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A Note on the Entropy for Heisenberg Group Actions on the Torus

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Abstract In this paper, the entropy of discrete Heisenberg group actions is considered. Let α be a discrete Heisenberg group action on a compact metric space X. Two types of entropies, $\widetilde{h}(\alpha)$ and $h(\alpha)$ are introduced, in which $\widetilde{h}(\alpha)$ is defined in Ruelle's way and $h(\alpha)$ is defined via the natural extension of α. It is shown that when X is the torus and α is induced by integer matrices then $\widetilde{h}(\alpha)$ is zero and $h(\alpha)$ can be expressed via the eigenvalues of the matrices.

Keywords Entropy, Heisenberg group action, natural extension, torus

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1 Introduction

Let (X, d_X) be a compact metric space, f a homeomorphism (resp. continuous map) on X. Via iteration, f induces a $\mathbb{Z}$ (resp. $\mathbb{Z}_+$)-action on X. The topological entropy $h(f)$ is an important invariant which measures the complexity of f via the exponential growth rate of the number of orbits distinguishable with limit precision. The theory of entropies, including topological entropy and measure-theoretical entropy and their applications, was well investigated (see the monographs [1, 12, 19] and [6], etc.).

Based on the need in the study of lattice statistical mechanics, Ruelle [17] introduced the concept of entropy for $\mathbb{Z}^k$-actions for $k > 1$. Let β be a $\mathbb{Z}^k$-action on X and $h(\beta)$ be the topological entropy in [17]. A necessary condition for this entropy to be positive is that each of the generators should have infinite entropy as a single transformation (see [18], for example). In another word, if one of the generators has finite entropy, then $h(\beta) = 0$. That means that Ruelle's entropy can not well characterize the complexity of a class of important $\mathbb{Z}^k$-actions, such as the smooth $\mathbb{Z}^k$-actions on compact finite dimensional manifolds, especially Lie groups. To appropriately describe the complexity of $\mathbb{Z}^k$-actions from different viewpoints, some other types of entropies, such as the entropy via the natural extension (see [7, 9, 10, 21, 23, 24], etc.) and the entropy of the system along certain directions (see [3, 14–16], etc.), were introduced and investigated. A natural question is how to extend the theory of these entropies to noncommutative group actions.

The discrete Heisenberg group $\mathcal{H}$ is a 2-step nilpotent group which is most closest to being abelian and whose generators A, B, C of $\mathcal{H}$ satisfy the property

$$AC = CA, \quad BC = CB, \quad AB = BAC. \tag{1.1}$$

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In this paper, we will consider the entropy of $\mathcal{H}$-actions on compact metric spaces, especially the torus. Let X be a compact metric space and $\alpha : \mathcal{H} \to \mathrm{Homeo}(X, X)$ be an $\mathcal{H}$-action on X with the generators $\alpha_A, \alpha_B, \alpha_C$, where $\mathrm{Homeo}(X, X)$ is the homeomorphism group of X. We first consider a version of entropy $\widetilde{h}(\alpha)$ in the way of Ruelle [17] and show that it is zero if $h(\alpha_C)$ is finite (Proposition 2.4). Hence if α is linearly induced on $\mathbb{T}^m$, that is, α is induced by the matrix Heisenberg group $\mathcal{H}$ with generators $A, B, C \in \mathrm{SL}(m, \mathbb{Z})$, then $\widetilde{h}(\alpha) = 0$. Therefore $\widetilde{h}(\alpha)$ is not a satisfactory quantity to measure the complexity of Heisenberg group actions. Then we turn to consider the entropy $h(\alpha)$ of the natural extension of α, just as it had been done for $\mathbb{Z}^k$-actions by Friedland [9].

To obtain the formula of $h(\alpha)$ for a linearly induced $\mathcal{H}$-action α on $\mathbb{T}^m$, we adopt the strategy in [24], in which the formula of the entropy for the natural extension of a linearly induced $\mathbb{Z}^k$-action on torus was obtained. For the systematic investigation, including the calculation of entropy, for general algebraic $\mathbb{Z}^k$-actions of entropy rank one, namely those for which each element has finite entropy, we refer to Einsiedler and Lind's work [7]. We will first show that there is a decomposition of $\mathbb{R}^m$ which is $\mathcal{H}$-invariant (Theorem A). Then we relate natural extension of α to a skew product transformation where the fiber maps are the generators of α. The fiber entropy of the skew product can be explicitly computed in terms of the eigenvalues of generators of $\mathcal{H}$. Finally, using the relative variational principle of Ledrappier and Walters [13], we get the entropy formula of the skew product and hence obtain the formula of $h(\alpha)$ (Theorem B).

This paper is organized as follows. The notations and statements of results are given in Section 2. The proof of Theorem A is given in Section 3. Section 4 is devoted to the proofs of Proposition 2.4 and Theorem B.

2 Notations and Statements of Results

Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Let (X, d_X) be a compact metric space and $\mathcal{H}$ be the *discrete Heisenberg group with generators* A, B *and* C satisfying the relations (1.1). Then for every $K \in \mathcal{H}$, there is a unique triple $(n_1, n_2, n_3) \in \mathbb{Z}^3$ such that $K = A^{n_1} B^{n_2} C^{n_3}$. Clearly, $\mathcal{H}$ is a 2-step nilpotent group with center $\langle C \rangle$. Let $\alpha : \mathcal{H} \to \mathrm{Homeo}(X, X)$ be a continuous *Heisenberg group action on* X. Then α has the generators $\alpha_A = \alpha(A)$, $\alpha_B = \alpha(B)$ and $\alpha_C = \alpha(C)$ with the property

$$\alpha_A \alpha_C = \alpha_C \alpha_A, \quad \alpha_B \alpha_C = \alpha_C \alpha_B, \quad \alpha_A \alpha_B = \alpha_B \alpha_A \alpha_C. \tag{2.1}$$

This indicates that α_C is *the center element* of Heisenberg group action α.

The Heisenberg group $\mathcal{H}$ naturally has an action on $\mathbb{T}^3$ since $\mathcal{H}$ embeds into $\mathrm{SL}(3, \mathbb{Z})$ (see [8], for example). We can obtain more general examples such as the following.

