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Trudinger–Moser Inequalities on a Closed Riemann Surface with a Symmetric Conical Metric

Yu FANG
College of Teacher Education, Quzhou University, Quzhou 324003*, P. R. China*
E-mail: *fangyu@qzc.edu.cn*

Yun Yan YANG1)
Department of Mathematics, Renmin University of China, Beijing 100872*, P. R. China*
E-mail: *yunyanyang@ruc.edu.cn*

Abstract This is a continuation of our previous work (*Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **20**, 1295–1324, 2020). Let (Σ, g) be a closed Riemann surface, where the metric g has conical singularities at finite points. Suppose $\mathbf{G}$ is a group whose elements are isometries acting on (Σ, g). Trudinger–Moser inequalities involving $\mathbf{G}$ are established via the method of blow-up analysis, and the corresponding extremals are also obtained. This extends previous results of Chen (*Proc. Amer. Math. Soc.*, 1990), Iula–Manicini (*Nonlinear Anal.*, 2017), and the authors (2020).

Keywords Trudinger–Moser inequality, blow-up analysis, conical singularity

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1 Introduction and Main Results

Let $\mathbb{S}^2$ be the 2-dimensional sphere $x_1^2+x_2^2+x_3^2=1$ endowed with the metric $g_1=dx_1^2+dx_2^2+dx_3^2$ for all $x=(x_1,x_2,x_3)\in\mathbb{R}^3$. It was proved by Moser [18] that there exists a universal constant C satisfying

$$\int_{\mathbb{S}^2} \mathrm{e}^{4\pi u^2} dv_{g_1} \leq C \tag{1.1}$$

for all smooth functions u with $\int_{\mathbb{S}^2} |\nabla_{g_1} u|^2 dv_{g_1} \leq 1$ and $\int_{\mathbb{S}^2} u dv_{g_1} = 0$, where ∇_{g_1} and dv_{g_1} stand for the gradient operator and the volume element on $(\mathbb{S}^2, g_1)$ respectively. Here 4π is the best constant in the sense that when 4π is replaced by any $\alpha > 4\pi$, the integrals are still finite, but the universal constant C no longer exists. It was also remarked by Moser [19] that if one considers even functions u, say $u(x)=u(-x)$ for all $x\in\mathbb{S}^2$, then the constant 4π in (1.1) would double. Namely there exists an absolute constant C such that

$$\int_{\mathbb{S}^2} \mathrm{e}^{8\pi u^2} dv_{g_1} \leq C \tag{1.2}$$

for all even functions u satisfying $\int_{\mathbb{S}^2} |\nabla_{g_1} u|^2 dv_{g_1} \leq 1$, $\int_{\mathbb{S}^2} u dv_{g_1} = 0$.

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1) Corresponding author

A general manifold version of (1.1) was established by Fontana [14] via the estimation on Green functions and by O'Neil's lemma [20]. This comes from a Euclidean scheme designed by Adams [1]. However, Li [15] was able to prove the inequality (1.1) by the method of blow-up analysis. In a recent work [13], we extended (1.2) to the case of a closed Riemann surface with a smooth "symmetric" metric. In the current paper, we consider the case of closed Riemann surface with a "symmetric" singular metric.

Now we recall some notation from differential geometry. Let (Σ, g_0) be a closed Riemann surface, and $d_{g_0}(\cdot,\cdot)$ be the geodesic distance between two points of Σ. A smooth metric g defined on $\Sigma\setminus\{p_1,\dots,p_L\}$ is said to have conical singularity of order $\beta_i > -1$ at p_i, $i = 1,\dots,L$, if

$$g = \rho g_0, \tag{1.3}$$

where $\rho \in C^\infty(\Sigma \setminus \{p_1,\dots,p_L\}, g_0)$ satisfies $\rho > 0$ on $\Sigma \setminus \{p_1,\dots,p_L\}$ and

$$0 < C \leq \frac{\rho(x)}{d_{g_0}(x,p_i)^{2\beta_i}} \in C^0(\Sigma, g_0) \tag{1.4}$$

for some constant C and $i = 1,\dots,L$. Here we write the righthand side of (1.4) in the sense that $\rho/d_{g_0}(x,p_i)^{2\beta_i}$ can be continuously extended to the whole surface (Σ, g_0). With (1.3) and (1.4), (Σ, g) is called a closed Riemann surface having conical singularities of the divisor $\mathbf{b} = \sum_{i=1}^L \beta_i p_i$. For more details on singular surfaces, we refer the reader to Troyanov [23]. For compact singular surface (Σ,g) with conical singularities $\{p_1,\dots,p_{i_0}\}$ each of order β_i-th order, $(i = 1,\dots,i_0)$. Still let ∇_g and Δ_g be its gradient operator and Laplace–Beltrami operator respectively, dv_g be its volume element. On a closed Riemann surface (Σ, g) with singular metric g as above, Stefano–Gabriele [21, Theorem 1.3] have proved that $\forall p > 1$,

$$\sup_{\int_\Sigma u dv_g = 0, \int_\Sigma |\nabla u|^2 dv_g \leq 1} \int_\Sigma \mathrm{e}^{4\pi\beta u^2(1+\alpha\|u\|^2_{L^q(\Sigma,g)})} dx < \infty \tag{1.5}$$

can be obtained, if $\beta < (1 + \min_i \beta_i)$ while

$$\alpha < \lambda_{1,p}(\Sigma) = \inf_{\int_\Sigma u dv_g = 0, \int_\Sigma |\nabla u|^2 dv_g \leq \infty,\, \int_\Sigma u^p dv_g = 1} \int_\Sigma |\nabla_g u|^2 \mathrm{d}v_g,$$

or if $\beta = (1 + \min_i \beta_i)$ while α small significantly. For earlier works on Trudinger–Moser inequalities involving singular metrics, we refer the reader to Troyanov [23], Chen [7], Adimurthi–Sandeep [3], Adimurthi–Yang [5], Li–Yang [17], Csato–Roy [10], Yang–Zhu [26] and the references therein.

It's also necessary to introduce finite isometric group to describe symmetric metric as in [7] and [13]. We say that $\mathbf{G} = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$ is a finite isometric group acting on (Σ, g), if each smooth map $\sigma_k : \Sigma \to \Sigma$ satisfies

$$(\sigma_k^* g_0)_x = g_{0_{\sigma_k(x)}} \quad \text{and} \quad \rho(\sigma_k(x)) = \rho(x) \quad \text{for all } x \in \Sigma. \tag{1.6}$$

This in particular implies

$$\sigma^* g_x = g_{\sigma(x)} \quad \text{for all } x \in \Sigma. \tag{1.7}$$

Note that $\mathbf{G}$ is a geometric structure on special Riemann surface (Σ, g). It is clear that $\mathbf{G}(p_j) = \{\sigma_i(p_j)\}_{i=1}^N \subset \{p_1,\dots,p_L\}$ for all j, and that $\beta_k = \beta_j$ provided that $p_k \in \mathbf{G}(p_j)$ for some j.

Denote for any $x \in \Sigma$,

$$I(x) = \sharp\mathbf{G}(x) \tag{1.8}$$

and

$$\beta(x) = \begin{cases} 0, & x \notin \{p_1, \ldots, p_L\}, \\ \beta_j, & x = p_j, 1 \le j \le L, \end{cases} \tag{1.9}$$

where $\sharp\mathbf{A}$ is the number of all distinct elements in the set $\mathbf{A}$. Noting that $1 \le I(x) \le N$ and $\beta(x) > -1$ for all $x \in \Sigma$, one defines

$$\ell = \min_{x\in\Sigma} \min\left\{I(x), I(x)(1+\beta(x))\right\}. \tag{1.10}$$

Let $W^{1,2}(\Sigma, g)$ be the completion of $C^\infty(\Sigma, g_0)$ under the norm

$$\|u\|_{W^{1,2}(\Sigma,g)} = \left(\int_\Sigma \left(|\nabla_g u|^2 + u^2\right) dv_g\right)^{1/2}. \tag{1.11}$$

For convenience, we introduce the following subspace of $W^{1,2}(\Sigma, g)$

$$\mathscr{H}_{\mathbf{G}} = \left\{u \in W^{1,2}(\Sigma, g) : \int_\Sigma u dv_g = 0,\ u(x) = u(\sigma(x)) \text{ for a.e. } x \in \Sigma \text{ and all } \sigma \in \mathbf{G}\right\}. \tag{1.12}$$

Clearly, $\mathscr{H}_{\mathbf{G}}$ is a Hilbert space with inner product

$$\langle u, v\rangle_{\mathscr{H}_{\mathbf{G}}} = \int_\Sigma \langle\nabla_g u, \nabla_g v\rangle dv_g.$$

The first eigenvalue of Δ_g on $\mathscr{H}_{\mathbf{G}}$ is defined by

$$\lambda_1^{\mathbf{G}} = \inf_{u\in\mathscr{H}_{\mathbf{G}},\, \int_\Sigma u^2 dv_g = 1} \int_\Sigma |\nabla_g u|^2 \mathrm{d}v_g, \tag{1.13}$$

where Δ_g is the Laplace–Beltrami operator with respect to the conical metric g. A direct method of variation leads to $\lambda_1^{\mathbf{G}} > 0$. For any α strictly less than $\lambda_1^{\mathbf{G}}$, we can define an equivalent norm of (1.11) on $\mathscr{H}_{\mathbf{G}}$ by

$$\|u\|_{1,\alpha} = \left(\int_\Sigma |\nabla_g u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g\right)^{1/2}. \tag{1.14}$$

The first eigenfunction space with respect to $\lambda_1^{\mathbf{G}}$ reads as

$$E_{\lambda_1^{\mathbf{G}}} = \left\{u \in \mathscr{H}_{\mathbf{G}} : \Delta_g u = \lambda_1^{\mathbf{G}} u\right\}. \tag{1.15}$$

According to Chen [7], there holds

$$\sup_{u\in\mathscr{H}_{\mathbf{G}}, \int_\Sigma |\nabla_g u|^2 dv_g \le 1} \int_\Sigma \mathrm{e}^{4\pi\ell u^2} dv_g < \infty, \tag{1.16}$$

where ℓ is given as in (1.10), and $4\pi\ell$ is the best constant for (1.16) in the sense that if $4\pi\ell$ is replaced by any $\gamma > 4\pi\ell$, then the supremum in (1.16) is infinity. Our main concern is the attainability of the above supremum. We have the following more general result:

Theorem 1.1 *Let (Σ, g) be a closed Riemann surface with conical singularities of the divisor $\mathbf{b} = \sum_{i=1}^L \beta_i p_i$, where p_i belongs to Σ and*

$$-1 < \beta_i \le 0, \quad i = 1, \ldots, L. \tag{1.17}$$

Suppose that $\mathbf{G}=\{\sigma_1,\sigma_2,\ldots,\sigma_N\}$ *is a group of isometries given in* (1.6)*, and that* ℓ, $\mathscr{H}_{\mathbf{G}}$ *and* $\lambda_1^{\mathbf{G}}$ *are defined as in* (1.10)*,* (1.12) *and* (1.13) *respectively. Then for any* $\alpha<\lambda_1^{\mathbf{G}}$*, the supremum*

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\le 1}\int_\Sigma \mathrm{e}^{4\pi\ell u^2}dv_g \tag{1.18}$$

is attained by some function $u_0\in C^1(\Sigma\setminus\{p_1,\ldots,p_L\},g_0)\cap C^0(\Sigma,g_0)\cap\mathscr{H}_{\mathbf{G}}$ *satisfying* $\|u_0\|_{1,\alpha}=1$*, where* g_0 *is a smooth metric given as in* (1.3) *and* $\|\cdot\|_{1,\alpha}$ *is defined as in* (1.14)*.*

When $N=1$, Theorem 1.1 reduces to one of results of Iula–Mancini [21]. While if $\beta(x)\equiv 0$ for all $x\in\Sigma$, then Theorem 1.1 is exactly our earlier result [13]. To prove Theorem 1.1, we use the method of blow-up analysis designed by Li [15]. Early groundbreaking works go back to Carleson–Chang [6], Ding–Jost–Li–Wang [12] and Adimurthi–Struwe [4].

