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GUE via Frobenius Manifolds. I. From Matrix Gravity to Topological Gravity and Back

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Abstract Dubrovin establishes a certain relationship between the GUE partition function and the partition function of Gromov–Witten invariants of the complex projective line. In this paper, we give a direct proof of Dubrovin's result. We also present in a diagram the recent progress on topological gravity and matrix gravity.

Keywords Frobenius manifold, Dubrovin–Zhang hierarchy, GUE, Toda lattice hierarchy, jet space, topological gravity, matrix gravity

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1 Introduction

For $n \geq 1$ being an integer, denote by $\mathcal{H}(n)$ the space of $n \times n$ hermitian matrices. The *normalized Gaussian Unitary Ensemble* (*GUE*) *partition function of size* n is defined by

$$Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon) = 2^{-n/2}(\pi\epsilon)^{-n^2/2}\int_{\mathcal{H}(n)} \mathrm{e}^{-\frac{1}{\epsilon}\mathrm{tr}\,Q(M;\mathbf{s})}dM, \tag{1}$$

where $\mathbf{s} := (s_1, s_2, \ldots)$, $Q(y;\mathbf{s})$ is a power series in y of the form

$$Q(y;\mathbf{s}) = \frac{1}{2}y^2 - \sum_{j\geq 1} s_j y^j, \tag{2}$$

and $dM = \prod_{1\leq i\leq n} dM_{ii} \prod_{1\leq i<j\leq n} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}$. For the interest of the present paper, we understand this integral in the way of first expanding the integrand as a power series of $\mathbf{s}$ then integrating the coefficient of each monomial of $\mathbf{s}$ with respect to the measure dM, and we note that the factor $2^{-n/2}(\pi\epsilon)^{-n^2/2}$ in front of this integral is a normalization factor so that $Z_n^{\mathrm{GUE1}}(\mathbf{0};\epsilon) \equiv 1$.

The integral in (1) is closely related to the enumeration of ribbon graphs (cf. [6–8, 15, 37, 47–51, 58, 65, 71]). Denote by $\mathcal{R}_{f;j_1,\ldots,j_k}$ the set of oriented not-necessarily connected ribbon graphs having f faces and k vertices with valencies $j_1, \ldots, j_k$, and by $\mathcal{R}^{\mathrm{conn}}_{f;j_1,\ldots,j_k} \subset \mathcal{R}_{f;j_1,\ldots,j_k}$ the subset consisting of the *connected* ones. Then the partition function $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$ has the expression:

$$Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon) = 1 + \sum_{f,k\geq 1}\sum_{j_1,\ldots,j_k\geq 1} b(f;\mathbf{j}) s_{j_1}\cdots s_{j_k} n^f \epsilon^{\frac{|\mathbf{j}|}{2}-k}, \tag{3}$$

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where $\mathbf{j}=(j_1,\ldots,j_k)$, $|\mathbf{j}|=j_1+\cdots+j_k$, and

$$b(f;\mathbf{j})=\sum_{G\in\mathcal{R}_{f;\mathbf{j}}}\frac{j_1\cdots j_k}{|\mathrm{Aut}(G)|}. \tag{4}$$

Applying Euler's formula to ribbon graphs, we see that $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)\in\mathbb{Q}[n,\epsilon,\epsilon^{-1}][[\mathbf{s}]]$. By further taking the logarithms on both sides of (3) we obtain

$$\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)=\sum_{f,k\geq1}\sum_{j_1,\ldots,j_k\geq1}a(f;\mathbf{j})s_{j_1}\cdots s_{j_k}n^f\epsilon^{\frac{|\mathbf{j}|}{2}-k}\in\mathbb{Q}[n,\epsilon,\epsilon^{-1}][[\mathbf{s}]], \tag{5}$$

where

$$a(f;\mathbf{j})=\sum_{G\in\mathcal{R}^{\mathrm{conn}}_{f;\mathbf{j}}}\frac{j_1\cdots j_k}{|\mathrm{Aut}(G)|}. \tag{6}$$

Following t'Hooft [48, 49], introduce

$$x:=n\epsilon. \tag{7}$$

Define

$$Z^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)=Z^{\mathrm{GUE1}}_{x/\epsilon}(\mathbf{s};\epsilon),\quad \mathcal{F}^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)=\log Z^{\mathrm{GUE1}}_{x/\epsilon}(\mathbf{s};\epsilon), \tag{8}$$

and we have

$$Z^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)=1+\sum_{k,f\geq1}\sum_{j_1,\ldots,j_k\geq1}b(f;\mathbf{j})s_{j_1}\cdots s_{j_k}x^f\epsilon^{\frac{|\mathbf{j}|}{2}-k-f}\in\mathbb{Q}[x](\epsilon^2)[[\mathbf{s}]], \tag{9}$$

$$\mathcal{F}^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)=\sum_{k\geq1}\sum_{\substack{g\geq0,\,j_1,\ldots,j_k\geq1\\ 2-2g-k+|\mathbf{j}|/2\geq1}}a_g(\mathbf{j})s_{j_1}\cdots s_{j_k}\epsilon^{2g-2}x^{2-2g-k+\frac{|\mathbf{j}|}{2}}\in\epsilon^{-2}\mathbb{Q}[x,\epsilon^2][[\mathbf{s}]], \tag{10}$$

where $a_g(\mathbf{j}):=a(2-2g-k+|\mathbf{j}|/2;\mathbf{j})$, and we used Euler's formula

$$k-\frac{|\mathbf{j}|}{2}+f=2-2g \tag{11}$$

for a connected ribbon graph of genus g. We call $Z^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)$ the *normalized GUE partition function*, and $\mathcal{F}^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)$ the *normalized GUE free energy*.

As in e.g. [21, 27], define the *corrected GUE free energy* $\mathcal{F}(x,\mathbf{s};\epsilon)$ by

$$\mathcal{F}(x,\mathbf{s};\epsilon):=\mathcal{F}^{\mathrm{GUE1}}(x,\mathbf{s};\epsilon)+\frac{x^2}{2\epsilon^2}\left(\log x-\frac{3}{2}\right)-\frac{\log x}{12}+\zeta'(-1)+\sum_{g\geq2}\frac{\epsilon^{2g-2}B_{2g}}{4g(g-1)x^{2g-2}}, \tag{12}$$

where $\zeta(s)$ denotes the Riemann zeta function, and B_m denotes the mth Bernoulli number. The *corrected GUE partition function* $Z(x,\mathbf{s};\epsilon)$ is defined as $\mathrm{e}^{\mathcal{F}(x,\mathbf{s};\epsilon)}$. Clearly,

$$\epsilon^2\mathcal{F}(x,\mathbf{s};\epsilon)\in\mathbb{C}[[\epsilon^2]][[x-1,\mathbf{s}]],\quad Z(x,\mathbf{s};\epsilon)\in\mathbb{C}((\epsilon^2))[[x-1,\mathbf{s}]]. \tag{13}$$

Equivalently, the corrected GUE partition function $Z(x,\mathbf{s};\epsilon)$ can be defined as

$$\frac{(2\pi)^{-n}\epsilon^{-\frac{1}{12}}}{\mathrm{Vol}(n)}\int_{\mathcal{H}(n)}\mathrm{e}^{-\frac{1}{\epsilon}\mathrm{tr}\,Q(M;\mathbf{s})}dM,\quad x=n\epsilon, \tag{14}$$

where

$$\mathrm{Vol}(n):=\mathrm{Vol}\left(U(n)/U(1)^n\right)=\frac{\pi^{\frac{n(n-1)}{2}}}{G(n+1)},\quad G(n+1)=\prod_{j=1}^{n-1}j!. \tag{15}$$

To see this equivalence, we view $G(n+1)$ as an analytic function (G denotes Barnes' G-function), then together with (9) we see that the coefficient of each monomial of $\mathbf{s}$ in $Z(x,\mathbf{s};\epsilon)$ defined by (14) is an analytic function of x,ϵ, and by taking the $n=x/\epsilon\to\infty$ asymptotics in these coefficients we obtain the equivalence. Here one needs to use the fact that Barnes' G-function (cf. [5, 39, 64]) has the asymptotic expansion:

$$\log G(z+1)\sim\frac{z^2}{2}\left(\log z-\frac{3}{2}\right)+\frac{z}{2}\log 2\pi-\frac{\log z}{12}+\zeta'(-1)+\sum_{\ell\geq 1}\frac{B_{2\ell+2}}{4\ell(\ell+1)z^{2\ell}}. \tag{16}$$

For simplicity of terminology, we refer to the corrected GUE partition function (resp., corrected GUE free energy) as the *GUE partition function* (resp., *GUE free energy*), as we do in e.g. [26–28]. Let $\mathcal{F}_g(x,\mathbf{s}):=\mathrm{Coef}(\epsilon^{2g-2},\mathcal{F}(x,\mathbf{s};\epsilon))$, $g\geq 0$. We call $\mathcal{F}_g(x,\mathbf{s})$ the *genus g part of the GUE free energy* (for short the *genus g GUE free energy*). It is also helpful to recall that the GUE partition function $Z(x,\mathbf{s};\epsilon)$ satisfies the following dilaton and string equations, respectively:

$$\sum_{j\geq 1}\left(s_j-\frac{1}{2}\delta_{j,2}\right)\frac{\partial Z(x,\mathbf{s};\epsilon)}{\partial s_j}+x\frac{\partial Z(x,\mathbf{s};\epsilon)}{\partial x}+\epsilon\frac{\partial Z(x,\mathbf{s};\epsilon)}{\partial \epsilon}+\frac{Z(x,\mathbf{s};\epsilon)}{12}=0, \tag{17}$$

$$\sum_{j\geq 1}j\left(s_j-\frac{1}{2}\delta_{j,2}\right)\frac{\partial Z(x,\mathbf{s};\epsilon)}{\partial s_{j-1}}+\frac{xs_1}{\epsilon^2}Z(x,\mathbf{s};\epsilon)=0. \tag{18}$$

The geometric way in understanding the GUE partition function is through the theory of integrable systems (Frobenius manifolds, tau-functions, bihamiltonian structures, etc.). Denote by $\Lambda=\mathrm{e}^{\epsilon\partial_x}$ the shift operator, and let

$$V(x,\mathbf{s};\epsilon)=\epsilon(\Lambda-1)\frac{\partial\log Z(x,\mathbf{s};\epsilon)}{\partial s_1},\quad W(x,\mathbf{s};\epsilon)=\epsilon^2\frac{\partial^2\log Z(x,\mathbf{s};\epsilon)}{\partial s_1\partial s_1}. \tag{19}$$

It is known (cf. e.g. [1]) that the power series $(V(x,\mathbf{s};\epsilon),W(x,\mathbf{s};\epsilon))$ is a particular solution to the *Toda lattice hierarchy* [40, 57], i.e., the difference operator L defined by

$$L=\Lambda+V(x,\mathbf{s};\epsilon)+W(x,\mathbf{s};\epsilon)\Lambda^{-1} \tag{20}$$

satisfies the following Lax-type equations:

$$\epsilon\frac{\partial L}{\partial s_j}=\left[(L^j)_+,L\right],\quad j\geq 1. \tag{21}$$

Moreover, $Z(x,\mathbf{s};\epsilon)$ is a tau-function of the solution $(V(x,\mathbf{s};\epsilon),W(x,\mathbf{s};\epsilon))$ to the Toda lattice hierarchy. Here and below, for a difference operator P in its normal form $P=\sum_{m\in\mathbb{Z}}P_m\Lambda^m$, $P_+:=\sum_{m\geq 0}P_m\Lambda^m$, $P_-:=\sum_{m<0}P_m\Lambda^m$, and $\operatorname{res}P:=P_0$. We note that the functions $V(x,\mathbf{s};\epsilon),W(x,\mathbf{s};\epsilon)$ can be uniquely determined by the Toda lattice hierarchy along with the initial data

$$V(x,\mathbf{0};\epsilon)=0,\quad W(x,\mathbf{0};\epsilon)=x. \tag{22}$$

The definition of a tau-function for the Toda lattice hierarchy as well as the proof of these statements will be reviewed in Section 2. Equations (17)–(18), the condition $\mathcal{F}(x,\mathbf{s};\epsilon)\in\epsilon^{-2}\mathbb{Q}[[\epsilon^2]][[x-1,\mathbf{s}]]$ and the tau-function statement all together uniquely determine the partition function $Z(x,\mathbf{s};\epsilon)=\mathrm{e}^{\mathcal{F}(x,\mathbf{s};\epsilon)}$ up to a pure constant factor.