Example 2.1 (Example 1.1 in [11]) Let

$$A = \begin{bmatrix} M_1 & I_m & O \\ O & M_1 & O \\ O & O & M_1 \end{bmatrix}, \quad B = \begin{bmatrix} M_2 & O & O \\ O & M_2 & I_m \\ O & O & M_2 \end{bmatrix}, \quad C = \begin{bmatrix} I_m & O & {M_1}^{-1}{M_2}^{-1} \\ O & I_m & O \\ O & O & I_m \end{bmatrix},$$

where $M_1, M_2 \in \mathrm{SL}(m, \mathbb{Z})$ with $M_1 M_2 = M_2 M_1$, and I_m and O are the $m \times m$ identity and zero matrix, respectively. It is easy to check that the condition (1.1) is satisfied. If $X = \mathbb{T}^{3m}$ then

A, B and C induce automorphisms on $\mathbb{T}^{3m}$, respectively, that generates a Heisenberg group action α on $\mathbb{T}^{3m}$.

Let α be an $\mathcal{H}$-action on X with generators α_A, α_B and α_C. Now we consider two types of topological entropies to measure the complexity of α. Firstly, we introduce the topological entropy in Ruelle's way (see [17] and [18], for example).

For $n > 0$, denote the cube

$$\Lambda_n = \{\vec{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3 \,|\, n_i \leq n, 1 \leq i \leq 3\}.$$

We denote

$$\alpha^{\vec{n}} = \alpha_A^{n_1}\alpha_B^{n_2}\alpha_C^{n_3}. \tag{2.2}$$

Remark 2.2 Recall that if β is a $\mathbb{Z}^3$-action on X with generators β_1, β_2 and β_3, then $\beta^{\vec{n}} = \beta_1^{n_1}\beta_2^{n_2}\beta_3^{n_3} = \beta_i^{n_i}\beta_j^{n_j}\beta_k^{n_k}$ for any pairwise different $1 \leq i, j, k \leq 3$. Since $\mathcal{H}$ is noncommutative, we have to specify an iteration order, such as in (2.2), for $\alpha^{\vec{n}}$.

A set $E \subset X$ is $(\alpha, \Lambda_n, \epsilon)$-*spanning* if for every $x \in X$ there exists a $y \in E$, such that $\max_{\vec{n}\in\Lambda_n} d_X(\alpha^{\vec{n}}x, \alpha^{\vec{n}}y) \leq \epsilon$. Let $r_{d_X}(\alpha, \Lambda_n, \epsilon, X)$ be the smallest cardinality of any $(\alpha, \Lambda_n, \epsilon)$-spanning set of α.

A set $F \subset X$ is $(\alpha, \Lambda_n, \epsilon)$-*separated* if for any $x, y \in F$, $\max_{\vec{n}\in\Lambda_n} d_X(\alpha^{\vec{n}}x, \alpha^{\vec{n}}y) > \epsilon$. Let $s_{d_X}(\alpha, \Lambda_n, \epsilon, X)$ be the largest cardinality of any $(\alpha, \Lambda_n, \epsilon)$-separated set of α.

Definition 2.3 *Let α be an $\mathcal{H}$-action on X. The* topological entropy $\widetilde{h}(\alpha)$ *is defined as*

$$\begin{aligned}\widetilde{h}(\alpha) &= \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^3}\log s_{d_X}(\alpha, \Lambda_n, \epsilon, X)\\ &= \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^3}\log r_{d_X}(\alpha, \Lambda_n, \epsilon, X).\end{aligned}$$

(*The second equality can be obtained by a standard discussion.*)

Proposition 2.4 *Let α be an $\mathcal{H}$-action on X with the generators $\alpha_A, \alpha_B, \alpha_C$. If either $h(\alpha_A)$ or $h(\alpha_C)$ is finite then $\widetilde{h}(\alpha) = 0$.*

In fact, if we replace (2.2) by $\alpha^{\vec{n}} = \alpha_B^{n_1}\alpha_A^{n_2}\alpha_C^{n_3}$, the condition "either $h(\alpha_A)$ or $h(\alpha_C)$ is finite" in the above proposition can be changed to "either $h(\alpha_B)$ or $h(\alpha_C)$ is finite". We will give the proof of this proposition in Section 4. By Proposition 2.4 we see that the topological entropy $\widetilde{h}(\alpha)$ cannot well measure the complexity of smooth Heisenberg group actions on compact finite dimensional Riemannian manifolds, especially the torus. Now we consider another type of topological entropy via the natural extension.

Define *the orbit space of* α by

$$X_\alpha = \left\{\overline{x} = \{x_n\}_{n\in\mathbb{Z}_+} \in \prod_{n\in\mathbb{Z}_+} X : \alpha_{i_n}(x_n) = x_{n+1} \text{ for some } \alpha_{i_n} \in \{\alpha_A, \alpha_B, \alpha_C\}\right\}.$$

This is a closed subset of the compact space $\prod_{n\in\mathbb{Z}_+} X$ and so is again compact. A natural metric on X_α is defined by

$$d_{X_\alpha}(\overline{x}, \overline{y}) = \sum_{n=0}^{\infty}\frac{d_X(x_n, y_n)}{2^n},$$

for $\overline{x} = \{x_n\}_{n\in\mathbb{Z}_+}, \overline{y} = \{y_n\}_{n\in\mathbb{Z}_+} \in X_\alpha$. We can define *a natural shift map* $\sigma_\alpha : X_\alpha \to X_\alpha$ by $\sigma_\alpha(\{x_n\}_{n\in\mathbb{Z}_+}) = \{x_{n+1}\}_{n\in\mathbb{Z}_+}$. We call $(X_\alpha, \sigma_\alpha)$ *the natural extension of* α.

Definition 2.5 *Let α be an $\mathcal{H}$-action on X and $\sigma_\alpha : X_\alpha \to X_\alpha$ be its natural extension. The* topological entropy $h(\alpha)$ *is defined as that of σ_α, i.e.,*

$$h(\alpha) := h(\sigma_\alpha) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{d_{X_\alpha}}(\sigma_\alpha, n, \epsilon, X_\alpha).$$

We can replace $s_{d_{X_\alpha}}(\sigma_\alpha, n, \epsilon, X_\alpha)$ by $r_{d_{X_\alpha}}(\sigma_\alpha, n, \epsilon, X_\alpha)$ in the above equation.

To calculate the entropy of the natural extension is not easy even for commutative group actions. In [7], a method to calculate the entropy of algebraic $\mathbb{Z}^k$-actions of entropy rank one was provided. Applying this method, a formula of the entropy of the natural extension of linearly induced $\mathbb{Z}^k$-actions on $\mathbb{T}^m$ was given in [24]. We will adapt this method to our case.

Let α be a Heisenberg group action on $\mathbb{T}^m$ which is induced by the matrix Heisenberg group $\mathcal{H}$ with generators $A, B, C \in \mathrm{SL}(m, \mathbb{Z})$. As we stated in the introduction, we first show that there is a decomposition of $\mathbb{R}^m$ which is $\mathcal{H}$-invariant.

