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The Orthogonal Bases of Exponential Functions Based on Moran–Sierpinski Measures

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Abstract Let $A_n \in M_2(\mathbb{Z})$ be integral matrices such that the infinite convolution of Dirac measures with equal weights

$$\mu_{\{A_n,n\geq 1\}} := \delta_{A_1^{-1}\mathcal{D}} * \delta_{A_1^{-1}A_2^{-2}\mathcal{D}} * \cdots$$

is a probability measure with compact support, where $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t\}$ is the Sierpinski digit. We prove that there exists a set $\Lambda \subset \mathbb{R}^2$ such that the family $\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal basis of $L^2(\mu_{\{A_n,n\geq 1\}})$ if and only if $\frac{1}{3}(1,-1)A_n \in \mathbb{Z}^2$ for $n \geq 2$ under some metric conditions on A_n.

Keywords Moran–Sierpinski measures; orthonormal basis of exponential functions; self-affine measures; spectral measures

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1 Introduction

Let μ be a Borel probability measure with compact support in plane. To study the Hilbert space $L^2(\mu)$, it is natural to investigate the Fourier analysis on it. For this aim, people consider the family of exponential functions decided by a set $\Lambda \subset \mathbb{R}^2$ as follows

$$E_\Lambda = \{e^{-2\pi i\langle x,\lambda\rangle} : x \in \mathbb{R}^2, \lambda \in \Lambda\}, \tag{1.1}$$

where $\langle x, \lambda\rangle$ is the standard inner product in $\mathbb{R}^2$. If E_Λ is an orthonormal basis of $L^2(\mu)$, like $E_\mathbb{Z}$ in $L^2[0,1]$, μ is called a *spectral measure* with a *spectrum* Λ, also (μ,Λ) is called a *spectral pair*.

It is known that μ is of pure type if it is a spectral measure, that is, μ is either a discrete measure with finite support, absolutely continuous or singularly continuous measure with respect to the Lebesgue's one (see [25]). Surprisingly, there are many differences between the absolutely continuous spectral measure and the singularly continuous one. For example, there are uncountable spectra Λ with $0 \in \Lambda$ for all known singularly spectral measures (see [9, 13] for example), but only finite many spectra Λ with $0 \in \Lambda$ for all known absolutely continuous spectral measures in one dimension (see [27] for example).

The first singularly spectral measure was found by Jorgensen and Pedersen in 1998 ([26]). They proved that the one forth Cantor measure μ_4, which satisfies

$$\mu_4(E) = \frac{1}{2}\mu_4(4E) + \frac{1}{2}\mu_4(4E-2), \quad \text{for all Borel set } E \subset \mathbb{R}$$

and is supported on the compact set $T = \{\sum_{k=1}^{\infty} d_k 4^{-k} : \text{all } d_k \in \{0,2\}\}$, is a spectral measure with a spectrum

$$\Lambda_4 = \{0,1\} + 4\{0,1\} + 4^2\{0,1\} + \cdots, \tag{1.2}$$

where all sums are finite. The measure μ_4 has many surprising properties. For example, there are infinitely many natural numbers k such that, for the Λ_4 given in (1.2), $k\Lambda_4$ is a spectrum of μ_4 (e.g. [15]). The mock Fourier expansion of any continuous function f for the basis E_{Λ_4} of $L^2(\mu_4)$ is uniformly convergence ([34]); however, there exists a continuous function whose mock Fourier expansion for the basis $E_{17\Lambda_4}$ of $L^2(\mu_4)$ is divergent at 0 ([16]).

From 1998 on, there is a lot of study on the spectrality of self-similar and self-affine measures, and many singular properties have been found (see [2, 6, 7, 10–12, 14, 15, 17, 19, 24, 33] and the references therein). The theory of singularly spectral measures mainly includes the sufficiency and the necessity of spectrality. The study also related with tilling (see [28] for example) and wavelet (see [21] for example).

Recently, the spectrality of a class of more complicated probability measures receives special interest. That is the so called "Moran self-affine measures", "infinite convolutions" or "tower measure" by Strichartz [34] (see Definition 1.1). Many important results have been proven (see [1–5, 20, 23, 34] and the references therein). Among these results, most of them associate the one dimensional case. In higher dimensional cases, an important work is due to Dutkay and Lai [20]. For Moran self-affine measures generated by random convolutions of finite atomic measures satisfying Hadamard triples, where the digits are chosen from a finite collections of digit sets, the authors showed that in dimension one, or in higher dimensions under certain conditions, "almost all" such measures are spectral measures, but, the Hadamard triples do not guarantee the spectrality of Moran self-affine measures in general cases (see [5]).

All the known results either give sufficient conditions or necessary conditions for some class of Moran self-affine measures to be spectral. Even for the self-affine measures, there are few characteristic conditions obtained for them to be spectral measures (only for the Sierpinski Measures by several authors, which is a special case of the main result in this paper). In this paper we study the Moran–Sierpinski Measures (see its definition below) and obtain a necessary and sufficient condition for them to be spectral.

Definition 1.1 *Let $\{A_n\}_{n=1}^{\infty} \subset M_d(\mathbb{R})$ be a sequence of nonsingular $d \times d$ matrices with real entries and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of finite sets in $\mathbb{R}^d$ (digit sets). Assume $T(A_n, \mathcal{D}_n) := \sum_{n=1}^{\infty} A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}_n$ is a compact set of $\mathbb{R}^d$. Then the sequence of discrete measures, generated by convolutions as follows,*

$$\mu_n = \delta_{A_1^{-1}\mathcal{D}_1} * \delta_{A_1^{-1}A_2^{-1}\mathcal{D}_2} * \cdots * \mu_{A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}_n}, \quad n \geq 1, \tag{1.3}$$

converges in weak sense to a Borel probability measure with compact support $T(A_n, \mathcal{D}_n)$, where $\delta_E = \frac{1}{\#E}\sum_{a\in E}\delta_a$, $\#E$ is the cardinality of E and δ_a is the Dirac measure at the point $a \in \mathbb{R}^d$. We denote the above limit measure by

$$\mu_{\{A_n,\mathcal{D}_n\}} = \delta_{A_1^{-1}\mathcal{D}_1} * \delta_{A_1^{-1}A_2^{-2}\mathcal{D}_2} * \cdots \ , \tag{1.4}$$

which is called a Moran self-affine measure. In particular $\mu_{\{A_n,\mathcal{D}_n\}}$ is called a self-affine measure and denoted by $\mu_{A,\mathcal{D}}$ if all $A_n = A$ and $\mathcal{D}_n = D$.

Remark By letting $\{X_n\}_n$ be a sequence of independent random variables such that X_n is uniformly distributed on $\mathcal{D}_n$ for all $n \geq 1$, it is easy to show that the sequence of measures $\{\mu_n\}_n$ defined in (1.3) converges in weak sense to a Borel probability measure with a compact support if and only if the set $T(A_n, \mathcal{D}_n)$ is a bounded compact set.

To study the spectrality of $\mu_{\{A_n,\mathcal{D}_n\}}$ in higher dimensions, up to now there has not been a characteristic criterion for $\mu_{\{A_n,\mathcal{D}_n\}}$ to be a spectral measure in general except for some special self-affine measures. In this paper we study the spectrality of $\mu_{\{A_n,\mathcal{D}_n\}}$ by assuming that all digit sets $\mathcal{D}_n$ are equal to the Sierpinski digit set and obtain a necessary condition and several easy-to-check sufficient conditions for it to be a spectral measure.

The Sierpinski digit set is defined by

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} := \{\mathbf{0}, e_1, e_2\}. \tag{1.5}$$

And associated to it we will use the set

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} := \{\mathbf{0}, e, -e\}. \tag{1.6}$$

In the sequel, $\mathcal{D}$ and $\mathcal{C}$ mean the above two subsets, respectively, except some mentioned cases. We always use e denote the vector $(1,-1)^t$ as in (1.6).

The Moran measure $\mu_{\{A_n,\mathcal{D}_n\}}$ is called a *Moran–Sierpinski Measure* if all $\mathcal{D}_n$ are the Sierpinski digit set $\mathcal{D}$ given by (1.5). For simplicity of notations we denote it by $\mu_{\{A_n,n\geq 1\}}$. Note that the compact support of $\mu_{\{A_n,n\geq 1\}}$ is the set $\sum_{n=1}^{\infty} A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}$, which in general is a fractal set and so the measure $\mu_{\{A_n,n\geq 1\}}$ is singular in general.

What we are interested in this paper is the following question:

Question 1 What is the necessary and sufficient condition for the Moran–Sierpinski Measure $\mu_{\{A_n,n\geq 1\}}$ to be spectral?

Recall that, as a special case of $\mu_{\{A_n,n\geq 1\}}$, the so called Sierpinski measure $\mu_{A,\mathcal{D}}$ satisfies the self-affine property, that is,

$$\mu_{A,\mathcal{D}}(E) = \frac{1}{\#\mathcal{D}}\sum_{d\in\mathcal{D}} \mu_{A,\mathcal{D}}(\phi_d^{-1}(E)), \quad \text{for any Borel set } E\subset\mathbb{R}^2,$$

where $A\in M_2(\mathbb{Z})$ is an expansive matrix (all its eigenvalues are larger than one in modulus) and $\phi_d(x) = A^{-1}(x+d)$ with $\mathcal{D}$ being defined by (1.5).

The Sierpinski measure attracted the most attention in the study of spectrality of self-affine measures (two dimension case). Early in 1998, Jorgensen and Pedersen ([26]) showed that the canonical Hausdorff measure on the Sierpinski gasket is not a spectral measure, but such measure on the Sierpinski tower is a spectral measure on $\mathbb{R}^3$. Later, in 2007, Dutkay and Jorgensen ([22]) proved a sufficient condition for the self-similar measure $\mu_{\rho,\mathcal{D}}$ on $\mathbb{R}^n$ to be spectral, that for $\rho\in(n+1)\mathbb{N}$, $\mu_{\rho,\mathcal{D}_n}$ admits some canonical spectra, where $\mathcal{D}_n = \{0, e_1, \ldots, e_n\}$ consisting of the zero vector and the standard basis of $\mathbb{R}^n$.

Assuming $A\in M_2(\mathbb{Z})$ to be an expansive matrix, Li ([29–32]) and An et al. ([2]) proved that $\mu_{A,\mathcal{D}}$ is a spectral measure if and only if $\frac{1}{3}(1,-1)A$ is an integral row vector.