The Frobenius manifold [17, 18, 33, 34] that corresponds to the Toda lattice hierarchy has the potential [18]:

$$F=\frac{1}{2}v^2u+\mathrm{e}^u, \tag{23}$$

where v, u are the flat coordinates with $\partial/\partial v$ being the unit vector field. More precisely, the differential of the generating function for the hamiltonian densities of the dispersionless Toda lattice hierarchy is a flat section of the Dubrovin connection [18] of the Frobenius manifold (23). It is helpful to note that this Frobenius manifold can also be obtained from the Gromov–Witten (GW) invariants of $\mathbb{P}^1$ [18, 53]. Indeed, the potential F equals, up to a quadratic function, the so-called genus zero primary free energy of these GW invariants. We often call the Frobenius manifold with potential (23) the $\mathbb{P}^1$-Frobenius manifold.

For a Frobenius manifold, Dubrovin [18] constructs an integrable hierarchy of tau-symmetric hamiltonian PDEs of hydrodynamic type, called the *principal hierarchy*. This integrable hierarchy has a particular solution called the *topological solution*. In [32] Dubrovin and Zhang prove that the tau-function of the topological solution to the principal hierarchy (exponential of the genus zero free energy for the topological solution) satisfies the genus zero Virasoro constraints (see also Liu and Tian [56]).

For a *semisimple* Frobenius manifold, by solving Virasoro constraints in the form of the so-called *loop equation*, Dubrovin and Zhang [33] (cf. [23]) construct the partition function of the Frobenius manifold, and use it to define the topological deformation of the principal hierarchy, now called the *Dubrovin–Zhang (DZ) integrable hierarchy* for the Frobenius manifold (cf. [10]). By their construction, the partition function of the Frobenius manifold is a particular tau-function called the *topological tau-function* for the DZ hierarchy, that is the tau-function of a particular solution, called the *topological solution*, to the DZ hierarchy. In particular, if the semisimple Frobenius manifold comes from the quantum cohomology of a smooth projective variety X, the partition function of the Frobenius manifold equals the partition function of the GW invariants of X [23, 33, 45, 46, 62]. A key notion in the construction of Dubrovin and Zhang is the *jet space* [16, 33, 44, 65], not only because the solution (the free energy in higher genera) to the DZ loop equation is represented in terms of jet variables (jets for short) leading to uniqueness, but also due to the validity at the level of integrable hierarchy (free property of jets). More precisely, firstly, the free energy in higher genera gives rise to the topological tau-function when the jet variables are subjected to the topological solution of the principal hierarchy. Secondly, by construction the DZ hierarchy is quasi-trivial, namely, it is obtained from the principal hierarchy under a quasi-Miura transformation, which is given by the higher genera free energy in terms of jets; and this is interesting, because the quasi-Miura transformation could be substituted by any monotone solution to the principal hierarchy and makes it become a solution to the DZ hierarchy [21, 23, 29, 33, 34, 69]. In [69], it is suggested to interpret this as a universality class of Dubrovin [19, 20, 22, 24]. A particular dense subset of monotone solutions to the principal hierarchy can be obtained by performing time shifts in the topological solution, but there are also other interesting monotone solutions. They lead to solutions to the DZ hierarchy: all these solutions are beautifully connected to the topological solution (GW invariants in the case of quantum cohomology).

The quantum cohomology of $\mathbb{P}^1$ gives a semisimple Frobenius manifold. As we have mentioned above, the potential of this Frobenius manifold is given by (23), and the dispersionless limit of the Toda lattice hierarchy (21) form a part of the principal hierarchy (often called the stationary flows) of this Frobenius manifold. According to Dubrovin and Zhang [34] the corresponding DZ hierarchy is *normal Miura equivalent* to the extended Toda lattice hierarchy [11], with an explicit formula of the normal Miura transformation. The DZ hierarchy is quasi-trivial, so is the extended Toda lattice hierarchy. According to the above discussion, their quasi-trivial transformations can be obtained from the free energy of GW invariants of $\mathbb{P}^1$. For example, in [69], the *dessins/LUE solution* to the Toda lattice hierarchy is considered and it is shown that this solution can be obtained by the application of quasi-triviality.

In this paper we study the *GUE solution* to the Toda lattice hierarchy (cf. [1, 14, 27, 66]) by using the DZ approach. Denote by $\mathbf{v}(x,\mathbf{s}) = (v(x,\mathbf{s}), u(x,\mathbf{s}))$ the unique power-series-in-$\mathbf{s}$ solution to the principal hierarchy (30) with the initial condition

$$v(x,\mathbf{0}) = 0, \quad u(x,\mathbf{0}) = \log x. \tag{24}$$

Based on the DZ approach [33, 34], we will give a new proof to the following theorem.

Theorem 1.1 (Dubrovin [21]) *The genus zero GUE free energy $\mathcal{F}_0(x,\mathbf{s})$ has the expression:*

$$\begin{aligned}\mathcal{F}_0(x,\mathbf{s}) = {} & \frac{1}{2}\sum_{p,q\geq 0}(p+1)!(q+1)!\left(s_{p+1}-\frac{1}{2}\delta_{p,1}\right)\left(s_{q+1}-\frac{1}{2}\delta_{q,1}\right)\Omega_{2,p;2,q}(\mathbf{v}(x,\mathbf{s}))\\ & + x\sum_{p\geq 0}(p+1)!\left(s_{p+1}-\frac{1}{2}\delta_{p,1}\right)\theta_{2,p}(\mathbf{v}(x,\mathbf{s})) + \frac{1}{2}x^2u(x,\mathbf{s}).\end{aligned} \tag{25}$$

For $g \geq 1$, the genus g GUE free energy $\mathcal{F}_g(x,\mathbf{s})$ can be represented by

$$\mathcal{F}_g(x,\mathbf{s}) = F_g^{\mathbb{P}^1}\left(\mathbf{v}(x,\mathbf{s}), \frac{\partial \mathbf{v}(x,\mathbf{s})}{\partial x}, \ldots, \frac{\partial^{3g-2}\mathbf{v}(x,\mathbf{s})}{\partial x^{3g-2}}\right) + \left(\zeta'(-1) - \frac{1}{24}\log(-1)\right)\delta_{g,1}. \tag{26}$$

Here, $F_g^{\mathbb{P}^1}(z_1,\ldots,z_{3g-2})$ $(g \geq 1)$ denotes the genus g free energy in jets of the $\mathbb{P}^1$-Frobenius manifold (*see* (77) *of Section* 3).

Originally, a proof of this theorem was outlined in [21], where the terminology of *vacuum tau-function* [33] is used. Dubrovin also found [21] that the GUE partition function can be identified with part of the partition function of the Frobenius manifold with potential

$$F = \frac{1}{2}(u^1)^2u^2 + \frac{1}{2}(u^2)^2\log u^2 - \frac{3}{4}(u^2)^2, \tag{27}$$

realizing the space/time duality in the concrete example (cf. [11, 42]; in genus zero: the Legendre-type transformation [18] of Dubrovin). This Frobenius manifold is often called an *NLS Frobenius manifold* [11, 12, 18], and will be discussed in details in the next of the article-series. Our proof given in Section 3 will be a relatively more direct one, which is similar to the one given recently in [69] for a result for Grothendieck's dessins d'enfant/Laguerre Unitary Ensemble (LUE).

According to Witten [65], the GUE partition function being restricted to the even couplings corresponds to the matrix gravity. We find that performing a further restriction given by a certain explicit and rigorous limit in the jet space for the higher genera parts for the even GUE partition function yields those for Witten's topological gravity (the celebrated Witten–Kontsevich partition function); see Corollary 4.6. That means that, at least for these higher genera parts, the matrix gravity contains all the information of the topological gravity. Usually, to come back to the matrix gravity, one needs a deformation theory [25, 26, 38, 46]. But, the recent studies [26, 28, 30, 68] all together show that one can start with Witten's topological gravity and come back to the matrix gravity *without* a deformation theory, again at least in the higher genera. More precisely, we first go to the special cubic Hodge partition function by a *space/time duality* [18, 67, 68] (see also [2, 3]) (in [68] this is revealed by the *Hodge-BGW correspondence*), and then go to the so-called modified even GUE partition function by the Hodge-GUE correspondence [26, 28], and finally back to the even GUE partition function via a product formula [30], again at least to the higher genera in jets. (We note that the genus zero parts for the above-mentioned models are relatively easy, so for us the non-trivial things are in higher genera.) As a summary, we draw the following diagram:

$$
\begin{array}{ccccc}
F_g^{\mathrm{WK}} & \longleftarrow & F_g^{\mathrm{even}} & \longleftarrow & F_g^{\mathbb{P}^1} \\
\Big\updownarrow {\scriptstyle via\, s./t.\, duality} & & \Big\updownarrow {\scriptstyle via\, a\, pro.\, formula} & & \\
H_g & \overset{\mathrm{Hodge-GUE}}{=\!=\!=\!=} & \widetilde{F}_g & &
\end{array}
$$

Here, $g \geq 1$, each of these functions lives in a certain jet space, and $F_g^{\mathbb{P}^1}$, F_g^{WK}, F_g^{even}, $\widetilde{F}_g$, H_g stand for the genus g free energies in jets for GW invariants of $\mathbb{P}^1$, the Witten–Kontsevich correlators, the even GUE correlators, the modified even GUE correlators, and certain special cubic Hodge integrals, respectively. Each one-direction arrow means taking a certain restriction or say limit (see (131), (146)–(147), (117)), the long "=" simply means equal (see (145)), the double-direction arrow between F_g^{WK} and H_g means the two are related by an invertible change of their independent jet-variables (see (119) or (123)) up to a scalar $(-4)^{g-1}$, and the double-direction arrow between F_g^{even}'s and $\widetilde{F}_g$'s means they are related by an invertible operation (see (137) or (138); note that in this case there is a shuffling in genus), which comes from an invertible *product formula* [30].

Organization of the paper In Section 2, we review Toda lattice hierarchy and GUE. In Section 3, we prove Theorem 1.1. In Section 4 we present a discussion on topological gravity and matrix gravity.

2 Frobenius Manifold, Toda Lattice Hierarchy and GUE

This section contains materials of several known results about $\mathbb{P}^1$-Frobenius manifold, Toda lattice hierarchy and GUE. We refer [18, 27, 66] to the reader for further interest.