Theorem A Let $\mathcal{H}$ be the matrix Heisenberg group with generators $A, B, C \in \mathrm{SL}(m, \mathbb{Z})$. Then the module of the eigenvalues of C is 1 and there exists an $\mathcal{H}$-invariant decomposition of $\mathbb{R}^m$ as

$$\mathbb{R}^m = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

such that for all $j = 1, 2, \ldots, s$, $V_j \neq \{0\}$, the modules of eigenvalues of $A|_{V_j}$ and $B|_{V_j}$ are $\lambda_{A,j}$ and $\lambda_{B,j}$ respectively.

In Hu, Shi and Wang [11], the ergodic and rigidity properties of smooth actions of the discrete Heisenberg group $\mathcal{H}$ were well investigated. In particular, the decomposition of the tangent space of any C^∞ compact Riemannian manifold M for Lyapunov exponents was established. Theorem A can be seen as a simple version of that result in our special case.

Theorem B Let α be the Heisenberg group action on the torus $\mathbb{T}^m$ induced by the matrix Heisenberg group $\mathcal{H}$ with generators $A, B, C \in \mathrm{SL}(m, \mathbb{Z})$. Then we have

$$h(\alpha) = \max_{J \subset \{1,2,\ldots,s\}} \log \left(\prod_{j\in J} \lambda_{A,j}^{d_j} + \prod_{j\in J} \lambda_{B,j}^{d_j} + 1 \right), \tag{2.3}$$

where $d_j = \dim V_j$.

To generalize Theorem B to the case of general smooth Heisenberg group actions is a meaningful work. In [21], the entropy formula for the natural extension of a smooth $\mathbb{Z}^k$-action on a compact Riemannian manifold was given via the Lyapunov exponents of the generators. To get that formula, the Pesin's theory for smooth $\mathbb{Z}^k$-actions and the technique of random dynamical systems were applied. We think the strategy in [21] still works in the case of smooth Heisenberg group actions under certain assumptions on the center element.

3 An Invariant Decomposition of $\mathbb{R}^m$ with Respect to the Matrix Heisenberg Group

At the beginning of this section, we recall some basic facts about matrices. Let A be an $m \times m$ complex matrix. If λ^A is an eigenvalue of A, let

$$V_{\lambda^A} = \{\vec{u} \in \mathbb{C}^m : \ (\lambda^A I_m - A)^i \vec{u} = 0 \text{ for some } i \in \mathbb{N}\}.$$

If A is real and λ^A is a real eigenvalue, let

$$V_{\lambda^A} = \mathbb{R}^m \cap V_{\lambda^A} = \{\vec{u} \in \mathbb{R}^m : \ (\lambda^A I_m - A)^i \vec{u} = 0 \text{ for some } i \in \mathbb{N}\}.$$

If A is real and λ^A, $\overline{\lambda^A}$ is a pair of complex eigenvalue, let

$$V_{\lambda^A, \overline{\lambda^A}} = \mathbb{R}^m \cap (V_{\lambda^A} \oplus V_{\overline{\lambda^A}}).$$

These spaces are invariant and are called *generalized eigenspaces*, and we can write $\mathbb{R}^m$ into the direct sum of these generalized eigenspaces.

Lemma 3.1 (Lemma 2.6.3 in [4]) *Let A be an $m \times m$ real matrix and λ^A an eigenvalue of A. Then for any $\delta > 0$ there is $C(\delta) > 0$ such that*

$$C(\delta)^{-1}(|\lambda^A| - \delta)^n \|\vec{u}\| \le \|A^n \vec{u}\| \le C(\delta)(|\lambda^A| + \delta)^n \|\vec{u}\|$$

for every $n \in \mathbb{N}$ and every $\vec{u} \in V_{\lambda^A}$ (if $\lambda^A \in \mathbb{R}$) or every $\vec{u} \in V_{\lambda^A, \overline{\lambda^A}}$ (if $\lambda^A \notin \mathbb{R}$).

By this lemma, we know that if the eigenvalue $\lambda^A \in \mathbb{R}$ (resp. $\notin \mathbb{R}$), then for all $0 \neq \vec{u} \in V_{\lambda^A}$ (resp. $V_{\lambda^A, \overline{\lambda^A}}$), we have $\lim_{n\to\infty} \frac{1}{n} \log \|A^n \vec{u}\| = \log |\lambda^A|$. Let $\lambda_1^A, \ldots, \lambda_{r(A)}^A$ be the different modules of the eigenvalues of A. Then we can write

$$\mathbb{R}^m = \bigoplus_{i=1}^{r(A)} V_i^A, \tag{3.1}$$

where V_i^A is the direct sum of the generalized eigenspaces whose eigenvalues with the same module. Hence for any nonzero vector $\vec{u} \in V_i^A, 1 \le i \le r(A)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n \vec{u}\| = \log \lambda_i^A. \tag{3.2}$$

Here we set $\log 0 = 0$.

In the remaining of this section, we always assume that A, B and C are $m \times m$ nonsingular real matrices with the property (1.1). Therefore, A, B and C generate a matrix Heisenberg group.

For B and C, let $\lambda_1^B, \ldots, \lambda_{r(B)}^B$ and $\lambda_1^C, \ldots, \lambda_{r(C)}^C$ be the different modules of the eigenvalues of B and C respectively. Similar to (3.1), there are decompositions

$$\mathbb{R}^m = \bigoplus_{j=1}^{r(B)} V_j^B \quad \text{and} \quad \mathbb{R}^m = \bigoplus_{k=1}^{r(C)} V_k^C$$

and with the property similar to (3.2), respectively. Since A, B and C are all nonsingular, we have

$$\lambda_i^A, \lambda_j^B, \lambda_k^C > 0 \tag{3.3}$$

for all $1 \le i \le r(A), 1 \le j \le r(B)$ and $1 \le k \le r(C)$. In this section, we investigate the internal relations among the three decompositions.

Firstly, it is easy to see that the decomposition $\mathbb{R}^m = \bigoplus_{k=1}^{r(C)} V_k^C$ (resp. $\mathbb{R}^m = \bigoplus_{i=1}^{r(A)} V_i^A$) is A (resp. C)-invariant since A and C are commuting. Hence we get a refined decomposition

$$\mathbb{R}^m = \bigoplus_{i=1}^{r(A)} \bigoplus_{k=1}^{r(C)} V_{i,k}^{A,C}, \tag{3.4}$$

where $V_{i,k}^{A,C} := V_i^A \cap V_k^C$ (note that $V_{i,k}^{A,C}$ may be $\{0\}$ for some pairs of i and k). Clearly, this decomposition is simultaneously invariant with respect to A and C, and by Lemma 3.1, we have the following property immediately.