Assuming $A = \mathrm{diag}(\rho^{-1},\rho^{-1})$ with $0<|\rho|<1$ being any real number, Deng and Lau ([14]) proved that $\mu_{A,\mathcal{D}}$ is a spectral measure if and only if ρ^{-1} belongs to $3\mathbb{Z}\setminus\{0\}$. Furthermore, assuming $A = \mathrm{diag}(b_1, b_2)$ to be a general real diagonal expansive matrix, Dai, Fu and Yan ([8]) showed that $\mu_{A,\mathcal{D}}$ is a spectral measure if and only if $b_1,\ b_2\in 3\mathbb{Z}\setminus\{0\}$.

Clearly, the Moran–Sierpinski Measure $\mu_{\{A_n,n\geq 1\}}$ is not the case at all. For the spectrality of the Moran–Sierpinski Measure $\mu_{\{A_n,n\geq 1\}}$, Wang and Dong ([35]) considered the case that all A_n are diagonal matrices and obtained some sufficient condition. Then Zhang ([36]) extended these results to some more general cases.

To answer the above question, the first key problem is the necessity part. In general, it is a difficult problem to give the necessary conditions for the spectrality and few results have been obtained up to now. Here we find the necessary condition for the Moran–Sierpinski Measure $\mu_{\{A_n,n\geq 1\}}$ and develop a new method to prove it. The idea is, after assuming there exists a spectrum Λ, to divide the assumed Λ into several parts and use a property of weighted sums to obtain a proof (The details are given in Sections 2–3). Then we obtain the necessary condition of Question 1 completely.

Theorem 1.2 *Let $\{A_n\}_{n=1}^{\infty}\subset M_2(\mathbb{Z})$ be a sequence of expasive matrices such that $T(A_n,\mathcal{D}_n) := \sum_{n=1}^{\infty} A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}_n$ is a compact set of $\mathbb{R}^d$. If $\mu_{\{A_n,n\geq 1\}}$ is a spectral measure, then $\frac{1}{3}(1,-1)A_n\in\mathbb{Z}^2$ for $n\geq 2$.*

Clearly, the condition $\frac{1}{3}(1,-1)A_n\in\mathbb{Z}^2$ is equivalent to that there exist $A_n'\in M_2(\mathbb{Z})$ and

$k_n, l_n \in \{-1, 0, 1\}$ such that

$$A_n = 3A_n' + \begin{pmatrix} k_n & l_n \\ k_n & l_n \end{pmatrix}, \quad \text{for each } n \geq 2.$$

For the sufficiency part of Question 1, we need to use the Euclidean norm $\|\cdot\|$ and some non Euclidean norm $\|\cdot\|'$ on $\mathbb{R}^2$ and the following notion: For $\eta, \delta \in (0, \frac{1}{4})$, write

$$\mathcal{B}_{\eta,\delta} := \left\{ B \in M_2(\mathbb{Z}) : B^{-1}\left[-\frac{1}{2} - \eta, \frac{1}{2} + \eta \right]^2 \cap \left(\pm \frac{1}{3}e + \mathbb{Z}^2 \right)_\delta = \emptyset \right\}, \tag{1.7}$$

where $[a,b]^2 = \{(x,y)^t : x, y \in [a,b]\}$ for any $a \leq b$ and $E_\delta = \{x : \sup_{y \in E} \|x - y\| < \delta\}$ is the δ-neighborhood of E under the Euclidean norm $\|\cdot\|$. Then we obtain the following result.

Theorem 1.3 *Let $A_n \in M_d(\mathbb{Z})$ for $n \geq 1$. Suppose*

$$\limsup_{n\to\infty} \|A_n^{-1}\|' \leq r < 1$$

for some norm $\|\cdot\|'$ on $\mathbb{R}^2$ and there exist two positive numbers η and δ such that

$$B_n B_{n+1} \cdots B_{n+p} \in \mathcal{B}_{\eta,\delta} \tag{1.8}$$

for all $p \geq 0$ and sufficient large n, where $B_n = A_n^t$ is the transpose of A_n for $n \geq 1$. Then $\mu_{\{A_n, n\geq 1\}}$ is a spectral measure if and only if $\frac{1}{3}(1,-1)A_n \in \mathbb{Z}^2$ for $n \geq 2$.

Remark The assumption $\limsup_{n\to\infty} \|A_n^{-1}\|' \leq r < 1$ guarantees that $\mu_{\{A_n, n\geq 1\}}$ is a Borel probablity measure and

$$\lim_{p\to\infty} B_n^{-1} B_{n+1}^{-1} \cdots B_{n+p}^{-1} = 0, \quad \text{uniformly for } n \geq N.$$

Then (1.8) holds for $n \geq N$ and sufficient large p. The key point of the assumption (1.8) is the arbitrariness of $p \geq 0$.

The following corollaries are the main results of this paper.

Corollary 1.4 *Let $A_n \in M_2(\mathbb{Z})$ for $n \geq 1$. Suppose all A_n^{-1} are contractive with a common ratio $r < \frac{2}{3}$ under the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^2$. Then $\mu_{\{A_n, n\geq 1\}}$ is a spectral measure if and only if $\frac{1}{3}(1,-1)A_n \in \mathbb{Z}^2$ for $n \geq 2$.*

Corollary 1.5 *Suppose all A_n^{-1} are contractive under a norm $\|\cdot\|'$ on $\mathbb{R}^2$ and $\sup_{n\geq 1} \|A_n\|' < \infty$. Then $\mu_{\{A_n, n\geq 1\}}$ is a spectral measure if and only if $\frac{1}{3}(1,-1)A_n \in \mathbb{Z}^2$ for $n \geq 2$.*

Corollary 1.6 *Let*

$$A_n = \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix}$$

be an integer expanding matrix for $n \geq 1$. Then $\mu_{\{A_n, n\geq 1\}}$ is a spectral measure if and only if $3 \mid p_n$ and $3 \mid q_n$ for $n \geq 2$.

To prove Theorem 1.3, one may consider

$$\Lambda = \frac{1}{3}B_1\mathcal{C} + \frac{1}{3}B_1B_2\mathcal{C} + \frac{1}{3}B_1B_2B_3\mathcal{C} + \cdots, \quad \text{all finite sums}, \tag{1.9}$$

where $B_n = A_n^t$ for all $n \geq 1$ and $\mathcal{C}$ is given by (1.6). It is easy to show that the associated family E_Λ is an orthogonal family of $L^2(\mu_{A_n, n\geq 1})$. The difficult part is the complete property of E_Λ in $L^2(\mu_{A_n, n\geq 1})$. But, unfortunately, we do not know how to prove it.

Up to now, there are four ways to show the complete property of an orthogonal family of exponentials for $L^2(\mu)$, where μ is a probability measure with compact support:

(1) Jorgensen and Pedensen ([26]) used the function space and fixed point theorem;

(2) Strichartz ([34]) used approximation by μ_n given in (1.3) and the Lebesgue dominated convergence theorem;

(3) Dai et al. ([6, 9, 10] etc.) used the recurrent ideas.

(4) Dutkay et al. ([18]) used the frame theory to prove the complete property.

However, following those ideas, we could not prove the complete property. Thus the following question arises:

Question 2 Is the Λ given by (1.9) a spectrum of the Moran–Sierpinski Measure $\mu_{\{A_n,n\geq 1\}}$?

If r is small enough, it is not difficult to answer Question 2 by following the known methods, and so the sufficiency of Theorem 1.3 holds. In our general case, motivated by the ideas of the above (2), (3), (4) and using some new ideas, we prove the sufficiency of Theorem 1.3 by constructing a "non standard" spectral set candidate (see (4.9)) and the detailed proof is given in Section 4.

At the end of this section we guess

Conjecture 1.7 *Let $A_n \in M_2(\mathbb{Z})$ be such that $\mu_{\{A_n,n\geq 1\}}$ is a Borel probability measure with a compact support. Then $\mu_{\{A_n,n\geq 1\}}$ is a spectral measure if and only if $\frac{1}{3}(1,-1)A_n \in \mathbb{Z}^2$ for $n \geq 2$.*

2 Preliminaries

Let μ be a Borel probability measure with compact support in $\mathbb{R}^2$. The Fourier transform of μ is defined by

$$\widehat{\mu}(\xi) := \mathcal{F}(\mu)(\xi) := \int \mathrm{e}^{2\pi \mathrm{i}\langle \xi, x\rangle}\mathrm{d}\mu(x), \quad \xi \in \mathbb{R}^2.$$

Then for any nonsingular 2×2-matrix $A \in M_2(\mathbb{R})$

$$\mathcal{F}(\mu \circ A)(\xi) = \mathcal{F}(\mu)(B^{-1}\xi),$$

where $\mu \circ A(E) = \mu(AE)$ for $E \subset \mathbb{R}^2$ and $B = A^t$, the transpose matrix of A. The following criterion is the main idea to judge whether a probability measure μ with compact support is a spectral one or not, which comes from Plancherel identity and Stone–Weierstrass theorem.

Theorem 2.1 ([26]) *Let μ be a Borel probability measure with compact support and $\Lambda \subset \mathbb{R}^2$. Then the following statements hold*:

(i) Λ *is an orthogonal set of μ if and only if*

$$Q_{\mu,\Lambda}(\xi) := \sum_{\lambda\in\Lambda} |\widehat{\mu}(\xi+\lambda)|^2 \leq 1, \quad \text{for } \xi \in \mathbb{R}^2;$$

(ii) Λ *is a spectrum of μ if and only if $Q_{\mu,\Lambda}(\xi) = 1$ for all $\xi \in \mathbb{R}^2$.*

We will use the following lemma which is easy to be proven (see [9, Lemma 2.2] for its proof).

Lemma 2.2 *Let $\mu = \nu * w$ be the convolution of the two measures ν, w in $\mathbb{R}^2$ with compact support. If ν is not a Dirac measure and Λ is an orthogonal set of w, then Λ is an orthogonal set of μ but not a spectrum of it.*

The following lemma was given in several papers, e.g. [13] and the reference therein. Let A be an expanding matrix with entries in $\mathbb{Z}$ and let $\mathcal{S}, \mathcal{M} \subset \mathbb{Z}^2$ be two digit sets with the same cardinality. We say that $(A^{-1}\mathcal{S}, \mathcal{M})$ is an *integral compatible pair* if the matrix

$$[\mathrm{e}^{2\pi \mathrm{i}\langle A^{-1}s, m\rangle}]_{s\in\mathcal{S}, m\in\mathcal{M}}$$

is unitary. In this case, the pair $(A, \mathcal{S})$ is said *admissible* and the triple $(A, \mathcal{S}, \mathcal{M})$ is said a *Hadamard triple*. The following conclusions are well known.