2.1 Principal Hierarchy and Genus Zero Free Energy

Consider the $\mathbb{P}^1$-Frobenius manifold, denoted by M, which has the potential (23). Denote by η the invariant flat metric, and denote $v^1 = v$, $v^2 = u$, $\mathbf{v} = (v, u)$. Following [33, 34] (see also [25]), we fix the calibration $\theta_{\alpha,p}(\mathbf{v})$ ($\alpha = 1, 2, p \geq 0$) for this Frobenius manifold via the generating series

$$\theta_1(\mathbf{v}; z) := \sum_{p\geq 0} \theta_{1,p}(\mathbf{v}) z^p = -2\mathrm{e}^{zv} \sum_{m\geq 0} \left(\gamma - \frac{1}{2}u + \psi(m+1)\right) \mathrm{e}^{mu} \frac{z^{2m}}{m!^2}, \tag{28}$$

$$\theta_2(\mathbf{v}; z) := \sum_{p\geq 0} \theta_{2,p}(\mathbf{v}) z^p = z^{-1} \left(\sum_{m\geq 0} \mathrm{e}^{mu+zv} \frac{z^{2m}}{m!^2} - 1\right), \tag{29}$$

where γ is the Euler constant and ψ denotes the digamma function. (We recall that the calibration is a choice of a family of tau-symmetric hamiltonian densities for the principal hierarchy.) The associated *principal hierarchy* [18] reads

$$\frac{\partial v^\alpha}{\partial T^{\beta,q}} = \sum_{\gamma=1}^{2} \eta^{\alpha\gamma}\partial_x\left(\frac{\partial\theta_{\beta,q+1}(\mathbf{v})}{\partial v^\gamma}\right), \quad q\geq 0,\ \alpha,\beta=1,2, \tag{30}$$

where $\eta^{\alpha\gamma}=\delta_{\alpha+\gamma,3}$, $\alpha,\gamma=1,2$. As in [18], define a family of holomorphic functions $\Omega^{[0]}_{\alpha,p;\beta,q}(\mathbf{v})$ on M, called the *genus zero two-point correlation functions*, via

$$\sum_{p,q\geq 0}\Omega^{[0]}_{\alpha,p;\beta,q}(\mathbf{v})z^p y^q = \frac{1}{z+y}\left(\sum_{\rho,\sigma=1}^{2}\frac{\partial\theta_\alpha(\mathbf{v};z)}{\partial v^\rho}\eta^{\rho\sigma}\frac{\partial\theta_\beta(\mathbf{v};y)}{\partial v^\sigma}-\eta_{\alpha\beta}\right), \quad \alpha,\beta=1,2. \tag{31}$$

For an arbitrary solution $\mathbf{v}(\mathbf{T})$ to the principal hierarchy (30), there exists a function $\mathcal{F}_0^M(\mathbf{T})$ such that

$$\frac{\partial^2\mathcal{F}_0^M(\mathbf{T})}{\partial T^{\alpha,p}\partial T^{\beta,q}} = \Omega^{[0]}_{\alpha,p;\beta,q}(\mathbf{v}(\mathbf{T})), \quad \alpha=1,2,\ p,q\geq 0. \tag{32}$$

We call $\mathcal{F}_0^M(\mathbf{T})$ the *genus zero free energy* of the solution $\mathbf{v}(\mathbf{T})$ to the principal hierarchy (30), and call the exponential $\exp(\mathcal{F}_0^M(\mathbf{T}))$ the *tau-function* of the solution $\mathbf{v}(\mathbf{T})$ to the principal hierarchy (30).

As we have briefly mentioned in the Introduction, the $T^{2,q}$-flows of the principal hierarchy (30) coincide with the dispersionless limit of the Toda lattice hierarchy (21) under

$$\frac{\partial}{\partial T^{2,p}} = \frac{1}{(p+1)!}\frac{\partial}{\partial s_{p+1}}, \quad p\geq 0. \tag{33}$$

We also mentioned in the Introduction that the potential F of the $\mathbb{P}^1$-Frobenius manifold equals, up to a quadratic function, the genus zero primary free energy of the GW invariants of $\mathbb{P}^1$. More details about the GW invariants will be given in Section 3.

2.2 Review on tau-functions for the Toda lattice hierarchy

Let

$$\mathcal{A} := \mathbb{Z}[V(x),W(x),V(x\pm\epsilon),W(x\pm\epsilon),\dots] \tag{34}$$

be the ring of polynomials with integer coefficients. The second-order difference operator $L=\Lambda+V(x)+W(x)\Lambda^{-1}$ (cf. (20)) can be written in the matrix form

$$\mathcal{L}=\Lambda+U(\lambda), \quad U(\lambda)=\begin{pmatrix} V(x)-\lambda & W(x)\\ -1 & 0\end{pmatrix}, \tag{35}$$

where we recall that Λ is the shift operator: $\Lambda=\mathrm{e}^{\epsilon\partial_x}$.

Lemma 2.1 ([27]) *There exists a unique* 2×2 *matrix series*

$$R(\lambda)=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\mathrm{O}(\lambda^{-1})\in\mathrm{Mat}(2,\mathcal{A}[[\lambda^{-1}]]) \tag{36}$$

satisfying the equation

$$\Lambda(R(\lambda))U(\lambda)-U(\lambda)R(\lambda)=0 \tag{37}$$

along with the normalization conditions

$$\operatorname{tr}R(\lambda)=1, \quad \det R(\lambda)=0. \tag{38}$$

The unique series $R(\lambda)$ in the above lemma is called the *basic matrix resolvent* of $\mathcal{L}$. Following [27, 66], define $\omega_{i,j} \in \mathcal{A}$ $(i, j \geq 1)$ via the generating series

$$\sum_{i,j\geq 1} \frac{\omega_{i,j}}{\lambda^{i+1}\mu^{j+1}} = \frac{\operatorname{tr} R(\lambda)R(\mu) - 1}{(\lambda-\mu)^2}, \tag{39}$$

and define $\varphi_j = \operatorname{Coef}(\lambda^{-j-1}, (\Lambda(R(\lambda)))_{21}) \in \mathcal{A}$, $j \geq 1$.

Lemma 2.2 ([27]) *For an arbitrary solution $(V(x,\mathbf{s};\epsilon), W(x,\mathbf{s};\epsilon))$ to the Toda lattice hierarchy* (21)*, there exists a function $\tau(x,\mathbf{s};\epsilon)$ such that*

$$\epsilon^2 \frac{\partial^2 \log \tau(x,\mathbf{s};\epsilon)}{\partial s_i \partial s_j} = \omega_{i,j}(x,\mathbf{s};\epsilon), \quad i,j \geq 1, \tag{40}$$

$$\frac{\partial}{\partial s_j} \log \frac{\tau(x+\epsilon,\mathbf{s};\epsilon)}{\tau(x,\mathbf{s};\epsilon)} = \varphi_j(x,\mathbf{s};\epsilon), \quad j \geq 1, \tag{41}$$

$$\frac{\tau(x+\epsilon,\mathbf{s};\epsilon)\tau(x-\epsilon,\mathbf{s};\epsilon)}{\tau(x,\mathbf{s};\epsilon)^2} = W(x,\mathbf{s};\epsilon), \tag{42}$$

where $\omega_{i,j}(x,\mathbf{s};\epsilon)$ and $\varphi_j(x,\mathbf{s};\epsilon)$ mean the substitution of $(V(x,\mathbf{s};\epsilon), W(x,\mathbf{s};\epsilon))$ in the corresponding elements in $\mathcal{A}$.

The function $\tau(x,\mathbf{s};\epsilon)$ is determined by the solution $(V(x,\mathbf{s};\epsilon), W(x,\mathbf{s};\epsilon))$ up to

$$\tau(x,\mathbf{s};\epsilon) \mapsto \mathrm{e}^{c+bn+\sum_{j\geq 1} a_{j-1}s_j} \tau(x,\mathbf{s};\epsilon), \tag{43}$$

where c, b and a's can depend on ϵ. We call the function $\tau(x,\mathbf{s};\epsilon)$ the *DZ type tau-function* of the solution $(V(x,\mathbf{s};\epsilon), W(x,\mathbf{s};\epsilon))$ to the Toda lattice hierarchy, for short a Toda tau-function of the solution $(V(x,\mathbf{s};\epsilon), W(x,\mathbf{s};\epsilon))$. The elements $\omega_{i,j} \in \mathcal{A}$ $(i, j \geq 1)$ are called two-point correlation functions of the Toda lattice hierarchy.

Our definition of a Toda tau-function agrees with the one given in [11, 34]. Indeed, firstly, recall from [27] that the two-point correlation functions $\omega_{i,j}$ are associated with the tau-symmetric hamiltonian densities for the Toda lattice hierarchy; secondly, by taking the dispersionless limit $\epsilon \to 0$ in the above definition (symbolically put $v = \lim_{\epsilon\to 0} V(x)$ and $w = \lim_{\epsilon \to 0} W(x)$) and by using Lemma 2.1 (see [27, 66] for some more details), one immediately obtains that

$$\sum_{i,j\geq 1} \frac{\omega^{[0]}_{i,j}}{\lambda^{i+1}\mu^{j+1}} = \frac{B(\lambda)B(\mu)((\lambda-v)(\mu-v)-4w)-1}{2\,(\lambda-\mu)^2}, \tag{44}$$

$$\frac{1}{\lambda} + \sum_{i\geq 1} \frac{\varphi^{[0]}_i}{\lambda^{i+1}} = B(\lambda), \tag{45}$$

where

$$B(\lambda) = \frac{1}{\sqrt{(\lambda-v)^2-4w}} = \frac{1}{\lambda} + \frac{v}{\lambda^2} + \frac{v^2+2w}{\lambda^3} + \frac{v^3+6vw}{\lambda^4} + \mathrm{O}(\lambda^{-5}), \tag{46}$$

and the verification in the dispersionless limit, i.e., of the following equalities

$$\omega^{[0]}_{i,j} = i!j!\Omega^{[0]}_{2,i-1;2,j-1}, \quad \varphi^{[0]}_i = i!\theta_{2,i-1}, \quad i,j \geq 1, \tag{47}$$

(cf. (29), (31), (33)) is straightforward.

2.3 GUE Partition Function as a Toda Tau-function

Recalling Gaussian integral

$$\int_{\mathcal{H}(n)} \mathrm{e}^{-\frac{1}{2\epsilon}\operatorname{tr} M^2} dM = 2^{\frac{n}{2}} (\pi\epsilon)^{\frac{n^2}{2}}, \quad n \geq 1, \tag{48}$$

we get $Z_n^{\mathrm{GUE1}}(\mathbf{0};\epsilon) \equiv 1$. For $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$, there is the well-known formula (cf. [14, 27, 58])

$$\int_{\mathcal{H}(n)} \mathrm{e}^{-\frac{1}{\epsilon}\operatorname{tr} Q(M;\mathbf{s})} dM = \frac{1}{n!}\mathrm{Vol}(U(n)/U(1)^n) \int_{\mathbb{R}^n} \Delta_n(\lambda_1,\dots,\lambda_n)^2 \mathrm{e}^{-\frac{1}{\epsilon}\sum_{k=1}^n Q(\lambda_k;\mathbf{s})} d\lambda_1 \cdots d\lambda_n,$$

where

$$\Delta_n(\lambda_1,\dots,\lambda_n) = \prod_{1\leq i<j\leq n} (\lambda_i - \lambda_j), \quad n \geq 1.$$

One can apply the theory of orthogonal polynomials (cf. e.g. [14]) for a further computation of $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$. Let $(\cdot,\cdot)$ be an inner product on the space of polynomials defined by

$$(f,g) = \int_{-\infty}^{+\infty} f(\lambda) g(\lambda) \mathrm{e}^{-\frac{1}{\epsilon} Q(\lambda;\mathbf{s})} d\lambda, \quad \forall\, f,g. \tag{49}$$