Lemma 3.2 *For every* (i,k) *with* $V_{i,k}^{A,C} \neq \{0\}$ *and each* $0 \neq \vec{u} \in V_{i,k}^{A,C}$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n \vec{u}\| = \log \lambda_i^A \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \|C^n \vec{u}\| = \log \lambda_k^C.$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \log \|C^n A^n \vec{u}\| = \lim_{n\to\infty} \frac{1}{n} \log \|A^n C^n \vec{u}\| = \log \lambda_i^A + \log \lambda_k^C.$$

Lemma 3.3 *For every* (i,k) *with* $V_{i,k}^{A,C} \neq \{0\}$ *and each* $0 \neq \vec{u} \in V_{i,k}^{A,C}$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n B^l \vec{u}\| = \log \lambda_i^A + l \log \lambda_k^C \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \|C^n B^l \vec{u}\| = \log \lambda_k^C$$

for each positive integer l.

Proof Since $AB = BAC$ and $BC = CB$, for positive integers n and l, we have

$$A^n B^l = B^l A^n C^{ln} \quad \text{and} \quad C^n B^l = B^l C^n.$$

Let $c = \|B\|^l$. Then for each positive integer n and $0 \neq \vec{u} \in V_{i,k}^{A,C}$, we have

$$c^{-1}\|C^{ln} A^n \vec{u}\| \leq \|A^n B^l \vec{u}\| \leq c\|C^{ln} A^n \vec{u}\|,$$
$$c^{-1}\|C^n \vec{u}\| \leq \|C^n B^l \vec{u}\| \leq c\|C^n \vec{u}\|.$$

Then by Lemma 3.2 we get that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n B^l \vec{u}\| = \lim_{n\to\infty} \frac{1}{n} \log \|C^{ln} A^n \vec{u}\| = \log \lambda_i^A + l \log \lambda_k^C,$$
$$\lim_{n\to\infty} \frac{1}{n} \log \|C^n B^l \vec{u}\| = \lim_{n\to\infty} \frac{1}{n} \log \|C^n \vec{u}\| = \log \lambda_k^C,$$

which are what we need. □

Proposition 3.4 *The eigenvalues of* C *are all of module* 1.

Proof Assume that there exists $\lambda_{k_0}^C \neq 1$ for some $1 \leq k_0 \leq r(C)$. Take $1 \leq i_0 \leq r(A)$ with $V_{i_0,k_0}^{A,C} \neq \{0\}$. By Lemma 3.3, for each positive integer l and nonzero vector $\vec{u} \in V_{i_0,k_0}^{A,C}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n B^l \vec{u}\| = \log \lambda_{i_0}^A + l \log \lambda_{k_0}^C.$$

By (3.3), $\lambda_{k_0}^C \neq 0$, hence

$$\lim_{l\to\infty} \lim_{n\to\infty} \frac{1}{n} \log \|A^n B^l \vec{u}\| = \begin{cases} +\infty & \text{if } \lambda_{k_0}^C > 1, \\ -\infty & \text{if } \lambda_{k_0}^C < 1. \end{cases}$$

However, by Lemma 3.2 we have

$$\min_{1\leq i\leq r(A)} \log \lambda_i^A \leq \lim_{n\to\infty} \frac{1}{n} \log \|A^n B^l \vec{u}\| \leq \max_{1\leq i\leq r(A)} \log \lambda_i^A$$

for every nonzero vector $\vec{u} \in \mathbb{R}^m$. A contradiction. □

Proof of Theorem A By Proposition 3.4, $r(C) = 1$. Hence the A, C-invariant decomposition (3.4) becomes (3.1). That is, the decomposition $\mathbb{R}^m = \bigoplus_{i=1}^{r(A)} V_i^A$ is A, C-invariant. It follows from Lemma 3.3 that for all $0 \neq \vec{u} \in V_i^A$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n B\vec{u}\| = \log \lambda_i^A.$$

It means that $BV_i^A = V_i^A$, $i = 1, 2, \ldots, r(A)$. Hence the decomposition $\mathbb{R}^m = \bigoplus_{i=1}^{r(A)} V_i^A$ is $\mathcal{H}$-invariant. Since the roles of B and A are similar, the decomposition $\mathbb{R}^m = \bigoplus_{j=1}^{r(B)} V_j^B$ is also $\mathcal{H}$-invariant. Therefore, the decomposition

$$\mathbb{R}^m = \bigoplus_{i=1}^{r(A)} \bigoplus_{j=1}^{r(B)} V_{i,j}^{A,B} \tag{3.5}$$

where $V_{i,j}^{A,B} := V_i^A \cap V_j^B$, is $\mathcal{H}$-invariant.

If $V_{i,j}^{A,B} \neq \{0\}$, then the common module of eigenvalues of $A|_{V_{i,j}^{A,B}}$ is equal to λ_i^A and the common module of eigenvalues of $B|_{V_{i,j}^{A,B}}$ is equal to λ_j^B.

For simplicity of notation, we relabel the subspaces in the decomposition (3.5) by V_j, $1 \leq j \leq s$, such that

$$\mathbb{R}^m = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

in which each $V_j \neq \{0\}$. Each V_j is $\mathcal{H}$-invariant. We rename the corresponding common module of eigenvalues of A with respect to V_j by $\lambda_{A,j}$ and the corresponding common module of eigenvalues of B with respect to V_j by $\lambda_{B,j}$. If we denote $d_j = \dim V_j$, then clearly $\sum_{j=1}^{s} d_j = m$.

This completes the proof of Theorem A. □

4 Entropies of Heisenberg Group Actions

Let α be a Heisenberg group action on compact metric space (X, d_X) with the generators $\alpha_A, \alpha_B, \alpha_C$ satisfying (2.1).

