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On Some Sums Involving Small Arithmetic Functions

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Abstract Let f be any arithmetic function and define $S_f(x) := \sum_{n\le x} f([x/n])$. If the function f is small, namely, $f(n) \ll n^{\varepsilon}$, then the error term $E_f(x)$ in the asymptotic formula of $S_f(x)$ has the form $O(x^{1/2+\varepsilon})$. In this paper, we shall study the mean square of $E_f(x)$ and establish some new results of $E_f(x)$ for some special functions.

Keywords Small arithmetic function, exponential sum, asymptotic formula

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1 Introduction

1.1 On a Special Sum

Let $f : \mathbb{N} \to \mathbb{C}$ be any arithmetic function. For any $x \ge 1$, define

$$S_f(x) := \sum_{n\le x} f([x/n]).$$

This sum was first studied in [2] and then was extensively studied by many authors. See, for example, [1, 5, 7–12, 14, 16, 17, 19, 20].

Wu [16] and Zhai [19] proved independently that if $f(n) \ll n^{\alpha}(\log n)^{\theta}$ for some $0 \le \alpha < 1$ and $\theta \ge 0$, then the asymptotic formula

$$S_f(x) = C_f x + O(x^{\frac{1+\alpha}{2}}(\log x)^{\theta}) \tag{1.1}$$

holds, where the constant C_f is defined by

$$C_f := \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

If $f(n) \ll n^{\varepsilon}$, then we say f is a *small* arithmetic function. From (1.1) we see that if f is any *small* arithmetic function, then

$$S_f(x) = C_f x + O(x^{1/2+\varepsilon}). \tag{1.2}$$

The exponent $1/2$ in the error term of (1.2) is a barrier for small functions f's. It is interesting and natural to ask the following question: for a given small arithmetic function f, is it possible to break the barrier $1/2$?

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Several special examples have been studied. For simplicity, define

$$E_f(x) := S_f(x) - C_f x$$

and let θ_f denote the infimum of α_f for which the estimate $E_f(x) \ll x^{\alpha_f+\varepsilon}$ holds. Ma and Wu [11] first proved that $\theta_\Lambda \le 35/71$, where Λ is the von Mangoldt function. The exponent 35/71 was improved to 97/203 and 9/19 in Bordellés [1] and Liu, Wu and Yang [7], independently. Ma and Sun [10] proved that if $f(n) = \tau(n)$, the Dirichlet divisor function, then $\theta_\tau \le 11/23$. The exponent 11/23 was improved to 19/40 and 5/11 in Bordellés [1] and Stucky [14], independently.

Bordellés [1] also studied some other arithmetic functions. He proved that

$$\theta_{\tau_3} \le \frac{283}{574}, \quad \theta_{\tau_k} \le \frac{1}{2} - \frac{1}{2(4k^3-k-1)}\ (k \ge 4), \quad \theta_\omega \le \frac{455}{914}, \quad \theta_{2^\omega} \le \frac{97}{202}, \tag{1.3}$$

where $\tau_k(n)$ denotes the number of ways n can be written as a product of k factors, $\omega(n)$ denotes the number of distinct prime divisors of n, respectively. All results in (1.3) were improved by Liu, Wu and Yang [8], where they proved that

$$\theta_{\tau_k} \le \frac{5k-1}{10k-1}\ (k \ge 3), \quad \theta_\omega \le \frac{53}{110}, \quad \theta_{2^\omega} \le \frac{9}{19}. \tag{1.4}$$

Recently, Zhai [20] proved that if a small function f satisfies a good *binary additive* property, then the barrier 1/2 in the asymptotic formula (1.2) can be broken.

1.2 Some New Results of $S_f(x)$

We first study the mean square of $E_f(x)$ for arbitrary f. We have the following Theorem 1.1.

Theorem 1.1 *Let f be any small arithmetic function and $T \ge 10$ be a large parameter. Then we have*

$$\int_T^{2T} |E_f(x)|^2 dx \ll_{f,\varepsilon} T^{9/5+\varepsilon}.$$

Remark 1.2 From (1.2) we have $E_f(x) \ll x^{1/2+\varepsilon}$. Theorem 1.1 implies that the estimate $E_f(x) \ll x^{2/5+\varepsilon}$ holds on average.

From Theorem 1.1 we propose the following conjecture.

Conjecture 1.3 *Let f be any small arithmetic function. Then we have*

$$S_f(x) = C_f x + O(x^{2/5+\varepsilon}).$$

Now we study some special small arithmetic functions such that we can break the barrier 1/2. For any arithmetic function $f(n)$, we define

$$f_\ell(n) := \sum_{n=n_1\cdots n_\ell} f(n_1)\cdots f(n_\ell), \quad f_1(n) = f(n). \tag{1.5}$$

Theorem 1.4 *Let $k \ge 2$ and $\ell \ge 1$ be two fixed integers, and suppose f is any function in the set $\{\tau_k, \Lambda_\ell, \mu_\ell, \omega_\ell\}$, where $\mu(\cdot)$ denotes the Möbius function. Then we have the asymptotic formula*

$$S_f(x) = C_f x + O(x^{8/17+\varepsilon}).$$

Let $\mathbb{P}$ denote the set of all prime numbers and let $\mathbf{1}_{\mathbb{P}}$ denote its characteristic function. Heyman [5] proved that

$$S_{\mathbf{1}_{\mathbb{P}}}(x) := \sum_{n \le x} \mathbf{1}_{\mathbb{P}}\left(\left[\frac{x}{n}\right]\right) = C_{\mathbf{1}_{\mathbb{P}}} x + O(x^{1/2}). \tag{1.6}$$

Ma and Wu [12] proved that the exponent 1/2 in (1.6) can be replaced by $9/19+\varepsilon$. By the arguments of [12] and Theorem 1.4 with the case $f=\Lambda_1$ we get the following Corollary 1.5.

Corollary 1.5 *We have the asymptotic formula*

$$S_{\mathbf{1}_{\mathbb{P}}}(x)=C_{\mathbf{1}_{\mathbb{P}}}x+O(x^{8/17+\varepsilon}). \tag{1.7}$$

Remark 1.6 Theorem 1.4 is uniform for $k\geq 2$ and $\ell\geq 1$, which improves all results listed in (1.4). Although our result for τ_2 is weaker than that of Stucky [14], we keep it for completeness and the following Theorem 1.7.

Theorem 1.7 *Let $k\geq 2$ and $\ell\geq 1$ be two fixed integers, and suppose f is any function in the set $\{\tau_k,\Lambda_\ell,\mu_\ell,\omega_\ell\}$. Suppose $g:\mathbb{N}\to\mathbb{C}$ is any small arithmetic function and define $F(n):=\sum_{n=d^2m}f(m)g(d)$. We have the asymptotic formula*

$$S_F(x)=C_Fx+O(x^{73/155+\varepsilon}).$$

It is well-known that

$$2^{\omega(n)}=\sum_{n=d^2m}\tau(m)\mu(d),\quad \tau(n^2)=\sum_{n=md^2}\tau_3(m)\mu(d),\quad \tau^2(n)=\sum_{n=md^2}\tau_4(m)\mu(d).$$

The Liouville function $\lambda(n)$ satisfies

$$\lambda(n)=(-1)^{\Omega(n)}=\sum_{n=md^2}\mu(m),$$

where $\Omega(m)$ denotes the total number of prime divisors of n. Define the divisor functions

$$t_1(n):=\sum_{n=n_1n_2n_3^2}1,\quad t_2(n):=\sum_{n=n_1n_2n_3^2n_4^2}1.$$

These two functions $t_1(n)$ and $t_2(n)$ are important when counting subgroups of finite abelian groups; see, for example, [18, 21, 22].

From Theorem 1.7 we get the following

Corollary 1.8 *Let $f(n)\in\{2^{\omega(n)},\tau(n^2),\tau^2(n),\lambda(n),t_1(n),t_2(n)\}$ and $\ell\geq 1$ be a fixed integer. Then we have the asymptotic formula*

$$S_{f_\ell}(x)=C_{f_\ell}x+O(x^{73/155+\varepsilon}),$$

where f_ℓ was defined in (1.5).

Remark 1.9 From the proofs of Theorem 1.4 and Theorem 1.7 we see that if $f_1,f_2\in\bigcup_{\ell\geq 1}\{\tau_{\ell+1},\Lambda_\ell,\mu_\ell,\omega_\ell\}$, then Theorem 1.4 and Theorem 1.7 also hold for $f(n)=f_1*f_2(n)=\sum_{n=n_1n_2}f_1(n_1)f_2(n_2)$.

The structure of this paper is as follows. In Section 2 we give some lemmas needed for the proofs. In Section 3 we give the proof of Theorem 1.1. In Section 4 we give some estimates of exponential sums, which are important for the proofs of our theorems. We give the proofs of Theorem 1.4 and Theorem 1.7 in Section 5 and Section 6, respectively.

Notation Throughout this paper, $\tau_k(n)$ denotes the general divisor function, which counts the number of ways n can be written as a product of k factors, $\tau_2(n)=\tau(n)$, $\Lambda(n)$ denotes the von Mangoldt function, $\mu(n)$ denotes the Möbius function, $\omega(n)$ denotes the number of distinct prime divisors of n, $\Omega(m)$ denotes the total number of prime divisors of n, respectively. We use

$\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the set of complex numbers, respectively. For a real number t, $[t]$ denotes its integer part, $\{t\}$ denotes its fractional part, $\psi(t) = \{t\} - 1/2$, $\|t\| = \min(\{t\}, 1 - \{t\})$ and $e(t) = \exp(2\pi i t)$. The symbol $n \sim N$ means that the summation condition of n is $N < n \le 2N$ and $n \asymp N$ means that $c_1 N \le n \le c_2 N$ for two absolute positive constants $0 < c_1 < c_2$. We always use ε to denote a small positive constant, which may be different at different places. In Lemma 2.6 we use the symbol $|\cdot|^*$, which means

$$\Big|\sum_{M<m\le 2M} z_m\Big|^* = \max_{M<u\le 2M}\Big|\sum_{M<m\le u} z_m\Big|.$$

2 Some Lemmas

In order to prove theorems, we need the following lemmas.