Let

$$p_m = p_m(\lambda;\mathbf{s};\epsilon) = \lambda^m + a_{1m}(\mathbf{s};\epsilon)\lambda^{m-1} + \cdots + a_{mm}(\mathbf{s};\epsilon), \quad m \geq 0 \tag{50}$$

be a system of monic polynomials orthogonal with respect to $(\cdot,\cdot)$, i.e.,

$$(p_{m_1}(\lambda;\mathbf{s};\epsilon), p_{m_2}(\lambda;\mathbf{s};\epsilon)) =: h_{m_1}(\mathbf{s};\epsilon)\delta_{m_1 m_2}, \quad \forall\, m_1, m_2 \geq 0. \tag{51}$$

Observing that $\Delta_n(\lambda_1,\dots,\lambda_n)$ can be written into the form

$$\Delta_n(\lambda_1,\dots,\lambda_n) = \det \begin{pmatrix} p_0(\lambda_1;\mathbf{s};\epsilon) & p_0(\lambda_2;\mathbf{s};\epsilon) & \dots & p_0(\lambda_n;\mathbf{s};\epsilon) \\ p_1(\lambda_1;\mathbf{s};\epsilon) & p_1(\lambda_2;\mathbf{s};\epsilon) & \dots & p_1(\lambda_n;\mathbf{s};\epsilon) \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ p_{n-1}(\lambda_1;\mathbf{s};\epsilon) & p_{n-1}(\lambda_2;\mathbf{s};\epsilon) & \dots & p_{n-1}(\lambda_n;\mathbf{s};\epsilon) \end{pmatrix}_{n\times n}, \tag{52}$$

we find

$$\int_{\mathcal{H}(n)} \mathrm{e}^{-\frac{1}{\epsilon}\operatorname{tr} Q(M;\mathbf{s})} dM = \mathrm{Vol}(U(n)/U(1)^n) h_0(\mathbf{s};\epsilon) \cdots h_{n-1}(\mathbf{s};\epsilon), \quad n \geq 1. \tag{53}$$

At $\mathbf{s} = 0$ the orthogonal polynomials $p_m(\lambda;\mathbf{0};\epsilon)$ have the explicit expressions

$$p_m(\lambda;\mathbf{0}) = \epsilon^{\frac{m}{2}} \mathrm{He}_m(\lambda/\epsilon^{1/2}), \quad m \geq 0, \tag{54}$$

where $\mathrm{He}_m(s)$ are the hermite polynomials. Recalling that

$$\int_{-\infty}^{\infty} \mathrm{He}_{m_1}(t)\mathrm{He}_{m_2}(t) \mathrm{e}^{-\frac{1}{2}t^2} dt = \sqrt{2\pi} m_1! \delta_{m_1 m_2}, \tag{55}$$

we find

$$h_m(\mathbf{s} = \mathbf{0};\epsilon) = \epsilon^{m+\frac{1}{2}} \sqrt{2\pi} m!. \tag{56}$$

Using (48), (53) and (56), we obtain

$$\mathrm{Vol}(U(n)/U(1)^n) = \frac{\pi^{\frac{n^2-n}{2}}}{\prod_{j=1}^{n-1} j!}. \tag{57}$$

Therefore,

$$Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon) = \frac{(2\pi)^{-\frac{n}{2}}\epsilon^{-\frac{n^2}{2}}}{\prod_{j=1}^{n-1} j!} h_0(\mathbf{s};\epsilon)\cdots h_{n-1}(\mathbf{s};\epsilon), \quad n \geq 1. \tag{58}$$

We also define $Z_0^{\mathrm{GUE1}}(\mathbf{s};\epsilon) \equiv 1$.

The orthogonal polynomials $p_m(\lambda;\mathbf{s};\epsilon)$ satisfy the the *three-term recurrence relation*:

$$p_{m+1}(\lambda;\mathbf{s};\epsilon) + V_m(\mathbf{s};\epsilon)p_m(\lambda;\mathbf{s};\epsilon) + W_m(\mathbf{s};\epsilon)p_{m-1}(\lambda;\mathbf{s};\epsilon) = \lambda p_m(\lambda;\mathbf{s};\epsilon), \quad m \geq 0, \tag{59}$$

for some functions $V_m(\mathbf{s};\epsilon)$ and $W_m(\mathbf{s};\epsilon)$ $(m \geq 0)$, with $p_{-1}(\lambda;\mathbf{s};\epsilon) :\equiv 0$ and $W_0(\mathbf{s};\epsilon) :\equiv 0$. The functions $V_m(\mathbf{s};\epsilon)$ and $W_m(\mathbf{s};\epsilon)$ are well known to satisfy

$$V_m(\mathbf{0};\epsilon) = 0, \quad W_m(\mathbf{0};\epsilon) = \epsilon m, \quad m \geq 0. \tag{60}$$

The equality

$$(\lambda p_{m_1}, p_{m_2}) = (p_{m_1}, \lambda p_{m_2}) \tag{61}$$

then implies that

$$W_m(\mathbf{s};\epsilon) = \frac{h_m(\mathbf{s};\epsilon)}{h_{m-1}(\mathbf{s};\epsilon)} = \frac{Z_{m+1}^{\mathrm{GUE1}}(\mathbf{s};\epsilon)Z_{m-1}^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{Z_m^{\mathrm{GUE1}}(\mathbf{s};\epsilon)^2} + \text{correction}, \quad m \geq 1. \tag{62}$$

The three-term recurrence relation tells that $p_0, p_1, \ldots$ are eigenvectors of the difference operator $L = \Lambda + V_n(\mathbf{s};\epsilon) + W_n(\mathbf{s};\epsilon)\Lambda^{-1}$ with $\Lambda : f(n) \mapsto f(n+1)$ being the shift operator. Denote again by L the corresponding tri-diagonal matrix, and denote $\mathbf{p} = (p_0, p_1, \ldots)$. For a square matrix $X = (X_{i,j})$, denote

$$X_- = (X_{i,j})_{i<j}, \quad X_+ = (X_{i,j})_{i\geq j}, \quad X = X_+ + X_-.$$

Lemma 2.3 *For arbitrary $n \geq 0$, the following two statements are true: a) the polynomial $p_n(\lambda;\mathbf{s};\epsilon)$ satisfies*

$$\epsilon\frac{\partial p_n(\lambda;\mathbf{s};\epsilon)}{\partial s_j} - (A_j\mathbf{p})_n(\lambda;\mathbf{s};\epsilon) = 0, \quad A_j := -(L^j)_-, \quad j \geq 1; \tag{63}$$

b) we have

$$\epsilon\frac{\partial}{\partial s_j}\log\frac{Z_{n+1}^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)} = (L^j)_{nn}, \quad j \geq 1. \tag{64}$$

Proof Write

$$\frac{\partial p_n(\lambda;\mathbf{s};\epsilon)}{\partial s_j} = \sum_{m=0}^{n-1} A_{mn}^{(j)}(\mathbf{s};\epsilon)p_m(\lambda;\mathbf{s};\epsilon)$$

for some coefficients $A_{mn}^{(j)} = A_{mn}^{(j)}(\mathbf{s};\epsilon)$. Differentiating the orthogonality $(p_m, p_n) \equiv h_n\delta_{mn}$ with respect to s_j, we find that for $m < n$

$$A_{mn}^{(j)}h_m + \frac{1}{\epsilon}(\lambda^j p_n, p_m) = 0, \tag{65}$$

and for $m = n$,

$$\frac{1}{\epsilon}(\lambda^j p_n, p_n) = \frac{\partial h_n}{\partial s_j} = \frac{\partial}{\partial s_j}\log\frac{Z_{n+1}^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}. \tag{66}$$

Then by using (59) we obtain (63) and (64).

In particular, from (64) we see that for all $n \geq 0$,

$$\epsilon\frac{\partial}{\partial s_1}\log\frac{Z_{n+1}^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)} = V_n(\mathbf{s};\epsilon), \quad j \geq 1. \tag{67}$$

It follows from Lemma 2.3 that the difference operator L satisfies the Toda lattice hierarchy (21). Indeed, it immediately follows from (63) the Lax equations for the square matrix L; by using (62), (67) and the facts mentioned in the introduction that $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$ and $\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$ belong to $\mathbb{Q}[n,\epsilon,\epsilon^{-1}][[\mathbf{s}]]$, we know that $W_n(\mathbf{s};\epsilon), V_n(\mathbf{s};\epsilon)$ belong to $\mathbb{Q}[n,\epsilon,\epsilon^{-1}][[\mathbf{s}]]$; we therefore conclude that the Lax equations (21) hold for the difference operator L (namely with n being viewed as an indeterminate), i.e., $(V_n(\mathbf{s};\epsilon), W_n(\mathbf{s};\epsilon))$ is a solution to the Toda lattice hierarchy.

We note that equalities (62), (67), (64) hold true in $\mathbb{Q}[n,\epsilon,\epsilon^{-1}][[\mathbf{s}]]$, where the right-hand side of (64) should be viewed as $\operatorname{res} L^j$. By using (64) and the compatibility between (40), (41) we have

$$\sum_{i,j\geq 1}\frac{1}{\lambda^{i+1}\mu^{j+1}}(\Lambda-1)\left(\frac{\partial^2\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{\partial s_i\partial s_j}\right) = (\Lambda-1)\left(\frac{\operatorname{tr}R_n(\lambda)R_n(\mu)-1}{(\lambda-\mu)^2}\right). \tag{68}$$

Since $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon) \in \mathbb{Q}[n,\epsilon,\epsilon^{-1}][[\mathbf{s}]]$ and since $\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$ is divisible by n, we have

$$\sum_{i,j\geq 1}\frac{1}{\lambda^{i+1}\mu^{j+1}}\frac{\partial^2\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{\partial s_i\partial s_j} = \frac{\operatorname{tr}R_n(\lambda)R_n(\mu)-1}{(\lambda-\mu)^2}. \tag{69}$$

In particular, $W_n(\mathbf{s};\epsilon) = \epsilon^2\frac{\partial^2\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)}{\partial s_1\partial s_1}$. Since $\log Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$ and $\log Z(x,\mathbf{s};\epsilon)$ differ by a function that only depends on x,ϵ, we see that $W_n(\mathbf{s};\epsilon)$ defined in this section coincides with $W(x,\mathbf{s};\epsilon)$ defined in the Introduction.

We see also from (62), (64), (69) that $Z_n^{\mathrm{GUE1}}(\mathbf{s};\epsilon)$ almost satisfies the definition of a Toda tau-function of the solution $(V_n(\mathbf{s};\epsilon), W_n(\mathbf{s};\epsilon))$, except in (62) an extra term appears. It is easy to show (cf. [4], [64]) that the definition for the correction GUE partition function (cf. (12), (14)) eliminates the extra term and keep the other properties hold. We therefore arrive at the following proposition summarizing the above.

Proposition 2.4 (cf. [1, 27]) *The vector-valued function* $(V(x,\mathbf{s};\epsilon), W(x,\mathbf{s};\epsilon))$ *defined in* (19) *is the unique solution to the Toda lattice hierarchy* (21) *specified by the initial condition* (22), *and the GUE partition function* $Z(x,\mathbf{s};\epsilon)$ *is the tau-function of this solution to the Toda lattice hierarchy.*

3 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. To this end, we will first need to recall the definition of $F_g^{\mathbb{P}^1}$ $(g \geq 1)$ that appear in the context of Theorem 1.1.