4.1 The Proof of Proposition 2.4

We first give an equivalent definition of the entropy $\widetilde{h}(\alpha)$ using open covers. Let $\mathcal{U}$ be an open cover and $\mathcal{N}(\mathcal{U})$ be the number of elements in the smallest subcover of $\mathcal{U}$. Put

$$\widetilde{h}_{\text{cov}}(\alpha) = \sup_{\mathcal{U} \text{ is open cover}} \widetilde{h}_{\text{cov}}(\alpha, \mathcal{U}),$$

where

$$\widetilde{h}_{\text{cov}}(\alpha, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n^3} \log \mathcal{N}\Big(\bigvee_{\vec{n}\in\Lambda_n} (\alpha^{\vec{n}})^{-1}\mathcal{U} \Big).$$

If we denote $\mathcal{U}_\epsilon$ ($\epsilon > 0$) an open cover consisting of all ϵ-balls, then it is clear that

$$\widetilde{h}_{\text{cov}}(\alpha) = \lim_{\epsilon\to 0} \widetilde{h}_{\text{cov}}(\alpha, \mathcal{U}_\epsilon).$$

Proposition 4.1 *Let α be an $\mathcal{H}$-action on X. Then we have*

$$\widetilde{h}(\alpha) = \widetilde{h}_{\text{cov}}(\alpha). \tag{4.1}$$

Proof Since

$$\mathcal{N}\Big(\bigvee_{\vec{n}\in\Lambda_n} (\alpha^{\vec{n}})^{-1}\mathcal{U}_\epsilon \Big) \leq r_{d_X}(\alpha, \Lambda_n, \epsilon, X)$$

$$\leq s_{d_X}(\alpha, \Lambda_n, \epsilon, X)$$
$$\leq \mathcal{N}\Big(\bigvee_{\vec{n}\in\Lambda_n} (\alpha^{\vec{n}})^{-1}\mathcal{U}_{\frac{\epsilon}{2}}\Big)$$

for every $\epsilon > 0$ and

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^3}\log\mathcal{N}\Big(\bigvee_{\vec{n}\in\Lambda_n}(\alpha^{\vec{n}})^{-1}\mathcal{U}_\epsilon\Big) = \widetilde{h}_{\rm cov}(\alpha),$$

we obtain (4.1). □

Proof of Proposition 2.4 For any open cover $\mathcal{U}$,

$$\begin{aligned}
\widetilde{h}_{\rm cov}(\alpha,\mathcal{U}) &= \limsup_{n\to\infty}\frac{1}{n^3}\log\mathcal{N}\Big(\bigvee_{\vec{n}\in\Lambda_n}(\alpha^{\vec{n}})^{-1}\mathcal{U}\Big)\\
&= \limsup_{n\to\infty}\frac{1}{n^3}\log\mathcal{N}\Big(\bigvee_{n_2=0}^{n}\alpha_B^{-n_2}\Big(\bigvee_{n_1,n_3=0}^{n}\alpha_A^{-n_1}\alpha_C^{-n_3}\mathcal{U}\Big)\Big) \quad \text{(by (2.1))}\\
&\leq \limsup_{n\to\infty}\frac{1}{n^3}\log\Big(\mathcal{N}\Big(\bigvee_{n_1,n_3=0}^{n}\alpha_A^{-n_1}\alpha_C^{-n_3}\mathcal{U}\Big)\Big)^{n+1}\\
&= \limsup_{n\to\infty}\frac{1}{n^2}\log\mathcal{N}\Big(\bigvee_{n_1,n_3=0}^{n}\alpha_A^{-n_1}\alpha_C^{-n_3}\mathcal{U}\Big)\\
&\leq h(\alpha_{A,C}),
\end{aligned}$$

where $\alpha_{A,C}$ is the $\mathbb{Z}^2$-action generated by α_A and α_C. Since either $h(\alpha_A) < \infty$ or $h(\alpha_C) < \infty$, by Proposition 13.2 of [18] we have $h(\alpha_{A,C}) = 0$ and hence $\widetilde{h}_{\rm cov}(\alpha,\mathcal{U}) = 0$. Since $\mathcal{U}$ is arbitrary, we get $\widetilde{h}(\alpha) = 0$. □

Corollary 4.2 *Let α be an $\mathcal{H}$-action on the torus $\mathbb{T}^m$ with the generators induced by $A, B, C \in SL(m,\mathbb{Z})$ satisfying* (1.1)*, then we have $\widetilde{h}(\alpha) = 0$.*

Remark 4.3 Since Heisenberg groups are amenable, then the topological entropy of α using the Følner sequences is given by

$$h_{top}(\alpha) = \sup_{\mathcal{U}\text{ is open cover}}\lim_{n\to\infty}\frac{1}{|F_n|}\log\mathcal{N}\Big(\bigvee_{h\in F_n}h^{-1}\mathcal{U}\Big),$$

where $\{F_n\}$ is the Følner sequence of α which is asymptotically invariant (see [20], for example), i.e.,

$$\lim_{n\to\infty}\frac{|F_n \triangle \alpha^{\vec{m}}F_n|}{|F_n|} = 0,$$

for all $\vec{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$.

It is easy to check that $\{\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda_n}$ is not a Følner sequence of α. In fact, take $\vec{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3_+$, then $\alpha^{\vec{m}} = \alpha_A^{m_1}\alpha_B^{m_2}\alpha_C^{m_3}$. Without loss of generality, we may assume $n > \max\{m_1, m_2, m_3\}$ and $m_2 > 2$. Hence, for any $\vec{n} \in \Lambda_n$, $0 \leq n_i \leq n, 1 \leq i \leq 3$, we have

$$\alpha^{\vec{m}}\alpha^{\vec{n}} = \alpha_A^{m_1+n_1}\alpha_B^{m_2+n_2}\alpha_C^{m_3+n_3}\alpha_C^{-m_2n_1}.$$

It follows that $\{\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda_n} \cap \{\alpha^{\vec{m}}\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda_n} = \emptyset$ whenever $n_1 > \lceil\frac{m_3+n}{m_2}\rceil$. Hence, we obtain

$$\limsup_{n\to\infty}\frac{|\{\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda_n} \triangle \{\alpha^{\vec{m}}\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda_n}|}{|\{\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda_n}|} \geq \limsup_{n\to\infty}\frac{n^2(n - \lceil\frac{m_3+n}{m_2}\rceil)}{n^3} \geq \frac{m_2-2}{m_2} > 0.$$

By formula (4.1), we have that

$$\widetilde{h}(\alpha) = \sup_{\mathcal{U} \text{ is open cover}} \limsup_{n\to\infty} \frac{1}{n^3} \log \mathcal{N}\Big(\bigvee_{\vec{n}\in\Lambda_n} (\alpha^{\vec{n}})^{-1}\mathcal{U}\Big).$$

This means that we can not get the relationship between the topological entropies $\widetilde{h}(\alpha)$ and $h_{top}(\alpha)$ in general. Indeed, we can obtain that $\{\alpha^{\vec{n}}\}_{\vec{n}\in\Lambda'_n}$ is a Flner sequence of α, where

$$\Lambda'_n = \{\vec{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3 \mid n_i \le n, i \le 2 \text{ and } 1 \le n_3 \le n^2\} \text{ (see [5], for example).}$$

For the case that α is an $\mathcal{H}$-action on the torus $\mathbb{T}^m$ given in the Proposition 2.4, if either $h(\alpha_A)$ or $h(\alpha_C)$ is finite, we can also obtain that

$$h_{\text{top}}(\alpha) = \sup_{\mathcal{U} \text{ is open cover}} \lim_{n\to\infty} \frac{1}{n^4} \log \mathcal{N}\Big(\bigvee_{h\in\Lambda'_n} (\alpha^{\vec{n}})^{-1}\mathcal{U}\Big) = \widetilde{h}(\alpha) = 0.$$

This shows that the topological entropy of α using the Følner sequences may be equal to $\widetilde{h}(\alpha)$ in some special cases.