As in our previous work [13, Theorem 2], we may also consider the effect of higher order eigenvalues of Δ_g on Trudinger–Moser inequalities. Set $E_0=\{0\}$, $E_0^\perp=\mathscr{H}_{\mathbf{G}}$, and $E_1=E_{\lambda_1^{\mathbf{G}}}$ defined as in (1.15). By induction, E_j and $E_j^\perp$ can be defined for any positive integer j. To be precise, for any $j\ge 1$, we set $E_j=E_{\lambda_1^{\mathbf{G}}}\oplus\cdots\oplus E_{\lambda_j^{\mathbf{G}}}$ and

$$E_j^\perp=\left\{u\in\mathscr{H}_{\mathbf{G}}:\int_\Sigma uv dv_g=0,\forall v\in E_j\right\}, \tag{1.19}$$

where $\lambda_j^{\mathbf{G}}$ is the j-th eigenvalue of Δ_g given by

$$\lambda_j^{\mathbf{G}}=\inf_{u\in E_{j-1}^\perp,\,\int_\Sigma u^2dv_g=1}\int_\Sigma|\nabla_g u|^2\mathrm{d}v_g, \tag{1.20}$$

and $E_{\lambda_j^{\mathbf{G}}}=\{u\in E_{j-1}^\perp:\Delta_g u=\lambda_j^{\mathbf{G}}u\}$ is the corresponding j-th eigenfunction space. Obviously for any fixed $\alpha<\lambda_{j+1}^{\mathbf{G}}$, $\|\cdot\|_{1,\alpha}$ is equivalent to $\|\cdot\|_{W^{1,2}(\Sigma,g)}$ on the space $E_j^\perp$.

Our second result reads as follows:

Theorem 1.2 *Let* (Σ,g) *be a closed Riemann surface with conical singularities of divisor* $\mathbf{b}=\sum_{i=1}^L\beta_i p_i$*, where* p_i *belongs to* Σ *and* $-1<\beta_i\le 0$ *for* $i=1,\ldots,L$*. Suppose that* $\mathbf{G}=\{\sigma_1,\sigma_2,\ldots,\sigma_N\}$ *is a group of isometries given as in* (1.6)*. Then for any integer* $j\ge 1$ *and any real number* α *satisfying* $\alpha<\lambda_{j+1}^{\mathbf{G}}$*, the supremum*

$$\sup_{u\in E_j^\perp,\,\|u\|_{1,\alpha}\le 1}\int_\Sigma \mathrm{e}^{4\pi\ell u^2}dv_g \tag{1.21}$$

can be attained by some function $u_0\in C^1(\Sigma\setminus\{p_1,\ldots,p_L\},g_0)\cap C^0(\Sigma,g_0)\cap E_j^\perp$ *with* $\|u_0\|_{1,\alpha}=1$*, where* $\lambda_{j+1}^{\mathbf{G}}$*,* $E_j^\perp$*,* ℓ *and* $\|\cdot\|_{1,\alpha}$ *are defined as in* (1.13)*,* (1.19)*,* (1.10)*, and* (1.14) *respectively, and* g_0 *is a smooth metric given as in* (1.3)*.*

The proof of Theorem 1.2 is similar to that of Theorem 1.1. The difference is that we work on the space $E_j^\perp$ instead of $\mathscr{H}_{\mathbf{G}}$. Note that $E_j^\perp$ is still a Hilbert space for any $j\ge 1$. For more details of Trudinger–Moser inequalities involving eigenvalues, we refer the reader to [2, 22, 25]. In both proofs of Theorems 1.1 and 1.2, to derive an upper bound of the Trudinger–Moser functional, we need a singular version of Carleson–Chang's estimate, which was in literature due to Csato–Roy [10] (see also Iula–Mancini [21] and Li–Yang [17]), namely

Lemma 1.3 *Let* $\mathbb{B}_r\subset\mathbb{R}^2$ *be a ball centered at* 0 *with radius* r*. If* $\phi_\epsilon\in W_0^{1,2}(\mathbb{B}_r)$ *satisfies*

$\int_{\mathbb{B}_r}|\nabla\phi_\epsilon|^2dx\leq 1$, *and* $\phi_\epsilon\rightharpoonup 0$ *weakly in* $W_0^{1,2}(\mathbb{B}_r)$, *then for any* β *with* $-1<\beta\leq 0$, *there holds*

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}_r}\mathrm{e}^{(1+\beta)4\pi\phi_\epsilon^2}|x|^{2\beta}dx\leq\int_{\mathbb{B}_r}|x|^{2\beta}dx+\frac{\pi e}{1+\beta}r^{2+2\beta}. \tag{1.22}$$

The proof of Lemma 1.3 is based on a rearrangement argument, Hardy–Littlewood inequality, and Carleson–Chang's estimate [6]. In the remaining part of this paper, we prove Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Throughout this paper, we do not distinguish sequence and subsequence. Constants are often denoted by the same C from line to line, even on the same line.

2 Trudinger–Moser Inequalities Involving the First Eigenvalue

In this section we shall prove Theorem 1.1 by using the method of blow-up analysis, which was originally used in this topic by Li [15, 16], and extensively used by Yang [24, 25], Li–Yang [17], de Souza–do Ó [11], Yang–Zhu [26], Iula–Mancini [21] and others. The proof is divided into several subsections below.

2.1 The Best Constant

Let ℓ be defined as in (1.10). It was proved by Chen [7] that

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\int_\Sigma|\nabla u|^2dv_g\leq 1}\int_\Sigma\mathrm{e}^{\gamma u^2}dv_g<\infty,\quad\forall\gamma\leq 4\pi\ell; \tag{2.1}$$

moreover, the above integrals are still finite for any $\gamma>4\pi\ell$, but the supremum

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\int_\Sigma|\nabla u|^2dv_g\leq 1}\int_\Sigma\mathrm{e}^{\gamma u^2}dv_g=\infty,\quad\forall\gamma>4\pi\ell. \tag{2.2}$$

We now take the first eigenvalue $\lambda_1^{\mathbf{G}}$ of Δ_g (see (1.13) above) into account and have the following:

Lemma 2.1 *For any* $\alpha<\lambda_1^{\mathbf{G}}$, *there exists a real number* $\gamma_0>0$ *such that*

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\leq 1}\int_\Sigma\mathrm{e}^{\gamma_0u^2}dv_g<\infty,$$

where $\|\cdot\|_{1,\alpha}$ *is defined as in* (1.14).

Proof Assume $\alpha<\lambda_1^{\mathbf{G}}$ and $\|u\|_{1,\alpha}\leq 1$. Then

$$\Big(1-\frac{\alpha}{\lambda_1^{\mathbf{G}}}\Big)\int_\Sigma|\nabla_gu|^2dv_g\leq\int_\Sigma|\nabla_gu|^2dv_g-\alpha\int_\Sigma u^2dv_g\leq 1.$$

This together with (2.1) implies the existence of γ_0, as desired. □

In view of Lemma 2.1, for any fixed $\alpha<\lambda_1^{\mathbf{G}}$, we set

$$\gamma^*=\sup\Big\{\gamma_0:\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\leq 1}\int_\Sigma\mathrm{e}^{\gamma_0u^2}dv_g<\infty\Big\}.$$

Lemma 2.2 *There holds* $\gamma^*\geq 4\pi\ell$.

Proof Suppose $\gamma^*<4\pi\ell$. Then there exists a real number γ_1 with $\gamma^*<\gamma_1<4\pi\ell$ and a function sequence $(u_j)\subset\mathscr{H}_{\mathbf{G}}$ such that $\|u_j\|_{1,\alpha}\leq 1$ and

$$\int_\Sigma\mathrm{e}^{\gamma_1u_j^2}dv_g\to\infty\quad\text{as } j\to\infty. \tag{2.3}$$

Since $\alpha < \lambda_1^{\mathbf{G}}$, we have that (u_j) is bounded in $W^{1,2}(\Sigma, g)$. Thus, u_j converges to some u_0 weakly in $W^{1,2}(\Sigma, g)$, strongly in $L^2(\Sigma, g)$ and almost everywhere in Σ. This particularly leads to

$$\|u_j - u_0\|_{1,\alpha}^2 = \|u_j\|_{1,\alpha}^2 - \|u_0\|_{1,\alpha}^2 + o_j(1).$$

Clearly $u_0 \in \mathscr{H}_{\mathbf{G}}$. We now claim that $u_0 \equiv 0$. For otherwise, since $\|u_j\|_{1,\alpha} \leq 1$, there must hold

$$\int_\Sigma |\nabla_g (u_j - u_0)|^2 dv_g \leq 1 - \frac{1}{2}\|u_0\|_{1,\alpha}^2 \tag{2.4}$$

for sufficiently large j. Noting that $u_j^2 \leq (1+\nu)(u_j - u_0)^2 + (1+\nu^{-1})u_0^2$ for any $\nu > 0$, and that $\mathrm{e}^{u_0^2} \in L^q(\Sigma, g)$ for all $q > 1$, we conclude from (2.1) and (2.4),

$$\int_\Sigma \mathrm{e}^{\gamma_1 u_j^2} dv_g \leq C \tag{2.5}$$

for some constant C depending only on γ_1, ℓ and u_0. This contradicts (2.3) and confirms our claim $u_0 \equiv 0$. As a consequence

$$\int_\Sigma |\nabla_g u_j|^2 dv_g \leq 1 + \alpha \int_\Sigma u_j^2 dv_g = 1 + o_j(1).$$

This together with (2.1) gives (2.5), which again contradicts (2.3) and thus completes the proof of the lemma. □

More precisely we have

Lemma 2.3 *There holds $\gamma^* = 4\pi\ell$.*

Proof By Lemma 2.2, $\gamma^* \geq 4\pi\ell$. Suppose $\gamma^* > 4\pi\ell$. Fix some γ_2 with $4\pi\ell < \gamma_2 < \gamma^*$. In view of (2.2), there exists a sequence of functions $(M_k) \subset \mathscr{H}_{\mathbf{G}}$ such that

$$\int_\Sigma |\nabla_g M_k|^2 dv_g \leq 1 \tag{2.6}$$

and

$$\int_\Sigma \mathrm{e}^{\gamma_2 M_k^2} dv_g \to \infty. \tag{2.7}$$

Obviously (M_k) is bounded in $W^{1,2}(\Sigma, g)$. With no loss of generality, we assume M_k converges to M_0 weakly in $W^{1,2}(\Sigma, g)$, strongly in $L^2(\Sigma, g)$, and almost everywhere in Σ. Using the same argument as in the proof of Lemma 2.2, we have $M_0 \equiv 0$. It then follows that

$$\|M_k\|_{1,\alpha}^2 = \int_\Sigma |\nabla_g M_k|^2 dv_g - \alpha \int_\Sigma M_k^2 dv_g = 1 + o_k(1). \tag{2.8}$$

Combining (2.7) and (2.8), we have for some γ_3 with $\gamma_2 < \gamma_3 < \gamma^*$,

$$\sup_{u \in \mathscr{H}_{\mathbf{G}},\, \|u\|_{1,\alpha} \leq 1} \int_\Sigma \mathrm{e}^{\gamma_3 u^2} dv_g = \infty.$$

This contradicts the definition of γ^*. Therefore γ^* must be $4\pi\ell$. □

2.2 Maximizers for Subcritical Functionals

In this subsection, using a direct method of variation, we show existence of maximizers for subcritical Trudinger–Moser functionals. Let $\alpha < \lambda_1^{\mathbf{G}}$ be fixed. Then we have

Lemma 2.4 *For any $0<\epsilon<4\pi\ell$, there exists some $u_\epsilon\in C^1(\Sigma\backslash\{p_1,\dots,p_L\},g_0)\cap C^0(\Sigma,g_0)\cap\mathscr{H}_{\mathbf{G}}$ with $\|u_\epsilon\|_{1,\alpha}=1$ satisfying*

$$\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g=\sup_{u\in\mathscr{H}_{\mathbf{G}},\|u\|_{1,\alpha}\le 1}\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u^2}dv_g. \tag{2.9}$$

Moreover u_ϵ satisfies the Euler–Lagrange equation

$$\begin{cases}\triangle_g u_\epsilon-\alpha u_\epsilon=\dfrac{1}{\lambda_\epsilon}u_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}-\dfrac{\mu_\epsilon}{\lambda_\epsilon}\quad\text{in }\Sigma,\\ \displaystyle\int_\Sigma u_\epsilon dv_g=0,\\ \lambda_\epsilon=\displaystyle\int_\Sigma u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g,\\ \mu_\epsilon=\dfrac{1}{\mathrm{Vol}_g(\Sigma)}\displaystyle\int_\Sigma u_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g,\end{cases} \tag{2.10}$$

where Δ_g is the Laplace–Beltrami operator on (Σ,g).