Denote by $\mathcal{F}^{\mathbb{P}^1}(\mathbf{T};\epsilon)$ the *free energy of GW invariants of* $\mathbb{P}^1$, and by $Z^{\mathbb{P}^1}(\mathbf{T};\epsilon) := \exp(\mathcal{F}^{\mathbb{P}^1}(\mathbf{T};\epsilon))$ the *partition function* of these GW invariants:

$$\mathcal{F}^{\mathbb{P}^1}(\mathbf{T};\epsilon) = \sum_{d,k\geq 0}\frac{1}{k!}\sum_{\substack{1\leq\alpha_1,\ldots,\alpha_k\leq 2,\\ i_1,\ldots,i_k\geq 0}} T^{\alpha_1,i_1}\cdots T^{\alpha_k,i_k}\sum_{g\geq 0}\epsilon^{2g-2}\langle\tau_{i_1}(\alpha_1)\cdots\tau_{i_k}(\alpha_k)\rangle_{g,d}, \tag{70}$$

where $\mathbf{T} := (T^{\alpha,i})_{\alpha=1,2,\,i\geq 0}$ and $\langle \tau_{i_1}(\alpha_1)\cdots\tau_{i_k}(\alpha_k)\rangle_{g,d}$ denote the genus g and degree d GW invariants of $\mathbb{P}^1$ (cf., e.g., [34, 59, 60]). We denote by $\mathcal{F}_g^{\mathbb{P}^1}(\mathbf{T}) := \mathrm{Coef}(\epsilon^{2g-2}, \mathcal{F}^{\mathbb{P}^1}(\mathbf{T};\epsilon))$ the genus g part of $\mathcal{F}^{\mathbb{P}^1}(\mathbf{T};\epsilon)$, sometimes called for short *the genus g free energy of GW invariants of* $\mathbb{P}^1$. It was conjectured by Dubrovin [17], Eguchi–Yang [36] (cf., [43, 70]), and proved in [34, 59, 60] that the functions

$$V^{\mathbb{P}^1}(\mathbf{T};\epsilon) := \epsilon(\Lambda-1)\frac{\partial \log Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)}{\partial T^{2,0}}, \quad W^{\mathbb{P}^1}(\mathbf{T};\epsilon) := \epsilon^2\frac{\partial^2 \log Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)}{\partial T^{2,0}\partial T^{2,0}}, \tag{71}$$

satisfy the Toda lattice hierarchy (21) with

$$\frac{\partial}{\partial T^{2,p}} = \frac{1}{(p+1)!}\frac{\partial}{\partial s_{p+1}}, \quad p\geq 0, \tag{72}$$

and with $L := \Lambda + V^{\mathbb{P}^1}(\mathbf{T};\epsilon) + W^{\mathbb{P}^1}(\mathbf{T};\epsilon)\Lambda^{-1}$, and, moreover, $Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)$ is a tau-function of the solution $(V^{\mathbb{P}^1}(\mathbf{T};\epsilon), W^{\mathbb{P}^1}(\mathbf{T};\epsilon))$ to (21). Here $\Lambda = \exp(\epsilon\partial_x)$ and $x = T^{1,0}$. The partition function $Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)$ also satisfies the following dilaton and string equations:

$$\sum_{\alpha=1}^{2}\sum_{p\geq 0}(T^{\alpha,p} - \delta^{\alpha,1}\delta^{p,1})\frac{\partial Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)}{\partial T^{\alpha,p}} + \epsilon\frac{\partial Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)}{\partial\epsilon} + \frac{1}{12}Z^{\mathbb{P}^1}(\mathbf{T};\epsilon) = 0, \tag{73}$$

$$\sum_{\alpha=1}^{2}\sum_{p\geq 1}(T^{\alpha,p} - \delta^{\alpha,1}\delta^{p,1})\frac{\partial Z^{\mathbb{P}^1}(\mathbf{T};\epsilon)}{\partial T^{\alpha,p-1}} + \frac{T^{1,0}T^{2,0}}{\epsilon^2}Z^{\mathbb{P}^1}(\mathbf{T};\epsilon) = 0. \tag{74}$$

Let $\mathbf{v}^{\mathbb{P}^1}(\mathbf{T})$ be the *topological solution* to (30), that is the unique power series in $T^{\alpha,q}$, $\alpha = 1,2$, $q>0$, satisfying (30) and

$$v^{\alpha,\mathbb{P}^1}(\mathbf{T})\big|_{T^{\beta,q}=0,q>0,\beta=1,2} = T^{\alpha,0}, \quad \alpha = 1,2. \tag{75}$$

As it was mentioned in the Introduction, the genus zero free energy of GW invariants of $\mathbb{P}^1$ equals [18, 34] the one of the topological solution to the principal hierarchy, i.e.,

$$\mathcal{F}_0^{\mathbb{P}^1}(\mathbf{T}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{2}\sum_{p,q\geq 0}(T^{\alpha,p} - \delta^{\alpha,1}\delta^{p,1})(T^{\beta,q} - \delta^{\beta,1}\delta^{q,1})\Omega^{[0]}_{\alpha,p;\beta,q}(\mathbf{v}^{\mathbb{P}^1}(\mathbf{T})). \tag{76}$$

The higher genus free energies $\mathcal{F}_g^{\mathbb{P}^1}(\mathbf{T})$, $g\geq 1$, admit the *jet-variable representation* (cf. [16, 33–35, 44]). Namely, there exist functions $F_g^{\mathbb{P}^1}(\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_{3g-2})$ $(g\geq 1)$ with $\mathbf{v}_m = (v_m, u_m) = (v_m^1, v_m^2)$ and $\mathbf{v}_0 = \mathbf{v}$, such that

$$\mathcal{F}_g^{\mathbb{P}^1}(\mathbf{T}) = F_g^{\mathbb{P}^1}\left(\mathbf{v}^{\mathbb{P}^1}(\mathbf{T}), \frac{\partial \mathbf{v}^{\mathbb{P}^1}(\mathbf{T})}{\partial x},\ldots,\frac{\partial^{3g-2}\mathbf{v}^{\mathbb{P}^1}(\mathbf{T})}{\partial x^{3g-2}}\right). \tag{77}$$

By using Virasoro constraints, Dubrovin and Zhang [33, 34] obtained the following loop equation:

$$\begin{aligned}&\sum_{r\geq 0}\left(\frac{\partial\Delta F}{\partial v_r}\left(\frac{v-\lambda}{D}\right)_r - 2\frac{\partial\Delta F}{\partial u_r}\left(\frac{1}{D}\right)_r\right)\\ &\quad+\sum_{r\geq 1}\sum_{k=1}^{r}\binom{r}{k}\left(\frac{1}{\sqrt{D}}\right)_{k-1}\left(\frac{\partial\Delta F}{\partial v_r}\left(\frac{v-\lambda}{\sqrt{D}}\right)_{r-k+1} - 2\frac{\partial\Delta F}{\partial u_r}\left(\frac{1}{\sqrt{D}}\right)_{r-k+1}\right)\\ &\quad= -D^{-2}\mathrm{e}^u\end{aligned}$$

$$- \epsilon^2 \sum_{k,l\geq 0} \left(\frac{1}{4} S(\Delta F, v_k, v_l) \left(\frac{v-\lambda}{\sqrt{D}}\right)_{k+1} \left(\frac{v-\lambda}{\sqrt{D}}\right)_{l+1} - S(\Delta F, v_k, u_l) \left(\frac{v-\lambda}{\sqrt{D}}\right)_{k+1} \left(\frac{1}{\sqrt{D}}\right)_{l+1} + S(\Delta F, u_k, u_l) \left(\frac{1}{\sqrt{D}}\right)_{k+1} \left(\frac{1}{\sqrt{D}}\right)_{l+1} \right)$$
$$- \frac{\epsilon^2}{2} \sum_{k\geq 0} \Bigg(\frac{\partial \Delta F}{\partial v_k} \partial^{k+1} \left(e^u \frac{4e^u (v-\lambda) u_1 - ((v-\lambda)^2 + 4e^u) v_1}{D^3} \right) + \frac{\partial \Delta F}{\partial u_k} \partial^{k+1} \left(e^u \frac{4(v-\lambda)\, v_1 - ((v-\lambda)^2 + 4e^u) u_1}{D^3} \right) \Bigg), \tag{78}$$

where $\Delta F := \sum_{g\geq 1} \epsilon^{2g-2} F_g^{\mathbb{P}^1}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{3g-2})$, $D = (v-\lambda)^2 - 4e^u$, $S(f, a, b) := \frac{\partial^2 f}{\partial a \partial b} + \frac{\partial f}{\partial a} \frac{\partial f}{\partial b}$, and f_r stands for $\partial^r(f)$ with

$$\partial := \sum_{\alpha=1,2} \sum_{m\geq 0} v^\alpha_{m+1} \frac{\partial}{\partial v^\alpha_m}. \tag{79}$$

It is also shown in [33, 34] the solution ΔF to (78) is unique up to a sequence of additive constants for $F_g^{\mathbb{P}^1}$ $(g \geq 1)$, that for $g \geq 2$ can be fixed by the following equation:

$$\sum_{\alpha=1}^{2} \sum_{m=1}^{3g-2} m v^\alpha_m \frac{\partial F_g^{\mathbb{P}^1}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{3g-2})}{\partial v^\alpha_m} = (2g-2) F_g^{\mathbb{P}^1}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{3g-2}) + \frac{\delta_{g,1}}{12}, \; g \geq 1. \tag{80}$$

Moreover, for $g \geq 2$, $F_g^{\mathbb{P}^1}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{3g-2})$ are polynomials of $\mathbf{v}_2, \ldots, \mathbf{v}_{3g-2}$ and have rational dependence in $\mathbf{v}_1$ (cf., e.g., [33, 34]). In particular, for $g = 1$,

$$F_1^{\mathbb{P}^1}(\mathbf{v}_0, \mathbf{v}_1) = \frac{1}{24} \log((v_1)^2 - e^u (u_1)^2) - \frac{1}{24} u. \tag{81}$$

These unique functions $F_g^{\mathbb{P}^1}$ $(g \geq 1)$ are the ones used in the context of Theorem 1.1.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Start with genus zero. Let $(v(x, \mathbf{s}), u(x, \mathbf{s}))$ be the unique solution to the initial value problem (30), (24). The Riemann invariants for the principal hierarchy (30) are given by

$$R_1(\mathbf{v}) = v + 2e^{u/2}, \quad R_2(\mathbf{v}) = v - 2e^{u/2}. \tag{82}$$

Since $(R_i)_x$ $(i = 1, 2)$ do not vanish at generic $x = x_0$, the solution $(v(x, \mathbf{s}), u(x, \mathbf{s}))$ belongs to the class of *monotone solutions*. Therefore, it could be obtained by the hodograph method [22, 24, 63], yielding the following genus zero Euler–Lagrange equation:

$$x \delta_{\beta,2} + \sum_{p\geq 0} (T^{2,p} - \delta_{p,1}) \frac{\partial \theta_{2,p}}{\partial v^\beta}(\mathbf{v}(x, \mathbf{s})) = 0, \quad \beta = 1, 2, \tag{83}$$

where $T^{2,p} = (p+1)! s_{p+1}$, $p \geq 0$.