Lemma 2.1 *For any $\mathcal{H} \ge 3$, we have*

$$\psi(t) = -\sum_{1\le|h|\le\mathcal{H}} \frac{e(ht)}{2\pi h i} + O\bigg(\min\bigg(1, \frac{1}{\mathcal{H}\|t\|}\bigg)\bigg), \tag{2.1}$$

$$\min\bigg(1, \frac{1}{\mathcal{H}\|t\|}\bigg) = \sum_{h=-\infty}^{\infty} b(h)e(ht), \tag{2.2}$$

where

$$b(0) \ll \frac{\log\mathcal{H}}{\mathcal{H}}, \quad b(h) \ll \min(1/|h|, \mathcal{H}/h^2).$$

Proof See Heath-Brown [4]. □

Lemma 2.2 *Suppose $3 < a < b \le 2a$ and the function $f(u)$ is at least 6 times differentiable on the interval $[a, b]$ such that the estimate $|f^{(j)}| \asymp Fa^{-j}$ $(a \le u \le b)$ holds for some $F > 0$. Then for any exponent pair (κ, λ), we have the estimate*

$$\sum_{a<n\le b} e(f(n)) \ll \frac{a}{F} + F^\kappa a^{\lambda-\kappa}.$$

Proof See, for example, Graham and Kolesnik [6]. □

Lemma 2.3 *Let $M > 0, N > 0, u_m > 0, v_n > 0, A_m > 0, B_n > 0$ $(1 \le m \le M, 1 \le n \le N)$, and let Q_1 and Q_2 be given non-negative numbers, $Q_1 \le Q_2$. Then there is a Q such that $Q_1 \le Q \le Q_2$ and*

$$\sum_{m=1}^{M} A_m Q^{u_m} + \sum_{n=1}^{N} B_n Q^{-v_n} \ll \sum_{m=1}^{M}\sum_{n=1}^{N}(A_m^{v_n}B_n^{u_m})^{\frac{1}{u_m+v_n}} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-v_n}.$$

Proof This is Lemma 2.4 of Graham and Kolesnik [6]. □

Lemma 2.4 *Suppose $x^{1/2} < y < x^{2/3}$ is a parameter. Then*

$$E_f(x) = -\sum_{n\le y} k_f(n)\psi\Big(\frac{x}{n}\Big) + O(x^{1+\varepsilon}y^{-1}),$$

where $k_f(n) := f(n) - f(n-1)$ with $f(0) = 0$.

Proof This is contained in Formula (3.7) of Zhai [20]. □

Lemma 2.5 *Suppose $M, N \geq 1$ are real numbers, $\alpha > 0, \beta > 0$ are fixed constants, $\Delta > 0$. Let $\mathcal{G}(M, N; \Delta)$ denote the number of solutions of the inequality*

$$\left|\left(\frac{n_1}{n_2}\right)^{\alpha} - \left(\frac{m_1}{m_2}\right)^{\beta}\right| \leq \Delta, \quad n_1, n_2 \sim N, \quad m_1, m_2 \sim M.$$

Then we have

$$\mathcal{G}(M, N; \Delta) \ll MN \log 2MN + \Delta M^2 N^2,$$

where the $\ll$ constant is absolute.

Proof This is Lemma 1 of Fouvry and Iwaniec [3]. □

Lemma 2.6 *Let*

$$S_1 = \sum_{H<h\leq 2H} \sum_{N<n\leq 2N} a(h,n) \left| \sum_{M<m\leq 2M} e\left(U \frac{h^{\beta} n^{\gamma} m^{\alpha}}{H^{\beta} N^{\gamma} M^{\alpha}}\right) \right|^{*},$$

where H, N, M are positive integers, U is a real number greater than one, $a(h,n)$ is a complex number of modulus at most one; moreover, α, β, γ are fixed real numbers such that $\alpha(\alpha-1)\beta\gamma \neq 0$. Then we have

$$S_1 \ll (HNM)^{1+\varepsilon}\left(\left(\frac{U}{HNM^2}\right)^{1/4} + \frac{1}{M^{1/2}} + \frac{1}{U}\right).$$

Proof This is Theorem 3 of Robert and Sargos [13]. □

Lemma 2.7 *Let $\mathcal{H} \geq 3$. Then $\psi(t)$ can be written as the form*

$$\psi(t) = \sum_{1\leq |h|\leq \mathcal{H}} \alpha(h) e(ht) + O\left(\sum_{|h|\leq \mathcal{H}} \beta(h) e(ht)\right),$$

where $\alpha(h) \ll 1/|h|$, $\beta(h) \ll 1/\mathcal{H}$.

Proof See Vaaler [15], or the appendix of Graham and Kolesnik [6]. □

3 Proof of Theorem 1.1

Suppose $T \leq x \leq 2T$ and $T^{1/2} \ll y = y(T) \ll T^{2/3}$ is a parameter to be determined. By Lemma 2.4 we have

$$\int_T^{2T} |E_f(x)|^2 dx \ll \int_1 + T^{3+\varepsilon} y^{-2}, \tag{3.1}$$

where

$$\int_1 := \int_T^{2T} \left| \sum_{n\leq y} k_f(n) \psi\left(\frac{x}{n}\right) \right|^2 dx.$$

Take $\mathcal{H} = T^2$ in Lemma 2.1. We have

$$\sum_{n\leq y} k_f(n) \psi\left(\frac{x}{n}\right) = \Sigma_1(x) + O(\Sigma_2(x)), \tag{3.2}$$

$$\Sigma_1(x) = -\sum_{1\leq |h|\leq \mathcal{H}} \frac{1}{2\pi \mathrm{i} h} \sum_{n\leq y} k_f(n) e\left(\frac{hx}{n}\right),$$

$$\Sigma_2(x) = \sum_{n\leq y} |k_f(n)| \min\left(1, \frac{1}{\mathcal{H}\|\frac{x}{n}\|}\right).$$

By Cauchy's inequality we have (note that $f(n) \ll n^{\varepsilon}$)

$$\begin{aligned}\int_T^{2T} |\Sigma_2(x)|^2 dx &\ll \int_T^{2T} \sum_{n\le y} |k_f(n)|^2 \sum_{n\le y} \min\Big(1, \frac{1}{\mathcal{H}^2\|\frac{x}{n}\|^2}\Big) dx \\ &\ll y^{1+\varepsilon} \sum_{n\le y} n \int_{T/n}^{2T/n} \min\Big(1, \frac{1}{\mathcal{H}^2\|u\|^2}\Big) du \\ &\ll Ty^{1+\varepsilon} \int_{-1/2}^{1/2} \min\Big(1, \frac{1}{\mathcal{H}^2 u^2}\Big) du \\ &\ll Ty^{1+\varepsilon}\mathcal{H}^{-1} \ll 1. \end{aligned} \tag{3.3}$$

By a splitting argument we have

$$\Sigma_1(x) \ll \frac{\log^2 T}{H} \Big| \sum_{h\sim H} \sum_{n\sim N} k_f(n) e\Big(\frac{hx}{n}\Big)\Big| \tag{3.4}$$

for $1 \ll H \ll \mathcal{H}$ and $1 \ll N \ll y$. So

$$\begin{aligned}|\Sigma_1(x)|^2 &\ll \frac{\log^4 T}{H^2} \sum_{h_1,h_2\sim H} \sum_{n_1,n_2\sim N} k_f(n_1)k_f(n_2) e\Big(x\Big(\frac{h_1}{n_1} - \frac{h_2}{n_2}\Big)\Big) \\ &= \frac{\log^4 T}{H^2} (\Sigma_{11}(x) + \Sigma_{12}(x)),\end{aligned} \tag{3.5}$$

where

$$\Sigma_{11}(x) := \sum_{\substack{h_1,h_2\sim H; n_1,n_2\sim N \\ h_1n_2 = h_2n_1}} k_f(n_1)k_f(n_2),$$

$$\Sigma_{12}(x) := \sum_{\substack{h_1,h_2\sim H; n_1,n_2\sim N \\ h_1n_2 \ne h_2n_1}} k_f(n_1)k_f(n_2) e\Big(x\Big(\frac{h_1}{n_1} - \frac{h_2}{n_2}\Big)\Big)$$

Obviously we have

$$\int_T^{2T} \Sigma_{11}(x) dx \ll T^{1+\varepsilon} HN. \tag{3.6}$$

By the first derivative test we have

$$\begin{aligned}\int_T^{2T} \Sigma_{12}(x) dx &\ll \sum_{\substack{h_1,h_2\sim H; n_1,n_2\sim N \\ h_1n_2 \ne h_2n_1}} \frac{|k_f(n_1)k_f(n_2)|}{|\frac{h_1}{n_1} - \frac{h_2}{n_2}|} \\ &= \sum_{\substack{h_1,h_2\sim H; n_1,n_2\sim N \\ h_1n_2 \ne h_2n_1}} \frac{|k_f(n_1)k_f(n_2)n_1n_2|}{|h_1n_2 - h_2n_1|} \\ &\ll HN^{3+\varepsilon}.\end{aligned} \tag{3.7}$$

From (3.4)–(3.7) we get

$$\int_T^{2T} |\Sigma_1(x)|^2 dx \ll T^{1+\varepsilon}N + N^{3+\varepsilon} \ll T^{1+\varepsilon}y + y^{3+\varepsilon}. \tag{3.8}$$

Now Theorem 1.1 follows from (3.1)–(3.3) and (3.8) by choosing $y = T^{3/5}$.

4 Estimates of Some Exponential Sums

Let x and N be large real numbers with $x^{1/3} \ll N \ll x^{2/3}$. Suppose $H \geq 1, M_1 \geq 1, M_2 \geq 2$ are real numbers. Define

$$S_{\delta,I}(H, M_1, M_2) := \sum_{h\sim H} c_h \sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1m_2\asymp N}} a_{m_1} e\bigg(\frac{hx}{m_1m_2+\delta}\bigg),$$

$$S_{\delta,II}(H, M_1, M_2) := \sum_{h\sim H} c_h \sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1m_2\asymp N}} a_{m_1} b_{m_2} e\bigg(\frac{hx}{m_1m_2+\delta}\bigg),$$

where $c_h, a_{m_1}, b_{m_2} \in \mathbb{C}$ such that $c_h \ll 1, a_{m_1} \ll m_1^{\varepsilon}, b_{m_2} \ll m_2^{\varepsilon}$, and $0 \leq \delta \leq 1$.