Proof Fix $\alpha<4\pi\ell$ and $0<\epsilon<4\pi\ell$. Take a maximizing function sequence $(u_j)\subset\mathscr{H}_{\mathbf{G}}$ verifying that $\|u_j\|_{1,\alpha}\le 1$, and that as $j\to\infty$,

$$\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_j^2}dv_g\to\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\le 1}\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u^2}dv_g.$$

Clearly (u_j) is bounded in $W^{1,2}(\Sigma,g)$. With no loss of generality we assume u_j converges to u_ϵ weakly in $W^{1,2}(\Sigma,g)$, strongly in $L^s(\Sigma,g)$ for any $s>1$, and almost everywhere in Σ. This implies $u_\epsilon\in\mathscr{H}_{\mathbf{G}}$ and $\|u_\epsilon\|_{1,\alpha}\le 1$. By Lemma 2.3, we have that $\mathrm{e}^{(4\pi\ell-\epsilon)u_j^2}$ converges to $\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}$ in $L^1(\Sigma,g)$ as $j\to\infty$. Thus (2.9) holds. It is easy to see that $\|u_\epsilon\|_{1,\alpha}=1$.

By a simple calculation, u_ϵ is a distributional solution of the Euler–Lagrange equation (2.10). In view of $g=\rho g_0$, applying elliptic estimates to (2.10), we conclude $u_\epsilon\in C^1(\Sigma\setminus\{p_1,\dots,p_L\},g_0)\cap C^0(\Sigma,g_0)$. □

Using the same argument as [25, p. 3184], we get

$$\liminf_{\epsilon\to 0}\lambda_\epsilon>0,\quad |\mu_\epsilon|/\lambda_\epsilon\le C. \tag{2.11}$$

2.3 Blow-up Analysis

Since u_ϵ is bounded in $W^{1,2}(\Sigma,g)$, we assume u_ϵ converges to some u^* weakly in $W^{1,2}(\Sigma,g)$, strongly in $L^s(\Sigma,g)$ for any $s>1$, and almost everywhere in Σ. Obviously $\|u^*\|_{1,\alpha}\le 1$. If u_ϵ is uniformly bounded, then by the Lebesgue dominated convergence theorem,

$$\int_\Sigma \mathrm{e}^{4\pi\ell u^{*2}}dv_g=\lim_{\epsilon\to 0}\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g=\sup_{u\in\mathscr{H}_{\mathbf{G}},\|u\|_{1,\alpha}\le 1}\int_\Sigma \mathrm{e}^{4\pi\ell u^2}dv_g. \tag{2.12}$$

Thus u^* is the desired maximizer. In the following we assume $\max_\Sigma|u_\epsilon|\to\infty$ as $\epsilon\to 0$. Since $-u_\epsilon$ still satisfies (2.9) and (2.10), we assume with no loss of generality

$$c_\epsilon=u_\epsilon(x_\epsilon)=\max_\Sigma|u_\epsilon|\to\infty \tag{2.13}$$

and

$$x_\epsilon\to x_0\in\Sigma \tag{2.14}$$

as $\epsilon\to 0$. To begin with, we have

Lemma 2.5 *u_ϵ converges to 0 weakly in $W^{1,2}(\Sigma, g)$, strongly in $L^s(\Sigma, g)$ for any $s > 1$, and almost everywhere in Σ.*

Proof Since u_ϵ is bounded in $W^{1,2}(\Sigma, g)$, we assume u_ϵ converges to u_0 weakly in $W^{1,2}(\Sigma, g)$, strongly in $L^s(\Sigma, g)$ for any $s > 1$, and almost everywhere in Σ. Suppose $u_0 \not\equiv 0$. Then

$$\|u_\epsilon - u_0\|_{1,\alpha}^2 = \|u_\epsilon\|_{1,\alpha}^2 - \|u_0\|_{1,\alpha}^2 + o_\epsilon(1) \leq 1 - \frac{1}{2}\|u_0\|_{1,\alpha}^2$$

for sufficiently small $\epsilon > 0$. Using the Young inequality, the Hölder inequality and Lemma 2.3, we have that $\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}$ is bounded in $L^q(\Sigma, g)$ for some $q > 1$. Noting (2.11) and applying elliptic estimate to (2.10), we obtain u_ϵ is uniformly bounded. This contradicts (2.13). Hence $u_0 \equiv 0$. □

Recalling the definitions of $I(x)$, $\beta(x)$ and ℓ, namely (1.8)–(1.10), under the assumptions (2.13) and (2.14), we obtain the following energy concentration phenomenon. From now on, we write $I_0 = I(x_0)$ and $\beta_0 = \beta(x_0)$ for short, where x_0 is introduced in (2.14).

Lemma 2.6 (i) $\lim_{r\to 0}\lim_{\epsilon\to 0}\int_{B_{g_0,r}(x_0)} |\nabla_{g_0} u_\epsilon|^2 dv_{g_0} = 1/I_0$, *where $B_{g_0,r}(x_0)$ denotes the geodesic ball centered at x_0 with radius r with respect to the metric g_0*; (ii) $I_0(1 + \beta_0) = \ell$.

Proof We first prove the assertion (i). With no loss of generality, we assume $\sigma_1(x_0), \dots, \sigma_{I_0}(x_0)$ are all distinct points in $\mathbf{G}(x_0)$. Choose some $r_0 > 0$ such that $B_{g_0,r_0}(\sigma_j(x_0)) \cap B_{g_0,r_0}(\sigma_i(x_0)) = \varnothing$ for every $1 \leq i < j \leq I_0$. Since $\int_\Sigma |\nabla_{g_0} u_\epsilon|^2 dv_{g_0} = \int_\Sigma |\nabla_g u_\epsilon|^2 dv_g = 1 + o_\epsilon(1)$ and $B_{g_0,r_0}(\sigma_k(x_0)) = \sigma_k(B_{g_0,r_0}(x_0))$ for $k = 1, \dots, I_0$, we have

$$\int_{B_{g_0,r_0}(x_0)} |\nabla_{g_0} u_\epsilon|^2 dv_{g_0} \leq \frac{1}{I_0} + o_\epsilon(1). \tag{2.15}$$

Suppose (i) does not hold. There would exist a constant $\nu_0 > 0$ and $0 < r_1 < r_0$ such that

$$\int_{B_{g_0,r_1}(x_0)} |\nabla_{g_0} u_\epsilon|^2 dv_{g_0} \leq \frac{1}{I_0} - \nu_0 \tag{2.16}$$

for all sufficiently small $\epsilon > 0$. Since $\ell \leq \min\{I_0, I_0(1+\beta_0)\} \leq I_0$, one finds a $p > 1$ such that $\mathrm{e}^{4\pi\ell u_\epsilon^2}$ is bounded in $L^p(B_{g_0,r_1/2}(x_0))$. In view of (2.11) and Lemma 2.5, one has by applying elliptic estimates to (2.10) that u_ϵ is bounded in $L^\infty(B_{g_0,r_1/4}(x_0))$, which contradicts the assumption (2.13). This confirms (i).

(ii) Suppose not. Obviously $\ell < I_0(1 + \beta_0)$. By (1.17) and (1.9), we have $\beta_0 \leq 0$. This together with (i) and an inequality of Adimurthi–Sandeep [3, p. 587] implies that there exist $r_0 > 0$, $p > 1$ and $C > 0$ satisfying

$$\int_{B_{g_0,r_0}(x_0)} \mathrm{e}^{4\pi\ell p u_\epsilon^2} \rho dv_{g_0} \leq C.$$

Applying elliptic estimates to (2.10), we conclude that u_ϵ is bounded in $L^\infty(B_{g_0,r_0/2}(x_0))$, contradicting the assumption (2.13). Therefore (ii) holds. □

Set

$$r_\epsilon = \sqrt{\lambda_\epsilon} c_\epsilon^{-1} \mathrm{e}^{-(2\pi\ell-\epsilon/2)c_\epsilon^2}. \tag{2.17}$$

Using the same argument as that of derivation of [13, the equation (42)], we have for any $0 < a < 4\pi\ell$,

$$r_\epsilon^2 c_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon-a)c_\epsilon^2} = o_\epsilon(1). \tag{2.18}$$

In particular, $r_\epsilon \to 0$ as $\epsilon \to 0$. And it follows from (2.18) that

$$r_\epsilon^2 c_\epsilon^q \to 0, \quad \forall q > 1. \tag{2.19}$$

Keep in mind that g and g_0 satisfy (1.3), (1.4), and (1.6). For any $1 \le k \le N$, we take an isothermal coordinate system $(U_{\sigma_k(x_0)}, \psi_k; \{y_1, y_2\})$ near $\sigma_k(x_0)$ such that $\psi_k : U_{\sigma_k(x_0)} \to \Omega \subset \mathbb{R}^2$ is a homomorphism, $\psi_k(\sigma_k(x_0)) = 0$, and

$$g_0 = \mathrm{e}^{2f_k}(dy_1^2 + dy_2^2), \tag{2.20}$$

where $f_k \in C^1(\Omega, \mathbb{R})$ satisfies $f_k(0) = 0$. If g has a conical singularity of the order β_0 at x_0, then in the above coordinate system, g can be represented by

$$g = V_k e^{2f_k}|y|^{2\beta_0}(dy_1^2 + dy_2^2), \tag{2.21}$$

where $V_k \in C^0(\Omega, \mathbb{R})$. It follows from (1.4) and (1.6) that

$$V_k(0) = \lim_{d_{g_0}(x,x_0)\to 0} \frac{\rho(x)}{d_{g_0}(x, \sigma_k(x_0))^{2\beta_0}} = V_0, \tag{2.22}$$

where V_0 is a positive constant independent of k. In particular, if $\beta_0 = 0$, with no loss of generality, one can take $V_k(y) \equiv 1$, and (2.21) reduces to (2.20). Writing $\widetilde{x}_\epsilon = \psi_k^{-1}(x_\epsilon)$, we have the following:

Lemma 2.7 *If $\beta_0 < 0$, then $|\widetilde{x}_\epsilon|^{1+\beta_0}/r_\epsilon$ is uniformly bounded.*

