)\chi_{2t}\tilde\chi dr - \int_I \big(\rho_1^2 u_1\tilde\chi_r\tilde\chi + \rho_1^2\tilde u\chi_{2r}\tilde\chi\big)\,dr\\ &\qquad - \int_I (\tilde\rho(\rho_1+\rho_2)\chi_{2r}u_2\tilde u)\,dr + \int_I \frac{2}{r}\tilde\chi_r\tilde\chi dr - \int_I \rho_1\tilde\chi(\chi_1^2 + \chi_1\chi_2 + \chi_2^2)\tilde\chi dr\\ &\qquad - \int_I \tilde\rho\chi_2^3\tilde\chi dr + \int_I \rho_1\tilde\chi^2 + \int_I \tilde\rho\chi_2\tilde\chi dr\end{aligned}$$

$$
\begin{aligned}
&\le \|\tilde\chi\|_{L^\infty(I)}\|\rho_1\tilde\chi\|_{L^2(I)}\|\rho_{1t}\|_{L^2(I)} + \|\tilde\chi\|_{L^\infty(I)}\|\rho_1+\rho_2\|_{L^\infty(Q_T)}\|\tilde\rho\|_{L^2(I)}\|\chi_{2t}\|_{L^2(I)}\\
&\quad + \|\rho_1\|_{L^\infty(Q_T)}\|u_1\|_{L^\infty(I)}\|\tilde\chi_r\|_{L^2(I)}\|\rho_1\tilde\chi\|_{L^2(I)}\\
&\quad + \|\rho_1\|^{\frac12}_{L^\infty(Q_T)}\|\chi_{2r}\|_{L^\infty(I)}\|\sqrt{\rho_1}\tilde u\|_{L^2(I)}\|\rho_1\tilde\chi\|_{L^2(I)}\\
&\quad + \left\|\frac{\rho_1+\rho_2}{\sqrt{\rho_1}}\right\|_{L^\infty(Q_T)}\|\chi_{2r}\|_{L^\infty(I)}\|u_2\|_{L^\infty(I)}\|\sqrt{\rho_1}\tilde u\|_{L^2(I)}\|\tilde\rho\|_{L^2(I)}\\
&\quad + c\left\|\frac{1}{\rho_1}\right\|_{L^\infty(Q_T)}\|\tilde\chi_r\|_{L^2(I)}\|\rho_1\tilde\chi\|_{L^2(I)} + \left\|\frac{\chi_1^2+\chi_1\chi_2+\chi_2^2}{\rho_1}\right\|_{L^\infty(Q_T)}\|\rho_1\tilde\chi\|^2_{L^2(I)}\\
&\quad + \|\chi_2\|^3_{L^\infty(I)}\left\|\frac{1}{\rho_1}\right\|_{L^\infty(Q_T)}\|\tilde\rho\|_{L^2(I)}\|\rho_1\tilde\chi\|_{L^2(I)} + \left\|\frac{1}{\rho_1}\right\|_{L^\infty(Q_T)}\|\rho_1\tilde\chi\|^2_{L^2(I)}\\
&\quad + \|\chi_2\|_{L^\infty(I)}\|\frac{1}{\rho_1}\|_{L^\infty(Q_T)}\|\tilde\rho\|_{L^2(I)}\|\rho_1\tilde\chi\|_{L^2(I)}.
\end{aligned}
$$

From the equations $(1.1)_1$, $(1.1)_{3,4}$ and the estimates (2.5), (2.8), (2.9), (2.17), (2.20) we can deduce that

$$
\int_I \rho_{it}^2 dr \le c, \quad \int_I \chi_{it}^2 dr \le c, \quad i=1,2.
$$

Moreover, the Sobolev embedding theorem implies that

$$
\|\tilde\chi\|_{L^\infty(I)} \le c\|\tilde\chi\|_{H^1(I)} \le c(\|\tilde\chi\|_{L^2(I)} + \|\tilde\chi_r\|_{L^2(I)}).
$$

Hence together with the regularities for (ρ_i, u_i, χ_i), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_I \rho_1^2\tilde\chi^2 dr + \int_I \tilde\chi_r^2 dr\\
&\quad \le c\left(1+\int_I u_{2r}^2 dr\right)\int_I \tilde\rho^2 dr + c\left(1+\int_I u_{1r}^2\right)\int_I \rho_1^2\tilde\chi^2 dr + c\int_I \chi_{2rr}^2 dr\int_I \rho_1\tilde u^2 dr. \qquad (3.28)
\end{aligned}
$$