We first prove the following Lemma 4.1, which plays an important role in the proofs of both Theorem 1.4 and Theorem 1.7.

Lemma 4.1 *Suppose $x^{1/3} \ll N \ll x^{2/3}$, $N^{1/3} \ll M_1 \ll N^{1/2}$ and $H \ll N^{1/2-\varepsilon}$. For any exponent pair (κ, λ) we have the estimate*

$$\begin{aligned} S_{\delta,II}(H, M_1, M_2)N^{-\varepsilon} \ll{}& H^{1/2}M_1^{1/2}M_2 + Hx^{\frac{\kappa}{2}}M_1^{1-\kappa}M_2^{\frac{1+\lambda-2\kappa}{2}} \\ &+ Hx^{\frac{\kappa}{2+2\kappa}}M_1^{\frac{2}{2+2\kappa}}M_2^{\frac{1+\lambda}{2+2\kappa}} + \frac{H^2x}{N^2}. \end{aligned} \tag{4.1}$$

Especially we have

$$S_{\delta,II}(H, M_1, M_2)N^{-\varepsilon} \ll H^{1/2}M_1^{1/2}M_2 + Hx^{\frac14}M_1^{\frac12}M_2^{\frac14} + Hx^{\frac16}M_1^{\frac23}M_2^{\frac12} + \frac{H^2x}{N^2}. \tag{4.2}$$

Proof The idea of the proof of this lemma comes from [4]. By Taylor's formula we have

$$\frac{hx}{m_1m_2+\delta} = \frac{hx}{m_1m_2} - \frac{\delta hx}{m_1^2m_2^2} + O\bigg(\frac{hx}{M_1^3M_2^3}\bigg),$$

which implies that

$$S_{\delta,II}(H, M_1, M_2) = \sum_{h\sim H} c_h \sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1m_2\asymp N}} a_{m_1} b_{m_2} e\bigg(\frac{hx}{m_1m_2} - \frac{\delta hx}{m_1^2m_2^2}\bigg) + O\bigg(\frac{H^2x^{1+\varepsilon}}{N^2}\bigg). \tag{4.3}$$

We only need to bound the sum

$$S^*_{\delta,II}(H, M_1, M_2) := \sum_{h\sim H} c_h \sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1m_2\asymp N}} a_{m_1} b_{m_2} e\bigg(\frac{hx}{m_1m_2} - \frac{\delta hx}{m_1^2m_2^2}\bigg).$$

Obviously $0 < h/m_1 \leq 2H/M_1$. Suppose $1 \ll Q \ll HM_1$ is a positive integer parameter to be determined later. For each $1 \leq q \leq Q$, let

$$E_q = \bigg\{(h, m_1) : h \sim H, m_1 \sim M_1, \frac{2(q-1)H}{QM_1} < \frac{h}{m_1} \leq \frac{2qH}{QM_1}\bigg\}.$$

We can write $S^*_{\delta,II}(H, M_1, M_2)$ in the form

$$S^*_{\delta,II}(H, M_1, M_2) = \sum_{m_2\sim M_2} b_{m_2} \sum_{q=1}^{Q} \sum_{\substack{(h,m_1)\in E_q\\ m_1m_2\sim N}} c_h a_{m_1} e\bigg(\frac{hx}{m_1m_2} - \frac{\delta hx}{m_1^2m_2^2}\bigg).$$

By Cauchy's inequality we get

$$|S^*_{\delta,II}(H, M_1, M_2)|^2$$

$$\ll M_2^{1+\varepsilon} Q \sum_{m_2\sim M_2}\sum_{q=1}^{Q}\Bigg|\sum_{\substack{(h,m_1)\in E_q\\ m_1m_2\asymp N}} c_h a_{m_1} e\left(\frac{hx}{m_1m_2}-\frac{\delta hx}{m_1^2m_2^2}\right)\Bigg|^2$$

$$= M_2^{1+\varepsilon} Q \sum_{m_2\sim M_2}\sum_{q=1}^{Q}\sum_{\substack{(h_1,m_{11})\in E_q\\ m_{11}m_2\asymp N}}\sum_{\substack{(h_2,m_{12})\in E_q\\ m_{12}m_2\asymp N}} c_{h_1}a_{m_{11}}\overline{c_{h_2}a_{m_{12}}}e(r(m_2))$$

$$= M_2^{1+\varepsilon} Q \sum_{q=1}^{Q}\sum_{(h_1,m_{11})\in E_q}\sum_{(h_2,m_{12})\in E_q} c_{h_1}a_{m_{11}}\overline{c_{h_2}a_{m_{12}}}\sum_{m_2\in I} e(r(m_2)), \tag{4.4}$$

where $I=I(m_{11},m_{12})$ is a subinterval of $(M_2,2M_2]$, and

$$r(m_2)=r(m_2;h_1,h_2,m_{11},m_{12}):=\frac{x}{m_2}\left(\frac{h_1}{m_{11}}-\frac{h_2}{m_{12}}\right)-\frac{\delta x}{m_2^2}\left(\frac{h_1}{m_{11}^2}-\frac{h_2}{m_{12}^2}\right).$$

Let

$$\eta=\frac{h_1}{m_{11}}-\frac{h_2}{m_{12}}.$$

It is easy to see that the contribution of $\eta=0$ to $|S^*_{\delta,II}(H,M_1,M_2)|^2$ is $O(QM_1HM_2^2N^\varepsilon)$. Now we consider the case $\eta\neq 0$. From the conditions $H\ll N^{1/2-\varepsilon}$, $N^{1/3}\ll M_1\ll N^{1/2}$, $M_1M_2\asymp N$, we see easily that

$$|r^{(j)}(u)|\asymp \frac{x|\eta|}{M_2^{j+1}},\quad j=1,2,3,4,5,6.$$

By Lemma 2.2 we see that the estimate

$$\sum_{m_2\in I} e(r(m_2))\ll \min\left(M_2,\frac{M_2^2}{x|\eta|}\right)+\left(\frac{x|\eta|}{M_2^2}\right)^\kappa M_2^\lambda \tag{4.5}$$

holds for any exponent pair (κ,λ).

By the definition of E_q we have $|\eta|\le 2H/QM_1$. So from (4.4), (4.5) and the above discussions we get that (we use a splitting argument to $1/|\eta|$)

$$\begin{aligned}|S^*_{\delta,II}(H,M_1,M_2)|^2 &\ll QM_1HM_2^2N^\varepsilon + QM_2^{2+\varepsilon}\sum_{\substack{h_1,h_2\sim H;m_{11},m_{12}\sim M_1\\ |\eta|\le M_2x^{-1}}}1,\\ &\quad + QM_2^{3+\varepsilon}x^{-1}\sum_{\substack{h_1,h_2\sim H;m_{11},m_{12}\sim M_1\\ \frac{M_2}{x}\ll|\eta|\le\frac{H}{QM_1}}}\frac{1}{|\eta|}\\ &\quad + Q^{1-\kappa}x^\kappa H^\kappa M_1^{-\kappa}M_2^{1+\lambda-2\kappa}\sum_{\substack{h_1,h_2\sim H;m_{11},m_{12}\sim M_1\\ |\eta|\le HQ^{-1}M_1^{-1}}}1\\ &\ll QM_1HM_2^2N^\varepsilon + QM_2^{2+\varepsilon}\mathcal{A}(H,M_1;M_2x^{-1})\\ &\quad + QM_2^{3+\varepsilon}x^{-1}\max_{\frac{M_2}{x}\ll\Delta\le\frac{H}{QM_1}}\frac{1}{\Delta}\mathcal{A}(H,M_1;\Delta)\\ &\quad + Q^{1-\kappa}x^\kappa H^\kappa M_1^{-\kappa}M_2^{1+\lambda-2\kappa}\mathcal{A}(H,M_1;HQ^{-1}M_1^{-1}),\end{aligned} \tag{4.6}$$

where $\mathcal{A}(H,M_1;\Delta)$ denotes the number of solutions of the inequality

$$|\eta|\le\Delta,\quad h_1,h_2\sim H,\quad m_{11},m_{12}\sim M_1.$$

By Lemma 2.5 we get

$$\mathcal{A}(H, M_1; \Delta) \ll HM_1 \log HM_1 + \Delta HM_1^3. \tag{4.7}$$

From (4.7) we have

$$\begin{aligned} QM_2^{2+\varepsilon}\mathcal{A}(H, M_1; M_2x^{-1}) &\ll (QHM_2^2M_1 + QH(M_1M_2)^3x^{-1})N^\varepsilon \\ &\ll QHM_2^2M_1N^\varepsilon \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &QM_2^{3+\varepsilon}x^{-1} \max_{\frac{M_2}{x} \ll |\Delta| \le \frac{H}{QM_1}} \frac{1}{\Delta}\mathcal{A}(H, M_1; \Delta) \\ &\quad \ll (QHM_2^2M_1 + QH(M_1M_2)^3x^{-1})N^\varepsilon \ll QHM_2^2M_1N^\varepsilon \end{aligned} \tag{4.9}$$

by noting that $N \ll x^{2/3}$, $M_1 \ll N^{1/2}$, $M_1M_2 \asymp N$. From (4.7) again we get

$$\begin{aligned} &Q^{1-\kappa}x^\kappa H^\kappa M_1^{-\kappa} M_2^{1+\lambda-2\kappa}\mathcal{A}(H, M_1; HQ^{-1}M_1^{-1}) \\ &\quad \ll Q^{1-\kappa}x^\kappa H^\kappa M_1^{-\kappa} M_2^{1+\lambda-2\kappa}(HM_1 \log HM_1 + Q^{-1}H^2M_1^2) \\ &\quad \ll Q^{-\kappa}x^\kappa H^{2+\kappa}M_1^{2-\kappa}M_2^{1+\lambda-2\kappa} \end{aligned} \tag{4.10}$$

by recalling our assumption $Q \ll HM_1$.