Proof For otherwise, up to a subsequence, we have

$$|\widetilde{x}_\epsilon|^{1+\beta_0}/r_\epsilon \to \infty. \tag{2.23}$$

For $y \in \Omega_{1,\epsilon} := \{y \in \mathbb{R}^2 : \widetilde{x}_\epsilon + r_\epsilon|\widetilde{x}_\epsilon|^{-\beta_0} y \in \Omega\}$, we denote

$$w_\epsilon(y) = c_\epsilon^{-1}(u_\epsilon \circ \psi_k^{-1})(\widetilde{x}_\epsilon + r_\epsilon|\widetilde{x}_\epsilon|^{-\beta_0}y), \quad v_\epsilon(y) = c_\epsilon\left((u_\epsilon \circ \psi_k^{-1})(\widetilde{x}_\epsilon + r_\epsilon|\widetilde{x}_\epsilon|^{-\beta_0}y) - c_\epsilon\right).$$

By (2.10), we calculate on $\Omega_{1,\epsilon}$,

$$\begin{aligned} -\Delta_{\mathbb{R}^2} w_\epsilon &= V_k(\widetilde{x}_\epsilon + r_\epsilon y)\mathrm{e}^{2f_k(\widetilde{x}_\epsilon + r_\epsilon y)}|\widetilde{x}_\epsilon \\ &\quad + r_\epsilon y|^{2\beta_0}|\widetilde{x}_\epsilon|^{-2\beta_0}(\alpha r_\epsilon^2 w_\epsilon + c_\epsilon^{-2} w_\epsilon \mathrm{e}^{(4\pi\ell-\epsilon)c_\epsilon^2(w_\epsilon^2-1)} - c_\epsilon^{-1} r_\epsilon^2 \mu_\epsilon \lambda_\epsilon^{-1}), \\ -\Delta_{\mathbb{R}^2} v_\epsilon &= V_k(\widetilde{x}_\epsilon + r_\epsilon y)\mathrm{e}^{2f_k(\widetilde{x}_\epsilon + r_\epsilon y)}|\widetilde{x}_\epsilon \\ &\quad + r_\epsilon y|^{2\beta_0}|\widetilde{x}_\epsilon|^{-2\beta_0}(\alpha c_\epsilon^2 r_\epsilon^2 w_\epsilon + w_\epsilon \mathrm{e}^{(4\pi\ell-\epsilon)(1+w_\epsilon)v_\epsilon} - c_\epsilon r_\epsilon^2 \mu_\epsilon \lambda_\epsilon^{-1}), \end{aligned}$$

where $\Delta_{\mathbb{R}^2}$ stands for the standard Laplacian operator on $\mathbb{R}^2$. It follows from (2.23) that both $\mathrm{e}^{2f(\tilde{x}_\epsilon + r_\epsilon y)}$ and $|\tilde{x}_\epsilon + r_\epsilon y|^{2\beta_0} \cdot |\tilde{x}_\epsilon|^{-2\beta_0}$ are $1 + o_\epsilon(1)$ in $\mathbb{B}_R$ for any fixed $R > 0$. Combining (2.11) with (2.19), we have $c_\epsilon r_\epsilon^2 \mu_\epsilon \lambda_\epsilon^{-1}$ is $o_\epsilon(1)$. Applying elliptic estimates to the above two equations, we obtain

$$w_\epsilon \to 1 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2) \tag{2.24}$$

and $v_\epsilon \to v_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where v_0 satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2} v_0 = V_0 \mathrm{e}^{8\pi\ell v_0} & \text{in } \mathbb{R}^2, \\ v_0(0) = 0 = \sup_{\mathbb{R}^2} v_0 \end{cases} \tag{2.25}$$

in the is distributional sense. With the transformation of coordinate, for any fixed $R > 0$,

$$\int_{\mathbb{B}_R(0)} \mathrm{e}^{8\pi\ell v_0} dy = \lim_{\epsilon\to 0} \int_{\mathbb{B}_R(0)} \mathrm{e}^{(4\pi\ell-\epsilon)\tilde{u}_\epsilon^2(\tilde{x}_\epsilon + r_\epsilon y))} \mathrm{e}^{-(4\pi\ell-\epsilon)c_\epsilon^2} dy$$

$$= \lim_{\epsilon\to 0}\lambda_\epsilon^{-1}\int_{\mathbb{B}_{Rr_\epsilon|\tilde{x}_\epsilon|^{\beta_0}}(\tilde{x}_\epsilon)}|\tilde{x}_\epsilon|^{2\beta_0}c_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)\tilde{u}_\epsilon^2}dy$$
$$= \lim_{\epsilon\to 0}(V_0\lambda_\epsilon)^{-1}\int_{\mathbb{B}_{Rr_\epsilon|\tilde{x}_\epsilon|^{\beta_0}}(\tilde{x}_\epsilon)}V_0\mathrm{e}^{2f}|y|^{2\beta_0}\tilde{u}_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)\tilde{u}_\epsilon^2}dy$$
$$= \lim_{\epsilon\to 0}(I_0V_0\lambda_\epsilon)^{-1}\int_{\sum_{k=1}^{I_0}\phi_k^{-1}(\mathbb{B}_{Rr_\epsilon|\tilde{x}_\epsilon|^{\beta_0}}(\tilde{x}_{(k)}))}u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g$$
$$\le \frac{1}{I_0V_0}.$$

Pass the limit $R\to+\infty$,

$$\int_{\mathbb{R}^2}\mathrm{e}^{8\pi\ell v_0}dy \le \frac{1}{I_0V_0}. \tag{2.26}$$

In view of (2.25) and (2.26), we have by a classification theorem of Chen–Li [8],

$$v_0(y) = -\frac{1}{4\pi\ell}\log(1+\pi\ell V_0|y|^2).$$

It then follows that

$$\int_{\mathbb{R}^2}\mathrm{e}^{8\pi v_0}dy = \frac{1}{\ell V_0}. \tag{2.27}$$

Since $\beta_0 < 0$, it follows from (ii) of Lemma 2.6 that $\ell < I_0$. As a consequence, there is a contradiction between (2.27) and (2.26). This ends the proof of the lemma. □

We now define two sequences of functions

$$\psi_\epsilon(y) = c_\epsilon^{-1}\widetilde{u}_\epsilon(\widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y), \quad \varphi_\epsilon(y) = c_\epsilon(\widetilde{u}_\epsilon(\widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y) - c_\epsilon) \tag{2.28}$$

for $y\in\Omega_{2,\epsilon} := \{y\in\mathbb{R}^2 : \widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y\in\Omega\}$. Then there holds the following:

Lemma 2.8 *If $\beta_0 < 0$, then* (i) $\psi_\epsilon\to 1$ *in* $C^0_{\mathrm{loc}}(\mathbb{R}^2)\cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$; (ii) $\varphi_\epsilon\to\varphi$ *in* $C^0_{\mathrm{loc}}(\mathbb{R}^2)\cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, *where*

$$\varphi(y) = -\frac{1}{4\pi\ell}\log\left(1+\frac{\pi I_0V_0}{1+\beta_0}|y|^{2(1+\beta_0)}\right). \tag{2.29}$$

Proof By (2.10) and (2.17), we have on $\Omega_{2,\epsilon}$,

$$\begin{aligned}
-\Delta_{\mathbb{R}^2}\psi_\epsilon &= V_k(\widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y)\mathrm{e}^{2f_k(\widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y)}|y + r_\epsilon^{-1/(1+\beta_0)}\widetilde{x}_\epsilon|^{2\beta_0}(\alpha r_\epsilon^2\psi \\
&\quad + c_\epsilon^{-2}\psi_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)c_\epsilon^2(\psi_\epsilon^2-1)} - c_\epsilon^{-1}r_\epsilon^2\mu_\epsilon\lambda_\epsilon^{-1}),
\end{aligned} \tag{2.30}$$
$$\begin{aligned}
-\Delta_{\mathbb{R}^2}\varphi_\epsilon &= V_k(\widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y)\mathrm{e}^{2f_k(\widetilde{x}_\epsilon + r_\epsilon^{1/(1+\beta_0)}y)}|y + r_\epsilon^{-1/(1+\beta_0)}\widetilde{x}_\epsilon|^{2\beta_0}(\alpha c_\epsilon^2 r_\epsilon^2\psi_\epsilon \\
&\quad + \psi_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon} - c_\epsilon r_\epsilon^2\mu_\epsilon\lambda_\epsilon^{-1}).
\end{aligned} \tag{2.31}$$

In view of Lemma 2.7, $r_\epsilon^{-1/(1+\beta_0)}\widetilde{x}_\epsilon$ is a bounded sequence of points. We may assume with no loss of generality that $r_\epsilon^{-1/(1+\beta_0)}\widetilde{x}_\epsilon\to p\in\mathbb{R}^2$ as $\epsilon\to 0$. Note that $\beta > -1$. Applying elliptic estimates to (2.30), we obtain $\psi_\epsilon\to\psi$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)\cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, where ψ is a distributional harmonic function. Then the Liouville theorem leads to $\psi\equiv 1$. Further application of elliptic estimates on (2.31) implies that

$$\varphi_\epsilon\to\varphi \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^2)\cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2), \tag{2.32}$$

where φ is a distributional solution of

$$\begin{cases} -\Delta_{\mathbb{R}^2}\varphi = |y+p|^{2\beta_0}V_0\mathrm{e}^{8\pi I_0(1+\beta_0)\varphi} & \text{in } \mathbb{R}^2, \\ \varphi(0) = 0 = \max_{\mathbb{R}^2}\varphi. \end{cases} \tag{2.33}$$

For any fixed $R > |p| + 1$, by Fatou's lemma and lemma 2.7, we have

$$\begin{aligned}\int_{\mathbb{B}_R(-p)} & V_0|x+p|^{2\beta_0}\mathrm{e}^{8\pi\ell\varphi_0}dx \\ &\leq \limsup_{\epsilon\to 0}\int_{\mathbb{B}_R(-p)} V_0|x+r_\epsilon^{-1/(1+\beta_0)}\tilde{x}_\epsilon|^{2\beta_0}\mathrm{e}^{(4\pi\ell-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon}dx \\ &\leq \limsup_{\epsilon\to 0}\frac{c_\epsilon^2}{\lambda_\epsilon}\int_{2\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}} V_0\mathrm{e}^{2f}|y|^{2\beta_0}\mathrm{e}^{(4\pi\ell-\epsilon)\tilde{u}_\epsilon^2}dx \\ &\leq \limsup_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{\phi^{-1}(\mathbb{B}_{2Rr_\epsilon^{1/(1+\beta_0)}})} u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g \\ &\leq \lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_\Sigma u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g \leq 1.\end{aligned}$$

This suggests

$$\int_{\mathbb{R}^2} V_0|y+p|^{2\beta_0}\mathrm{e}^{8\pi\ell\varphi}dy \leq 1. \tag{2.34}$$

In view of (2.33) and (2.34), a classification theorem of Chen–Li [9] suggests the representation:

$$\varphi(y) = -\frac{1}{4\pi\ell}\log\left(1+\frac{\pi I_0 V_0}{1+\beta_0}|y+p|^{2(1+\beta_0)}\right). \tag{2.35}$$

Since $\varphi(0) = 0$, we have $p = 0$. By a straightforward calculation,

$$\int_{\mathbb{R}^2} V_0|y|^{2\beta_0}\mathrm{e}^{8\pi\ell\varphi(y)}dy = \frac{1}{I_0}, \tag{2.36}$$

as desired. □

In the case $\beta_0 = 0$, we have an analog of Lemma 2.8, namely

Lemma 2.9 *Let ψ_ϵ and φ_ϵ be defined as in* (2.28). *If $\beta_0 = 0$, then $\psi_\epsilon \to 1$ and $\varphi_\epsilon \to \varphi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where $\varphi(y) = -\frac{1}{4\pi I_0}\log\left(1+\pi I_0\rho(x_0)|y|^2\right)$, ρ is given as in* (1.3) *and* (1.4).