Lemma 2.3 *Let $(A^{-1}\mathcal{S}, \mathcal{M})$ be an integral compatible pair. Then the following statements hold.*

(i) *$(A^{-1}(\mathcal{S}+d), \mathcal{M}+c)$ is also an integral compatible pair for $d, c \in \mathbb{Z}^2$;*

(ii) *$(A^{-n}\mathcal{S}_n, \mathcal{M}_n)$ is an integral compatible pair for $n \geq 1$, where $\mathcal{S}_n = \mathcal{S} + A\mathcal{S} + \cdots + A^{n-1}\mathcal{S}$ and $\mathcal{M}_n = \mathcal{M} + A^t\mathcal{M} + \cdots + (A^t)^{n-1}\mathcal{M}$;*

(iii) *All elements in $\mathcal{S}$ (resp., $\mathcal{M}$) are in different coset of the group $\mathbb{Z}^2/A\mathbb{Z}^2$ (resp., $\mathbb{Z}^2/A^t\mathbb{Z}^2$);*

(iv) *$(A^{-1}\widetilde{\mathcal{S}}, \widetilde{\mathcal{M}})$ is an integral compatible pair for $\widetilde{\mathcal{S}} \equiv \mathcal{S} \pmod{A}$ and $\widetilde{\mathcal{M}} \equiv \mathcal{M} \pmod{A^t}$.*

Denote the mask of $\mathcal{S}$ by $m_{\mathcal{S}}(\xi)$, i.e.,

$$m_{\mathcal{S}}(\xi) = \frac{1}{\#\mathcal{S}} \sum_{s\in\mathcal{S}} \mathrm{e}^{2\pi\mathrm{i}\langle s,\xi\rangle}, \quad \xi \in \mathbb{R}^2. \tag{2.1}$$

It is easy to check the following known facts.

Lemma 2.4 *Let $A \in M_2(\mathbb{Z})$ be an expanding matrix with integral entries and $\mathcal{S}, \mathcal{M} \subset \mathbb{Z}^2$ with the same cardinality. Then the following statements are equivalent:*

(i) *$(A^{-1}\mathcal{S}, \mathcal{M})$ is an integral compatible pair;*

(ii) *$m_{\mathcal{S}}((A^t)^{-1}(m_1 - m_2)) = 0$ for any $m_1 \neq m_2 \in \mathcal{M}$;*

(iii) *$(\delta_{A^{-1}\mathcal{S}}, \mathcal{M})$ is a spectral pair, i.e.,*

$$\sum_{m\in\mathcal{M}} |\widehat{\delta}_{A^{-1}\mathcal{S}}(\xi+m)|^2 = \sum_{m\in\mathcal{M}} |m_{\mathcal{S}}((A^t)^{-1}(\xi+m))|^2 \equiv 1, \quad \forall \xi \in \mathbb{R}^2.$$

We extend some of the above results so that they suit for our setting.

Lemma 2.5 *Suppose that all $(A_n^{-1}\mathcal{S}_n, \mathcal{M}_n)$ are integral compatible pairs for $n \geq 1$. Then $((A_nA_{n-1}\cdots A_1)^{-1}\widetilde{\mathcal{S}_n}, \widetilde{\mathcal{M}_n})$ is an integral compatible pair for each $n \geq 1$, where*

$$\widetilde{\mathcal{S}_n} = \mathcal{S}_n + A_n\mathcal{S}_{n-1} + \cdots + A_nA_{n-1}\cdots A_2\mathcal{S}_1,$$
$$\widetilde{\mathcal{M}_n} = \mathcal{M}_1 + B_1\mathcal{M}_2 + \cdots + B_1B_2\cdots B_{n-1}\mathcal{M}_n,$$

where $B_n = A_n^t$ for $n \geq 1$.

Proof Write

$$\mu_n = \delta_{A_1^{-1}\mathcal{S}_1} * \delta_{A_1^{-1}A_2^{-1}\mathcal{S}_2} * \cdots \delta_{A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{S}_n} = \delta_{(A_nA_{n-1}\cdots A_1)^{-1}\widetilde{\mathcal{S}_n}}.$$

Then we have

$$\widehat{\mu_n}(\xi) = \prod_{k=1}^{n} m_{\mathcal{S}_k}(B_k^{-1}B_{k-1}^{-1}\cdots B_1^{-1}\xi).$$

According to (iii) in Lemma 2.3, it is easy to show that $\#\widetilde{\mathcal{S}_n} = \#\widetilde{\mathcal{M}_n}$. For any two different elements $c = c_1 + B_1c_2 + \cdots + B_1B_2\cdots B_{n-1}c_n$, $c' = c_1' + B_1c_2' + \cdots + B_1B_2\cdots B_{n-1}c_n' \in \widetilde{\mathcal{M}_n}$, where $c_k, c'_k \in \mathcal{M}_k$ for $1 \le k \le n$, let s be the integer such that $c_k = c'_k$ for $1 \le k < s$ and $c_s \neq c'_s$. Then

$$c - c' = B_1 B_2 \cdots B_{s-1}(c_s - c'_s + B_s w)$$

for some $w \in \mathbb{Z}^2$. Consequently,

$$m_{\mathcal{S}_s}(B_s^{-1}B_{s-1}^{-1}\cdots B_1^{-1}(c-c')) = m_{\mathcal{S}_s}(B_s^{-1}(c_s - c'_s) + w) = m_{\mathcal{S}_s}(B_s^{-1}(c_s - c'_s)) = 0.$$

Hence $\widehat{\mu_n}(c - c') = 0$. And the assertion holds by Lemma 2.4 (ii). □

In this paper we always write

$$m(\xi) = \frac{1}{3}\sum_{d\in\mathcal{D}} \mathrm{e}^{2\pi\mathrm{i}\langle\xi,d\rangle} = \frac{1}{3} + \frac{1}{3}\mathrm{e}^{2\pi\mathrm{i}\xi_1} + \frac{1}{3}\mathrm{e}^{2\pi\mathrm{i}\xi_2}. \tag{2.2}$$

Then

$$\widehat{\mu}_{\{A_n,n\ge1\}} = \prod_{k=1}^{\infty} \widehat{\delta}_{A_1^{-1}A_2^{-1}\cdots A_k^{-1}\mathcal{D}}(\xi) = \prod_{k=1}^{\infty} m(B_k^{-1}B_{k-1}^{-1}\cdots B_1^{-1}\xi), \tag{2.3}$$

when $\mu_{\{A_n,n\ge1\}}$ has compact support, where $B_k = A_k^t$ is the transpose of A_k for $k \ge 1$.

Denote the zero set of a function f by Ł(f), i.e.,

$$Ł(f) = \{x \in \mathbb{R}^2, f(x) = 0\}.$$

In the following part of this section, we assume $T(A_n, \mathcal{D}_n) := \sum_{n=1}^{\infty} A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}_n$ is a compact set of $\mathbb{R}^d$ (this means $\mu_{\{A_n,n\ge1\}}$ has compact support). By an elementary induction, we get

$$Ł(m) = \frac{1}{3}\left\{\pm e + 3\mathbb{Z}^2\right\},$$

where

$$e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, by (2.3) we have

$$\begin{aligned} Ł(\widehat{\mu}_{\{A_n,n\ge1\}}) &= \bigcup_{k=1}^{\infty} Ł(\widehat{\delta}_{A_1^{-1}A_2^{-1}\cdots A_k^{-1}\mathcal{D}}) = \bigcup_{k=1}^{\infty} Ł(\widehat{\delta}_{\mathcal{D}} \circ B_k^{-1}B_{k-1}^{-1}\cdots B_1^{-1}) \\ &= \bigcup_{k=1}^{\infty} \frac{1}{3} B_1 B_2 B_3 \cdots B_k(\pm e + 3\mathbb{Z}^2). \end{aligned} \tag{2.4}$$

Similar to the notation $\mu_{\{A_n,n\ge1\}}$, we write

$$\mu_{\{A_n,n\ge2\}} = \delta_{A_2^{-1}\mathcal{D}} * \delta_{A_2^{-1}A_3^{-1}\mathcal{D}} * \cdots * \mu_{A_2^{-1}\cdots A_n^{-1}\mathcal{D}} * \cdots \tag{2.5}$$

and

$$\mu_{\{M,A_n,n\ge2\}} = \delta_{M^{-1}\mathcal{D}} * \delta_{M^{-1}A_2^{-1}\mathcal{D}} * \cdots * \mu_{M^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}} * \cdots, \tag{2.6}$$

i.e., the measure with A_1 replaced by a nonsingular matrix M.

Lemma 2.6 *Let $M \in M_2(\mathbb{Z})$ be nonsingular and let $\mu_{\{M,A_n,n\ge2\}}$ be the measure defined in* (2.6)*. Then*

$$\mu_{\{A_n,n\ge1\}} = \mu_{\{M,A_n,n\ge2\}} \circ (M^{-1}A_1)$$

and $(\mu_{\{A_n,n\ge1\}}, \Lambda)$ is a spectral pair if and only if $(\mu_{\{M,A_n,n\ge2\}}, M^t B_1^{-1}\Lambda)$ is a spectral pair.