From (4.6) and (4.8)–(4.10) we get

$$|S^*_{\delta,II}(H, M_1, M_2)|^2N^{-\varepsilon} \ll QHM_2^2M_1 + Q^{-\kappa}x^\kappa H^{2+\kappa}M_1^{2-\kappa}M_2^{1+\lambda-2\kappa}.$$

Choosing a best $Q \in [1, HM_1]$ via Lemma 2.3 we get

$$S^*_{\delta,II}(H, M_1, M_2)N^{-\varepsilon} \ll H^{1/2}M_1^{1/2}M_2 + Hx^{\frac{\kappa}{2}}M_1^{1-\kappa}M_2^{\frac{1+\lambda-2\kappa}{2}} + Hx^{\frac{\kappa}{2+2\kappa}}M_1^{\frac{2}{2+2\kappa}}M_2^{\frac{1+\lambda}{2+2\kappa}}. \tag{4.11}$$

which combining (4.3) gives (4.1). The estimate (4.2) follows from (4.1) by taking the exponent pair $(1/2, 1/2)$. □

Lemma 4.2 *Suppose $x^{8/17} \ll N \ll x^{13/25}$. If $N^{1/3} \ll M_1 \ll N^{1/2}$ and $H \ll N^{1/2-\varepsilon}$, then*

$$S_{\delta,II}(H, M_1, M_2) \ll Hx^{\frac{47}{100}+\varepsilon}.$$

Proof From (4.2) and $N^{1/3} \ll M_1 \ll N^{1/2}$ we see that

$$S_{\delta,II}(H, M_1, M_2)N^{-\varepsilon} \ll H^{1/2}N^{5/6} + Hx^{\frac{1}{4}}N^{\frac{3}{8}} + Hx^{\frac{1}{6}}N^{\frac{7}{12}},$$

which implies Lemma 4.2 by noting that $N \ll x^{13/25}$. □

Lemma 4.3 *Suppose $x^{13/25} \ll N \ll x^{9/17}$. If $N^{2/9} \ll M_1 \ll N^{4/9}$ and $H \ll N^{1/2-\varepsilon}$, then*

$$S_{\delta,II}(H, M_1, M_2) \ll Hx^{\frac{8}{17}+\varepsilon}.$$

Proof From (4.2) and $N^{2/9} \ll M_1 \ll N^{4/9}$ we see that

$$S_{\delta,II}(H, M_1, M_2)N^{-\varepsilon} \ll H^{1/2}N^{8/9} + Hx^{\frac{1}{4}}N^{\frac{13}{36}} + Hx^{\frac{1}{6}}N^{\frac{31}{54}},$$

which implies Lemma 4.3 by noting that $N \ll x^{9/17}$. □

Lemma 4.4 *Suppose $x^{13/25} \ll N \ll x^{9/17}$. If $M_1 \ll N^{5/9}$ and $H \ll Nx^{-8/17}$, then*

$$S_{\delta,I}(H, M_1, M_2) \ll Hx^{\frac{8}{17}+\varepsilon}.$$

Proof We consider three cases: $M_1 \ll N^{2/9}$, $N^{2/9} \ll M_1 \ll N^{4/9}$, $N^{4/9} \ll M_1 \ll N^{5/9}$.

If $M_1 \ll N^{2/9}$, then using the exponent pair $(2/7, 4/7)$ to the sum over m_2 we get

$$\begin{aligned} S_{\delta,I}(H, M_1, M_2) &\ll HH^{2/7}x^{2/7+\varepsilon}M_1^{5/7} \ll HH^{2/7}x^{2/7+\varepsilon}N^{10/63} \\ &\ll H(Nx^{-8/17})^{2/7}x^{2/7+\varepsilon}N^{10/63} \ll Hx^{18/119+\varepsilon}N^{4/9} \\ &\ll Hx^{54/119+\varepsilon} \ll Hx^{8/17+\varepsilon}. \end{aligned}$$

When $N^{2/9} \ll M_1 \ll N^{4/9}$, by Lemma 4.3 we get

$$S_{\delta,I}(H, M_1, M_2) \ll Hx^{8/17+\varepsilon}.$$

Finally suppose $N^{4/9} \ll M_1 \ll N^{5/9}$. Define

$$g(n) := \sum_{\substack{n=m_1m_2 \\ m_1\sim M_1, m_2\sim M_2}} a_{m_1}.$$

Then $S_{\delta,I}(H, M_1, M_2)$ can be written as

$$\begin{aligned} S_{\delta,I}(H, M_1, M_2) &= \sum_{h\sim H} c_h \sum_{M_1M_2<n\le 4M_1M_2} g(n)e\left(\frac{hx}{n+\delta}\right) \\ &= \sum_{h\sim H} c_h \sum_{M_1M_2<n\le 4M_1M_2} g(n)e\left(\frac{hx}{n}\right)\Phi(n, h), \end{aligned} \tag{4.12}$$

where

$$\Phi(u, h) := e\left(\frac{hx}{u+\delta} - \frac{hx}{u}\right), \quad M_1M_2 < u \le 4M_1M_2.$$

It is easy to see that

$$\Phi(u, h) \ll 1, \quad \frac{\partial}{\partial u}\Phi(u, h) \ll \frac{hx}{u^3}. \tag{4.13}$$

Let

$$A(u, h) := \sum_{M_1M_2<n\le u} g(n)e\left(\frac{hx}{n}\right).$$

By partial integration we have

$$\begin{aligned} \sum_{M_1M_2<n\le 4M_1M_2} g(n)e\left(\frac{hx}{n}\right)\Phi(n, h) &= \int_{M_1M_2}^{4M_1M_2} \Phi(u, h)dA(u, h) \\ &= \Phi(4M_1M_2, h)A(4M_1M_2, h) \\ &\quad - \int_{M_1M_2}^{4M_1M_2} A(u, h) \times \frac{\partial}{\partial u}\Phi(u, h)du \end{aligned} \tag{4.14}$$

From (4.12)–(4.14) we have

$$\begin{aligned} S_{\delta,I}(H, M_1, M_2) &= \sum_{h\sim H} c_h\Phi(4M_1M_2, h)A(4M_1M_2, h) \\ &\quad - \int_{M_1M_2}^{4M_1M_2} \sum_{h\sim H} c_hA(u, h) \times \frac{\partial}{\partial u}\Phi(u, h)du \\ &\ll \left(1 + \frac{Hx}{N^2}\right) \sum_{h\sim H} \sum_{m_1\sim M_1} a^*_{m_1} \left| \sum_{m_2\sim M_2} e\left(\frac{hx}{m_1m_2}\right) \right|^*, \end{aligned} \tag{4.15}$$

where $a^*_{m_1} = |a_{m_1}|$. By Lemma 2.6 we have

$$x^{-\varepsilon} \sum_{h\sim H} \sum_{m_1\sim M_1} a^*_{m_1} \left| \sum_{m_2\sim M_2} e\left(\frac{hx}{m_1m_2}\right)\right|^* \ll Hx^{\frac14}M_1^{\frac12}M_2^{\frac14} + HM_1M_2^{\frac12} + \frac{N^2}{x}. \tag{4.16}$$

From (4.15), (4.16) and the condition $N^{4/9} \ll M_2 \ll N^{5/9}$ we get

$$\begin{aligned} x^{-\varepsilon}S_{\delta,I}(H, M_1, M_2) &\ll Hx^{\frac14}M_1^{\frac12}M_2^{\frac14} + HM_1M_2^{\frac12} + \frac{N^2}{x} \\ &\quad + \frac{H^2x^{\frac54}}{N^2}M_1^{\frac12}M_2^{\frac14} + \frac{H^2x}{N^2}M_1M_2^{\frac12} + H \\ &\ll Hx^{\frac14}N^{\frac{7}{18}} + HN^{\frac79} + \frac{N^2}{x} + \frac{H^2x^{\frac54}}{N^{\frac{29}{18}}} + \frac{H^2x}{N^{\frac{11}{9}}} \\ &\ll Hx^{\frac{7063}{15300}} \\ &\ll Hx^{\frac{8}{17}} \end{aligned} \tag{4.17}$$

by noting that $x^{13/25} \ll N \ll x^{9/17}$ and $H \ll Nx^{-8/17}$. This completes the proof of Lemma 4.4. □

Now we consider another exponential sum. Let x and N be large real numbers with $x^{1/3} \ll N \ll x^{2/3}$. Suppose $H \geq 1, D \geq 1, M_1 \geq 1, M_2 \geq 2$ are real numbers such that $D^2M_1M_2 \asymp N$. Define

$$T_\delta(H, D, M_1, M_2) := \sum_{h\sim H} c_h \sum_{d\sim D} \rho_d \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ d^2m_1m_2\asymp N}} a_{m_1}b_{m_2}e\left(\frac{hx}{d^2m_1m_2+\delta}\right),$$

where $c_h, \rho_d, a_{m_1}, b_{m_2} \in \mathbb{C}$ such that $c_h \ll 1, \rho_d \ll d^\varepsilon, a_{m_1} \ll m_1^\varepsilon, b_{m_2} \ll m_2^\varepsilon$, and $0 \leq \delta \leq 1$. We will prove the following Lemma 4.5, which plays an important role in the proof of Theorem 1.7.