4.2 The Proof of Theorem B

In the following, we will consider a skew product transformation and use it to evaluate the entropy of Heisenberg group action on X. We denote $\alpha_1 = \alpha_A$, $\alpha_2 = \alpha_B$ and $\alpha_3 = \alpha_C$.

Let $\Sigma_3 = \prod_{n\in\mathbb{Z}_+}\{1, 2, 3\}$ be the standard symbolic space with product topology. A *natural metric* d_{Σ_3} *on* Σ_3 is defined by

$$d_{\Sigma_3}(\{i_n\}_{n\in\mathbb{Z}_+}, \{j_n\}_{n\in\mathbb{Z}_+}) = \sum_{n=0}^{\infty} \frac{d(i_n, j_n)}{2^n}$$

for $\{i_n\}_{n\in\mathbb{Z}_+}, \{j_n\}_{n\in\mathbb{Z}_+} \in \Sigma_3$, where $d(i_n, j_n) = 0$ when $i_n = j_n$, and $d(i_n, j_n) = 1$ when $i_n \neq j_n$.

Define a map $\widetilde{\sigma} : \Sigma_3 \times X \to \Sigma_3 \times X$ by

$$\widetilde{\sigma}(\{i_n\}_{n\in\mathbb{Z}_+}, x) = (\{i_{n+1}\}_{n\in\mathbb{Z}_+}, \alpha_{i_0}x),$$

where $\alpha_{i_n} \in \{\alpha_A, \alpha_B, \alpha_C\}$, $n = 0, 1, 2, \ldots$. This is a *skew product over the shift transformation* $\sigma_3 : \Sigma_3 \to \Sigma_3$ by $\sigma_3(\{i_n\}_{n\in\mathbb{Z}_+}) = \{i_{n+1}\}_{n\in\mathbb{Z}_+}$.

There are two factors of $\widetilde{\sigma} : \Sigma_3 \times X \to \Sigma_3 \times X$. The first one is $\sigma_\alpha : X_\alpha \to X_\alpha$. Define a map $\widetilde{\pi} : \Sigma_3 \times X \to X_\alpha$ by

$$\widetilde{\pi}(\{i_n\}_{n\in\mathbb{Z}_+}, x) = \{x_n\}_{n\in\mathbb{Z}_+},$$

where $x_0 = x$ and $x_n = \alpha_{i_{n-1}} \circ \cdots \circ \alpha_{i_1}\alpha_{i_0}(x)$ for $n \ge 1$. And $h(\alpha) = h(\sigma_\alpha) \le h(\widetilde{\sigma})$ (see Proposition 2.5 of [23]). The second factor is $\sigma_3 : \Sigma_3 \to \Sigma_3$ via the natural semi-conjugacy, i.e., the projection $\pi : \Sigma_3 \times X \to \Sigma_3$. By Bowen's entropy inequality (Theorem 17 of [2]) and Ledrappier and Walter's relative variational principle (formula (1.2) in [13]), we have the following relations between the entropies of σ_3 and $\widetilde{\sigma}$.

Proposition 4.4 *Let* $\widetilde{\sigma} : \Sigma_3 \times X \to \Sigma_3 \times X$, $\sigma_3 : \Sigma_3 \to \Sigma_3$ *and* $\pi : \Sigma_3 \times X \to \Sigma_3$ *be as above. Then*

(1)

$$h(\widetilde{\sigma}) \le \log 3 + \sup_{\{i_n\}\in\Sigma_3} h(\widetilde{\sigma}, \pi^{-1}\{i_n\})$$

(*note that* $h(\sigma_3)=\log 3$);

(2) *for any* σ_3*-invariant probability measure* ν *on* Σ_3,

$$\sup_{\mu\in\pi^{-1}(\nu)} h_\mu(\widetilde{\sigma}) = h_\nu(\sigma_3) + \int_{\Sigma_3} h(\widetilde{\sigma},\pi^{-1}\{i_n\})d\nu(\{i_n\}), \tag{4.2}$$

where the supremum is taken over all $\widetilde{\sigma}$*-invariant probability measures* μ *that project to* ν.

Now we are ready to prove Theorem B. In the remaining of this section, we always assume that $X=\mathbb{T}^m$ and α is the Heisenberg group action on the torus $\mathbb{T}^m$ induced by the matrix Heisenberg group $\mathcal{H}$ with generators $A,B,C\in \mathrm{SL}(m,\mathbb{Z})$.

Now let $d_{\mathbb{R}^m}$ be the Euclidean metric on $\mathbb{R}^m$ and $d_{\mathbb{T}^m}$ be the standard metric on $\mathbb{T}^m$ induced by $d_{\mathbb{R}^m}$ via the covering map. Precisely, let $\pi':\mathbb{R}^m\to\mathbb{T}^m=\mathbb{R}^m/\mathbb{Z}^m$ be *the canonical projection* defined by

$$\pi'(x_1,\ldots,x_m)=(x_1,\ldots,x_m) \pmod{\mathbb{Z}^m}.$$

Then for $x,y\in\mathbb{T}^m$,

$$d_{\mathbb{T}^m}(x,y):=\min\{d_{\mathbb{R}^m}(\vec{u}_x,\vec{u}_y):\vec{u}_x\in{\pi'}^{-1}x,\vec{u}_y\in{\pi'}^{-1}y\},$$

where $d_{\mathbb{R}^m}(\vec{u}_x,\vec{u}_y)=\|\vec{u}_x-\vec{u}_y\|$.