Proof According to the uniqueness of Fourier transform, it is sufficient to show that

$$\mathcal{F}(\mu_{\{M,A_n,n\geq 2\}}\circ(M^{-1}A_1))(\xi)=\widehat{\mu}_{\{A_n,n\geq 1\}}(\xi)$$

for the first assertion. In fact, by the definition and Theorem 2.1, one has that

$$\begin{aligned}\mathcal{F}(\mu_{\{M,A_n,n\geq 2\}}\circ(M^{-1}A_1))(\xi)&=\mathcal{F}(\mu_{\{M,A_n,n\geq 2\}})(M^tB_1^{-1}\xi)\\&=m((M^t)^{-1}M^tB_1^{-1}\xi)\prod_{k=2}^{\infty}m(B_k^{-1}B_{k-1}^{-1}\cdots B_2^{-1}(M^t)^{-1}M^tB_1^{-1}\xi)\\&=\prod_{k=1}^{\infty}m(B_k^{-1}B_{k-1}^{-1}\cdots B_2^{-1}B_1^{-1}\xi)\\&=\widehat{\mu}_{\{A_n,n\geq 1\}}(\xi).\end{aligned}$$

To prove the second assertion, we have for $\xi\in\mathbb{R}^2$,

$$\begin{aligned}Q_{\mu_{\{A_n,n\geq 1\}},\Lambda}(\xi)&=\sum_{\lambda\in\Lambda}|\widehat{\mu}_{\{A_n,n\geq 1\}}(\xi+\lambda)|^2\\&=\sum_{\lambda\in\Lambda}|\widehat{\mu}_{\{M,A_n,n\geq 2\}}(M^tB_1^{-1}(\xi+\lambda))|^2\\&=\sum_{\lambda\in\Lambda}|\widehat{\mu}_{\{M,A_n,n\geq 2\}}(M^tB_1^{-1}\xi+M^tB_1^{-1}\lambda)|^2\\&=Q_{\mu_{\{M,A_n,n\geq 2\}},M^tB_1^{-1}\Lambda}(M^tB_1^{-1}\xi).\end{aligned}$$

Then the second assertion follows by Theorem 2.1. □

Lemma 2.7 *Let $p_{i,j}$ be positive numbers such that $\sum_{j=1}^n p_{i,j}=1$ and let $q_{i,j}$ be nonnegative numbers such that $\sum_{i=1}^n \max_{1\leq j\leq n} q_{i,j}\leq 1$. Then $\sum_{i=1}^n\sum_{j=1}^n p_{i,j}q_{i,j}=1$ if and only if $q_{i,j}=q_i$ for $1\leq i,j\leq n$ and $\sum_{i=1}^n q_i=1$.*

Proof Since

$$1-\sum_{i=1}^n\sum_{j=1}^n p_{i,j}q_{i,j}$$

can be rewritten as

$$\Big[1-\sum_{i=1}^n\max_{1\leq j\leq n}q_{i,j}\Big]+\sum_{i=1}^n\sum_{j=1}^n p_{i,j}\Big[\max_{1\leq j\leq n}q_{i,j}-q_{i,j}\Big],$$

By the assumptions, each term in middle brackets are nonnegative. Then the conclusion follows. □

3 Proof of Theorem 1.2

In this section, we always assume that $A_n\in M_2(\mathbb{Z})$ is a sequence of expansive matrices such that $T(A_n,\mathcal{D}_n):=\sum_{n=1}^{\infty}A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}_n$ is a compact set of $\mathbb{R}^d$. According to Lemma 2.6, without loss of generality, we can assume $A_1=\mathrm{diag}(3,3)$. Then (2.4) shows that $\mathrm{Ł}(\widehat{\mu}_{\{A_n,n\geq 1\}})\subseteq\mathbb{Z}^2$.

For simple notations we write

$$\mu=\mu_{\{\mathrm{diag}[3,3],A_n,n\geq 2\}}\quad\text{and}\quad\nu=\mu_{\{A_n,n\geq 2\}},\tag{3.1}$$

where $\operatorname{diag}[3,3] = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Then

$$\mu = \delta_{\frac{1}{3}\mathcal{D}} * (\nu \circ 3) \quad \text{and} \quad \widehat{\mu}(\xi) = m\left(\frac{\xi}{3}\right)\widehat{\nu}\left(\frac{\xi}{3}\right). \tag{3.2}$$

And by (2.3), one has that

$$Ł(\widehat{\mu}) = \left(\pm e + 3\mathbb{Z}^2\right) \cup \bigcup_{k=2}^{\infty} B_2 B_3 \cdots B_k \left(\pm e + 3\mathbb{Z}^2\right) \subset \mathbb{Z}^2. \tag{3.3}$$

Let

$$\nabla = \left\{ \begin{pmatrix} i \\ j \end{pmatrix} : i, j \in \{-1, 0, 1\} \right\}. \tag{3.4}$$

It is clear that ∇ is the complete residue system mod $\operatorname{diag}[3,3]$. Then, for any $w \in \mathbb{Z}^2$, there exists $c_k \in \nabla$ for $k \geq 1$ and $c_k = 0$ for sufficient large k such that

$$w = \sum_{k=1}^{\infty} 3^{k-1} c_k, \tag{3.5}$$

and the expression is unique.

Let Λ be a spectrum of μ with $0 \in \Lambda$. Then $\Lambda \subset \mathbb{Z}^2$ by $\Lambda - \Lambda \subset \{0\} \cup Ł(\widehat{\mu})$ and (3.3). For $\lambda \in \Lambda$, by (3.5) there exists a unique $\gamma \in \nabla$ such that $\lambda = \gamma + 3w$ for some $w \in \mathbb{Z}^2$. Define

$$\Lambda_\gamma = \{w \in \mathbb{Z}^2 : \gamma + 3w \in \Lambda\}. \tag{3.6}$$

Thus we have the following decomposition

$$\Lambda = \bigcup_{\gamma \in \nabla} (\gamma + 3\Lambda_\gamma), \tag{3.7}$$

where $\gamma + 3\Lambda_\gamma = \emptyset$ if $\Lambda_\gamma = \emptyset$. Moreover, the union is disjoint. Since $0 \in \Lambda$, it is clear

$$\Lambda_0 \neq \emptyset. \tag{3.8}$$

Lemma 3.1 *Let Λ be a spectrum of μ with $0 \in \Lambda$. Then Λ_γ is either an empty set or orthogonal set of ν for each $\gamma \in \nabla$.*

Proof Suppose that Λ_γ is a nonempty set ($\gamma \in \nabla$) and $\lambda \neq \beta \in \Lambda_\gamma$. Then $\gamma + 3\lambda, \gamma + 3\beta \in \Lambda$. This leads to

$$0 = \widehat{\mu}(3(\lambda - \beta)) = m(\lambda - \beta)\widehat{\nu}(\lambda - \beta) = \widehat{\nu}(\lambda - \beta),$$

which is equivalent to that Λ_γ is an orthogonal set of ν. □

Write

$$\mathcal{C}_0 = \{0, e, -e\},$$

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{C}_{-1} = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

It is easy to check the following further decomposition

$$\Lambda=\bigcup_{\gamma\in\nabla}(\gamma+3\Lambda_\gamma)=\bigcup_{i\in\{-1,0,1\}}\bigcup_{c\in\mathcal{C}_i}(c+3\Lambda_c)\,. \tag{3.9}$$

Since

$$\mathcal{C}_1\equiv\mathcal{C}_0+\begin{pmatrix}1\\0\end{pmatrix}\left(\bmod\begin{pmatrix}3&0\\0&3\end{pmatrix}\right)$$

and

$$\mathcal{C}_{-1}\equiv\mathcal{C}_0+\begin{pmatrix}-1\\0\end{pmatrix}\left(\bmod\begin{pmatrix}3&0\\0&3\end{pmatrix}\right),$$

from Lemma 2.3 (i) it follows that $(A^{-1}\mathcal{D},\frac{1}{3}A^t\mathcal{C}_i)$ are compatible pairs for any nonsingular matrix A and $i\in\{-1,0,1\}$.

Lemma 3.2 *Let Λ be a spectrum of μ with $0\in\Lambda$. Then, for any $c_i\in\mathcal{C}_i, i=-1,0,1$,*

$$\Lambda_{c_{-1},c_0,c_1}:=\bigcup_{i\in\{-1,0,1\}}\left(\frac{c_i}{3}+\Lambda_{c_i}\right)$$

is either an empty set or an orthogonal set of ν. When it is an orthogonal set of ν, we have

$$\sum_{i\in\{-1,0,1\}}\sum_{\lambda\in\Lambda_{c_i}}\left|\widehat{\nu}\left(\xi+\frac{c_i}{3}+\lambda\right)\right|^2\le1,\quad \text{for } \xi\in\mathbb{R}^2,$$

where the term $\sum_{\lambda\in\Lambda_{c_i}}|\widehat{\nu}(\xi+\frac{c_i}{3}+\lambda)|^2$ is equal to 0 if $\Lambda_{c_i}=\emptyset$ for some $i\in\{-1,0,1\}$.

Proof By Lemma 3.1, it is sufficient to prove that, for $\alpha\in\Lambda_{c_0}$ and $\beta\in\Lambda_{c_1}$, one has that

$$\widehat{\nu}\left(\frac{c_0}{3}+\alpha-\frac{c_1}{3}-\beta\right)=0.$$

In fact, by (3.9) and (3.2) one has that

$$\begin{aligned}0=\widehat{\mu}(c_0+3\alpha-c_1-3\beta)&=m\left(\frac{c_0}{3}+\alpha-\frac{c_1}{3}-\beta\right)\widehat{\nu}\left(\frac{c_0}{3}+\alpha-\frac{c_1}{3}-\beta\right)\\&=m\left(\frac{c_0}{3}-\frac{c_1}{3}\right)\widehat{\nu}\left(\frac{c_0}{3}+\alpha-\frac{c_1}{3}-\beta\right).\end{aligned}$$

From the definition of $\mathcal{C}_0$ and $\mathcal{C}_1$, it follows that $\frac{c_0}{3}-\frac{c_1}{3}\notin \text{Ł}(m)$, which implies $m(\frac{c_0}{3}-\frac{c_1}{3})\ne0$. We obtain the desired result. □

The following lemma is the rewrite of Lemma 2.7.

Lemma 3.3 *Let $\mathcal{C}_{-1},\mathcal{C}_0,\mathcal{C}_1$ be defined as above. Assume all p_c are positive numbers such that $\sum_{c\in\mathcal{C}_i}p_c=1$ for $i\in\{-1,0,1\}$ and all q_c are nonnegative numbers such that $\sum_{i\in\{-1,0,1\}}\max_{c\in\mathcal{C}_i}\{q_c\}\le1$. Then $\sum_{i\in\{-1,0,1\}}\sum_{c\in\mathcal{C}_i}p_cq_c=1$ if and only if all q_c are equal for $c\in\mathcal{C}_i$ ($i\in\{-1,0,1\}$), and $\sum_{i=1}^3q_{c_i}=1$ for any choice of $c_i\in\mathcal{C}_i$.*

Lemma 3.4 *Let Λ be a spectrum of μ with $0\in\Lambda$. Then, for any choosing $c_i\in\mathcal{C}_i$ for $i=-1,0,1$, the set Λ_{c_{-1},c_0,c_1} is a spectrum of ν.*