Lemma 4.5 *Suppose $x^{1/3} \ll N \ll x^{2/3}, 1 \ll D \ll N^{1/6}, H \ll (N/D^2)^{1/2-\varepsilon}$ and $(N/D^2)^{1/3} \ll M_1 \ll (N/D^2)^{1/2}$. Then for any exponent pair (κ, λ) we have*

$$T_\delta(H, D, M_1, M_2)N^{-\varepsilon} \ll \frac{H^{\frac12}N^{\frac56}}{D^{\frac76}} + \frac{Hx^{\frac{\kappa}{2}}N^{\frac{3+\lambda-4\kappa}{4}}}{D^{\frac{1+\lambda-\kappa}{2}}} + \frac{Hx^{\frac{\kappa}{2+2\kappa}}N^{\frac{3+\lambda}{4+4\kappa}}}{D^{\frac{1+\lambda+\kappa}{2+2\kappa}}} + \frac{H^2x}{DN^2}. \tag{4.18}$$

Especially we have

$$T_\delta(H, D, M_1, M_2)N^{-\varepsilon} \ll \frac{H^{\frac12}N^{\frac56}}{D^{\frac76}} + \frac{Hx^{\frac14}N^{\frac38}}{D^{\frac12}} + \frac{Hx^{\frac16}N^{\frac{7}{12}}}{D^{\frac23}} + \frac{H^2x}{DN^2}. \tag{4.19}$$

Proof Similar to (4.3) we have

$$T_\delta(H, D, M_1, M_2) = T^*_\delta(H, D, M_1, M_2) + O\left(\frac{H^2x^{1+\varepsilon}}{D^5M_1^2M_2^2}\right), \tag{4.20}$$

where

$$T^*_\delta(H, D, M_1, M_2) := \sum_{h\sim H} c_h \sum_{d\sim D} \rho_d \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ m_1m_2\asymp N}} a_{m_1}b_{m_2}e\left(\frac{hx}{d^2m_1m_2} - \frac{\delta hx}{d^4m_1^2m_2^2}\right).$$

Obviously $0 < \frac{h}{d^2m_1} \leq \frac{2H}{D^2M_1}$. Suppose $1 \ll Q \ll DHM_1$ is a positive parameter to be determined later. For each $1 \leq q \leq Q$, let

$$E^*_q = \left\{(h, d, m_1) : h \sim H, d \sim D, m_1 \sim M_1, \frac{2(q-1)H}{QD^2M_1} < \frac{h}{d^2m_1} \leq \frac{2qH}{QD^2M_1}\right\}.$$

We now write $T_\delta^*(H,D,M_1,M_2)$ in the form

$$T_\delta^*(H,D,M_1,M_2)=\sum_{m_2\sim M_2} b_{m_2}\sum_{q=1}^{Q}\sum_{\substack{(h,d,m_1)\in E_q\\ d^2m_1m_2\sim N}} c_h\rho_d a_{m_1} e\left(\frac{hx}{d^2m_1m_2}-\frac{\delta hx}{d^4m_1^2m_2^2}\right).$$

Similar to (4.4) we can get

$$|T_\delta^*(H,D,M_1,M_2)|^2\ll M_2^{1+\varepsilon}Q\sum_{\substack{h_j\sim H,d_j\sim D,m_{1j}\sim M_1(j=1,2)\\ |\eta^*|\le\frac{2H}{QD^2M_1}}}\left|\sum_{m_2\in I^*} e(r_*(m_2))\right|, \tag{4.21}$$

where $I^*=I^*(m_{11},m_{12})$ is a subinterval of $(M_2,2M_2]$, and

$$r_*(m_2)=r(m_2;h_1,h_2,d_1,d_2,m_{11},m_{12}):=\frac{x}{m_2}\eta^*-\frac{\delta x}{m_2^2}\left(\frac{h_1}{d_1^4m_{11}^2}-\frac{h_2}{d_2^4m_{12}^2}\right),$$

$$\eta^*=\frac{h_1}{d_1^2m_{11}}-\frac{h_2}{d_2^2m_{12}}.$$

It is easy to see that the contribution of $\eta^*=0$ to $|T_\delta^*(H,D,M_1,M_2)|^2$ is at most $O(QDM_1HM_2^2N^\varepsilon)$. Now we consider the case $\eta^*\neq 0$. From the conditions $1\ll D\ll N^{1/6}, H\ll (N/D^2)^{1/2-\varepsilon},\ D^2M_1M_2\asymp N$, we see easily that

$$|r_*^{(j)}(u)|\asymp\frac{x|\eta^*|}{M_2^{j+1}}\ (M_2\le u\le 2M_2),\quad j=1,2,3,4,5,6.$$

Suppose that (κ,λ) is any exponent pair. By Lemma 2.2 we have the estimate

$$\sum_{m_2\in I^*} e(r_*(m_2))\ll\min\left(M_2,\frac{M_2^2}{x|\eta^*|}\right)+\left(\frac{x|\eta^*|}{M_2^2}\right)^\kappa M_2^\lambda. \tag{4.22}$$

By the definition of E_q^* we have $|\eta^*|\le 2H/QD^2M_1$. So from (4.21), (4.22) and the above discussions we get that (we use a splitting argument to $1/|\eta^*|$)

$$\begin{aligned}
&|T_\delta^*(H,D,M_1,M_2)|^2\\
&\quad\ll QM_1DHM_2^2N^\varepsilon+QM_2^{2+\varepsilon}\mathcal{B}(H,D,M_1;M_2x^{-1})\\
&\qquad+QM_2^{3+\varepsilon}x^{-1}\max_{\frac{M_2}{x}\ll\Delta\le\frac{2H}{QD^2M_1}}\frac{1}{\Delta}\mathcal{B}(H,D,M_1;\Delta)\\
&\qquad+Q^{1-\kappa}x^\kappa H^\kappa D^{-2\kappa}M_1^{-\kappa}M_2^{1+\lambda-2\kappa}\mathcal{B}(H,D,M_1;HQ^{-1}D^{-2}M_1^{-1}),
\end{aligned}\tag{4.23}$$

where $\mathcal{B}(H,D,M_1;\Delta)$ denotes the number of solutions of the inequality

$$\left|\frac{h_1}{d_1^2m_{11}}-\frac{h_2}{d_2^2m_{12}}\right|\le\Delta,\quad h_1,h_2\sim H,\quad d_1,d_2\sim D,\quad m_{11},m_{12}\sim M_1. \tag{4.24}$$

If (4.24) holds, then we have

$$\left|\frac{h_1m_{12}}{h_2m_{11}}-\frac{d_1^2}{d_2^2}\right|\le\frac{8\Delta D^2M_1}{H},$$

which combining Lemma 2.5 gives

$$\mathcal{B}(H,D,M_1;\Delta)=\sum_{\substack{h_1,h_2\sim H,d_1,d_2\sim D,m_{11},m_{12}\sim M_1\\ |\frac{h_1}{d_1^2m_{11}}-\frac{h_2}{d_2^2m_{12}}|\le\Delta}}1$$

$$\ll \sum_{\substack{h_1,h_2\sim H,d_1,d_2\sim D,m_{11},m_{12}\sim M_1\\ |\frac{h_1m_{12}}{h_2m_{11}}-\frac{d_1^2}{d_2^2}|\le\frac{8\Delta D^2M_1}{H}}} 1$$
$$\ll \sum_{\substack{n_1,n_2\sim HM_1,d_1,d_2\sim D\\ |\frac{n_1}{n_2}-\frac{d_1^2}{d_2^2}|\le\frac{8\Delta D^2M_1}{H}}} \tau(n_1)\tau(n_2)$$
$$\ll N^\varepsilon \sum_{\substack{n_1,n_2\sim HM_1,d_1,d_2\sim D\\ |\frac{n_1}{n_2}-\frac{d_1^2}{d_2^2}|\le\frac{8\Delta D^2M_1}{H}}} 1$$
$$\ll (HDM_1+\Delta HM_1^3D^4)N^\varepsilon, \tag{4.25}$$

where we used the well-known bound $\tau(n)\ll n^\varepsilon$.

From the conditions $1\ll D\ll N^{1/6}, x^{1/3}\ll N\ll x^{2/3},\ D^2M_1M_2\asymp N$ we get

$$(M_1M_2)^3D^4x^{-1}\ll M_2^2DM_1,$$

which combining (4.25) gives

$$\begin{aligned}QM_2^{2+\varepsilon}\mathcal{B}(H,D,M_1;M_2x^{-1}) &\ll (QHM_2^2DM_1+QH(M_1M_2)^3D^4x^{-1})N^\varepsilon\\ &\ll QHM_2^2DM_1N^\varepsilon \end{aligned}\tag{4.26}$$

and

$$\begin{aligned}&QM_2^{3+\varepsilon}x^{-1}\max_{\frac{M_2}{x}\ll|\Delta|\le\frac{H}{QD^2M_1}}\frac{1}{\Delta}\mathcal{B}(H,D,M_1;\Delta)\\ &\ll (QHM_2^2DM_1+QH(M_1M_2)^3D^4x^{-1})N^\varepsilon\\ &\ll QHM_2^2DM_1N^\varepsilon.\end{aligned}\tag{4.27}$$

From (4.25) again we get

$$\begin{aligned}&Q^{1-\kappa}x^\kappa H^\kappa D^{-2\kappa}M_1^{-\kappa}M_2^{1+\lambda-2\kappa}\mathcal{B}(H,D,M_1;HQ^{-1}D^{-2}M_1^{-1})\\ &\ll Q^{1-\kappa}x^\kappa H^\kappa D^{-2\kappa}M_1^{-\kappa}M_2^{1+\lambda-2\kappa}(HM_1D+Q^{-1}H^2D^2M_1^2)N^\varepsilon\\ &\ll Q^{-\kappa}x^\kappa H^{2+\kappa}D^{2-2\kappa}M_1^{2-\kappa}M_2^{1+\lambda-2\kappa}\end{aligned}\tag{4.28}$$

by recalling our assumption $Q\ll HDM_1$.

From (4.23) and (4.26)–(4.28) we get

$$|T_\delta^*(H,D,M_1,M_2)|^2N^{-\varepsilon}\ll QHM_2^2DM_1+Q^{-\kappa}x^\kappa H^{2+\kappa}D^{2-2\kappa}M_1^{2-\kappa}M_2^{1+\lambda-2\kappa}.$$

Choosing a best $Q\in[1,HDM_1]$ via Lemma 2.3 we get

$$\begin{aligned}&T_\delta^*(H,D,M_1,M_2)N^{-\varepsilon}\\ &\ll H^{1/2}M_2M_1^{1/2}D^{1/2}+Hx^{\frac{\kappa}{2}}D^{1-\frac{3\kappa}{2}}M_1^{1-\kappa}M_2^{\frac{1+\lambda-2\kappa}{2}}\\ &\quad+Hx^{\frac{\kappa}{2+2\kappa}}D^{\frac{2-\kappa}{2+2\kappa}}M_1^{\frac{2}{2+2\kappa}}M_2^{\frac{1+\lambda}{2+2\kappa}}\\ &\ll \frac{H^{\frac12}N^{\frac56}}{D^{\frac76}}+\frac{Hx^{\frac{\kappa}{2}}N^{\frac{3+\lambda-4\kappa}{4}}}{D^{\frac{1+\lambda-\kappa}{2}}}+\frac{Hx^{\frac{\kappa}{2+2\kappa}}N^{\frac{3+\lambda}{4+4\kappa}}}{D^{\frac{1+\lambda+\kappa}{2+2\kappa}}}\end{aligned}\tag{4.29}$$

by recalling that $(N/D^2)^{1/3} \ll M_1 \ll (N/D^2)^{1/2}$ and $D^2M_1M_2 \asymp N$. Now (4.18) follows from (4.20) and (4.29). The estimate (4.19) follows from (4.18) by taking the exponent pair $(1/2, 1/2)$. □

Remark 4.6 When we use Lemma 4.1 and Lemma 4.5 to prove Theorem 1.4 and Theorem 1.7, we choose the exponent pair $(1/2, 1/2)$. If we choose exponent pairs as well as we can, then we can slightly improve both the exponent 8/17 in Theorem 1.4 and the exponent 73/155 in Theorem 1.7. Especially, if $(\varepsilon, 1/2+\varepsilon)$ is an exponent pair, then the exponent 8/17 in Theorem 1.4 can be improved to 7/15.