Adding (3.26), (3.27) and (3.28) together, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_I \left(\tilde\rho^2 + \rho_1\tilde u^2 + c\rho_1^2\tilde\chi^2\right) dr\\
&\quad \le c\left(\int_I u_{2t}^2 dr + \|u_2\|_{H^2(I)} + \int_I \rho_{1r}^2 dr + 1\right)\int_I \tilde\rho^2 dr\\
&\qquad + c\left(\int_I u_{1r}^2 dr + \|u_2\|_{H^2(I)} + \int \chi_{2rr}^2 dr + 1\right)\int_I \rho_1\tilde u^2 dr + c\left(1+\int_I u_{1r}^2 dr\right)\int_I \rho_1^2\tilde\chi^2 dr\\
&\quad \le c\left(\int_I u_{2t}^2 dr + \|u_2\|_{H^2(I)} + \|u_1\|_{H^1(I)} + \|\chi_2\|_{H^2(I)} + \int_I \rho_{1r}^2 dr + 1\right)\int_I (\tilde\rho^2 + \rho_1\tilde u^2 + c\rho_1^2\tilde\chi^2) dr\\
&\quad = K(t)\int_I \left(\tilde\rho^2 + \rho_1\tilde u^2 + c\rho_1^2\tilde\chi^2\right) dr, \qquad (3.29)
\end{aligned}
$$

where $K(t) = c\left(\int_I u_{2t}^2 dr + \|u_2\|_{H^2(I)} + \|u_1\|_{H^1(I)} + \|\chi_2\|_{H^2(I)} + \int_I \rho_{1r}^2 dr + 1\right)$. From equation $(1.1)_2$ and estimates (2.8), (2.17), (2.20), we can get $\int_0^T\int_I u_{it}^2 drds \le +\infty$ $(i=1,2)$. Combining Lemmas 2.1–2.8 we see that $\int_0^T K(t)dt < +\infty$. Applying the Gronwall inequality to (3.29) we arrive at

$$
\int_I (\tilde\rho^2 + \rho_1\tilde u^2 + c\rho_1^2\tilde\chi^2) dr \le \int_I (\tilde\rho^2 + \rho_1\tilde u^2 + c\rho_1^2\tilde\chi^2)|_{t=0} dr \exp\left(\int_0^T K(t)dt\right) = 0.
$$

Noticing that the density ρ_1 and the constant c are positive, thus we obtain $(\tilde\rho, \tilde u, \tilde\chi) = 0$. The proof is complete.

We are in preparation to deal with the case of the viscosity coefficient ν depending on the concentration χ. In fact, in this case, the global Hölder estimates for the solution involve the Hölder estimates of χ_{rt} which is nontrivial.

Conflict of Interest The authors declare no conflict of interest.

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References

[1] Abels, H.: On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. *Arch. Ration. Mech. Anal.*, **194**, 463–506 (2009)

[2] Abels, H., Feireisl, E.: On a diffuse interface model for a two-phase flow of compressible viscous fluids. *Indiana Univ. Math. J.*, **57**(2), 659–698 (2008)

[3] Anderson, D. M., McFadden, G. B., Wheeler, A. A.: Diffuse-interface method in fluid mechanics. *Annu. Rev. Fluid Mech.*, **30**, 139–165 (1998)

[4] Babak, S., Stefan, T., Matthias, M., et al.: Isogeometric analysis of the Navier–Stokes/Cahn–Hilliard. *J. Comput. Phys.*, **348**, 171–194 (2017)

[5] Blesgen, T.: A generalization of the Navier–Stokes equations to two-phase flow. *J. Phys. D Appl. Phys.*, **32**, 1119–1123 (1999)

[6] Boyer, F.: A theoretical and numerical model for the study of incompressible mixture flows. *Comput. & Fluids*, **31**, 41–68 (2002)

[7] Boyer, F.: Mathematical study of multi-phase flow under shear through order parameter formulation. *Asymptot. Anal.*, **20**(2), 175–212 (1999)

[8] Chen, Y., He, Q., Huang, B., et al.: Global strong solution to a thermodynamic compressible diffuse interface model with temperature dependent heat-conductivity in 1-D. *Math. Methods Appl. Sci.*, **44**(17), 12945–12962 (2021)

[9] Chen, Y., He, Q., Huang, B., et al.: The Cauchy problem for non-isentropic compressible Navier–Stokes/Allen–Cahn system with degenerate heat-conductivity. arXiv:2005.11205 (2020)

[10] Chen, M., Guo, X.: On the existence and uniqueness of solution to a stochastic 2D Navier–Stokes/Cahn–Hilliard model. *J. Differential Equations*, **263**, 1028–1054 (2017)

[11] Chen, M., Guo, X.: Global large solutions for a coupled compressible Navier–Stokes/Allen–Cahn system with initial vacuum. *Nonlinear Anal. Real World Appl.*, **37**, 350–373 (2017)

[12] Chen, S., Zhu, C.: Blow-up criterion and the global existence of strong/classical solutions to Navier–Stokes/Allen–Cahn system. *Z. Angew. Math. Phys.*, **72**(1), Article number: 14 (2021)

[13] Ding, S., Li, Y., Tang, Y.: Strong solutions to 1D compressible Navier–Stokes/Allen–Cahn system with free boundary. *Math. Methods Appl. Sci.*, **42**(14), 4780–4794 (2019)

[14] Deugoué, G., Tachim, M. T.: Convergence of the solution of the stochastic 3D globally modified Navier–Stokes/Cahn–Hilliard equations. *J. Differential Equations*, **265**, 545–592 (2018)

[15] Ding, S., Li, Y., Luo, W.: Global solutions for a coupled compressible Navier–Stokes/Allen–Cahn system in 1D. *J. Math. Fluid Mech.*, **15**, 335–360(2013)