Proof By Theorem 2.1, one has for any $\xi \in \mathbb{R}^2$,

$$\begin{aligned}1 \equiv \sum_{\lambda\in\Lambda} |\widehat{\mu}(\xi+\lambda)|^2 &= \sum_{\gamma\in\nabla}\sum_{\lambda\in\Lambda_\gamma} |\widehat{\mu}(\xi+\gamma+3\lambda)|^2 \quad \text{by (3.9)}\\ &= \sum_{\gamma\in\nabla}\sum_{\lambda\in\Lambda_\gamma} \Big|m\Big(\frac{\xi+\gamma}{3}+\lambda\Big)\Big|^2 \Big|\widehat{\nu}\Big(\frac{\xi+\gamma}{3}+\lambda\Big)\Big|^2 \quad \text{by (3.2)}\\ &= \sum_{\gamma\in\nabla} \Big|m\Big(\frac{\xi+\gamma}{3}\Big)\Big|^2 \sum_{\lambda\in\Lambda_\gamma} \Big|\widehat{\nu}\Big(\frac{\xi+\gamma}{3}+\lambda\Big)\Big|^2\\ &= \sum_{i\in\{-1,0,1\}}\sum_{c_i\in\mathcal{C}_i} \Big|m\Big(\frac{\xi+c_i}{3}\Big)\Big|^2 \sum_{\lambda\in\Lambda_{c_i}} \Big|\widehat{\nu}\Big(\frac{\xi+c_i}{3}+\lambda\Big)\Big|^2.\end{aligned}$$

Choose ξ with irrational entries. Write

$$p_c = \Big|m(\frac{\xi+c}{3})\Big|^2, \quad q_c = \sum_{\lambda\in\Lambda_c} \Big|\widehat{\nu}(\frac{\xi+c}{3}+\lambda)\Big|^2, \quad c\in\nabla.$$

Then $\sum_{i\in\{-1,0,1\}}\sum_{c\in\mathcal{C}_i} p_c q_c = 1$ and all $p_{c_i} > 0$. By Theorem 2.1 (i) and Lemma 3.2, one has

$$\sum_{i\in\{-1,0,1\}} \max_{c\in\mathcal{C}_i} q_c \le 1.$$

Since $(\mathcal{D}, \frac{1}{3}\mathcal{C}_i)$ is a compatible pair for $i\in\{-1,0,1\}$, from Lemma 2.4 it follows $\sum_{c_i\in\mathcal{C}_i} p_{c_i} = 1$. Then, according to Lemma 3.3, we have $q_c = \max\{q_x : x\in\mathcal{C}_i\} := q_i$ for any $c\in\mathcal{C}_i$ ($i\in\{-1,0,1\}$), and

$$1 = \sum_{i\in\{-1,0,1\}} q_i = \sum_{i\in\{-1,0,1\}}\sum_{\lambda\in\Lambda_{c_i}} \Big|\widehat{\nu}\Big(\frac{\xi+c_i}{3}+\lambda\Big)\Big|^2 = \sum_{\lambda\in\Lambda_{c_{-1},c_0,c_1}} \Big|\widehat{\nu}\Big(\frac{\xi}{3}+\lambda\Big)\Big|^2.$$

Hence Λ_{c_{-1},c_0,c_1} is a spectrum of ν by Theorem 2.1 (ii). □

Corollary 3.5 *Let Λ be a spectrum of μ with $0\in\Lambda$. Then there exist z_+ and $z_-\in\mathbb{Z}^2$ such that both $e+3z_+$ and $-e+3z_-$ lie in Λ.*

Proof According to Lemma 3.4 and its proof, one has that $q_0 = q_e = q_{-e}$. By $q_0 > 0$ (because (3.8)), it is clear that both Λ_e and Λ_{-e} are nonempty. The assertion follows by (3.9). □

Now we are in the place to prove Theorem 1.2. For convenience of readers, we rewrite Theorem 1.2 as

Theorem 3.6 *Let $A_n \in M_2(\mathbb{Z})$ be a sequence of expansive matrices. Assume $T(A_n,\mathcal{D}_n) := \sum_{n=1}^{\infty} A_1^{-1}A_2^{-1}\cdots A_n^{-1}\mathcal{D}_n$ is a compact set of $\mathbb{R}^d$. If $\mu_{\{A_n,n\ge1\}}$ is a spectral measure, then $\frac{1}{3}(1,-1)A_n\in\mathbb{Z}^2$ for $n\ge2$.*

Proof According to Lemma 2.6, without loss of generality, we can assume $A_1 = \mathrm{diag}(3,\ 3)$. Assume that μ has a spectrum Λ with $\mathbf{0}\in\Lambda$, then (2.4) shows that $\Lambda\subseteq\{0\}\cup Ł(\widehat{\mu}_{\{A_n,n\ge1\}})\subset\mathbb{Z}^2$.

Choose $c_0 = \mathbf{0}\in\mathcal{C}_0$, $c_{-1} = (-1,0)^t\in\mathcal{C}_{-1}$ and $c_1 = (1,1)^t\in\mathcal{C}_1$. From Lemma 3.4, it follows that $\mathbf{0}\in\Lambda_{c_{-1},c_0,c_1}$ is a spectrum of $\nu = \mu_{\{A_n,n\ge2\}}$. Then, we have $3B_2^{-1}\Lambda_{c_{-1},c_0,c_1}$ is a spectrum of $\mu_{\{\mathrm{diag}[3,3],A_n,n\ge3\}}$ by Lemma 2.6 again. According to Corollary 3.5, there exist $z_+, z_-\in\mathbb{Z}^2$ such that

$$\frac{1}{3}B_2e + B_2z_+, -\frac{1}{3}B_2e + B_2z_- \in \Lambda_{c_{-1},c_0,c_1}. \tag{3.10}$$

Suppose that $\frac{1}{3}B_2e$ is not in $\mathbb{Z}^2$. Recall that

$$\Lambda_{c_{-1},c_0,c_1} = \Lambda_0 \cup \left(\frac{1}{3}c_1 + \Lambda_{c_1}\right) \cup \left(\frac{1}{3}c_{-1} + \Lambda_{c_{-1}}\right)$$

and all $\Lambda_\gamma \subset \mathbb{Z}^2$ for $\gamma \in \nabla$. Then $\frac{1}{3}B_2e \notin \Lambda_0$.

If both $\frac{1}{3}B_2e + B_2z_+$ and $-\frac{1}{3}B_2e + B_2z_-$ belong to the same set in the union of Λ_{c_{-1},c_0,c_1}, then $\pm\frac{1}{3}B_2e - \frac{1}{3}c_1 \in \mathbb{Z}^2$ or $\pm\frac{1}{3}B_2e - \frac{1}{3}c_{-1} \in \mathbb{Z}^2$. Consequently, $-\frac{2}{3}c_1 \in \mathbb{Z}^2$ or $-\frac{2}{3}c_{-1} \in \mathbb{Z}^2$, which is impossible.

If not, without loss of generality we assume that $\frac{1}{3}B_2e + B_2z_+ \in \frac{1}{3}c_1 + \Lambda_{c_1}$ and $-\frac{1}{3}B_2e + B_2z_- \in \frac{1}{3}c_{-1} + \Lambda_{c_{-1}}$. Then $\frac{1}{3}B_2e - \frac{1}{3}c_1 \in \mathbb{Z}^2$ and $-\frac{1}{3}B_2e - \frac{1}{3}c_{-1} \in \mathbb{Z}^2$. This implies that $-\frac{1}{3}(c_1 + c_{-1}) \in \mathbb{Z}^2$, which contradicts the known fact $-\frac{1}{3}(c_1 + c_{-1}) = (0, -\frac{1}{3})^t$. Hence, $\frac{1}{3}B_2e$ lies in $\mathbb{Z}^2$, that is, $\frac{1}{3}(1,-1)A_2 \in \mathbb{Z}^2$.

If one replaces $\mu_{\{A_n,n\geq 1\}}$ by ν in the above argument. Then one obtains that $\frac{1}{3}(1,-1)A_3 \in \mathbb{Z}^2$. Hence the assertion follows by induction. □

4 Proof of Theorem 1.3

The necessity of Theorem 1.3 follows from Theorem 1.2. In this section we prove the sufficiency of Theorem 1.3. We rewrite it as the following statement. In the sequel, we write $\mathcal{C} = \mathcal{C}_0 = \{0,\ e,\ -e\}$ for simplicity.

Theorem 4.1 *Let $\{A_n\}_{n=1}^\infty$ be a sequence of nonsingular matrices in $M_2(\mathbb{Z})$ satisfying that $\|A_n^{-1}\|' \leq r < 1$ for $n \geq 1$. Suppose that there exist two positive numbers $\eta, \delta \in (0, \frac{1}{4})$ and an integer N such that*

$$B_{k+1} \cdots B_{k+p} \in \mathcal{B}_{\eta,\delta} := \left\{ B : B^{-1}\left[-\frac{1}{2} - \eta, \frac{1}{2} + \eta\right]^2 \cap \left(\pm\frac{1}{3}e + \mathbb{Z}^2\right)_\delta = \emptyset \right\} \tag{4.1}$$

for $p \geq 1$ and $k \geq N$. If $\frac{1}{3}(1,-1)A_n \in \mathbb{Z}^2$ for $n \geq 2$, then $\mu_{\{A_n,n\geq 1\}}$ is a spectral measure.

Remark To prove Theorem 4.1, we first use the technique in Lemma 2.6. Then, from now on, we can assume that $\frac{1}{3}(1,-1)A_1 \in \mathbb{Z}^2$.

In this case it is easy to see that the set

$$\Lambda = \frac{1}{3}B_1\mathcal{C} + \frac{1}{3}B_1B_2\mathcal{C} + \frac{1}{3}B_1B_2B_3\mathcal{C} + \cdots,$$

where all the sums are finite, is a subset of $\mathbb{Z}^2$ and E_Λ is an orthogonal set of $\mu_{\{A_n,n\geq 1\}}$. Unfortunately, we cannot prove it is a spectrum of $\mu_{\{A_n,n\geq 1\}}$ by following all known methods. To construct a spectrum for $\mu_{\{A_n,n\geq 1\}}$, we need the following conclusions.

It is well known that all norms on $\mathbb{R}^2$ are equivalent. Then there exists a number $\alpha \geq 1$ such that

$$\frac{1}{\alpha}\|\cdot\|' \leq \|\cdot\| \leq \alpha\|\cdot\|'. \tag{4.2}$$

We will use the simple fact:

$$\|B_n^{-1}\|' \leq r, \quad \text{for } 1 \leq n \leq k \implies \|B_1^{-1}B_2^{-1}\cdots B_n^{-1}\| \leq \alpha^2 r^n, \quad \text{for } 1 \leq n \leq k \tag{4.3}$$

several times in the following proofs.