5 Proof of Theorem 1.4

5.1 Proof of Theorem 1.4

Suppose $k \geq 2$ and $\ell \geq 1$ are fixed integers and $f \in \{\tau_k, \Lambda_\ell, \mu_\ell, \omega_\ell\}$. Let $y = x^{9/17}$. Then $x/y = x^{8/17}$. By Lemma 2.4 we have

$$\begin{aligned} E_f(x) &= -\sum_{n\leq x^{9/17}} k_f(n)\psi\left(\frac{x}{n}\right) + O(x^{8/17+\varepsilon}) \\ &= -\sum_{x^{8/17}<n\leq x^{9/17}} f(n)\psi\left(\frac{x}{n}\right) + \sum_{x^{8/17}<n\leq x^{9/17}} f(n)\psi\left(\frac{x}{n+1}\right) + O(x^{8/17+\varepsilon}). \end{aligned} \tag{5.1}$$

So we only need to bound the sum

$$R_{f,\delta}(N;x) := \sum_{n\sim N} f(n)\psi\left(\frac{x}{n+\delta}\right) \quad (\delta = 0, 1)$$

for $x^{8/17} \ll N \ll x^{9/17}$. Let $\mathcal{H} := Nx^{-8/17}$. By (2.1) of Lemma 2.1 we have

$$R_{f,\delta}(N;x) = \Sigma_{f,\delta 1} + O(\Sigma_{f,\delta 2}), \tag{5.2}$$

where

$$\begin{aligned} \Sigma_{f,\delta 1} &= -\sum_{1\leq|h|\leq\mathcal{H}} \frac{1}{2\pi \mathrm{i} h} \sum_{n\sim N} f(n)e\left(\frac{hx}{n+\delta}\right), \\ \Sigma_{f,\delta 2} &= \sum_{n\sim N} |f(n)| \min\left(1, \frac{1}{\mathcal{H}\|\frac{x}{n+\delta}\|}\right). \end{aligned}$$

By (2.2) of Lemma 2.1 and Lemma 2.2 with the exponent pair $(2/7, 4/7)$ we have

$$\begin{aligned} \Sigma_{f,\delta 2} &\ll N^\varepsilon \sum_{n\sim N} \min\left(1, \frac{1}{\mathcal{H}\|\frac{x}{n+\delta}\|}\right) \\ &\ll N^\varepsilon \left(\frac{N}{\mathcal{H}} + \sum_{h=1}^{\infty} \min(h^{-1}, \mathcal{H}h^{-2}) \sum_{n\sim N} e\left(\frac{hx}{n+\delta}\right)\right) \\ &\ll N^\varepsilon \left(\frac{N}{\mathcal{H}} + \mathcal{H}^{2/7}x^{2/7} + \frac{N^2}{x}\right) \\ &\ll x^{8/17+\varepsilon}. \end{aligned} \tag{5.3}$$

By a splitting argument we get

$$\Sigma_{f,\delta 1} \ll \frac{1}{H}\left|\sum_{h\sim H}\sum_{n\sim N} f(n)e\left(\frac{hx}{n+\delta}\right)\right| \log x \tag{5.4}$$

for some $1 \ll H \ll \mathcal{H}$. So Theorem 1.4 follows from (5.1)–(5.4) and the estimate

$$S_{f,\delta}(H,N) := \sum_{h\sim H}\sum_{n\sim N} f(n)e\left(\frac{hx}{n+\delta}\right) \ll Hx^{8/17+\varepsilon}. \tag{5.5}$$

We only need to prove that (5.5) holds for $f \in \{\tau_k, \Lambda_\ell, \mu_\ell, \omega_\ell\}$. We can prove the following two lemmas.

Lemma 5.1 *The sum $S_{f,\delta}(H,N)$ can be written as a sum of* $(\log^{\nu_f} x)$ *expressions of the form* (*Type* I *sum*)

$$S_{f;I}(H,M_1,M_2) := \sum_{h\sim H} c_h \sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1m_2\asymp N}} a_{m_1} e\left(\frac{hx}{m_1m_2+\delta}\right) \tag{5.6}$$

with $M_1 \ll N^{1/3}$ and expressions of the form (*Type* II *sum*)

$$S_{f;II}(H,M_1,M_2) := \sum_{h\sim H} c_h \sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1m_2\asymp N}} a_{m_1} b_{m_2} e\left(\frac{hx}{m_1m_2+\delta}\right) \tag{5.7}$$

with $N^{1/3} \ll M_1 \ll N^{1/2}$, where $\nu_f \geq 1$ is a fixed integer, and $c_h \ll 1$, $a_{m_1} \ll m_1^{\varepsilon}$, $b_{m_2} \ll m_2^{\varepsilon}$. Note that when $f = \omega_\ell$, an additional term $HN^{2/3}$ should be added to the above sums.

Lemma 5.2 *The sum $S_{f,\delta}(H,N)$ can be written as a sum of* $(\log^{\nu_f} x)$ *expressions of the form* (5.6) *with $M_1 \ll N^{5/9}$ and expressions of the form* (5.7) *with $N^{2/9} \ll M_1 \ll N^{4/9}$, where $\nu_f \geq 1$ is a fixed integer, and $c_h \ll 1$, $a_{m_1} \ll m_1^{\varepsilon}$, $b_{m_2} \ll m_2^{\varepsilon}$. Note that when $f = \omega_\ell$, an additional term $HN^{2/3}$ should be added to the above sums.*

First suppose $x^{8/17} \ll N \ll x^{13/25}$. By Lemma 4.2 we have

$$S_{f;II}(H,M_1,M_2) \ll Hx^{\frac{47}{100}+\varepsilon}. \tag{5.8}$$

Using Lemma 2.2 with the exponent pair $(2/7, 4/7)$ to estimate the sum over m_2 and estimate the sums over h and m_1 trivially we get

$$\begin{aligned} x^{-\varepsilon} S_{f;I}(H,M_1,M_2) &\ll HH^{\frac{2}{7}}x^{\frac{2}{7}}M_1^{\frac{5}{7}} + \frac{N^2}{x} \\ &\ll H(Nx^{-8/17})^{\frac{2}{7}}x^{\frac{2}{7}}N^{\frac{5}{21}} + \frac{N^2}{x} \\ &\ll Hx^{3/7} \\ &\ll Hx^{8/17} \end{aligned} \tag{5.9}$$

by noting that $H \ll Nx^{-8/17}$ and $M_1 \ll N^{1/3}$. So from (5.8) and (5.9) we get an estimate better than (5.5) in the range $x^{8/17} \ll N \ll x^{13/25}$.

When $x^{13/25} \ll N \ll x^{9/17}$, the estimate (5.5) follows from Lemma 4.3, Lemma 4.4 and Lemma 5.2. This completes the proof of Theorem 1.4.

We will prove Lemma 5.1 and Lemma 5.2 in the next two subsections.

5.2 Proof of Lemma 5.1.

In this subsection we will prove Lemma 5.1. We consider only $f \in \{\tau_k, \Lambda_\ell, \omega_\ell\}$, since the proofs for Λ_ℓ and μ_ℓ are the same.

We first prove the following decomposition formula.

Lemma 5.3 *Suppose $K \geq 2$ is a fixed integer, $N_1 \geq 1, \ldots, N_K \geq 1$ are natural numbers such that $N_1 \cdots N_K \asymp N$, $W(u)$ is any function defined on $(N, 2N]$, $a_j(n_j) \in \mathbb{C}$ such that $a_j(n_j) \ll n_j^{\varepsilon}$ $(j = 1, \ldots, K)$. If $\max(N_1, \ldots, N_K) \ll N^{2/3}$, then the sum*

$$\sum_{n_1 \sim N_1} a_1(n_1) \cdots \sum_{n_K \sim N_K} a_K(n_K) W(n_1 \cdots n_K) \tag{5.10}$$

can be written as the form

$$\sum_{m_1 \sim M_1} a_{m_1} \sum_{m_2 \sim M_2} b_{m_2} W(m_1 m_2) \tag{5.11}$$

such that $N^{1/3} \ll M_1 \ll N^{1/2}$ and $a_{m_1} \ll m_1^{\varepsilon}$, $b_{m_2} \ll m_2^{\varepsilon}$.

Proof Without loss of generality, suppose $N_1 \leq N_2 \leq \cdots \leq N_K$.

If $K = 2$, then $N_1 \leq N_2 \ll N^{2/3}$ and $N_1 N_2 \asymp N$ implies that $N^{1/3} \ll N_1 \ll N^{1/2}$. So we see that (5.10) is of the form (5.11) by taking $m_1 = n_1, m_2 = n_2$.

Now suppose $K \geq 3$. We consider three cases: $N_K < N^{1/3}$, $N^{1/3} \leq N_K \ll N^{1/2}$, $N^{1/2} \ll N_K \ll N^{2/3}$.