Let $D\in\mathrm{SL}(m,\mathbb{Z})$ and f_D be the induced homeomorphism on $\mathbb{T}^m$. For any $x\in\mathbb{T}^m$ and sufficiently small $\epsilon>0$, we have

$$\mathrm{Vol}_m(f_D(B_{d_{\mathbb{T}^m}}(x,\epsilon)))=\mathrm{Vol}_m(D(B_{d_{\mathbb{R}^m}}(0,\epsilon))), \tag{4.3}$$

where Vol_m denotes the m-dimensional volume.

Recall we have denoted $\alpha_1=\alpha_A$, $\alpha_2=\alpha_B$ and $\alpha_3=\alpha_C$. For uniformity of the notations, we denote $\widetilde{\alpha}_1=A$, $\widetilde{\alpha}_2=B$ and $\widetilde{\alpha}_3=C$.

Proof of Theorem B Let $\widetilde{\sigma}:\Sigma_3\times X\to\Sigma_3\times X$, $\sigma_3:\Sigma_3\to\Sigma_3$, $\sigma_\alpha:X_\alpha\to X_\alpha$, $\pi:\Sigma_3\times X\to\Sigma_3$ and $\widetilde{\pi}:\Sigma_3\times X\to X_\alpha$ be as above. We will prove the entropy formula (2.3) in two steps. Firstly, we show that

$$h(\widetilde{\sigma})=\max_{J\subset\{1,2,\ldots,s\}}\log\Big(\prod_{j\in J}\lambda_{A,j}^{d_j}+\prod_{j\in J}\lambda_{B,j}^{d_j}+1\Big). \tag{4.4}$$

Note that for any $\{i_n\}_{n\in\mathbb{Z}_+}\in\Sigma_3$, the restriction $\widetilde{\sigma}|_{\pi^{-1}\{i_n\}}$ is a nonautonomous dynamical system on $\mathbb{T}^m$ induced by the compositions of the sequence of maps $\{\alpha_{i_n}\}_{n\in\mathbb{Z}_+}$. According to [24] and [22], we have that for any $\epsilon>0$,

$$h(\widetilde{\sigma},\pi^{-1}\{i_n\}):=h(\{\alpha_{i_n}\})=\limsup_{l\to\infty}\Big[-\frac{1}{l}\log\mathrm{Vol}_m(D_l(0,\epsilon,\{\alpha_{i_n}\}))\Big], \tag{4.5}$$

where $D_l(0,\epsilon,\{\alpha_{i_n}\})=\bigcap_{t=0}^{l-1}(\alpha_{i_t}\circ\cdots\circ\alpha_{i_0})^{-1}B_{d_{\mathbb{T}^m}}(0,\epsilon)$.

Let

$$\mathbb{R}^m=V_1\oplus V_2\oplus\cdots\oplus V_s$$

be the $\mathcal{H}$-invariant decomposition of $\mathbb{R}^m$ as in Theorem A. By (4.3), we have

$$\limsup_{l\to\infty}\Big[-\frac{1}{l}\log\mathrm{Vol}_m(D_l\big(0,\epsilon,\{\alpha_{i_n}\}\big))\Big]=\limsup_{l\to\infty}\Big[-\frac{1}{l}\log\prod_{j=1}^{s}\mathrm{Vol}_{d_j}\big(\widetilde{D}_l^{(j)}(0,\epsilon,\{\widetilde{\alpha}_{i_n}\})\big)\Big],$$

where $\widetilde{D}_l^{(j)}(0,\epsilon,\{\widetilde{\alpha}_{i_n}\}) = \bigcap_{t=0}^{l-1}\left[(\widetilde{\alpha}_{i_t}\circ\cdots\circ\widetilde{\alpha}_{i_0})|_{V_j}\right]^{-1}B_{d_{\mathbb{R}^{d_j}}}(0,\epsilon)$. By Theorem A, we have

$$\limsup_{l\to\infty}\left[-\frac{1}{l}\log \mathrm{Vol}_{d_j}\left(\widetilde{D}_l^{(j)}(0,\epsilon,\{\widetilde{\alpha}_{i_n}\})\right)\right] = \limsup_{l\to\infty}\frac{1}{l}\left(\max_{0\le t\le l-1}\sum_{r=0}^{t}\log\lambda^{d_j}_{\widetilde{\alpha}_{i_r},j}\right). \tag{4.6}$$

Now for any $1\le j\le s$, define a function

$$\eta^{(j)}:\Sigma_3\to\mathbb{R},\quad \{i_n\}_{n\in\mathbb{Z}_+}\longmapsto \log\lambda^{d_j}_{\widetilde{\alpha}_{i_0},j}.$$

Assume ν is a σ_3-ergodic measure on Σ_3, then by the Birkhoff Ergodic Theorem, we have

$$\lim_{l\to\infty}\frac{1}{l}\sum_{t=0}^{l-1}\log\lambda^{d_j}_{\widetilde{\alpha}_{i_t},j} = \int_{\Sigma_3}\eta^{(j)}d\nu,\ \ \nu\text{-a.e. }\{i_n\}_{n\in\mathbb{Z}_+}.$$

It is well known that for any sequence of real numbers $\{a_l\}$ with $\lim_{l\to\infty}\frac{a_l}{l}=a$, we have $\lim_{l\to\infty}\frac{\max_{1\le t\le l}a_t}{l}=\max\{a,0\}$ (see Lemma 9.2 of [7], for example). Based on this fact and by the equality (4.6), we get the following formula

$$h(\{\alpha_{i_n}\}) = \sum_{j=1}^{s}\max\left\{\int_{\Sigma_3}\eta^{(j)}d\nu,0\right\} = \max_{J\subset\{1,2,\dots,s\}}\int_{\Sigma_3}\sum_{j\in J}\eta^{(j)}d\nu \tag{4.7}$$

for ν-almost all $\{i_n\}_{n\in\mathbb{Z}_+}$.