For simplicity, we will use the following notations frequently:

$$\mu = \mu_{\{A_n,n\geq 1\}};$$

$$\begin{aligned}
&\mu_{\{A_n,q\}} = \delta_{A_1^{-1}\mathcal{D}} * \delta_{A_1^{-1}A_2^{-1}\mathcal{D}} * \cdots * \delta_{A_1^{-1}A_2^{-1}\cdots A_q^{-1}\mathcal{D}};\\
&\mu_{\{A_n,q<n\le m\}} = \delta_{A_{q+1}^{-1}\mathcal{D}} * \delta_{A_{q+1}^{-1}A_{q+2}^{-1}\mathcal{D}} * \cdots * \delta_{A_{q+1}^{-1}A_{q+2}^{-1}\cdots A_m^{-1}\mathcal{D}};\\
&\mu_{\{A_n,n>q\}} = \delta_{A_{q+1}^{-1}\mathcal{D}} * \delta_{A_{q+1}^{-1}A_{q+2}^{-1}\mathcal{D}} * \cdots .
\end{aligned} \tag{4.4}$$

Lemma 4.2 *For the sequence $\{A_n\}_{n=1}^{\infty}$ given in Theorem* 4.1*, there exists $M \ge N$ (N is given by Theorem* 4.1*) and a sequence β_k of positive numbers (for $k \ge M$), which depend only on the r and the α given in* (4.2)*, such that $\beta_k \uparrow 1$ as $k \to +\infty$ and*

$$|\widehat{\mu}_{\{A_n,n>q+k\}}(B_{q+k}^{-1}\cdots B_{q+1}^{-1}\xi)| \ge \beta_k$$

for all $k \ge M$, $q \ge 0$ and $\xi \in [-1,1]^2$.

Proof By simple calculations, it is easy to obtain that $|m(\xi)-1| \le \frac{4\pi}{3}\|\xi\|$ and

$$-\ln x \le 2(1-x), \quad \text{for } \frac{1}{2} \le x \le 1.$$

By (4.3), one has

$$\|B_{q+k}^{-1}\cdots B_{q+1}^{-1}\xi\| \le \sqrt{2}\alpha^2 r^k, \quad \text{for } k \ge 1,\ q \ge 0 \text{ and } \xi \in [-1,1]^2.$$

Then the continuity of $m(x)$ and the fact $m(0)=1$ show that there exists $M \ge N$ depending only on r, α such that

$$|m(B_{q+k}^{-1}\cdots B_{q+1}^{-1}\xi)| \ge \frac{1}{2}, \quad \text{for } k \ge M,\ q \ge 0 \text{ and } \xi \in [-1,1]^2.$$

Hence, by $1-|m(\xi)| \le |1-m(\xi)| \le \frac{4\pi}{3}\|\xi\|$, one has that

$$\begin{aligned}
-\ln|\widehat{\mu}_{\{A_n,n>q+k\}}(B_{q+k}^{-1}\cdots B_{q+1}^{-1}\xi)| &= -\ln\prod_{n=1}^{\infty}|m(B_{q+k+n}^{-1}\cdots B_{q+1}^{-1}\xi)|\\
&= \sum_{n=1}^{\infty} -\ln|m(B_{q+k+n}^{-1}\cdots B_{q+1}^{-1}\xi)|\\
&\le 2\sum_{n=1}^{\infty}(1-|m(B_{q+k+n}^{-1}\cdots B_{q+1}^{-1}\xi)|)\\
&\le 2\sum_{n=1}^{\infty}|1-m(B_{q+k+n}^{-1}\cdots B_{q+1}^{-1}\xi)|\\
&\le \frac{8\pi}{3}\sum_{n=1}^{\infty}\|B_{q+k+n}^{-1}\cdots B_{q+1}^{-1}\xi\|\\
&\le \frac{8\sqrt{2}\alpha^2\pi}{3}\frac{r^{k+1}}{1-r}
\end{aligned}$$

for $k \ge M$, $q \ge 0$ and $\xi \in [-1,1]^2$. Consequently, there exists a positive number $\beta_k := \exp\{-\frac{8\sqrt{2}\alpha^2\pi}{3}\frac{r^{k+1}}{1-r}\}$ depended only on r, α and k such that

$$|\widehat{\mu}_{\{A_n,n>q+k\}}(B_{q+k}^{-1}\cdots B_{q+1}^{-1}\xi)| = \prod_{n=1}^{\infty}|m(B_{q+k+n}^{-1}\cdots B_{q+1}^{-1}\xi)| \ge \beta_k, \quad k = M, M+1, \ldots$$

uniformly for $\xi \in [-1,1]^2$ and $q \ge 0$. Clearly $\beta_k \uparrow 1$ as $k \to +\infty$, then the proof is completed. □

Lemma 4.3 *For the sequence $\{A_n\}_{n=1}^{\infty}$ given in Theorem 4.1, there exists a constant $\gamma > 0$, which depends only on η, δ, such that*

$$|\widehat{\mu}_{\{A_n, q<n\le q+k\}}(\xi)| = \prod_{n=1}^{k} |m(B_{q+n}^{-1} \cdots B_{q+1}^{-1}\xi)| \ge \gamma^k$$

uniformly for $\xi \in [-\frac{1}{2}-\eta, \frac{1}{2}+\eta]^2$, $k \ge 1$ and $q \ge N$, where N is given in Theorem 4.1.

Proof Noticing that $\|B_{q+n}^{-1} \cdots B_{q+1}^{-1}\| \le \alpha^2 r^n$ and $0 < \eta < \frac{1}{4}$, one has

$$\|B_{q+n}^{-1} \cdots B_{q+1}^{-1}\xi\| \le \frac{3\sqrt{2}}{4}\alpha^2 r^n < 2\alpha^2, \quad \text{for } \xi \in \left[-\frac{1}{2}-\eta, \frac{1}{2}+\eta\right]^2. \tag{4.5}$$

According to the definition of $\mathcal{B}_{\eta,\delta}$, for any $q \ge N$ and $n \ge 1$, one has

$$B_{q+n}^{-1} \cdots B_{q+1}^{-1}\left[-\frac{1}{2}-\eta, \frac{1}{2}+\eta\right]^2 \subset B(0, 2\alpha^2) \setminus \left(\pm\frac{1}{3}e + \mathbb{Z}^2\right)_\delta,$$

which implies that $B_{q+n}^{-1} \cdots B_{q+1}^{-1}\xi$ belongs to a fixed compact set $T \subset \mathbb{R}^2 \setminus (\pm\frac{1}{3}e + \mathbb{Z}^2)_\delta$ (which does not intersect the set of zeros of $m(x)$). Write

$$\gamma = \inf\{|m(x)| : x \in T\}.$$

By the continuity of $m(x)$, it is clear that $\gamma > 0$. We finish the proof. $\square$

Corollary 4.4 *For the sequence $\{A_n\}_{n=1}^{\infty}$ given in Theorem 4.1 and M given in Lemma 4.2, we have*

$$|\widehat{\mu}_{\{A_n, n>q\}}(\xi)| \ge \beta_k \gamma^k, \quad \textit{uniformly for } \min\{k, q\} \ge M \textit{ and } \xi \in \left[-\frac{1}{2}-\eta, \frac{1}{2}+\eta\right]^2,$$

where β_k and γ^k are given by Lemma 4.2 and Lemma 4.3.

Proof Note that

$$|\widehat{\mu}_{\{A_n, n>q\}}(\xi)| = |\widehat{\mu}_{\{A_n, q<n\le q+k\}}(\xi)||\widehat{\mu}_{\{A_n, n>q+k\}}(B_{q+k}^{-1} \cdots B_{q+1}^{-1}\xi)|.$$

Then the assertion follows by Lemma 4.2 and Lemma 4.3. $\square$

To prove that $\mu_{\{A_n, n\ge1\}}$ is a spectral measure, the key step is to choose a suitable spectrum candidate. To do it, we rewrite $\mu_{\{A_n, n\ge1\}}$ by combining its K-factors of convolutions as follows,

$$\begin{aligned}
\mu &= (\delta_{A_1^{-1}\mathcal{D}} * \cdots * \delta_{A_1^{-1}A_2^{-1}\cdots A_K^{-1}\mathcal{D}})* \\
&\quad (\delta_{(A_1^{-1}A_2^{-1}\cdots A_K^{-1})A_{K+1}^{-1}\mathcal{D}} * \cdots * \delta_{(A_1^{-1}A_2^{-1}\cdots A_K^{-1})A_{K+1}^{-1}A_{K+2}^{-1}\cdots A_{2K}^{-1}\mathcal{D}}) * \cdots \\
&:= \delta_{R_0^{-1}\mathcal{D}_0} * \delta_{R_0^{-1}R_1^{-1}\mathcal{D}_1} * \cdots * \delta_{R_0^{-1}R_1^{-1}\cdots R_n^{-1}\mathcal{D}_n} * \cdots
\end{aligned}$$

for some suitable integer K, where $\mathcal{D}_k$ and R_k are defined as

$$\mathcal{D}_k = \mathcal{D} + A_{kK+K}\mathcal{D} + A_{kK+K}A_{kK+K-1}\mathcal{D} + \cdots + A_{kK+K}A_{kK+K-1}\cdots A_{kK+2}\mathcal{D},$$

and

$$R_k = A_{kK+K}A_{kK+K-1}\cdots A_{kK+1}$$

for $k \ge 0$. Therefore

$$\widehat{\mu}(\xi) = \prod_{j=0}^{\infty} m_{\mathcal{D}_j}((R_0^{-1}R_1^{-1}\cdots R_j^{-1})^t\xi) = \prod_{k=1}^{\infty} m(B_k^{-1}B_{k-1}^{-1}\cdots B_1^{-1}\xi), \tag{4.6}$$

where $m_{\mathcal{D}_j}(\xi)$ is the mask of the set $\mathcal{D}_j$ defined by (2.1). Corresponding to these notations, we set

$$\mathcal{C}_k = \frac{1}{3}B_{kK+1}\mathcal{C} + \frac{1}{3}B_{kK+1}B_{kK+2}\mathcal{C} + \cdots + \frac{1}{3}B_{kK+1}B_{kK+2}\cdots B_{kK+K}\mathcal{C}$$

for $k \geq 0$. Then $(R_k^{-1}\mathcal{D}_k, \mathcal{C}_k)$ is a compatible pair for each $k \geq 0$ by Lemma 2.5. In this case, similar to (4.4), we write

$$\mu = \mu_{\{A_n, n\geq 1\}} = \mu_{\{R_n, \mathcal{D}_n, n\geq 0\}} \quad \text{and} \quad \mu_{\{R_i, \mathcal{D}_i, n\}} = \delta_{R_0^{-1}\mathcal{D}_0} * \delta_{R_0^{-1}R_1^{-1}\mathcal{D}_1} * \cdots * \delta_{R_0^{-1}R_1^{-1}\cdots R_n^{-1}\mathcal{D}_n}$$

for $n \geq 0$.