Following [18], define $\widehat{\mathcal{F}}_0(x, \mathbf{s})$ as the right-hand side of (25). By using the well-known properties

$$\Omega^{[0]}_{\alpha,p;\beta,q}(\mathbf{v}) = \Omega^{[0]}_{\beta,q;\alpha,p}(\mathbf{v}), \quad \partial_{t^{\gamma,s}}(\Omega^{[0]}_{\alpha,p;\beta,q}(\mathbf{v})) = \partial_{t^{\beta,q}}(\Omega^{[0]}_{\alpha,p;\gamma,s}(\mathbf{v})), \quad \forall\, p, q, s \geq 0, \tag{84}$$

$$\theta_{\alpha,p}(\mathbf{v}) = \Omega^{[0]}_{\alpha,p;1,0}(\mathbf{v}), \quad \forall\, p \geq 0, \tag{85}$$

one can verify the validity of the following equalities:

$$\frac{\partial^2 \widehat{\mathcal{F}}_0(x, \mathbf{s})}{\partial T^{2,p} \partial T^{2,q}} = \Omega^{[0]}_{2,p;2,q}(\mathbf{v}(x, \mathbf{s})), \quad \forall\, p, q \geq 0, \tag{86}$$

$$\frac{\partial^2 \widehat{\mathcal{F}}_0(x,\mathbf{s})}{\partial x \partial x} = u(x,\mathbf{s}), \tag{87}$$

$$\frac{\partial^2 \widehat{\mathcal{F}}_0(x,\mathbf{s})}{\partial x \partial T^{2,p}} = \Omega^{[0]}_{1,0;2,p}(\mathbf{v}(x,\mathbf{s})), \quad \forall\, p \geq 0. \tag{88}$$

From these equalities we see that $\exp(\epsilon^{-2}\widehat{\mathcal{F}}_0(x,\mathbf{s}))$ is the tau-function for the solution $(v(x,\mathbf{s}), u(x,\mathbf{s}))$ to the $\partial_x, \partial_{T^{2,q}}$-flows of the principal hierarchy (30) (cf. (32)).

It is not difficult to verify that $\widehat{\mathcal{F}}_0(x,\mathbf{s})$ also satisfies the following linear equations:

$$\sum_{j\geq 1}\left(s_j - \frac{1}{2}\delta_{j,2}\right)\frac{\partial \widehat{\mathcal{F}}_0(x,\mathbf{s})}{\partial s_j} + x\frac{\widehat{\mathcal{F}}_0(x,\mathbf{s})}{\partial x} = 2\widehat{\mathcal{F}}_0(x,\mathbf{s}), \tag{89}$$

$$\sum_{j\geq 2} j\left(s_j - \frac{1}{2}\delta_{j,2}\right)\frac{\partial \widehat{\mathcal{F}}_0(x,\mathbf{s})}{\partial s_{j-1}} + x s_1 = 0. \tag{90}$$

We conclude from (86)–(88) and (90) that $\widehat{\mathcal{F}}_0(x,\mathbf{s})$ could differ from $\mathcal{F}_0(x,\mathbf{s})$ only possibly by adding a function of x (actually with at most linear dependence in x). Taking $\mathbf{s} = \mathbf{0}$ in $\widehat{\mathcal{F}}_0(x,\mathbf{s})$ and in $\mathcal{F}_0(x,\mathbf{s})$, we find that they both give

$$\frac{x^2}{2}\left(\log x - \frac{3}{2}\right). \tag{91}$$

Hence formula (25) is proved.

Similarly as we do for the LUE case in [69], we proceed with the higher genera by using *quasi-triviality*. According to [33, 34], the following quasi-trivial map

$$\widehat{V} = \frac{\Lambda - 1}{\epsilon \partial_x}(v) + (\Lambda - 1)\circ \partial_{t^{2,0}}\left(\sum_{g\geq 1}\epsilon^{2g-1}F_g^{\mathbb{P}^1}\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial x}, \ldots, \frac{\partial^{3g-2}\mathbf{v}}{\partial x^{3g-2}}\right)\right), \tag{92}$$

$$\widehat{W} = \frac{(\Lambda + \Lambda^{-1} - 2)}{\epsilon^2 \partial_x^2}(u) + (\Lambda + \Lambda^{-1} - 2)\left(\sum_{g\geq 1}\epsilon^{2g-2}F_g^{\mathbb{P}^1}\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial x}, \ldots, \frac{\partial^{3g-2}\mathbf{v}}{\partial x^{3g-2}}\right)\right), \tag{93}$$

transforms the principal hierarchy (30) to the extended Toda hierarchy [11, 34]. The quasi-Miura map (92)–(93) transforms a monotone solution of the principal hierarchy (30) to a solution of the extended Toda hierarchy (see Theorem 1.1 of [34]). As we just mentioned above, the particular solution $(v(x,\mathbf{s}), u(x,\mathbf{s}))$ of interest to the $\partial_{t^{2,p}}$-flows ($p \geq 0$) in the principal hierarchy (30) specified by the initial data (24) is monotone. Therefore, the function $(\widehat{V}(x,\mathbf{s};\epsilon), \widehat{U}(x,\mathbf{s};\epsilon))$ defined by

$$\widehat{V}(x,\mathbf{s};\epsilon) := \widehat{V}|_{v_k\mapsto \partial_x^k(v(x,\mathbf{s};\epsilon)),\, u_k\mapsto \partial_x^k(u(x,\mathbf{s};\epsilon)),\, k\geq 0}, \tag{94}$$

$$\widehat{U}(x,\mathbf{s};\epsilon) := \widehat{U}|_{v_k\mapsto \partial_x^k(v(x,\mathbf{s};\epsilon)),\, u_k\mapsto \partial_x^k(u(x,\mathbf{s};\epsilon)),\, k\geq 0} \tag{95}$$

is a particular solution to the Toda lattice hierarchy (21). What is more, since $\mathrm{e}^{\epsilon^{-2}\mathcal{F}_0(x,\mathbf{s})} = \mathrm{e}^{\epsilon^{-2}\widehat{\mathcal{F}}_0(x,\mathbf{s})}$ is the tau-function of the solution $(v(x,\mathbf{s}), u(x,\mathbf{s}))$ to the dispersionless Toda lattice hierarchy and using again Theorem 1.1 of [34], we find that

$$\tau(x,\mathbf{s};\epsilon) := \exp\left(\epsilon^{-2}\widehat{\mathcal{F}}_0(x,\mathbf{s}) + \sum_{g\geq 1}\epsilon^{2g-2}F_g^{\mathbb{P}^1}|_{v_k\mapsto \partial_x^k(v(x,\mathbf{s})),\, u_k\mapsto \partial_x^k(u(x,\mathbf{s})),\, k\geq 0}\right) \tag{96}$$

is the tau-function of the solution $(\widehat{V}(x,\mathbf{s};\epsilon), \widehat{U}(x,\mathbf{s};\epsilon))$ to the Toda lattice hierarchy. Note that the functions $F_g^{\mathbb{P}^1}$, $g \ge 2$, satisfy the following equation:

$$\frac{\partial F_g^{\mathbb{P}^1}}{\partial v} = 0, \tag{97}$$

which follows from the string equation (74) for the GW invariants of $\mathbb{P}^1$. By using (89), (80), (90), (97) one can verify that this tau-function $\tau(x, \mathbf{s}; \epsilon)$ satisfies the following two relations:

$$\sum_{j\ge 1}\left(s_j - \frac{1}{2}\delta_{j,2}\right)\frac{\partial \tau(x,\mathbf{s};\epsilon)}{\partial s_j} + \epsilon\frac{\partial \tau(x,\mathbf{s};\epsilon)}{\partial \epsilon} + x\frac{\partial \tau(x,\mathbf{s};\epsilon)}{\partial x} + \frac{1}{12}\tau(x,\mathbf{s};\epsilon) = 0, \tag{98}$$

$$\sum_{j\ge 2} j\left(s_j - \frac{1}{2}\delta_{j,2}\right)\frac{\partial \tau(x,\mathbf{s};\epsilon)}{\partial s_{j-1}} + \frac{x s_1}{\epsilon^2}\tau(x,\mathbf{s};\epsilon) = 0, \tag{99}$$

which agree with the linear equations (17), (18). The theorem is proved. □

Several applications of Theorem 1.1 can be found in [26–28, 31]; some of the details are also given in the next section.

4 Topological Gravity and Matrix Gravity

In the previous sections, we studied the GUE partition function and give in Theorem 1.1 a jet representation for the genus g GUE free energy $\mathcal{F}_g(x, \mathbf{s})$ for $g \ge 1$, obtained by the one for the genus g free energy of GW invariants of $\mathbb{P}^1$. In this section, we consider the restriction to *even couplings*, and revisit its connection to GW invariants of *a point* and the associated Hodge integrals.

4.1 Identification in Topological Gravity

In his seminal work [65], Witten proposed two versions of two-dimensional quantum gravity: topological gravity and matrix gravity. In this subsection, let us consider the topological one, that is, following Witten [65], the partition function of psi-class integrals on Deligne–Mumford's moduli space of curves [13]. To be precise, let $\mathcal{F}_{\mathrm{WK}}(\mathbf{t}; \epsilon)$, $g \ge 0$, be the following generating series for psi-class integrals:

$$\mathcal{F}_{\mathrm{WK}}(\mathbf{t};\epsilon) := \sum_{g\ge 0}\epsilon^{2g-2}\sum_{k\ge 0}\sum_{i_1,\dots,i_k\ge 0}\frac{t_{i_1}\cdots t_{i_k}}{k!}\int_{\overline{\mathcal{M}}_{g,k}}\psi_1^{i_1}\cdots\psi_k^{i_k}, \tag{100}$$

called the *free energy*. Here, $\mathbf{t} = (t_0, t_1, t_2, \dots)$ and ϵ are indeterminates, $\overline{\mathcal{M}}_{g,k}$ denotes the moduli space of stable algebraic curves of genus g with k distinct marked points, and ψ_i ($1 \le i \le k$) denotes the first Chern class of the ith cotangent line bundle on $\overline{\mathcal{M}}_{g,k}$. Let

$$\mathcal{F}_{\mathrm{WK}}(\mathbf{t};\epsilon) := \sum_{g\ge 0}\epsilon^{2g-2}\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t}). \tag{101}$$

We call $\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t})$ the *genus g part of the free energy* $\mathcal{F}_{\mathrm{WK}}(\mathbf{t};\epsilon)$. The exponential

$$\exp(\mathcal{F}_{\mathrm{WK}}(\mathbf{t};\epsilon)) =: Z_{\mathrm{WK}}(\mathbf{t};\epsilon) \tag{102}$$

is called the *partition function of psi-class integrals*. It was conjectured by Witten [65] and proved by Kontsevich [52] that the partition function $Z_{\mathrm{WK}}(\mathbf{t};\epsilon)$ is a particular tau-function for the Korteweg–de Vries (KdV) integrable hierarchy. We also refer to $Z_{\mathrm{WK}}(\mathbf{t};\epsilon)$ as the *partition function for the topological quantum gravity*.