Case I $N_K < N^{1/3}$, which implies that $N_j < N^{1/3}$ $(j = 1, \ldots, K)$. Since $\prod_{j=1}^{K} N_j \asymp N$, we find that there is a j such that $2 \leq j < K$ and

$$N_1 \cdots N_{j-1} < N^{1/3}, \quad N_1 \cdots N_j \gg N^{1/3}.$$

Thus we have

$$N^{1/3} \ll N_1 \cdots N_j = N_1 \cdots N_{j-1} \times N_j \ll N^{2/3}.$$

If $N_1 \cdots N_j \ll N^{1/2}$, then the sum (5.10) can be written as (5.11) by taking

$$\begin{aligned} &m_1 = n_1 \cdots n_j, \quad m_2 = n_{j+1} \cdots n_K, \quad M_1 = N_1 \cdots N_j, \quad M_2 = N_{j+1} \cdots N_K, \\ &a_{m_1} = a_1(n_1) \cdots a_j(n_j) \ll m_1^{\varepsilon}, \quad b_{m_2} = a_{j+1}(n_{j+1}) \cdots a_K(n_K) \ll m_2^{\varepsilon}. \end{aligned}$$

If $N_1 \cdots N_j \gg N^{1/2}$, then the sum (5.10) can be written as (5.11) by taking

$$m_1 = n_{j+1} \cdots n_K, \quad m_2 = n_1 \cdots n_j.$$

Case II $N^{1/3} \ll N_K \ll N^{1/2}$.

In this case the sum (5.10) can be written as (5.11) by taking

$$\begin{aligned} &m_1 = n_K, \quad m_2 = n_1 \cdots n_{K-1}, \quad M_1 = N_K, \quad M_2 = N_1 \cdots N_{K-1}, \\ &a_{m_1} = a_K(n_K) \ll m_1^{\varepsilon}, \quad b_{m_2} = a_1(n_1) \cdots a_{K-1}(n_{K-1}) \ll m_2^{\varepsilon}. \end{aligned}$$

Case III $N^{1/2} \ll N_K \ll N^{2/3}$.

In this case the sum (5.10) can be written as (5.11) by taking

$$\begin{aligned} &m_1 = n_1 \cdots n_{K-1}, \quad m_2 = n_K, \quad M_1 = N_1 \cdots N_{K-1}, \quad M_2 = N_K, \\ &a_{m_1} = a_1(n_1) \cdots a_{K-1}(n_{K-1}) \ll m_1^{\varepsilon}, \quad b_{m_2} = a_K(n_K) \ll m_2^{\varepsilon}. \end{aligned}$$

This completes the proof of Lemma 5.3. □

5.2.1 Proof of Lemma 5.1 for τ_k

Suppose $k \geq 2$ is a fixed integer. By a splitting argument we see that $S_{\tau_k,\delta}(H,N)$ can be written as a sum of $O(\log^{k-1} N)$ exponential sums of the form

$$S_{\tau_k,\delta}(H,N_1,\ldots,N_k) := \sum_{h\sim H} \sum_{\substack{n_1\sim N_1,\ldots,n_k\sim N_k\\ n_1\cdots n_k\asymp N}} e\left(\frac{hx}{n_1\cdots n_k+\delta}\right),$$

where $N_1 \geq 1,\ldots,N_k \geq 1$ are natural numbers such that $N_1\cdots N_k \asymp N$. Without loss of generality, we suppose $N_1 \leq N_2 \leq \cdots \leq N_k$.

If $N_k \ll N^{2/3}$, then from Lemma 5.3 we see that $S_{\tau_k,\delta}(H,N_1,\ldots,N_k)$ can be written as the form (5.7) with $N^{1/3} \ll M_1 \ll N^{1/2}$. If $N_k \gg N^{2/3}$, then $N_1\cdots N_{k-1} \ll N^{1/3}$. So $S_{\tau_k,\delta}(H,N_1,\ldots,N_k)$ can be written as the form (5.6) with $M_1 \ll N^{1/3}$.

5.2.2 Proof of Lemma 5.1 for $f=\Lambda_\ell, \ell=1$

We first consider the function Λ_ℓ for the case $\ell=1$. Let $u\geq 3$ be a fixed integer. If $2v^u \geq n$, then we have Heath-Brown's identity

$$\Lambda(n) = \sum_{j=1}^{u}(-1)^j\tbinom{u}{j} \sum_{\substack{n=\prod_{t=1}^{j} n_t \prod_{t=1}^{j} n_{j+t}\\ n_1,\ldots,n_j<v}} \mu(n_1)\cdots\mu(n_j)\log n_{2u}.$$

Using Heath-Brown's identity with $u=4$ we find that $S_{\Lambda,\delta}(H,N)$ can be written as a sum of $O(\log^7 N)$ exponential sums of the form

$$S_{\Lambda,\delta}(H,N_1,\ldots,N_8) := \sum_{h\sim H} \sum_{\substack{n_1\sim N_1,\ldots,n_8\sim N_8\\ n_1\cdots n_8\asymp N}} \log n_8 \prod_{j=1}^{4}\mu(n_j) e\left(\frac{hx}{n_1\cdots n_8+\delta}\right),$$

where $N_1 \geq 1,\ldots,N_8\geq 1$ are natural numbers such that $N_1\cdots N_8 \asymp N,\ N_j \leq N^{1/4}\,(j=1,2,3,4)$.

If $\max(N_1,\ldots,N_8) \ll N^{2/3}$, then from Lemma 5.3 we see that $S_{\Lambda,\delta}(H,N_1,\ldots,N_8)$ can be written as the form (5.7) with $N^{1/3}\ll M_1 \ll N^{1/2}$. Suppose now $N_j=\max(N_1,\ldots,N_8)\gg N^{2/3}$, then it is easy to see that $j\in\{5,6,7,8\}$. So the sum $S_{\Lambda,\delta}(H,N_1,\ldots,N_8)$ can be written as the form (5.6) with $M_1 \ll N^{1/3}$.

5.2.3 Proof of Lemma 5.1 for $\Lambda_\ell, \ell\geq 2$

Since $\Lambda_\ell(n) = \sum_{n=n_1\cdots n_\ell}\Lambda(n_1)\cdots\Lambda(n_\ell)$, the sum $S_{\Lambda_\ell,\delta}(H,N)$ can be written as a sum of $O(\log^\ell N)$ exponential sums of the form

$$S_{\Lambda_\ell,\delta}(H,N_1,\ldots,N_\ell) := \sum_{h\sim H} \sum_{\substack{n_1\sim N_1,\ldots,n_\ell\sim N_\ell\\ n_1\cdots n_\ell\asymp N}} \prod_{j=1}^{\ell}\Lambda(n_j) e\left(\frac{hx}{n_1\cdots n_\ell+\delta}\right),$$

where $N_1\geq 1,\ldots,N_\ell\geq 1$ are natural numbers such that $N_1\cdots N_\ell\asymp N$. Without loss of generality, suppose $N_1\leq N_2\cdots\leq N_\ell$.

If $N_\ell \leq N^{2/3}$, then from Lemma 5.3 we see that $S_{\Lambda_\ell,\delta}(H,N_1,\ldots,N_\ell)$ can be written as the form (5.7) with $N^{1/3}\ll M_1\ll N^{1/2}$.

Now suppose $N_\ell > N^{2/3}$. From $N_1\cdots N_\ell\asymp N$, we get that $N_1N_2\cdots N_{\ell-1}\ll N^{1/3}$. Applying Heath-Brown's identity with $u=4$ to $\Lambda(n_\ell)$ again we find that $S_{\Lambda_\ell,\delta}(H,N_1,\ldots,N_\ell)$ can be written as a sum of $O(\log^7 N)$ exponential sums of the form

$$S_{\Lambda_\ell,\delta}(H,\mathbf{N}) := \sum_{h\sim H} \sum_{\substack{n_j\sim N_j\\ 1\le j\le \ell-1}} \prod_{j=1}^{\ell-1} \Lambda(n_j) \sum_{\substack{n_{\ell 1}\sim N_{\ell 1},\dots,n_{\ell 8}\sim N_{\ell 8}\\ n_{\ell 1}\cdots n_{\ell 8}\asymp N_\ell}} \prod_{j=1}^{4} \mu(n_{\ell j}) \log n_{\ell 8} \times e\biggl(\frac{hx}{n_1\cdots n_{\ell-1}n_{\ell 1}\cdots n_{\ell 8}+\delta}\biggr),$$

where $\mathbf{N}=(N_1,\dots,N_{\ell-1},N_{\ell 1},\dots,N_{\ell 8})$, $N_{\ell 1}\ge 1,\dots,N_{\ell 8}\ge 1$ are natural numbers such that $N_{\ell 1}\cdots N_{\ell 8}\asymp N_\ell$, and $N_{\ell j}\le N_\ell^{1/4}$ $(j=1,2,3,4)$.

If there exists an $N_{\ell j}$ such that $N_{\ell j}\gg N^{2/3}$, then we have $5\le j\le 8$. So the sum $S_{\Lambda_\ell,\delta}(H,\mathbf{N})$ can be written as the form (5.6) with $M_1\ll N^{1/3}$. If $N_{\ell j}\ll N^{2/3}$ $(1\le j\le 8)$, then by Lemma 5.3 the sum $S_{\Lambda_\ell,\delta}(H,\mathbf{N})$ can be written as the form (5.7) with $N^{1/3}\ll M_1\ll N^{1/2}$.

5.2.4 Proof of Lemma 5.1 for $f=\omega_\ell, \ell=1$

Since $\omega(n)=\sum_{n=pn_1}1$, the sum $S_{\omega,\delta}(H,N)$ can be written as a sum of $O(\log N)$ exponential sums of the form

$$S_{\omega,\delta}(H,P,N_1) := \sum_{h\sim H} \sum_{\substack{n_1\sim N_1,p\sim P\\ n_1p\asymp N}} e\biggl(\frac{hx}{n_1p+\delta}\biggr),$$

where $N_1\ge 1, P\ge 1$ are natural numbers such that $N_1P\asymp N$.

If $N_1\gg N^{2/3}$, then the sum $S_{\omega,\delta}(H,P,N_1)$ can be written as the form (5.6) by taking $m_1=p,\ m_2=n_1$. If $N^{1/2}\ll N_1\ll N^{2/3}$, then $S_{\omega,\delta}(H,P,N_1)$ can be written as the form (5.7) by taking $m_1=p,\ m_2=n_1$. If $N^{1/3}<N_1\ll N^{1/2}$, then $S_{\omega,\delta}(H,P,N_1)$ can be written as the form (5.7) by taking $m_1=n_1,\ m_2=p$.