Lemma 4.5 *With the above notations, assume that $\frac{1}{3}(1,\ -1)A_n \in \mathbb{Z}^2$ and $\|A_n^{-1}\|' \leq r$ for $n \geq 1$. Then there exists $\mathcal{N}_k \subset \mathbb{Z}^2$ with $0 \in \mathcal{N}_k$ such that $(R_k^{-1}\mathcal{D}_k,\ \mathcal{N}_k)$ is an integral compatible pair for each $k \geq 0$ and, for the $\eta \in (0, \frac{1}{4})$ given by Theorem 4.1, there exists $K \geq M$ (M is given by Lemma 4.2) depended only on η and r such that*

$$(R_0^t R_1^t \cdots R_k^t)^{-1}\mathcal{N}_0 + \cdots + (R_{k-1}^t R_k^t)^{-1}\mathcal{N}_{k-1} + (R_k^t)^{-1}\mathcal{N}_k \subset \left[-\frac{1}{2} - \frac{1}{4}\eta, \frac{1}{2} + \frac{1}{4}\eta \right]^2 \tag{4.7}$$

for $k \geq 0$.

Proof Write

$$\mathcal{M}_k = R_k^t \left(-\frac{1}{2}, \frac{1}{2} \right]^2 \cap \mathbb{Z}^2, \quad \text{for } k \geq 0.$$

Then the set $\mathcal{M}_k$ is a complete residue set of mod R_k^t for $k \geq 0$. Let $E_j = B_j(-\frac{1}{2},\ \frac{1}{2}]^2 \cap \mathbb{Z}^2$ for $j \geq 1$. It is easy to check that

$$E_{kK+1} + B_{kK+1}E_{kK+2} + \cdots + B_{kK+1}B_{kK+2}\cdots B_{kK+K-1}E_{kK+K}$$

is also a complete residue set of mod R_k^t for $k \geq 0$. Therefore

$$E_{kK+1} + B_{kK+1}E_{kK+2} + \cdots + B_{kK+1}B_{kK+2}\cdots B_{kK+K-1}E_{kK+K} \equiv \mathcal{M}_k \pmod{R_k^t}.$$

Noticing that $\frac{1}{3}B_j\mathcal{C} \subset E_j$ by the assumption $\frac{1}{3}(1, -1)A_j \in \mathbb{Z}^2$ for $j \geq 1$, we see that there exists a subset $\mathcal{N}_k \subset \mathcal{M}_k$ with $0 \in \mathcal{N}_k$ such that $\mathcal{C}_k \equiv \mathcal{N}_k \pmod{R_k^t}$.

On the other hand, Lemma 2.5 shows that $(R_k^{-1}\mathcal{D}_k,\ \mathcal{C}_k)$ is an integral compatible pair for each $k \geq 0$. According to Lemma 2.3 (iv), $(R_k^{-1}\mathcal{D}_k,\ \mathcal{N}_k)$ is also an integral compatible pair for each $k \geq 0$.

The remaining thing is to prove (4.7). The definition of $\mathcal{M}_k$ shows

$$(R_k^t)^{-1}\mathcal{M}_k \subset \left(-\frac{1}{2},\ \frac{1}{2} \right]^2 \tag{4.8}$$

for all $k \geq 0$ and so, using (4.3), the diameter of $(R_0^t R_1^t \cdots R_k^t)^{-1}\mathcal{N}_0 + \cdots + (R_{k-1}^t R_k^t)^{-1}\mathcal{N}_{k-1}$ is less than

$$\frac{\alpha^2}{\sqrt{2}}(r^K + r^{2K} + \cdots + r^{kK}) < \frac{\alpha^2}{\sqrt{2}} \frac{r^K}{1 - r^K}.$$

Therefore, there exists $K \geq M$ (M is given by Lemma 4.2) depending only on η and r such that the set

$$(R_0^t R_1^t \cdots R_k^t)^{-1}\mathcal{N}_0 + \cdots + (R_{k-1}^t R_k^t)^{-1}\mathcal{N}_{k-1} + (R_k^t)^{-1}\mathcal{N}_k$$

is a subset of $[-\frac{1}{2}-\frac{1}{4}\eta,\frac{1}{2}+\frac{1}{4}\eta]^2$ for $K\geq M$. The lemma is proven. $\square$

For the $\mathcal{N}_k$ defined by Lemma 4.5, let

$$\Lambda_n=\mathcal{N}_0+R_0^t\mathcal{N}_1+\cdots+R_0^t\cdots R_{n-1}^t\mathcal{N}_n,\quad \Lambda=\bigcup_{n=1}^{+\infty}\Lambda_n. \tag{4.9}$$

We will prove that the above Λ is a spectrum of μ. Clearly, E_Λ is an orthogonal family of $L^2(\mu)$ by Lemma 2.5 and Lemma 4.5. To prove the complete property of E_Λ, we write

$$Q_n(\xi)=\sum_{\lambda\in\Lambda_n}|\widehat{\mu}(\lambda+\xi)|^2,\quad \xi\in\mathbb{R}^2.$$

Note that $\Lambda_n\subset\Lambda_{n+1}$ for $n\geq 1$. One has that

$$\lim_{n\to\infty}Q_n(\xi)=\sum_{\lambda\in\Lambda}|\widehat{\mu}(\lambda+\xi)|^2=Q_\Lambda(\xi).$$

The complete property is equivalent to that $Q_\Lambda(\xi)\equiv 1$ for $\xi\in\mathbb{R}^2$ by Theorem 2.1. To show that $Q_\Lambda(\xi)\equiv 1$ for $\xi\in\mathbb{R}^2$, we remark that Lemma 4.5 says that

$$(R_0^tR_1^t\cdots R_n^t)^{-1}\Lambda_n\subset\left[-\frac{1}{2}-\frac{1}{4}\eta,\frac{1}{2}+\frac{1}{4}\eta\right]^2. \tag{4.10}$$

It is easy to show that $(\mu_{\{R_i,\mathcal{D}_i,n\}},\Lambda_n)$ is a spectral pair. Then by Theorem 2.1 we have

$$\sum_{\lambda\in\Lambda_n}|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n\}}(\xi+\lambda)|^2\equiv 1,\quad \forall\xi\in\mathbb{R}^2. \tag{4.11}$$

We will use the two representations of $\widehat{\mu}_{\{R_i,\mathcal{D}_i,n\}}$:

$$\widehat{\mu}_{\{R_i,\mathcal{D}_i,n\}}(\xi)=\prod_{j=0}^{n}\left|m_{\mathcal{D}_j}((R_0^{-1}R_1^{-1}\cdots R_j^{-1})^t(\xi))\right|^2=\prod_{j=1}^{(n+1)K}\left|m(B_j^{-1}B_{j-1}^{-1}\cdots B_1^{-1}\xi)\right|^2. \tag{4.12}$$

for $n\geq 0$ in the following argument.

Lemma 4.6 *Consider the* $\{A_n\}_{n=1}^{\infty}$ *and* $\mathcal{B}_{\eta,\delta}$ *given in Theorem* 4.1*, the* K *given in Lemma* 4.5 *satisfying* (4.7)*. For any* $\xi\in\mathbb{R}^2$*, there is an integer* N_ξ $(\geq K)$*, which depends only on* ξ *and* η*, so that*

$$|\widehat{\mu}(\lambda+\xi)|\geq\beta_{N_1}\gamma^{N_1}\left|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n\}}(\lambda+\xi)\right|$$

uniformly for $N_1\geq K$*,* $n\geq N_\xi$ *and* $\lambda\in\Lambda_n$*, where* β_{N_1} *and* γ *are given in Lemma* 4.2 *and Lemma* 4.3 *respectively.*

Proof Since

$$\widehat{\mu}_{\{R_i,\mathcal{D}_i,i\geq 0\}}(\lambda+\xi)=\widehat{\mu}_{\{R_i,\mathcal{D}_i,n\}}(\lambda+\xi)\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n\}}[(R_0^{-1}R_1^{-1}\cdots R_n^{-1})^t(\lambda+\xi)],$$

we need to show that there exists N_ξ such that

$$\left|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n\}}[(R_0^{-1}R_1^{-1}\cdots R_n^{-1})^t(\lambda+\xi)]\right|^2\geq\beta_{N_1}\gamma^{N_1},\quad \text{for } \lambda\in\Lambda_n,\ n\geq N_\xi \text{ and } N_1\geq K.$$

Indeed, according to (4.10) and the fact that $R_0^{-1}R_1^{-1}\cdots R_n^{-1}$ tends to the zero matrix as n tends to infinity, there exists $N_\xi\geq K$ such that

$$(R_0^{-1}R_1^{-1}\cdots R_n^{-1})^t(\lambda+\xi)\subset\left[-\frac{1}{2}-\eta,\frac{1}{2}+\eta\right]^2,\quad \text{for } n\geq N_\xi. \tag{4.13}$$

Note that

$$\left|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n\}}[(R_0^{-1}R_1^{-1}\cdots R_n^{-1})^t(\lambda+\xi)]\right| = \left|\widehat{\mu}_{\{A_k,k>K(n+1)\}}[(R_0^{-1}R_1^{-1}\cdots R_n^{-1})^t(\lambda+\xi)]\right|$$

and $K(n+1)\geq M$ for $n\geq 0$. Then by (4.13) and Corollary 4.4, one has that, for $N_1\geq K$,

$$\left|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n\}}[(R_0^{-1}R_1^{-1}\cdots R_n^{-1})^t(\lambda+\xi)]\right| \geq \beta_{N_1}\gamma^{N_1}.$$

Hence, the assertion follows. □

To prove Theorem 4.1, according to the above preparations and Lemma 2.6, it is sufficient to prove the following theorem:

Theorem 4.7 *Let $\{A_n\}_{n=1}^\infty$ and $\mathcal{B}_{\eta,\delta}$ be given in Theorem 4.1. Suppose further $A_1=\mathrm{diag}(3,3)$. Then $\mu_{\{A_n,n\geq 1\}}$ is a spectral measure with a spectrum Λ given by (4.9).*