Another important model regarding the intersection theory on $\overline{\mathcal{M}}_{g,k}$ is the partition function of certain special cubic Hodge integrals [25, 38, 54, 61], which from its definition is a deformation of the partition function $Z_{\mathrm{WK}}(\mathbf{t};\epsilon)$ and has important relation to the GUE partition function [26, 28]. To be precise, define $Z_{\mathrm{H}}(\mathbf{t};\epsilon)$ as follows:

$$Z_{\mathrm{H}}(\mathbf{t};\epsilon) = \mathrm{e}^{\mathcal{H}(\mathbf{t};\epsilon)}, \tag{103}$$

where

$$\mathcal{H}(\mathbf{t};\epsilon) := \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{H}_g(\mathbf{t}), \tag{104}$$

$$\mathcal{H}_g(\mathbf{t}) := \sum_{k\geq 0}\sum_{i_1,\dots,i_k\geq 0} \frac{t_{i_1}\cdots t_{i_k}}{k!}\int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1}\cdots\psi_k^{i_k}\Lambda(-1)^2\Lambda\left(\frac{1}{2}\right), \quad g\geq 0. \tag{105}$$

Here, $\Lambda(z) := \sum_{j=0}^{g}\lambda_j z^j$ is the Chern polynomial of the Hodge bundle $\mathbb{E}_{g,k}$ on $\overline{\mathcal{M}}_{g,k}$ with λ_j being the jth Chern class of $\mathbb{E}_{g,k}$. We call $\mathcal{H}(\mathbf{t};\epsilon)$ the *Hodge free energy* and $Z_{\mathrm{H}}(\mathbf{t};\epsilon)$ the *Hodge partition function*[1)]. Being suggested by the *Hodge-GUE correspondence* [26, 28] (see also Theorem 4.4 below), we refer to the Hodge partition function $Z_{\mathrm{H}}(\mathbf{t};\epsilon)$ defined in (103)–(105) as the *dual partition function for the topological quantum gravity.*

In genus zero, we have the obvious equality

$$\mathcal{H}_0(\mathbf{t}) = \mathcal{F}_0^{\mathrm{WK}}(\mathbf{t}). \tag{106}$$

The discrepancy between the two partition functions $Z_{\mathrm{H}}(\mathbf{t};\epsilon)$ and $Z_{\mathrm{WK}}(\mathbf{t};\epsilon)$ starts from their genus one parts. To understand this discrepancy, it will be convenient to look at their jet-representations [16, 29, 33, 35, 65]. Recall the following lemma.

Lemma 4.1 *Denote*

$$v_{\mathrm{WK}}(\mathbf{t}) := \frac{\partial^2\mathcal{F}_0^{\mathrm{WK}}(\mathbf{t})}{\partial t_0^2}. \tag{107}$$

For each $g\geq 1$, there exist elements

$$F_g^{\mathrm{WK}}(z_1,\dots,z_{3g-2}) \in \mathbb{Q}[z_2,\dots,z_{3g-2},z_1,z_1^{-1}], \tag{108}$$

$$H_g(b_0,b_1,\dots,b_{3g-2}) \in \mathbb{Q}[b_2,\dots,b_{3g-2},b_0,b_1,b_1^{-1}], \tag{109}$$

such that

$$\mathcal{F}_g^{\mathrm{WK}}(\mathbf{t}) = F_g^{\mathrm{WK}}\left(\frac{\partial v_{\mathrm{WK}}(\mathbf{t})}{\partial t_0},\dots,\frac{\partial^{3g-2}v_{\mathrm{WK}}(\mathbf{t})}{\partial t_0^{3g-2}}\right), \tag{110}$$

$$\mathcal{H}_g(\mathbf{t}) = H_g\left(v_{\mathrm{WK}}(\mathbf{t}),\frac{\partial v_{\mathrm{WK}}(\mathbf{t})}{\partial t_0},\dots,\frac{\partial^{3g-2}v_{\mathrm{WK}}(\mathbf{t})}{\partial t_0^{3g-2}}\right). \tag{111}$$

Moreover, for $g\geq 2$, $H_g(b_0,b_1,\dots,b_{3g-2})$ does not depend on b_0.

See for example [25, 29] for the proof of this lemma. For the reader's convenience, we list the first few $F_g^{\mathrm{WK}}(z_1,\dots,z_{3g-2})$, $H_g(b_0,b_1,\dots,b_{3g-2})$ as follows:

$$F_1^{\mathrm{WK}}(z_1) = \frac{1}{24}\log z_1, \quad F_2^{\mathrm{WK}}(z_1,z_2,z_3,z_4) = \frac{z_4}{1152z_1^2} - \frac{7z_3z_2}{1920z_1^3} + \frac{z_2^3}{360z_1^4}, \tag{112}$$

1) The Hodge partition function considered in this paper is a specialization of the one in [25]; geometric and topological significance of this specialization can be found, e.g., in [25, 26, 28].

$$H_1(b_0,b_1)=\frac{1}{24}\log b_1-\frac{b_0}{16}, \tag{113}$$

$$H_2(b_0,b_1,b_2,b_3,b_4)=\frac{7b_2}{2560}+\frac{11b_2^2}{3840b_1^2}-\frac{b_1^2}{11520}-\frac{b_3}{320b_1}+\frac{b_4}{1152b_1^2}-\frac{7b_3b_2}{1920b_1^3}+\frac{b_2^3}{360b_1^4}. \tag{114}$$

The elements $F_g^{\rm WK}(z_1,\dots,z_{3g-2})$ with $g\ge 1$ can be calculated recursively by solving the DZ loop equation [33]; the elements $H_g(b_0,b_1,\dots,b_{3g-2})$ can also be calculated recursively by solving the DZ type loop equation [26], or, they can be calculated by using the algorithm given in [25].

Introduce a gradation $\widetilde{\deg}$ in $\mathbb{Q}[b_2,\dots,b_{3g-2},b_0,b_1,b_1^{-1}]$ by assigning

$$\widetilde{\deg}\, b_k=1,\quad \forall\, k\ge 0. \tag{115}$$

Then for $g\ge 2$, $H_g(b_0,b_1,\dots,b_{3g-2})$ decomposes into the homogeneous parts with respect to $\widetilde{\deg}$ as follows:

$$H_g(b_0,b_1,\dots,b_{3g-2})=\sum_{d=1-g}^{2g-2}H_g^{[d]}(b_0,b_1,\dots,b_{3g-2}), \tag{116}$$

where $H_g^{[d]}(b_0,b_1,\dots,b_{3g-2})$ is homogeneous of degree d with respect to $\widetilde{\deg}$. We have (cf. [25, 29])

$$H_1(b_0,b_1)=F_1^{\rm WK}(b_1)-\frac{1}{16}b_0,\quad H_g^{[1-g]}(b_0,b_1,\dots,b_{3g-2})=F_g^{\rm WK}(b_1,\dots,b_{3g-2})\ (g\ge 2). \tag{117}$$

Namely, H_g can be viewed as a specific deformation of $F_g^{\rm WK}$; in the big phase space this is obvious (by definition), and we see the deformation in the jet space by equalities in (117).

The following proposition says that under a coordinate transformation in the jet space, *remarkably*, $H_g(b_0,b_1,\dots,b_{3g-2})$ becomes $F_g^{\rm WK}(z_1,\dots,z_{3g-2})$, $g\ge 1$.

Proposition 4.2 *Under the transformation $B:(z_0,z_1,\dots)\to(b_0,b_1,\dots)$ (i.e., $b_i=B_i(\mathbf{z})$, $i\ge 0$), defined inductively from*

$$B_0(\mathbf{z})=-\log z_0,\quad \partial''(z_0)=-\frac{1}{2}\frac{z_1}{\sqrt{z_0}},\quad [\partial',\partial'']=0, \tag{118}$$

we have the identities:

$$(-4)^{g-1}H_g(B_0(\mathbf{z}),B_1(\mathbf{z}),\dots,B_{3g-2}(\mathbf{z}))=F_g^{\rm WK}(z_1,\dots,z_{3g-2}),\quad g\ge 1. \tag{119}$$

Here, ∂' is the derivation on $\mathbb{Q}[z_0,z_1,z_1^{-1},z_2,z_3,\dots]$ such that $\partial'(z_i)=z_{i+1}$, and ∂'' is the derivation on $\mathbb{Q}[b_0,b_1,b_1^{-1},b_2,b_3,\dots]$ such that $\partial''(b_i)=b_{i+1}$.

The proof of Proposition 4.2 using the Hodge-BGW correspondence is given in [68]. An equivalent version of this proposition and the proof are given in [67]. For the reader's convenience, let us list the first few terms of the change of jet-variables in Proposition 4.2:

$$B_1(\mathbf{z})=\frac{z_1}{2z_0^{3/2}},\quad B_2(\mathbf{z})=\frac{z_1^2}{2z_0^3}-\frac{z_2}{4z_0^2},\quad B_3(\mathbf{z})=\frac{1}{8}\frac{z_3}{z_0^{5/2}}-\frac{15}{16}\frac{z_1z_2}{z_0^{7/2}}+\frac{35}{32}\frac{z_1^3}{z_0^{9/2}} \tag{120}$$

with $B_0(\mathbf{z})$ given already in (118). This transformation is invertible, and let us list also the first few terms of the inverse transformation:

$$(B^{-1})_0(\mathbf{b})={\rm e}^{-b_0},\quad (B^{-1})_1(\mathbf{b})=2{\rm e}^{-\frac{3}{2}b_0}b_1,\quad (B^{-1})_2(\mathbf{b})={\rm e}^{-2b_0}(-4b_2+8b_1^2), \tag{121}$$

$$(B^{-1})_3(\mathbf{b})={\rm e}^{-\frac{5}{2}b_0}(8b_3-60b_2b_1+50b_1^3). \tag{122}$$

A closed formula for the map B^{-1} is found with Don Zagier [67]. We have the identity

$$F_g^{\mathrm{WK}}((B^{-1})_1(\mathbf{b}),\dots,(B^{-1})_{3g-2}(\mathbf{b})) = (-4)^{g-1}H_g(b_0,b_1,\dots,b_{3g-2}),\quad g\ge 1. \tag{123}$$

In view of integrable systems, the relationship given in Proposition 4.2 reveals the space/time duality between the q-deformed KdV hierarchy (cf. [9, 41, 55]) and the KdV hierarchy.

4.2 Back to the Matrix Gravity

In the previous subsection, we recalled the identification between the partition function (102) and the dual partition function (103) for the topological quantum gravity: for genus zero, it is given in the big phase space by (106); for higher genera, it is given in the jet-space by Proposition 4.2.

In this subsection, following Witten [65], we look at a certain reduction of the GUE partition function, which is referred to as the *matrix gravity*. To be precise, define the *even GUE partition function* $Z_{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}})$ by

$$Z_{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}}) := \frac{(2\pi)^{-n}\epsilon^{-\frac{1}{12}}}{\mathrm{Vol}(n)}\int_{\mathcal{H}(n)} \mathrm{e}^{-\frac{1}{\epsilon}\mathrm{tr}\,Q_{\mathrm{even}}(M;\mathbf{s}_{\mathrm{even}})}dM,\quad x=n\epsilon, \tag{124}$$

where $\mathrm{Vol}(n)$ is defined in (15), and

$$Q_{\mathrm{even}}(y;\mathbf{s}_{\mathrm{even}}) := \frac{1}{2}y^2 - \sum_{j\in\mathbb{Z}^{\mathrm{even}}_{\ge 2}} s_j y^j. \tag{125}$$

Clearly, this partition function equals $Z(x,\mathbf{s})$ being restricted to $\mathbf{s}_{\mathrm{odd}}=\mathbf{0}$.

According to (10) and (12), the logarithm of $Z_{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}})$ has the expression

$$\log Z_{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}};\epsilon) =: \mathcal{F}_{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}};\epsilon) =: \sum_{g\ge 0}\epsilon^{2g-2}\mathcal{F}_g^{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}}) \tag{126}$$

$$= \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{\log x}{12} + \zeta'(-1) + \sum_{g\ge 2}\frac{\epsilon^{2g-2}B_{2g}}{4g(g-1)x^{2g-2}}$$
$$+\sum_{k\ge 1}\sum_{\substack{g\ge 0,\, j_1,\dots,j_k\in Z^{\mathrm{even}}_{\ge 2}\\ 2-2g-k+|\mathbf{j}|/2\ge 1}} a_g(\mathbf{j})s_{j_1}\cdots s_{j_k}\epsilon^{2g-2}x^{2-2g-k+|\mathbf{j}|/2}. \tag{127}$$

We call $\mathcal{F}_{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}};\epsilon)$ the *even GUE free energy*, and $\mathcal{F}_g^{\mathrm{even}}(x,\mathbf{s}_{\mathrm{even}})$ its genus g part.