[16] Ding, S., Lin, J., Wang, C., et al.: Compressible hydrodynamic flow liquid crystals in 1D. *Discrete Contin. Dyn. Syst.*, **32**(2), 539–563 (2012)

[17] Feireisl, E., Petzeltová, H., Rocca, E., et al.: Analysis of a phase-field model for two-phase compressible fluids. *Math. Methods Appl. Sci.*, **31**, 1972–1995 (2008)

[18] Feng, J. J., Liu, C., Shen, J., et al.: An energetic variational formulation with phase field methods for interfacial dynamics of complex fluids: advantages and challenges. *Modeling of Soft Matter IMA Vol Math. Appl.*, **141**, 1–26 (2005)

[19] Gal, C. G., Grasselli, M.: Asymptotic behavior of a Cahn–Hilliard-Navier–Stokes system in 2D. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27**, 401–436 (2010)

[20] Gal, C. G., Grasselli, M.: Trajectory attractors for binary fluid mixtures in 3D. *Chin. Ann. Math. Ser. B*, **31**(5), 655–678 (2010)

[21] Gal, C. G., Grasselli, M.: Instability of two-phase flows: a lower bound on the dimension of global attractor of the Cahn–Hilliard–Navier–Stokes system. *Phys. D*, **240**, 629–635 (2011)

[22] Giorgini, A., Miranville, A., Temam, R.: Uniqueness and regularity for the Navier–Stokes–Cahn–Hilliard system. *SIAM J. Math. Anal.*, **51**, 2535–2574 (2019)

[23] Kotschote, M.: Strong solutions of the Navier–Stokes equations for a compressible fluid of Allen–Cahn type. *Arch. Ration. Mech. Anal.*, **206**, 489–514 (2012)

[24] Kotschote, M., Zacher, R.: Strong solutions in the dynamical theory of compressible fluid mixtures. *Math. Models Methods Appl. Sci.*, **25**(7), 1217–1256 (2015)

[25] Li, Y., Huang, M.: Strong solutions for an incompressible Navier–Stokes/Allen–Cahn system with different densities. *Z. Angew. Math. Phys.*, **69**, Article number: 68 (2018)

[26] Liu, W., Bertozzi, A. L., Kolokolnikov, T.: Diffuse interface surface tension models in an expanding flow. *Commun. Math. Sci.*, **10**(1), 387–418 (2012)

[27] Lowengrub, J., Truskinovsky. L.: Quasi-incompressible Cahn–Hilliard fluids and topological transitions. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **454**, 2617–2654 (1998)

[28] Simon, J.: Nonhomogeneous viscous incompressible fluids: Existence of velocity, density and pressure. *SIAM J. Math. Anal.*, **21**, 1093–1117 (1990)

[29] Song, C., Wang, Y.: Spherically symmetric solutions for a coupled compressible Navier–Stokes/Allen–Cahn system. *J. Partial Differential Equations*, **31**(4), 374–384 (2018)

[30] Song, C., Zhang, J., Wang, Y.: Time-periodic solution to the compressible Navier–Stokes/Allen–Cahn system. *Acta Math. Sinica, Engl. Ser.*, **36**(4), 419–442 (2020)

[31] Tachim, M. T.: On the existence and uniqueness of solution to a stochastic 2D Navier–Stokes/Cahn–Hilliard model. *J. Differential Equations*, **263**, 1028–1054 (2017)

[32] Tan, Z., Lim, K. M., Khoo, B. C.: An adaptive mesh redistribution method for the incompressible mixture flows using phase-field model. *J. Comput. Phys.*, **225**, 1137–1158 (2007)

[33] Witterstein, G., Hoffmann, K. H.: On the existence of a solution for a model of stem cell differentiation. *Math. Methods Appl. Sci.*, **31**, 1972–1995 (2008)

[34] Xu, X., Zhao, L., Liu, C.: Axisymmetric solutions to coupled Navier–Stokes/Allen–Cahn equations. *SIAM J. Math. Anal.*, **41**(6), 2246–2282 (2010)

[35] Yan, Y., Ding, S., Li, Y.: Strong solutions for 1D compressible Navier–Stokes/Allen–Cahn system with phase variable dependent viscosity. arXiv:2109.14529 (2021)

[36] Yang, X., Feng, J., Liu, C., et al.: Numerical simulations of jet pinching-off and drop formation using an energetic variational phase-field method. *J. Comput. Phys.*, **218**, 417–428 (2006)

[37] Zhang, J.: A regularity criterion for the 3D incompressible density-dependent Navier–Stokes–Allen–Cahn equations. *J. Partial Differential Equations*, **29**(2), 117–124 (2016)

[38] Zhang, J.: Regularity of solutions to 1D compressible Navier–Stokes–Allen–Cahn system. *Appl. Anal.*, **100**(9), 1827–1842 (2021)

[39] Zhao, L., Guo, B., Huang, H.: Vanishing viscosity limit for a coupled Navier–Stokes/Allen–Cahn system. *J. Math. Anal. Appl.*, **384**, 232–245 (2011)