Proof Noting that if $\beta_0 = 0$, we have by applying elliptic estimates to (2.30) and (2.31) that $\psi_\epsilon \to 1$ and $\varphi_\epsilon \to \varphi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where φ satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2}\varphi = \rho(x_0)\mathrm{e}^{8\pi I_0\varphi} & \text{in } \mathbb{R}^2, \\ \varphi(0) = 0 = \max_{\mathbb{R}^2}\varphi, \\ \int_{\mathbb{R}^2}\rho(x_0)\mathrm{e}^{8\pi I_0\varphi}dy \leq 1. \end{cases}$$

Then a result of Chen–Li [8] leads to $\varphi(y) = -\frac{1}{4\pi I_0}\log\left(1+\pi I_0\rho(x_0)|y|^2\right)$. As a consequence,

$$\int_{\mathbb{R}^2}\rho(x_0)\mathrm{e}^{8\pi I_0\varphi(y)}dy = \frac{1}{I_0}, \tag{2.37}$$

which is an analog of (2.36). □

By (2.17), Lemmas 2.8 and 2.9, we have for any fixed $R > 0$,

$$\begin{aligned}\int_{\mathbb{B}_R(0)} V_0|y|^{2\beta_0}\mathrm{e}^{8\pi\ell\varphi}dy &= \lim_{\epsilon\to 0}\int_{\mathbb{B}_R(0)} V_0|y|^{2\beta_0}\mathrm{e}^{(4\pi\ell-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon}dy \\ &= \lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon)} V_0\mathrm{e}^{2f_k}|y|^{2\beta_0}\widetilde{u}_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{u}_\epsilon^2}dy\end{aligned}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))} u_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g,$$

where $V_0=\rho(x_0)$ and $\ell=I_0$ if $\beta_0=0$. This together with (2.36) and (2.37) implies that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))} u_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g=\frac{1}{I_0}. \tag{2.38}$$

Noting that

$$\lambda_\epsilon=\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))} u_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g+\int_{\Sigma\setminus\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))} u_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g,$$

we conclude from (2.38) that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\frac{1}{\lambda_\epsilon}\int_{\Sigma\setminus\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))} u_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g=0. \tag{2.39}$$

As in [15], we define $u_{\epsilon,\gamma}=\min\{u_\epsilon,\gamma c_\epsilon\}$ for any $0<\gamma<1$, and have

Lemma 2.10 *For any* $0<\gamma<1$*, there holds*

$$\lim_{\epsilon\to 0}\int_\Sigma|\nabla_g u_{\epsilon,\gamma}|^2dv_g=\gamma.$$

Proof For fixed $R>0$ and sufficiently small ϵ, in view of (2.10), we have by using integration by parts, (2.38) and (2.39) that

$$\begin{aligned}\int_\Sigma|\nabla_g u_{\epsilon,\gamma}|^2dv_g&=\int_\Sigma\nabla_g u_{\epsilon,\gamma}\nabla_g u_\epsilon dv_g\\&=\lambda_\epsilon^{-1}\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}u_\epsilon u_{\epsilon,\gamma}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g\\&\quad+\lambda_\epsilon^{-1}\int_{\Sigma\setminus\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}u_\epsilon u_{\epsilon,\gamma}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g+o_\epsilon(1)\\&=(1+o_\epsilon(1))\gamma{\lambda_\epsilon}^{-1}\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g+o(1)\\&=\gamma+o(1),\end{aligned}$$

where $o(1)\to 0$ as $\epsilon\to 0$ first, and then $R\to\infty$. The lemma follows immediately. □

Lemma 2.11 *There holds* $c_\epsilon/\lambda_\epsilon\to 0$ *as* $\epsilon\to 0$*.*

Proof For any fixed $0<\gamma<1$,

$$\begin{aligned}\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g&=\int_{u_\epsilon\le\gamma c_\epsilon}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g+\int_{u_\epsilon>\gamma c_\epsilon}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g\\&\le\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_{\epsilon,\gamma}^2}dv_g+\frac{\lambda_\epsilon}{\gamma^2c_\epsilon^2}.\end{aligned} \tag{2.40}$$

By Lemmas 2.5 and 2.10, we conclude

$$\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_{\epsilon,\gamma}^2}dv_g=\mathrm{Vol}_g(\Sigma)+o_\epsilon(1).$$

Passing to the limit $\epsilon\to 0$ first, and then $\gamma\to 1$ in (2.40), we have

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\le 1}\int_\Sigma e^{4\pi\ell u^2}dv_g=\lim_{\epsilon\to 0}\int_\Sigma e^{4\pi\ell u_\epsilon^2}dv_g\le \mathrm{Vol}_g(\Sigma)+\liminf_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}. \tag{2.41}$$

Since

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\le 1}\int_\Sigma e^{4\pi\ell u^2}dv_g>\mathrm{Vol}_g(\Sigma),$$

we have by (2.41) that $\liminf_{\epsilon\to 0}\lambda_\epsilon/c_\epsilon^2>0$. In particular $c_\epsilon/\lambda_\epsilon\to 0$ as $\epsilon\to 0$. □

Recall $\mathbf{G}(x_0)=\{\sigma_1(x_0),\dots,\sigma_{I_0}(x_0)\}$, and $\mathbf{S}=\{p_1,\dots,p_L\}$. The convergence of $c_\epsilon u_\epsilon$ is precisely described as follows.

Lemma 2.12 *For any $1<q<2$, we have $c_\epsilon u_\epsilon$ converges to G_α weakly in $W^{1,q}(\Sigma,g_0)$, strongly in $L^{2q/(2-q)}(\Sigma)$, and in $C^1(\Sigma\setminus\{\mathbf{G}(x_0)\cup\mathbf{S}\})$, where G_α is a Green function satisfying*

$$\begin{cases}\Delta_{g_0}G_\alpha-\alpha\rho G_\alpha=\dfrac{1}{I_0}\displaystyle\sum_{i=1}^{I_0}\delta_{\sigma_i(x_0)}-\dfrac{\rho}{\mathrm{Vol}_g(\Sigma)},\\ \displaystyle\int_\Sigma G_\alpha dv_g=0,\\ G_\alpha(\sigma_i(x))=G_\alpha(x),\quad x\in\Sigma\setminus\{\sigma_j(x_0)\}_{j=1}^{I_0},\ 1\le i\le I_0.\end{cases} \tag{2.42}$$

Proof In view of (2.10), one has

$$\begin{cases}\Delta_g(c_\epsilon u_\epsilon)-\alpha(c_\epsilon u_\epsilon)=f_\epsilon-b_\epsilon\quad\text{on }\Sigma,\\ \displaystyle\int_\Sigma c_\epsilon u_\epsilon dv_g=0,\\ f_\epsilon=\dfrac{1}{\lambda_\epsilon}c_\epsilon u_\epsilon e^{(4\pi\ell-\epsilon)u_\epsilon^2},\\ b_\epsilon=\dfrac{c_\epsilon\mu_\epsilon}{\lambda_\epsilon}.\end{cases} \tag{2.43}$$

Firstly we claim that

$$f_\epsilon dv_g\rightharpoonup\frac{1}{I_0}\sum_{i=1}^{I_0}\delta_{\sigma_i(x_0)} \tag{2.44}$$

weakly in the sense of measure, or equivalently, there holds

$$\int_\Sigma f_\epsilon\phi dv_g=\frac{1}{I_0}\sum_{i=1}^{I_0}\phi(\sigma_i(x_0))+o_\epsilon(1),\quad\forall\phi\in C^0(\Sigma,g_0).$$

To see it, we estimate for any fixed $0<\gamma<1$ and $R>0$

$$\begin{aligned}\int_\Sigma f_\epsilon\phi dv_g&=\int_{u_\epsilon\le\gamma c_\epsilon}f_\epsilon\phi dv_g+\int_{\{u_\epsilon>\gamma c_\epsilon\}\cap\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}f_\epsilon\phi dv_g\\&\quad+\int_{\{u_\epsilon>\gamma c_\epsilon\}\setminus\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}f_\epsilon\phi dv_g\\&:=\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned} \tag{2.45}$$

By Lemmas 2.5, 2.10 and 2.11, we have by the Hölder inequality

$$\mathrm{I}=\frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon\le\gamma c_\epsilon}u_\epsilon e^{(4\pi\ell-\epsilon)u_{\epsilon,\gamma}^2}\phi dv_g=o_\epsilon(1).$$

Note that $\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))\subset\{u_\epsilon>\gamma c_\epsilon\}$ for sufficiently small $\epsilon>0$. In view of Lemmas 2.8 and 2.9, we calculate by using (2.38) and the mean value theorem for integrals,

$$\begin{aligned}\mathrm{II}&=\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}f_\epsilon\phi dv_g\\&=\sum_{k=1}^{I_0}\phi(\sigma_k(x_0))(1+o_\epsilon(1))\int_{\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}\frac{c_\epsilon}{\lambda_\epsilon}u_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g\\&=\sum_{i=1}^{I_0}\phi(\sigma_i(x_0))\Big(\frac{1}{I_0}+o(1)\Big),\end{aligned}$$

and

$$\begin{aligned}\mathrm{III}&\le\int_{\{u_\epsilon>\gamma c_\epsilon\}\setminus\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}f_\epsilon|\phi|dv_g\\&\le\frac{\sup_\Sigma|\phi|}{\gamma}\int_{\{u_\epsilon>\gamma c_\epsilon\}\setminus\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}\lambda_\epsilon^{-1}u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g\\&\le\frac{\sup_\Sigma|\phi|}{\gamma}\Big(1-\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}\lambda_\epsilon^{-1}u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g\Big)\\&=o(1),\end{aligned}$$

where $o(1)\to0$ as $\epsilon\to0$ first, and then $R\to\infty$. Inserting the estimates of I–III into (2.45), we conclude our claim (2.44).

Secondly we calculate b_ϵ in (2.43). Similar to the estimate of (2.45), we have for any fixed $0<\gamma<1$,

$$\frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon\le\gamma c_\epsilon}u_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g=o_\epsilon(1)$$

and

$$\begin{aligned}\frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon>\gamma c_\epsilon}u_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g&=\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}}(\widetilde{x}_\epsilon))}\frac{1}{\lambda_\epsilon}u_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g+o(1)\\&=1+o(1).\end{aligned}$$

It then follows that

$$b_\epsilon=\frac{1}{\mathrm{Vol}_g(\Sigma)}\frac{c_\epsilon}{\lambda_\epsilon}\int_\Sigma u_\epsilon\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g=\frac{1}{\mathrm{Vol}_g(\Sigma)}+o_\epsilon(1).\tag{2.46}$$

Thirdly we prove that $c_\epsilon u_\epsilon$ is bounded in $L^1(\Sigma,g)$. Suppose on the contrary

$$\|c_\epsilon u_\epsilon\|_{L^1(\Sigma,g)}\to\infty.\tag{2.47}$$

Since for any fixed $0<\gamma<1$,

$$\int_\Sigma|f_\epsilon|dv_g=\int_{u_\epsilon\le\gamma c_\epsilon}|f_\epsilon|dv_g+\int_{u_\epsilon>\gamma c_\epsilon}f_\epsilon dv_g,$$

we have that f_ϵ is bounded in $L^1(\Sigma,g)$ by using a similar argument of the estimate of (2.45). Obviously b_ϵ is a bounded sequence of numbers due to (2.46). Define $w_\epsilon=c_\epsilon u_\epsilon/\|c_\epsilon u_\epsilon\|_{L^1(\Sigma,g)}$.