The power series $u(x,\mathbf{s})$, $v(x,\mathbf{s})$ (cf. (83)) being restricted to $\mathbf{s}_{\mathrm{odd}}=\mathbf{0}$, denoted by $u(x,\mathbf{s}_{\mathrm{even}})$, $v(x,\mathbf{s}_{\mathrm{even}})$, have the following explicit expressions [28]:

$$v(x,\mathbf{s}_{\mathrm{even}}) = 0, \tag{128}$$

$$\mathrm{e}^{u(x,\mathbf{s}_{\mathrm{even}})} = \sum_{k=1}\frac{1}{k}\sum_{\substack{j_1,\dots,j_k\in\mathbb{Z}^{\mathrm{even}}_{\ge 0},\\ j_1+\cdots+j_k=2k-2}} \mathrm{wt}(j_1)\cdots\mathrm{wt}(j_k)\binom{j_1}{j_1/2}\cdots\binom{j_k}{j_k/2}s_{j_1}\cdots s_{j_k}, \tag{129}$$

where we put $s_0=x$, and for $j\in\mathbb{Z}^{\mathrm{even}}_{\ge 0}$,

$$\mathrm{wt}(j) := \begin{cases} 1, & j=0,\\ j/2, & \text{otherwise}.\end{cases} \tag{130}$$

It is shown in [28] that one can take $v_0 = v_1 = v_2 = \cdots = 0$ in $F_g^{\mathbb{P}^1}(\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{3g-2})$, $g \geq 1$, yielding functions of $u_1, u_2, \ldots$, denoted by $F_g^{\rm even}(u_1, \ldots, u_{3g-2})$; explicitly,

$$F_g^{\rm even}(u_1, \ldots, u_{3g-2}) := F_g^{\mathbb{P}^1}(\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{3g-2})\big|_{v_0=v_1=\cdots=0} + \left(\zeta'(-1) - \frac{\log(-1)}{24}\right)\delta_{g,1}. \quad (131)$$

For example, $F_1^{\rm even} = \frac{1}{12}\log u_1 + \zeta'(-1)$. The expression for $F_g^{\rm even}$ with $g = 2, \ldots, 5$ can be found in [28]. The following theorem is then obtained.

Theorem 4.3 ([28]) *The genus zero part of the even GUE free energy $\mathcal{F}_0^{\rm even}(x, \mathbf{s}_{\rm even})$ has the expression:*

$$\mathcal{F}_0^{\rm even}(x, \mathbf{s}_{\rm even}) = \frac{1}{2}x^2 u(x, \mathbf{s}_{\rm even}) + x \sum_{j \in \mathbb{Z}^{\rm even}_{\geq 2}} \binom{j}{j/2}\left(s_j - \frac{1}{2}\delta_{j,2}\right) \mathrm{e}^{\frac{j_1+j_2}{2}u(x,\mathbf{s}_{\rm even})}$$
$$+ \frac{1}{4}\sum_{j_1,j_2 \in \mathbb{Z}^{\rm even}_{\geq 2}} \frac{j_1 j_2}{j_1+j_2}\binom{j_1}{j_1/2}\binom{j_2}{j_2/2}\left(s_{j_1} - \frac{1}{2}\delta_{j_1,2}\right)\left(s_{j_2} - \frac{1}{2}\delta_{j_2,2}\right)\mathrm{e}^{\frac{j_1+j_2}{2}u(x,\mathbf{s}_{\rm even})}, \quad (132)$$

where $u(x, \mathbf{s}_{\rm even}) = \log x + \cdots$ is given by (129)*. For $g \geq 1$, the genus g part of the even GUE free energy $\mathcal{F}_g^{\rm even}(x, \mathbf{s}_{\rm even})$ can be represented by*

$$\mathcal{F}_g^{\rm even}(x, \mathbf{s}_{\rm even}) = F_g^{\rm even}\left(u(x, \mathbf{s}_{\rm even}), \frac{\partial u(x, \mathbf{s}_{\rm even})}{\partial x}, \ldots, \frac{\partial^{3g-2}u(x, \mathbf{s}_{\rm even})}{\partial x^{3g-2}}\right), \quad (133)$$

where $F_g^{\rm even}(u, u_1, \ldots, u_{3g-2})$ are defined by (131)*.*

Let

$$\Lambda = \mathrm{e}^{\epsilon \partial_x} \quad (134)$$

denote the shift operator. Following [26] (cf., also [28]), define the *modified even GUE free energy* $\widetilde{\mathcal{F}}(x, \mathbf{s}_{\rm even}; \epsilon)$ by

$$\widetilde{\mathcal{F}}(x, \mathbf{s}_{\rm even}; \epsilon) := (\Lambda^{1/2} + \Lambda^{-1/2})^{-1}(\mathcal{F}_{\rm even}(x, \mathbf{s}_{\rm even}; \epsilon)) =: \sum_{g \geq 0} \epsilon^{2g-2}\widetilde{\mathcal{F}}_g(x, \mathbf{s}_{\rm even}). \quad (135)$$

We call $\widetilde{\mathcal{F}}_g(x, \mathbf{s}_{\rm even})$ the *genus g modified GUE free energy*, and call the exponential $\mathrm{e}^{\widetilde{\mathcal{F}}(x,\mathbf{s}_{\rm even})} =: \widetilde{Z}(x, \mathbf{s}_{\rm even})$ the *modified even GUE partition function.*

By definition (135) and by (133) we see the followings [28]: $\widetilde{\mathcal{F}}_0(x, \mathbf{s}_{\rm even}) = \mathcal{F}_0^{\rm even}(x, \mathbf{s}_{\rm even})/2$, and for $g \geq 1$ $\widetilde{\mathcal{F}}_g(x, \mathbf{s}_{\rm even})$ admits the jet representation:

$$\widetilde{\mathcal{F}}_g(x, \mathbf{s}_{\rm even}) = \widetilde{F}_g\left(u(x, \mathbf{s}_{\rm even}), \frac{\partial u(x, \mathbf{s}_{\rm even})}{\partial x}, \ldots, \frac{\partial^{3g-2}u(x, \mathbf{s}_{\rm even})}{\partial x^{3g-2}}\right), \quad (136)$$

where $\widetilde{F}_g(u, u_1, \ldots, u_{3g-2})$, $g \geq 1$, can be determined by

$$\widetilde{F}_g(u, u_1, \ldots, u_{3g-2})$$
$$= \frac{(-1)^g}{2}E_{2g}u_{2g-2} + \frac{1}{2}\sum_{g_1=1}^{g}(-1)^{g-g_1}E_{2g-2g_1}\partial^{2g-2g_1}(F_{g_1}^{\rm even}(u, u_1, \ldots, u_{3g_1-2})) \quad (137)$$

with E_k being the kth Euler number

$$\partial := \sum_{k \geq 0} u_{k+1}\partial_{u_k}.$$

Here the shuffling in genus phenomenon also appeared in [69]. We also have

$$F_g^{\rm even}(u_1,\dots,u_{3g-2}) = \frac{u_{2g-2}}{2^{2g}(2g)!} + \sum_{m=1}^{g} \frac{2^{3m-2g}}{(2g-2m)!}\partial^{2g-2m}(\widetilde{F}_m(u,u_1,\dots,u_{3m-2})), \tag{138}$$

where $g\geq 1$.

The *Hodge-GUE correspondence*, conjectured in [28] (cf. also [25]) and proved in [26], is given by the following theorem.

Theorem 4.4 ([26, 28]) *The identity*

$$\widetilde{Z}(x,\mathbf{s}_{\rm even};\epsilon) = \exp\left(\frac{A(x,\mathbf{s}_{\rm even})}{2\epsilon^2} + \frac{\zeta'(-1)}{2}\right) Z_{\rm H}(\mathbf{t}(x,\mathbf{s}_{\rm even});\sqrt{2}\epsilon), \tag{139}$$

holds true in $\mathbb{C}((\epsilon^2))[[x-1,\mathbf{s}]]$, *where*

$$A(x,\mathbf{s}_{\rm even}) = \frac{1}{4}\sum_{j_1,j_2\in\mathbb{Z}^{\rm even}_{\geq 2}} \frac{j_1 j_2}{j_1+j_2}\binom{j_1}{j_1/2}\binom{j_2}{j_2/2}\left(s_{j_1}-\frac{\delta_{j_1,2}}{2}\right)\left(s_{j_2}-\frac{\delta_{j_2,2}}{2}\right) + x\sum_{j\in\mathbb{Z}^{\rm even}_{\geq 2}}\binom{j}{j/2}\left(s_j-\frac{\delta_{j,2}}{2}\right), \tag{140}$$

and

$$t_i(x,\mathbf{s}) = \sum_{j\in\mathbb{Z}^{\rm even}_{\geq 2}} (j/2)^{i+1}\binom{j}{j/2}\left(s_j-\frac{\delta_{j,2}}{2}\right) + \delta_{i,1} + x\delta_{i,0}, \quad i\geq 0. \tag{141}$$

Taking logarithms on both sides of the identity (139), we find that it is equivalent to

$$\widetilde{\mathcal{F}}(x,\mathbf{s}_{\rm even};\epsilon) = \frac{A(x,\mathbf{s}_{\rm even})}{2\epsilon^2} + \frac{\zeta'(-1)}{2} + \mathcal{H}(\mathbf{t}(x,\mathbf{s}_{\rm even});\sqrt{2}\epsilon). \tag{142}$$

The $g=0$ part of this identity is proved in [28], and the higher genera parts are proved in [26]. To understand more the higher genera parts of (139), again, we go to the jet space. The following lemma recalls the important relationship between $v_{\rm WK}(\mathbf{t})$ and $u(x,\mathbf{s})$.

Lemma 4.5 ([28]) *Under the substitution* (141), *the following identity is true*:

$$v_{\rm WK}(\mathbf{t}(x,\mathbf{s})) = u(x,\mathbf{s}). \tag{143}$$

Using (143) and observing that

$$\frac{\partial}{\partial t_0} = \frac{\partial}{\partial x}, \tag{144}$$

we can rewrite the higher genera parts of identity (139) in the jet space as follows:

$$\widetilde{F}_g(b_0,b_1,\dots,b_{3g-2}) = H_g(b_0,b_1,\dots,b_{3g-2}), \quad g\geq 1. \tag{145}$$

Therefore, we have identified the higher genera parts in jets of the modified even GUE partition function with those of the Hodge partition function. Then by using (138), one comes back to the matrix gravity $F_g^{\rm even}$ in the higher genera with the topological gravity as a starting point, i.e., $F_g^{\rm WK} \leftrightarrow H_g = \widetilde{F}_g \leftrightarrow F_g^{\rm even}$, $g\geq 1$ (cf., the diagram of the Introduction).

Comparing the lowest degree part of the identity (145) with respect to $\widetilde{\deg}$, using (137), and noticing that the operator ∂ does not change $\widetilde{\deg}$, we arrive at the following corollary, which explains the starting arrow on top of the square of the diagram of the Introduction.

Corollary 4.6 *The following equalities are true:*

$$2F_1^{\mathrm{WK}}(z_0, z_1) = F_1^{\mathrm{even}}(z_1) - \zeta'(-1) = \frac{1}{12}\log z_1, \tag{146}$$

$$2^g F_g^{\mathrm{WK}}(z_1, \ldots, z_{3g-2}) = F_g^{\mathrm{even},[1-g]}(z_1, \ldots, z_{3g-2}), \quad g \geq 2. \tag{147}$$

Conflict of Interest

The authors declare no conflict of interest.

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