Proof Assume that the Λ defined by (4.9) is not a spectrum of μ. Then there is a number θ and a point $\xi\in\mathbb{R}^2$ such that $Q_\Lambda(\xi)<\theta<1$. We will fix this ξ in the following. We choose a sequence $\{n_k\}_{k=1}^\infty$ of increasing integers such that $n_1=1$ and n_k satisfying that $\beta_{n_2}\geq\frac{Q_\Lambda(\xi)}{\theta}$, $n_{k+1}-n_k\geq n_2\geq M$ (M is given in Lemma 4.2) for $k\geq 2$ and

$$(R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda)\subset\left[-\frac{1}{2}-\eta,\frac{1}{2}+\eta\right]^2,\quad\text{for } k\geq 2 \text{ and } \lambda\in\Lambda_{n_k}.$$

Then, by Lemma 4.2, we have

$$|\widehat{\mu}_{\{A_n,n>q+n_{k+1}-n_k\}}(B_{q+n_{k+1}-n_k}^{-1}\cdots B_{q+1}^{-1}\xi)|\geq\beta_{n_{k+1}-n_k}\geq\beta_{n_2}\geq\frac{Q_\Lambda(\xi)}{\theta}$$

uniformly for $q\geq 0$, $k\geq 2$. Consequently

$$|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>q+n_{k+1}-n_k\}}((R_{q+1}^{-1}R_{q+2}^{-1}\cdots R_{q+n_{k+1}-n_k}^{-1})^t\xi)|\geq\frac{Q_\Lambda(\xi)}{\theta}$$

uniformly for $k\geq 2$, $q\geq 0$. Hence, by choosing $q=n_k$, we have

$$\begin{aligned}
&|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n_{k+1}\}}((R_0^{-1}R_1^{-1}\cdots R_{n_{k+1}}^{-1})^t(\xi+\lambda))|\\
&\quad=|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n_{k+1}\}}((R_{n_k+1}^{-1}R_{n_k+2}^{-1}\cdots R_{n_{k+1}}^{-1})^t(R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda))|\\
&\quad\geq\frac{Q_\Lambda(\xi)}{\theta},\quad\forall\lambda\in\Lambda_{n_k},\ k\geq 2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_k<i\leq n_{k+1}\}}((R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda))|\\
&\quad=\frac{|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n_k\}}((R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda))|}{|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n_{k+1}\}}((R_0^{-1}R_1^{-1}\cdots R_{n_{k+1}}^{-1})^t(\xi+\lambda))|}\\
&\quad\leq\frac{\theta}{Q_\Lambda(\xi)}|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n_k\}}((R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda))|
\end{aligned}$$

for $\lambda\in\Lambda_{n_k}$ and $k\geq 2$. Hence it is clear

$$\begin{aligned}
|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_{k+1}\}}(\xi+\lambda)|&=|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_k\}}(\xi+\lambda)||\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_k<i\leq n_{k+1}\}}((R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda))|\\
&\leq\frac{\theta}{Q_\Lambda(\xi)}|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_k\}}(\xi+\lambda)||\widehat{\mu}_{\{R_i,\mathcal{D}_i,i>n_k\}}((R_0^{-1}R_1^{-1}\cdots R_{n_k}^{-1})^t(\xi+\lambda))|\\
&=\frac{\theta}{Q_\Lambda(\xi)}|\widehat{\mu}_{\{R_i,\mathcal{D}_i,i\geq 0\}}(\xi+\lambda)|
\end{aligned}$$

$$= \frac{\theta}{Q_\Lambda(\xi)} |\widehat{\mu}(\xi+\lambda)|, \quad \forall \lambda \in \Lambda_{n_k},\ k \geq 2. \tag{4.14}$$

According to the definition of $Q_n(\xi)$, we have

$$Q_{n_{k+1}}(\xi) - Q_{n_k}(\xi) = \sum_{\lambda \in \Lambda_{n_{k+1}} \setminus \Lambda_{n_k}} |\widehat{\mu}(\lambda+\xi)|^2,$$

Lemma 4.6 shows that there exists an integer N_ξ and a positive number $\beta := \beta_{N_1}\gamma^{N_1}$ such that

$$|\widehat{\mu}(\lambda+\xi)|^2 \geq \beta \left|\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_{k+1}\}}(\xi+\lambda)\right|^2, \quad \forall \lambda \in \Lambda_{n_{k+1}}, k \geq N_\xi.$$

Then by (4.11) and (4.14), we get

$$\begin{aligned} Q_{n_{k+1}}(\xi) - Q_{n_k}(\xi) &\geq \beta \sum_{\lambda \in \Lambda_{n_{k+1}} \setminus\ \Lambda_{n_k}} |\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_{k+1}\}}(\xi+\lambda)|^2 \\ &= \beta\Bigg(\sum_{\lambda \in \Lambda_{n_{k+1}}} |\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_{k+1}\}}(\xi+\lambda)|^2 - \sum_{\lambda \in \Lambda_{n_k}} |\widehat{\mu}_{\{R_i,\mathcal{D}_i,n_{k+1}\}}(\xi+\lambda)|^2 \Bigg) \\ &\geq \beta\Bigg(1 - \frac{\theta}{Q_\Lambda(\xi)} \sum_{\lambda \in \Lambda_{n_k}} |\widehat{\mu}(\xi+\lambda)|^2\Bigg) \\ &\geq \beta(1-\theta), \quad k \geq N_\xi. \end{aligned}$$

Hence

$$1 > Q_\Lambda(\xi) = \lim_{k\to\infty} Q_{n_{k+1}}(\xi) \geq \sum_{k=N_\xi}^{\infty} [Q_{n_{k+1}}(\xi) - Q_{n_k}(\xi)] \geq \sum_{k=N_\xi}^{\infty} [\beta\,(1-\theta)] = +\infty,$$

a contradiction. This contradiction yields the theorem. □

Proof of Corollary 1.4 By the assumption we have $\|B_n^{-1}\| = \|A_n^{-1}\| \leq r < \frac{2}{3}$ for $n \geq 1$. It is sufficient to show that there exist two positive numbers η and δ such that

$$B_k B_{k+1} \cdots B_{k+N} \in \mathcal{B}_{\eta,\delta}, \quad \text{for } k \geq 1,\ N \geq 0. \tag{4.15}$$

Since

$$\left\| B_k^{-1}\left[-\frac{1}{2} - \eta, \frac{1}{2} + \eta\right]^2 \right\| \leq r\sqrt{2}\left(\frac{1}{2}+\eta\right)$$

for $k \geq 1$, by $r < \frac{2}{3}$, we can find two positive numbers η and δ so that $\eta \in (0, \frac{1}{3r} - \frac{1}{2})$ and $\delta + r\sqrt{2}(\frac{1}{2}+\eta) < \frac{\sqrt{2}}{3}$. Then

$$\left\| B_k^{-1}\left[-\frac{1}{2} - \eta, \frac{1}{2} + \eta\right]^2 \right\| \leq r\sqrt{2}\left(\frac{1}{2}+\eta\right) < \frac{\sqrt{2}}{3},$$

and we have $B_k \in \mathcal{B}_{\eta,\delta}$ for $k \geq 1$. By noting that $\|(B_k B_{k+1} \cdots B_{k+N})^{-1}\| \leq \|B_k^{-1}\|$ for $N \geq 0$, then $B_k B_{k+1} \cdots B_{k+N} \in \mathcal{B}_{\eta,\delta}$ for $k \geq 1$ and $N \geq 0$. Hence (4.15) holds and the assertion follows. □

Proof of Corollary 1.5 According to the assumption $\sup_{n\geq 1} \|A_n\|' < \infty$, one see that the family $\{A_n : n \geq 1\}$ is a finite set by using the fact that A_n are integer matrices. Write $\{A_n : n \geq 1\} = \{M_1, M_2, \ldots, M_s\}$. Then $\|M_i^{-1}\|' \leq r < 1$ for $1 \leq i \leq s$ by the hypotheses. By Theorem 1.3 we need to show that $\mu_{\{A_n, n\geq 1\}}$ is a spectral measure if $\frac{1}{3}(1,-1)M_i \in \mathbb{Z}^2$ for

$1 \le i \le s$. Then it is sufficient to show that there exist two positive numbers η and δ such that (4.15) holds. According to (4.3), we have

$$\|(B_k B_{k+1} \cdots B_{k+p})^{-1}[-1,1]^2\|' \le \sqrt{2}\alpha^2 r^{p+1}, \quad \text{for } k \ge 1 \text{ and } p \ge 0.$$

Hence (4.2) shows that there exists an integer $N > 0$ such that

$$(B_k B_{k+1} \cdots B_{k+p})^{-1}\left[-\frac{3}{4}, \frac{3}{4}\right]^2 \subset \left[-\frac{1}{8}, \frac{1}{8}\right]^2, \quad \text{for } k \ge 1 \text{ and } p > N. \tag{4.16}$$

Set

$$\mathcal{A} = \{M_{i_1} M_{i_2} \cdots M_{i_j} : i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, s\} \text{ and } 1 \le j \le N\}.$$

Consider

$$(B_k B_{k+1} \cdots B_{k+p})^{-1}\left[-\frac{3}{4}, \frac{3}{4}\right]^2 \cap \left(\pm\frac{1}{3}e + \mathbb{Z}^2\right), \quad \text{for } k \ge 1 \text{ and } 0 \le p < N. \tag{4.17}$$

If one of the above intersections is a nonempty set, that is, there exist $\xi \in [-\frac{3}{4}, \frac{3}{4}]^2 \setminus \{0\}$ and $v \in \mathbb{Z}^2$ such that $(B_k B_{k+1} \cdots B_{k+p})^{-1}\xi = \pm\frac{1}{3}e + v$, which implies $\xi = B_k B_{k+1} \cdots B_{k+p}(\pm\frac{1}{3}e + v) \subseteq \mathbb{Z}^2$, a contradiction. Note that $\mathcal{A}$ is finite and $B_k B_{k+1} \cdots B_{k+p} \in \mathcal{A}$ for $k \ge 1$ and $0 \le p < N$. Then there exists $\delta > 0$ such that

$$(B_k B_{k+1} \cdots B_{k+p})^{-1}\left[-\frac{3}{4}, \frac{3}{4}\right]^2 \cap \left(\pm\frac{1}{3}e + \mathbb{Z}^2\right)_\delta = \emptyset, \quad \text{for } k \ge 1 \text{ and } 0 \le p < N.$$

Combining (4.16) and the above relationship shows that (4.15) holds for $\eta = \frac{1}{4}$ and the $\delta' = \min\{\delta, \frac{1}{12}\}$, and the assertion follows. □

Proof of Corollary 1.6 Note that $\frac{1}{3}(1,-1)\begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix} \in \mathbb{Z}^2$ if and only if $3 \mid p_n$ and $3 \mid q_n$ for $n \ge 1$. Then the assertion follows by Corollary 1.4. □

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