Then (2.43) gives

$$\begin{cases} \Delta_{g_0} w_\epsilon = h_\epsilon := \alpha\rho w_\epsilon + \rho \dfrac{f_\epsilon - b_\epsilon}{\|c_\epsilon u_\epsilon\|_{L^1(\Sigma,g)}} & \text{on } \Sigma, \\ \displaystyle\int_\Sigma w_\epsilon dv_g = 0, \\ \|w_\epsilon\|_{L^1(\Sigma,g)} = 1. \end{cases} \tag{2.48}$$

Clearly we have got

$$\int_\Sigma |h_\epsilon| dv_{g_0} \le C. \tag{2.49}$$

By the Green representation formula,

$$w_\epsilon(x) - \frac{1}{\mathrm{Vol}_{g_0}(\Sigma)} \int_\Sigma w_\epsilon dv_{g_0} = \int_\Sigma G(x,y) h_\epsilon(y) dv_{g_0,y}, \tag{2.50}$$

where $G(x,y)$ is the Green function for Δ_{g_0}. In particular there exists a constant C such that $|G(x,y)| \le C|\log \mathrm{dist}_{g_0}(x,y)|$ and $|\nabla_{g_0,x} G(x,y)| \le C(\mathrm{dist}_{g_0}(x,y))^{-1}$ for all $x, y \in \Sigma$. By (1.17), $\rho(x)$ has a positive lower bound on Σ. As a consequence

$$\frac{1}{\mathrm{Vol}_{g_0}(\Sigma)} \int_\Sigma |w_\epsilon| dv_{g_0} \le C \int_\Sigma |w_\epsilon| \rho dv_{g_0} = C. \tag{2.51}$$

Combining (2.49) and (2.50), we obtain for any $1 < q < 2$,

$$\int_\Sigma |\nabla_{g_0} w_\epsilon|^q dv_{g_0} \le C \int_\Sigma |h_\epsilon| dv_{g_0} \le C.$$

While (2.50) and (2.51) imply that for any $q > 1$, there holds $\|w_\epsilon\|_{L^q(\Sigma,g_0)} \le C$. Therefore w_ϵ is bounded in $W^{1,q}(\Sigma, g_0)$ for any $1 < q < 2$. The Sobolev embedding theorem leads to that w_ϵ converges to w weakly in $W^{1,q}(\Sigma, g_0)$, strongly in $L^r(\Sigma, g_0)$ for any $r < 2q/(2-q)$, and almost everywhere in Σ. Clearly w satisfies

$$\begin{cases} \Delta_{g_0} w = \alpha\rho w & \text{in } \Sigma, \\ \displaystyle\int_\Sigma w\rho dv_{g_0} = 0. \end{cases}$$

Since $\rho \in L^r(\Sigma \setminus \bigcup_{i=1}^L B_{g_0,\delta}(p_i), g_0)$ for any small $\delta > 0$ and some $r > 1$, we have $w \in C^1(\Sigma \setminus \mathbf{S}, g_0)$ and $u \in \mathscr{H}_{\boldsymbol{G}}$ by using elliptic estimates. Then integration by parts gives

$$\int_\Sigma |\nabla_g w|^2 dv_g = \alpha \int_\Sigma w^2 dv_g,$$

which leads to $w \equiv 0$ due to $\alpha < \lambda_1^{\boldsymbol{G}}$. This contradicts $\|w\|_{L^1(\Sigma,g)} = \lim_{\epsilon\to 0} \|w_\epsilon\|_{L^1(\Sigma,g)} = 1$. Therefore $c_\epsilon u_\epsilon$ is bounded in $L^1(\Sigma, g)$.

Fourthly we analyze the convergence of $c_\epsilon u_\epsilon$. Rewrite (2.43) as

$$\begin{cases} \Delta_{g_0}(c_\epsilon u_\epsilon) = \xi_\epsilon := \alpha\rho c_\epsilon u_\epsilon + \rho(f_\epsilon - b_\epsilon) & \text{on } \Sigma, \\ \displaystyle\int_\Sigma c_\epsilon u_\epsilon \rho dv_{g_0} = 0. \end{cases} \tag{2.52}$$

Now since ξ_ϵ is bounded in $L^1(\Sigma, g_0)$, we conclude that $c_\epsilon u_\epsilon$ is bounded in $W^{1,q}(\Sigma, g_0)$ for any $1 < q < 2$ similar to w_ϵ. Hence $c_\epsilon u_\epsilon$ converges to some G_α weakly in $W^{1,q}(\Sigma, g_0)$, strongly in $L^r(\Sigma, g_0)$ for any $r < 2q/(2-q)$, and almost everywhere in Σ. In view of (2.44) and (2.46), G_α satisfies (2.42) in the distributional sense. Applying elliptic estimates to (2.52), we have that $c_\epsilon u_\epsilon$ converges to G_α in $C^1(\Sigma \setminus \{\mathbf{G}(x_0) \cup \mathbf{S}\})$. This completes the proof of the lemma. □

2.4 Upper Bound Estimate

In this section, we aim to give an upper bound estimate of the functional in (1.18) under assumptions (2.13) and (2.14). The calculation bases on the convergence of u_ϵ and $c_\epsilon u_\epsilon$, which is precisely studied in the previous section. Recall the isothermal coordinate system $(U_{\sigma_k(x_0)}, \psi_k)$ near $\sigma_k(x_0)$ (here we only take k from 1 to I_0) given as in (2.20). Set

$$r_0 = \frac{1}{4} \min_{1 \le i < j \le I_0} d_{g_0}(\sigma_i(x_0), \sigma_j(x_0)).$$

For $\delta < r_0$ with $B_{g_0,2\delta}(x_0) \subset U_{x_0}$, there exist two positive constants $c_1(\delta)$ and $c_2(\delta)$ such that $B_{g_0,(1-c_1(\delta))\delta}(\sigma_k(x_0)) \subset \psi_k^{-1}(\mathbb{B}_\delta) \subset B_{g_0,(1+c_2(\delta))\delta}(\sigma_k(x_0))$. Moreover, both $c_1(\delta)$ and $c_2(\delta)$ converge to 0 as $\delta \to 0$. Hence, on $B_{g_0,2\delta}(\sigma_k(x_0))$, by using isothermal coordinate system (U_k, ψ_k), (2.42) can be rewritten as the equation $G_\alpha \circ \psi_k^{-1}$ satisfies on $\psi_k(B_{g_0,2\delta}(\sigma_k(x_0)))$. By using elliptic estimates to that equation, we obtain:

$$G_\alpha \circ \psi_k^{-1} = -\frac{1}{2\pi I_0} \log |y| + A_0 + \Psi_k(y), \tag{2.53}$$

where $\Psi_k \in C^1(\mathbb{B}_{\frac{5}{3}\delta})$ satisfies $\Psi_k(0) = 0$ for small δ, and A_0 is a constant defined by

$$A_0 = \lim_{y \to 0} \left(G_\alpha \circ \psi_k^{-1}(y) + \frac{1}{2\pi I_0} \log |y| \right) = \lim_{x \to x_0} \left(G_\alpha(x) + \frac{1}{2\pi I_0} \log d_{g_0}(x, x_0) \right). \tag{2.54}$$

By (2.53), G_α near x_0 can be locally presented by

$$G_\alpha(x) = -\frac{1}{2\pi I_0} \log d_{g_0}(x, x_0) + A_0 + \widetilde{\Psi}(x), \tag{2.55}$$

where $\widetilde{\Psi} \in C^1(B_{g_0,\frac{3}{2}\delta}(x_0))$ satisfies $\widetilde{\Psi}(x_0) = 0$. Furthermore, we obtain G_α near $\sigma_k(x_0)$ can be locally presented by

$$G_\alpha(x) = -\frac{1}{2\pi I_0} \log d_{g_0}(x, \sigma_k(x_0)) + A_0 + \widetilde{\Psi}(\sigma_k^{-1}(x)). \tag{2.56}$$

This conclusion is based on an observation that, for $x \in B_{g_0,\frac{3}{2}\delta}(\sigma_k(x_0))$, by (2.55)

$$\begin{aligned} G_\alpha(x) + \frac{1}{2\pi I_0} \log d_{g_0}(x, \sigma_k(x_0)) - A_0 &= \left(G_\alpha(\sigma_k^{-1}(x)) + \frac{1}{2\pi I_0} \log d_{g_0}(\sigma_k^{-1}(x), x_0) - A_0 \right) \\ &= \widetilde{\Psi}(\sigma_k^{-1}(x)). \end{aligned}$$

In the view of (2.10), integration by parts leads to

$$\begin{aligned} \int_{\Sigma \setminus \bigcup_{k=1}^{I_0} \psi_k^{-1}(\mathbb{B}_\delta)} |\nabla_g u_\epsilon|^2 dv_g &= \int_{\Sigma \setminus \bigcup_{k=1}^{I_0} \psi_k^{-1}(\mathbb{B}_\delta)} |\nabla_{g_0} u_\epsilon|^2 dv_{g_0} \\ &= -\sum_{k=1}^{I_0} \int_{\partial \psi_k^{-1}(\mathbb{B}_\delta)} u_\epsilon \frac{\partial u_\epsilon}{\partial n} ds_{g_0} + \int_{\Sigma \setminus \bigcup_{k=1}^{I_0} \psi_k^{-1}(\mathbb{B}_\delta)} u_\epsilon \triangle_{g_0} u_\epsilon dv_{g_0} \\ &= -\sum_{k=1}^{I_0} \int_{\partial \psi_k^{-1}(\mathbb{B}_\delta)} u_\epsilon \frac{\partial u_\epsilon}{\partial n} ds_{g_0} + \alpha \int_\Sigma u_\epsilon^2 dv_g + 1 + o_\delta(1). \end{aligned}$$

This together with (2.56), (2.42), and $c_\epsilon u_\epsilon \to G_\alpha$ in $L^2(\Sigma, g) \cap C^1(\Sigma \setminus \{\mathbf{G}(x_0) \cup \mathbf{S}\})$ shows

$$\int_{\Sigma \setminus \bigcup_{k=1}^{I_0} \psi_k^{-1}(\mathbb{B}_\delta)} |\nabla_g u_\epsilon|^2 dv_g = \frac{1}{c_\epsilon^2} \left(\frac{1}{2\pi I_0} \log \delta + A_0 + \alpha \int_\Sigma G_\alpha^2 dv_g + o_\epsilon(1) + o_\delta(1) \right).$$

We then calculate

$$\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_\delta)}|\nabla_g u_\epsilon|^2dv_g = 1-\frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi I_0}\log\delta + A_0 + o_\epsilon(1)+o_\delta(1)\right) := \tau_\epsilon.$$

Set $s_\epsilon = \sup_{\partial\psi^{-1}(\mathbb{B}_\delta)} u_\epsilon$ and $\hat{u}_\epsilon = (u_\epsilon - s_\epsilon)^+$. Clearly, $\hat{u}_\epsilon \in W_0^{1,2}(\psi_k^{-1}(\mathbb{B}_\delta))$. Moreover, we have

$$\int_{\mathbb{B}_\delta}|\nabla_g(\hat{u}_\epsilon\circ\phi^{-1})|^2dx = \int_{\psi_k^{-1}(\mathbb{B}_\delta)}|\nabla_g\hat{u}_\epsilon|^2dv_g \le \frac{1}{I_0}\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_\delta)}|\nabla_g u_\epsilon|^2dv_g \le \frac{\tau_\epsilon}{I_0}.$$