By the relative variational principle in Proposition 4.4, we can compute $h(\widetilde{\sigma})$ as the supremum of $h_\mu(\widetilde{\sigma})$ over all ergodic measure μ on $\Sigma_3\times X$. Such a measure projects under π to a σ_3-invariant ergodic measure on Σ_3. Hence

$$\begin{aligned}
h(\widetilde{\sigma}) &= \sup_{\nu}\{\sup_{\mu\in\pi^{-1}(\nu)}h_\mu(\widetilde{\sigma})\}\\
&= \sup_{\nu}\left\{h_\nu(\sigma_3)+\int_{\Sigma_3}h(\widetilde{\sigma},\pi^{-1}\{i_n\})d\nu(\{i_n\})\right\}\quad(\text{by (4.2)})\\
&= \sup_{\nu}\left\{h_\nu(\sigma_3)+\int_{\Sigma_3}h(\{\alpha_{i_n}\})d\nu(\{i_n\})\right\}\quad(\text{by (4.5)})\\
&= \max_{J\subset\{1,2,\dots,s\}}\sup_{\nu}\left\{h_\nu(\sigma_3)+\int_{\Sigma_3}\sum_{j\in J}\eta^{(j)}d\nu\right\}\quad(\text{by (4.7)})\\
&= \max_{J\subset\{1,2,\dots,s\}}P\left(\sigma_3,\sum_{j\in J}\eta^{(j)}\right),
\end{aligned}$$

where ν ranges over all σ_3-invariant ergodic measure on Σ_3. And in the last line we use the variational principle for the pressure of $\sum_{j\in J}\eta^{(j)}$. From Chapter 9 of [19], we can get that

$$h(\widetilde{\sigma}) = \max_{J\subset\{1,2,\dots,s\}}P\left(\sigma_3,\sum_{j\in J}\eta^{(j)}\right) = \max_{J\subset\{1,2,\dots,s\}}\log\left(\prod_{j\in J}\lambda^{d_j}_{A,j}+\prod_{j\in J}\lambda^{d_j}_{B,j}+1\right). \tag{4.8}$$

For any $J\subset\{1,2,\dots,s\}$, $\sum_{j\in J}\eta^{(j)}$ has a unique equilibrium state $\nu^\star$ which is the product measure defined by the measure on σ_3 which gives the points α_A, α_B and α_C, measure

$$P_{\alpha_A} = \frac{\prod_{j\in J}\lambda^{d_j}_{A,j}}{\prod_{j\in J}\lambda^{d_j}_{A,j}+\prod_{j\in J}\lambda^{d_j}_{B,j}+1},\quad P_{\alpha_B} = \frac{\prod_{j\in J}\lambda^{d_j}_{B,j}}{\prod_{j\in J}\lambda^{d_j}_{A,j}+\prod_{j\in J}\lambda^{d_j}_{B,j}+1}$$

and

$$P_{\alpha_C} = \frac{1}{\prod_{j\in J}\lambda^{d_j}_{A,j}+\prod_{j\in J}\lambda^{d_j}_{B,j}+1}.$$

Next, we show that $h(\alpha) = h(\widetilde{\sigma})$. We have known that $h(\alpha) = h(\sigma_\alpha) \le h(\widetilde{\sigma})$, then we only need to prove $h(\widetilde{\sigma}) \le h(\alpha)$. Note that from the process of the proof of (4.8), $\widetilde{\sigma}$ has a measure of maximal entropy of the form $\nu \times \omega$, where ν is a product measure on Σ_3 defined by some probability vector $(P_{\alpha_A}, P_{\alpha_B}, P_{\alpha_C})$ which is mentioned above and ω is the Haar measure on $\mathbb{T}^m$. That is $h(\widetilde{\sigma}) = h_{\nu\times\omega}(\widetilde{\sigma})$. Since A, B and C are pairwise different, we can construct a set $\widetilde{E}$ on $\mathbb{T}^m$ by

$$\widetilde{E} = \bigcup_{n=1}^{\infty} \bigcup_{1\le i_0,\dots,i_{n-1}\le 3} (\alpha_{i_0})^{-1} \circ \cdots \circ (\alpha_{i_{n-1}})^{-1} E,$$

where $E = \{x \in \mathbb{T}^m : \alpha_i x = \alpha_j x \text{ for some } 1 \le i, j \le 3, i \ne j\}$, hence $\omega(\widetilde{E}) = \omega(E) = 0$. Then by the definition of $\widetilde{\pi}$, we can know that $\widetilde{\pi}$ is one-to-one on a set $\Sigma_3 \times (X \setminus (E \cup \widetilde{E}))$ of full $\nu \times \omega$ measure. So

$$h(\widetilde{\sigma}) = h_{\nu\times\omega}(\widetilde{\sigma}) = h_{\widetilde{\pi}(\nu\times\omega)}(\sigma_\alpha) \le h(\sigma_\alpha) = h(\alpha). \tag{4.9}$$

Combining (4.4) and (4.9), we obtain the formula (2.3).

This completes the proof of Theorem B. □

Example 4.5 Let α be the Heisenberg group action on the torus $\mathbb{T}^4$ which is induced by the automorphisms

$$A = \begin{bmatrix} Y & O \\ O & -Y \end{bmatrix}, \quad B = \begin{bmatrix} O & Y \\ Y & O \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -I_2 & O \\ O & -I_2 \end{bmatrix},$$

where

$$Y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

and $A, B, C \in \mathrm{SL}(4, \mathbb{Z})$. By Theorem A, $\mathbb{R}^4 = V_1 \oplus V_2$, where V_1 and V_2 are $\mathcal{H}$-invariant subspaces and $\dim V_1 = \dim V_2 = 2$. The common module of eigenvalues of $A|_{V_1}$ and $B|_{V_1}$ is $\frac{3+\sqrt{5}}{2}$, the common module of eigenvalues of $A|_{V_2}$ and $B|_{V_2}$ is $\frac{3-\sqrt{5}}{2}$. Hence, by Theorem B

$$h(\alpha) = \log\left(\left(\frac{3+\sqrt{5}}{2}\right)^2 + \left(\frac{3+\sqrt{5}}{2}\right)^2 + 1\right) = \log(8 + 3\sqrt{5}).$$

In fact, from the process of the proof of (4.8), $h(\alpha) = h(\widetilde{\sigma}) = P(\sigma_3, \eta^{(1)})$. It means that $h(\alpha)$ is the pressure of the function $\eta^{(1)} : \Sigma_3 \to \mathbb{R}$ which is defined by

$$\eta^{(1)}(\{i_n\}_{n\in\mathbb{Z}_+}) = \begin{cases} \log \lambda_{A,1}^2 = 2\log\dfrac{3+\sqrt{5}}{2}, & i_0 = 1, \\ \log \lambda_{B,1}^2 = 2\log\dfrac{3+\sqrt{5}}{2}, & i_0 = 2, \\ \log \lambda_{C,1}^2 = 0, & i_0 = 3, \end{cases}$$

with respect to $\sigma_3 : \Sigma_3 \to \Sigma_3$. Moreover, $\eta^{(1)}$ has a unique equilibrium state $\nu^\star$ which is the product measure defined by the probability vector

$$(P_{\alpha_A}, P_{\alpha_B}, P_{\alpha_C}) = \left(\frac{11+3\sqrt{5}}{38}, \frac{11+3\sqrt{5}}{38}, \frac{16-6\sqrt{5}}{38}\right).$$

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