Then by using Lemma 1.3, we obtain

$$\begin{aligned}\limsup_{\epsilon\to 0}\int_{\psi^{-1}(\mathbb{B}_\delta)}(\mathrm{e}^{4\pi\ell\hat{u}_\epsilon^2/\tau_\epsilon}-1)dv_g &= \limsup_{\epsilon\to 0}\int_{\mathbb{B}_\delta}V(y)\mathrm{e}^{2f}|y|^{2\beta}(\mathrm{e}^{4\pi(1+\beta_0)I_0(\hat{u}_\epsilon\circ\phi^{-1})^2/\tau_\epsilon}-1)dy\\ &= \limsup_{\epsilon\to 0}\mathrm{e}^{o_\delta(1)}\int_{\mathbb{B}_\delta}V_0|y|^{2\beta_0}(\mathrm{e}^{4\pi(1+\beta_0)I_0(\hat{u}_\epsilon\circ\phi^{-1})^2/\tau_\epsilon}-1)dy\\ &\le \frac{\pi V_0\mathrm{e}^{1+o_\delta(1)}}{1+\beta_0}\delta^{2+2\beta_0}. \end{aligned}\tag{2.57}$$

For any fixed $R>0$, we have $u_\epsilon/c_\epsilon = 1+o_\epsilon(1)$ on $\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta)}})$ $(k=1,\ldots,I_0)$. Hence, using the definition of τ_ϵ, we obtain

$$\begin{aligned}(4\pi\ell-\epsilon)u_\epsilon^2 &\le 4\pi\ell(\hat{u}_\epsilon+s_\epsilon)^2\\ &= 4\pi\ell\hat{u}_\epsilon^2 + 8\pi\ell\hat{u}_\epsilon s_\epsilon + o_\epsilon(1)\\ &= 4\pi\ell\hat{u}_\epsilon^2 - 4(1+\beta_0)\log\delta + 8\pi\ell A_0 + o(1)\\ &= 4\pi\ell_0\hat{u}_\epsilon^2/\tau_\epsilon - 2(1+\beta_0)\log\delta + 4\pi\ell A_0 + o(1),\end{aligned}$$

where $o(1)\to 0$ as $\epsilon\to 0$ first, and then $\delta\to 0$. Combining this with (2.57), we have

$$\begin{aligned}\int_{\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}})}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g &\le \delta^{-2-2\beta}\mathrm{e}^{4\pi\ell A_0+o(1)}\int_{\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}})}\mathrm{e}^{(4\pi\ell-\epsilon)\hat{u}_\epsilon^2/\tau_\epsilon}dv_g\\ &= \delta^{-2-2\beta_0}\mathrm{e}^{4\pi\ell A_0+o(1)}\int_{\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}})}(\mathrm{e}^{(4\pi\ell-\epsilon)\hat{u}_\epsilon^2/\tau_\epsilon}-1)dv_g + o(1)\\ &\le \delta^{-2-2\beta_0}\mathrm{e}^{4\pi\ell A_0+o(1)}\int_{\psi_k^{-1}(\mathbb{B}_\delta)}(\mathrm{e}^{(4\pi\ell-\epsilon)\hat{u}_\epsilon^2/\tau_\epsilon}-1)dv_g + o(1)\\ &\le \frac{\pi V_0\mathrm{e}^{1+4\pi\ell A_0+o(1)}}{1+\beta_0}.\end{aligned}$$

Letting $\epsilon\to 0$ first, and then $\delta\to 0$, we obtain

$$\limsup_{\epsilon\to 0}\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}})}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g \le \frac{\pi I_0V_0\mathrm{e}^{1+4\pi\ell A_0}}{1+\beta_0}.\tag{2.58}$$

Also we have

$$\begin{aligned}\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}})}\mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g &= I_0(1+o_\epsilon(1))\int_{\mathbb{B}_{Rr_\epsilon^{1/(1+\beta)}}}V_0\mathrm{e}^{2f}|x|^{2\beta}\tilde{u}_\epsilon^2\mathrm{e}^{(4\pi\ell-\epsilon)\tilde{u}_\epsilon^2}dx\\ &= \frac{I_0\lambda_\epsilon}{c_\epsilon^2}(1+o_\epsilon(1))\left(\int_{\mathbb{B}_R(0)}V_0|y|^{2\beta}\mathrm{e}^{8\pi\ell\varphi_\epsilon}dy + o_\epsilon(1)\right)\\ &= \frac{\lambda_\epsilon}{c_\epsilon^2}(1+o(1)),\end{aligned}$$

where $o(1) \to 0$ as $\epsilon \to 0$ first, and then $R \to \infty$. This together with (2.58) and (2.41) leads to

$$\sup_{u\in\mathscr{H}_{\mathbf{G}},\,\|u\|_{1,\alpha}\le 1}\int_\Sigma \mathrm{e}^{4\pi\ell u^2}dv_g \le \mathrm{Vol}_g(\Sigma) + \lim_{R\to\infty}\limsup_{\epsilon\to 0}\int_{\bigcup_{k=1}^{I_0}\psi_k^{-1}(\mathbb{B}_{Rr_\epsilon^{1/(1+\beta_0)}})} \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2}dv_g$$
$$\le \mathrm{Vol}_g(\Sigma) + \frac{\pi I_0 V_0 \mathrm{e}^{1+4\pi\ell A_0}}{1+\beta_0}. \tag{2.59}$$

2.5 Existence of Extremal Functions

Recall that (Σ, g) has a conical singularity of the order β_0 at x_0 with $-1 < \beta_0 \le 0$, $I_0 = I(x_0)$ and $\beta_0 = \beta(x_0)$, where $I(x)$ and $\beta(x)$ are defined as in (1.8) and (1.9). In this section, we shall construct a sequence of functions $\widetilde{\Phi}_\epsilon \in \mathscr{H}_{\mathbf{G}}$ satisfying $\|\widetilde{\Phi}_\epsilon\|_{1,\alpha} = 1$, and

$$\int_\Sigma \mathrm{e}^{4\pi\ell\widetilde{\Phi}_\epsilon^2}dv_g > \mathrm{Vol}_g(\Sigma) + \frac{\pi I_0 V_0 \mathrm{e}^{1+4\pi\ell A_0}}{1+\beta_0}, \tag{2.60}$$

where A_0 and V_0 are constants defined as in (2.54) and (2.22). The contradiction between (2.59) and (2.60) implies that c_ϵ must be bounded, i.e., blow-up does not occur. This ends the proof of Theorem 1.1.

Set $R = (-\log\epsilon)^{1/(1+\beta_0)}$. It follows that $R \to \infty$ and $R\epsilon \to 0$ as $\epsilon \to 0$. Hence, when $\epsilon > 0$ is sufficiently small, $B_{g_0,2R\epsilon}(\sigma_i(x_0)) \cap B_{g_0,2R\epsilon}(\sigma_j(x_0)) = \emptyset$ for $1 \le i < j \le I_0$. We firstly define a cut-off function η on $B_{g_0,2R\epsilon}(x_0)$, which is radially symmetric with respect to x_0. Besides, we require $\eta \in C_0^\infty(B_{g_0,2R\epsilon}(x_0))$ to be a nonnegative function satisfying $\eta = 1$ on $B_{g_0,R\epsilon}(x_0)$ and $\|\nabla\eta\|_{L^\infty(B_{2R\epsilon})} = O(\frac{1}{R\epsilon})$. Then we define a sequence of functions Φ_ϵ on Σ for small $\epsilon > 0$ by

$$\Phi_\epsilon = \begin{cases} c + \dfrac{-\frac{1}{4\pi\ell}\log\big(1+\frac{\pi I_0}{1+\beta_0}\frac{d_{g_0}(x,\sigma_k(x_0))^{2(1+\beta_0)}}{\epsilon^{2(1+\beta_0)}}\big)+b}{c}, & x \in \overline{B_{g_0,R\epsilon}(\sigma_k(x_0))},\\ \dfrac{G_\alpha(x) - \eta(\sigma_k^{-1}(x))\widetilde{\Psi}(\sigma_k^{-1}(x))}{c}, & x \in B_{g_0,2R\epsilon}(\sigma_k(x_0)) \setminus \overline{B_{g_0,R\epsilon}(\sigma_k(x_0))},\\ \dfrac{G_\alpha}{c}, & x \in \Sigma \setminus \bigcup_{k=1}^{I_0}\sigma_k(B_{2R\epsilon}), \end{cases} \tag{2.61}$$

where k is taken from 1 to I_0, and $\widetilde{\Psi}$ is the function mentioned in (2.55), both b and c are constants depending on ϵ to be determined later.

Recall (2.42), $G_\alpha(\sigma(x)) = G_\alpha(x)$ for any $x \in \Sigma \setminus \bigcup_{k=1}^{I_0}\{\sigma_k(x_0)\}$ and all $\sigma \in \mathbf{G}$. Combining this with our premises that η is radially symmetric and any $\sigma \in \mathbf{G}$ is an isometric map, for sufficiently small ϵ, we conclude

$$\Phi_\epsilon(x) = \Phi_\epsilon(\sigma(x)), \quad \forall\sigma \in \mathbf{G},\ \text{a.e.}\, x \in \Sigma. \tag{2.62}$$

Set $\bar{\Phi}_\epsilon = \Phi_\epsilon - \frac{1}{\mathrm{Vol}_g(\Sigma)}\int_\Sigma \Phi_\epsilon dv_g$. We shall choose suitable b and c to make $\widetilde{\Phi}_\epsilon = \bar{\Phi}_\epsilon/\|\bar{\Phi}_\epsilon\|_{1,\alpha} \in \mathscr{H}_{\mathbf{G}}$. Since the calculation is very similar to [26, pp. 3365–3368], we omit the details but give its outline here. Integration by parts shows

$$\begin{aligned}&\int_{\Sigma\setminus\bigcup_{k=1}^{I_0}B_{g_0,R\epsilon}(\sigma_k(x_0))}|\nabla_g G_\alpha|^2 dv_g\\ &= -\sum_{k=1}^{I_0}\int_{\partial B_{g_0,R\epsilon}(\sigma_k(x_0))} G_\alpha\frac{\partial G_\alpha}{\partial n}ds_g + \int_{\Sigma\setminus\bigcup_{k=1}^{I_0}B_{g_0,R\epsilon}(\sigma_k(x_0))} G_\alpha\Delta_g G_\alpha dv_g\\ &= -\frac{1}{2\pi I_0}\log R\epsilon + A_0 + \alpha\int_\Sigma G_\alpha^2 dv_g + O(R\epsilon),\end{aligned}$$

and it follows that

$$\int_{\Sigma\setminus\bigcup_{k=1}^{I_0} B_{g_0,R\epsilon}(\sigma_k(x_0))} |\nabla_g \Phi_\epsilon|^2 dv_g = \frac{1}{c^2}\Big(-\frac{1}{2\pi I_0}\log R\epsilon + A_0 + \alpha\int_\Sigma G_\alpha^2 dv_g + O(R\epsilon)\Big).$$

Here we use estimates

$$\int_{B_{g_0,2R\epsilon}(\sigma_k(x_0))\setminus B_{g_0,R\epsilon}(\sigma_k(x_0))} |\nabla_g(\widetilde{\Psi}\eta)|^2 dv_g = O(R^2\epsilon^2),$$

and

$$\int_{B_{g_0,2R\epsilon}(\sigma_k(x_0))\setminus B_{g_0,R\epsilon}(\sigma_k(x_0))} \nabla_g G_\alpha \nabla_g(\widetilde{\Psi}\eta) dv_g = O(R\epsilon).$$

By a straightforward calculation, we obtain

$$\int_{\bigcup_{k=1}^{I_0} B_{g_0,R\epsilon}(\sigma_k(x_0))} |\nabla_g \Phi_\epsilon|^2 dv_g = \frac{1}{4\pi\ell c^2}\Big(\log\frac{\pi I_0}{1+\beta_0} - 1 + \log R^{2+2\beta_0} + O(R^{-2-2\beta_0})\Big).$$

Thus

$$\begin{aligned}\int_\Sigma |\nabla_g \Phi_\epsilon|^2 dv_g = \frac{1}{c^2}\Big(&-\frac{\log\epsilon}{2\pi I_0} + A_0 + \alpha\int_\Sigma G^2 dv_g - \frac{1}{4\pi\ell}\\ &+\frac{1}{4\pi\ell}\log\frac{\pi I_0}{1+\beta_0} + O(R^{-2-2\beta_0})\Big).\end{aligned}$$

Moreover, we have

$$\int_\Sigma |\Phi_\epsilon - \bar{\Phi}_\epsilon|^2 dv_g = \frac{1}{c^2}\left(\int_\Sigma G^2 dv_g + O(R^{-2-2\beta_0})\right).$$

In the view of $\widetilde{\Phi}_\epsilon \in W^{1,2}(\Sigma, g)$ and $\|\widetilde{\Phi}_\epsilon\|_{1,\alpha} = 1$, it follows from the above equations that

$$c^2 = -\frac{1}{2\pi I_0}\log\epsilon + A_0 - \frac{1}{4\pi\ell} + \frac{1}{4\pi\ell}\log\frac{\pi I_0}{1+\beta_0} + O(R^{-2(1+\beta_0)}),$$

and

$$b = \frac{1}{4\pi\ell} + O(R^{-2(1+\beta_0)}).$$

On $B_{g_0,R\epsilon}(\sigma_k(x_0))$, we have the following estimate:

$$\begin{aligned}4\pi\ell(1+\beta_0)\widetilde{\Phi}_\epsilon^2 \geq{}& -2\log\left(1+\frac{\pi I_0}{1+\beta_0}\frac{r^{2(1+\beta_0)}}{\epsilon^{2(1+\beta_0)}}\right) + 1 - 2(1+\beta_0)\log\epsilon\\ &+ 4\pi\ell A_0 + \log\frac{\pi I_0}{1+\beta_0} + O(R^{-2-2\beta_0}).\end{aligned}$$

Note that

$$\int_{\mathbb{B}_R} \frac{1}{(1+\frac{\pi I_0}{1+\beta_0}|y|^{2(1+\beta_0)})|y|^{2\beta_0}} = 1 - \frac{1}{1+\frac{\pi I_0}{1+\beta_0}R^{2+2\beta_0}}.$$

This leads to

$$\begin{aligned}\int_{\bigcup_{k=1}^{I_0} B_{g_0,R\epsilon}(\sigma_k(x_0))} \mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2} dv_g &= (1+O(R\epsilon))I_0\int_{\mathbb{B}_{R\epsilon}} V_0|x|^{2\beta_0}\mathrm{e}^{(4\pi\ell-\epsilon)(\widetilde{\Phi}_\epsilon^2(\exp_{x_0}x)^2} dx\\ &\geq (1+O(R\epsilon))\frac{\pi V_0 I_0 \mathrm{e}^{1+4\pi\ell A_0}}{1+\beta_0}.\end{aligned}$$

On the other hand, by $\mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2} \geq 1 + (4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2$, we obtain

$$\int_{\Sigma\setminus\bigcup_{k=1}^{I_0} B_{g_0,2R\epsilon}(\sigma_k(x_0)} \mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2} dv_g \geq \mathrm{Vol}_g(\Sigma) + \frac{4\pi\ell}{c^2}\int_\Sigma G^2 dv_g + O(R^{-2-2\beta_0}),$$

which immediately lead to

$$\int_{\Sigma} \mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2} dv_g = \int_{\bigcup_{k=1}^{I_0} B_{g_0,2R\epsilon}(\sigma_k(x_0))} \mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2} dv_g + \int_{\Sigma\setminus\bigcup_{k=1}^{I_0} B_{g_0,2R\epsilon}(\sigma_k(x_0))} \mathrm{e}^{(4\pi\ell-\epsilon)\widetilde{\Phi}_\epsilon^2} dv_g$$

$$\geq \mathrm{Vol}_g(\Sigma) + \frac{\pi V_0 I_0 \mathrm{e}^{1+4\pi\ell A_0}}{1+\beta_0} + \frac{4\pi\ell}{c^2}\int_{\Sigma} G^2 dv_g + O(R^{-2-2\beta_0}).$$

Note that $R = (-\log\epsilon)^{1/(1+\beta_0)}$, and $O(R^{-2(1+\beta_0)}) = o(1/c^2)$. If $\epsilon > 0$ is chosen sufficiently small, then we arrive at (2.60), as desired. □

3 Proof of Theorem 1.2

The method we use to proof of Theorem 1.2 is analogous to that of Theorem 1.1. Firstly, we conclude $4\pi\ell$ is the best constant for the inequality (1.21) by a discussion totally similar to that in Subsection 2.1. Then we introduce an orthonormal basis (e_j) $(1 \leq j \leq n_\ell)$ of E_ℓ satisfying

$$\begin{cases} E_\ell = \mathrm{span}\{e_1, \ldots, e_{n_\ell}\}, & \\ e_j \in C^0(\Sigma, g) \cap \mathscr{H}_{\mathbf{G}}, & \forall 1 \leq j \leq n_\ell, \\ \int_\Sigma |e_j|^2 dv_g = 1, & \forall 1 \leq j \leq n_\ell, \\ \int_\Sigma e_l e_m dv_g = 0, & m \neq l, \end{cases}$$

where $n_\ell = \dim E_\ell$. Under this orthonormal basis, $E_\ell^\perp$ is written as

$$E_\ell^\perp = \left\{ u \in \mathscr{H}_{\mathbf{G}} : \int_\Sigma u e_j dv_g = 0,\ 1 \leq j \leq n_\ell \right\}.$$

Secondly, we prove the existence of extremals for subcritical Trudinger–Moser functionals. Namely, for any $0 < \epsilon < 4\pi\ell$, there exists some $u_\epsilon \in E_\ell^\perp \cap C^1(\Sigma \setminus \{p_1, \ldots, p_L\}, g_0) \cap C^0(\Sigma, g_0)$ such that

$$\int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g = \sup_{u\in E_\ell^\perp\ \|u\|_{1,\alpha}\leq 1} \int_\Sigma \mathrm{e}^{(4\pi\ell-\epsilon)u^2} dv_g.$$

Clearly u_ϵ satisfies the Euler–Lagrange equation

$$\begin{cases} \triangle_g u_\epsilon - \alpha u_\epsilon = \dfrac{1}{\lambda_\epsilon} u_\epsilon \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2} - \dfrac{\mu_\epsilon}{\lambda_\epsilon} - \displaystyle\sum_{j=1}^{n_\ell} \omega_{j,\epsilon} e_j, \\ \|u_\epsilon\|_{1,\alpha} = 1, \\ \lambda_\epsilon = \displaystyle\int_\Sigma u_\epsilon^2 \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g, \\ \mu_\epsilon = \dfrac{1}{\mathrm{Vol}_g(\Sigma)} \displaystyle\int_\Sigma u_\epsilon \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g, \\ \omega_{j,\epsilon} = \dfrac{1}{\lambda_\epsilon} \displaystyle\int_\Sigma e_j u_\epsilon \mathrm{e}^{(4\pi\ell-\epsilon)u_\epsilon^2} dv_g. \end{cases} \tag{3.1}$$

Assume u_ϵ converges to u^* weakly in $W^{1,2}(\Sigma, g)$, strongly in $L^s(\Sigma, g)$ for any $s > 1$, and almost everywhere in Σ. If u_ϵ is uniformly bounded, then we have by the Lebesgue dominated convergence theorem

$$\int_\Sigma u^* e_j dv_g = \lim_{\epsilon\to 0} \int_\Sigma u_\epsilon e_j dv_g = 0, \quad \forall 1 \leq j \leq n_\ell,$$

and thus $u^* \in E_\ell^\perp \cap C^1(\Sigma \setminus \{p_1, \dots, p_L\}, g_0) \cap C^0(\Sigma, g_0)$ is the desired extremal function.

If blow-up happens, we still have analogs of Lemmas 2.8 and 2.9. For any $1 < q < 2$, we obtain $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $W^{1,q}(\Sigma, g)$, where G is green function satisfying

$$\begin{cases} \Delta_g G - \alpha G = \sum_{i=1}^{I_0} \frac{\delta_{\sigma_i(x_0)}}{I_0} - \frac{1}{\mathrm{Vol_g}(\Sigma)} - \sum_{j=1}^{n_\ell} e_j(x_0) e_j, \\ \int_\Sigma G e_j dv_g = 0, \quad 1 \le j \le n_\ell, \\ G(\sigma_i(x)) = G(x), \quad 1 \le i \le N, \, x \in \Sigma \setminus \{\sigma_i(x_0)\}_{i=1}^{I_0}. \end{cases}$$

As in the proof of (2.59), we can draw the conclusion that

$$\sup_{u \in E_\ell^\perp, \, \|u\|_{1,\alpha} \le 1} \int_\Sigma \mathrm{e}^{4\pi\ell u^2} dv_g \le \mathrm{Vol}_g(\Sigma) + \frac{\pi I_0 V_0}{1 + \beta_0} \mathrm{e}^{1+4\pi\ell A_0}, \tag{3.2}$$

where all the constants in (3.2) have the same definition as in the last section.

At last, we shall construct a sequence of functions to contradict (3.2). Denote

$$\omega_\epsilon = \Phi_\epsilon - \sum_{j=1}^{n_j} (\Phi_\epsilon, e_j) e_j,$$

where Φ_ϵ is defined as in (2.61), and

$$(\Phi_\epsilon, e_j) = \int_\Sigma \Phi_\epsilon e_j dv_g.$$

Set $\widetilde{\omega}_\epsilon = \omega_\epsilon - \frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma \omega_\epsilon dv_g$. We may choose suitable constants b and c to make $\widetilde{\omega}_\epsilon \in E_\ell^\perp$. A straightforward calculation shows

$$\begin{aligned} \int_\Sigma \mathrm{e}^{4\pi\ell \frac{\widetilde{\omega}_\epsilon^2}{\|\widetilde{\omega}_\epsilon\|_{1,\alpha}}} dv_g &= \int_\Sigma \mathrm{e}^{4\pi\ell \widetilde{\omega}_\epsilon^2 + o(\frac{1}{\log \epsilon})} dv_g \\ &\ge \left(1 + o\left(\frac{1}{\log \epsilon}\right)\right) \left(\mathrm{Vol}_g(\Sigma) + 4\pi I_0 \frac{\|G\|_2^2}{c^2} + \frac{\pi I_0 V_0 \mathrm{e}^{1+4\pi\ell A_0}}{1 + \beta_0}\right) \\ &\ge \mathrm{Vol}_g(\Sigma) + 4\pi I_0 \frac{\|G\|_2^2}{-\log \epsilon} + \frac{\pi I_0 V_0 \mathrm{e}^{1+4\pi\ell A_0}}{1 + \beta_0} + o\left(\frac{1}{\log \epsilon}\right). \end{aligned}$$

This indicates

$$\sup_{u \in E_\ell^\perp, \, \|u\|_{1,\alpha} \le 1} \int_\Sigma \mathrm{e}^{4\pi\ell u^2} dv_g > \mathrm{Vol}_g(\Sigma) + \frac{\pi I_0 V_0}{1 + \beta_0} \mathrm{e}^{1+4\pi\ell A_0},$$

which contradicts (3.2). Thus the proof of Theorem 1.2 is finished. □

Added in the proof: Note that similar results were also obtained in the paper (de Souza, Manassés X., Trudinger-Moser type inequalities with a symmetric conical metric and a symmetric potential. Nonlinear Analysis 223 (2022) Paper No. 113030, 23 pages).

Conflict of Interest The authors declare no conflict of interest.

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