Now suppose $N_1\ll N^{1/3}$, namely, $P\gg N^{2/3}$. Then we have

$$S_{\omega,\delta}(H,P,N_1) \ll \frac{1}{\log P}|S^*_{\omega,\delta}(H,P,N_1)| + HN_1P^{1/2}, \tag{5.12}$$

where

$$S^*_{\omega,\delta}(H,P,N_1) := \sum_{h\sim H} \sum_{\substack{n_1\sim N_1,n\sim P\\ n_1n\asymp N}} \Lambda(n) e\biggl(\frac{hx}{n_1n+\delta}\biggr).$$

We use Heath-Brown's identity with $u=4$ to $\Lambda(n)$. So we find that $S^*_{\omega,\delta}(H,P,N_1)$ can be written as a sum of $O(\log^7 N)$ exponential sums of the form

$$S^*_{\omega,\delta}(H,N_1,N_2,\dots,N_9) := \sum_{h\sim H} \sum_{\substack{n_1\sim N_1,\dots,n_9\sim N_9\\ n_1\cdots n_9\asymp N}} \log n_9 \prod_{j=2}^{5} \mu(n_j) e\biggl(\frac{hx}{n_1\cdots n_9+\delta}\biggr),$$

where $N_2\ge 1,\dots,N_9\ge 1$ are natural numbers such that $N_2\cdots N_9\asymp P$, $N_j\le P^{1/4}$ $(j=2,3,4,5)$.

If there exists an N_j for which $N_j\gg N^{2/3}$, then we have $6\le j\le 9$. Without loss of generality, suppose $j=9$. We see that the sum $S^*_{\omega,\delta}(H,N_1,N_2,\dots,N_9)$ can be written as the form (5.6) with $m_1\ll N^{1/3}$ by taking $m_1=n_1\cdots n_8,\ m_2=n_9$. If $N_j\ll N^{2/3}$ $(j=2\le j\le 9)$, then from Lemma 5.3 we see that $S^*_{\omega,\delta}(H,N_1,N_2,\dots,N_9)$ can be written as the form (5.7) with $N^{1/3}\ll m_1\ll N^{1/2}$.

Note that $HN_1P^{1/2}$ in (5.12) satisfies $HN_1P^{1/2}\ll HN^{2/3}$.

5.2.5 Proof of Lemma 5.1 for $f=\omega_\ell, \ell\geq 2$

Since $\omega_\ell(n) = \sum_{n=n_1\cdots n_\ell}\omega(n_1)\cdots\omega(n_\ell)$, the sum $S_{\omega_\ell,\delta}(H,N)$ can be written as a sum of $O(\log^\ell N)$ exponential sums of the form

$$S_{\omega_\ell,\delta}(H,N_1,\ldots,N_\ell) := \sum_{h\sim H}\sum_{\substack{n_1\sim N_1,\ldots,n_\ell\sim N_\ell\\ n_1\cdots n_\ell\asymp N}}\prod_{j=1}^{\ell}\omega(n_j)e\biggl(\frac{hx}{n_1\cdots n_\ell+\delta}\biggr),$$

where $N_1\geq 1,\ldots,N_\ell\geq 1$ are natural numbers such that $N_1\cdots N_\ell\asymp N$ and $N_1\leq N_2\cdots\leq N_\ell$.

If $N_\ell\leq N^{2/3}$, then by Lemma 5.3 we see that the sum $S_{\omega_\ell,\delta}(H,N_1,\ldots,N_\ell)$ can be written as the form (5.7) with $N^{1/3}\ll M_1\ll N^{1/2}$.

If $N_\ell\gg N^{2/3}$, then the sum $\sum_{n_\ell\sim N_\ell}$ can be written as sums of the form $\sum_{p\sim P_\ell}\sum_{n\sim N_\ell^*}$ with $P_\ell N_\ell^*\asymp N_\ell$. If $N_\ell^*\gg N^{2/3}$, we get a sum of the form (5.6) with $M_1\ll N^{1/3}$. If $N^{1/3}\ll N_\ell^*\ll N^{2/3}$, we can get a sum of the form (5.7) with $N^{1/3}\ll M_1\ll N^{1/2}$. Finally suppose $N_\ell^*\ll N^{1/3}$. If $P_\ell\ll N^{2/3}$, then we use Lemma 5.3 directly to get a sum (5.7) with $N^{1/3}\ll M_1\ll N^{1/2}$. If $P_\ell\gg N^{2/3}$, then we apply Henth-Brown's identity to p and then use the procedure of the proof of ω_1. We omit the details.

5.3 Proof of Lemma 5.2

In this subsection we will prove Lemma 5.2. Since the proof is similar to that of Lemma 5.1, we give only a short description.

We first give the following decomposition formula without proof.

Lemma 5.4 *Under the conditions of Lemma* 5.3, *if* $\max(N_1,\ldots,N_K)\ll N^{4/9}$, *then the sum* (5.10) *can be written as the form* (5.11) *such that* $N^{2/9}\ll M_1\ll N^{4/9}$ *and* $a_{m_1}\ll m_1^\varepsilon$, $b_{m_2}\ll m_2^\varepsilon$.

From the proof of Lemma 5.1 we see that if $f\in\{\tau_k,\Lambda_\ell,\mu_\ell,\omega_\ell\}$, then the sum $S_{f,\delta}(H,N)$ can be written as a sum of $O((\log x)^{\nu_f})$ expressions of the form

$$S_{f;\delta}(H,N_1,\ldots,N_K) := \sum_{h\sim H}c_h\sum_{n_1\sim N_1}a_{n_1}\cdots\sum_{n_K\sim N_K}a_{n_K}e\biggl(\frac{hx}{n_1\cdots n_K+\delta}\biggr),$$

where $\nu_f\geq 1$ and $K=K(f)\geq 2$ are fixed integers, $N_1\geq 1,\ldots,N_K\geq 1$ and $N_1\cdots N_K\asymp N$.

Let $N_j=\max(N_1,\ldots,N_K)$. If $N_j\gg N^{4/9}$, then $a_{n_j}=1$ or $a_{n_j}=\log n_j$. Hence the sum $S_{f;\delta}(H,N_1,\ldots,N_K)$ can be written as the form (5.6) with $m_1\ll N^{5/9}$. If $N_j\ll N^{4/9}$, then from Lemma 5.4 we see that the sum $S_{f;\delta}(H,N_1,\ldots,N_K)$ can be written as the form (5.7) with $N^{2/9}\ll m_1\ll N^{4/9}$.

6 Proof of Theorem 1.7

Suppose $k\geq 2$ and $\ell\geq 1$ are two fixed integers, and $f\in\{\tau_k,\Lambda_\ell,\mu_\ell,\omega_\ell\}$. Recall that $F(n)=\sum_{n=d^2m}g(d)f(m)$ with $g(d)\ll d^\varepsilon$.

Let $y=x^{82/155}$. Then $x/y=x^{73/155}$. Similar to (5.1) we have

$$\begin{aligned}E_F(x) = &-\sum_{x^{73/155}<n\leq x^{82/155}}F(n)\psi\Bigl(\frac{x}{n}\Bigr)+\sum_{x^{73/155}<n\leq x^{82/155}}F(n)\psi\Bigl(\frac{x}{n+1}\Bigr)\\ &+O(x^{73/155+\varepsilon}).\end{aligned}\tag{6.1}$$

It suffices to bound the sum

$$R_{F,\delta}(N;x) := \sum_{n\sim N} F(n)\psi\left(\frac{x}{n+\delta}\right) \quad (\delta = 0,1)$$

for $x^{73/155} \ll N \ll x^{82/155}$. By the expression $F(n) = \sum_{n=d^2m} g(d)f(m)$ we see that $R_{F,\delta}(N;x)$ can be written as a sum of $O(\log x)$ expressions of the form

$$R_{F,\delta}(D,M;x) := \sum_{\substack{d\sim D, m\sim M \\ d^2m\asymp N}} f(m)g(d)\psi\left(\frac{x}{d^2m+\delta}\right) \quad (\delta = 0,1),$$

where $D \geq 1, M \geq 1, D^2M \asymp N$. Taking $\mathcal{H} = MDx^{-73/155}$ in Lemma 2.7 we get that

$$R_{F,\delta}(D,M;x) \ll |R^*_{F,\delta}(D,M;x)| + O(x^{73/155+\varepsilon}), \tag{6.2}$$

where

$$R^*_{F,\delta}(D,M;x) := \sum_{1\leq h\leq \mathcal{H}} \alpha^*(h) \sum_{\substack{d\sim D, m\sim M \\ d^2m\asymp N}} f(m)g(d)e\left(\frac{hx}{d^2m+\delta}\right)$$

with $\alpha^*(h) \ll 1/h$.

By a splitting argument we get

$$R^*_{F,\delta}(D,M;x) \ll \frac{1}{H}\left|\sum_{h\sim H} c_h \sum_{\substack{d\sim D, m\sim M \\ d^2m\asymp N}} f(m)g(d)e\left(\frac{hx}{d^2m+\delta}\right)\right| \log x \tag{6.3}$$

for some $1 \ll H \ll \mathcal{H}$, where $c_h = \alpha^*(h)h \ll 1$. So Theorem 1.7 follows from (6.1)–(6.3) and the estimate

$$S_{F,\delta}(H,D,M) := \sum_{h\sim H} c_h \sum_{\substack{d\sim D, m\sim M \\ d^2m\asymp N}} f(m)g(d)e\left(\frac{hx}{d^2m+\delta}\right) \ll Hx^{73/155+\varepsilon}. \tag{6.4}$$

We consider three cases.

Case I $\quad D \gg N^{1/6}$.

In this case by trivial estimate we have

$$S_{F,\delta}(H,D,M) \ll HD^{1+\varepsilon}M \ll \frac{HN^{1+\varepsilon}}{D} \ll HN^{5/6+\varepsilon} \ll Hx^{41/93+\varepsilon} \ll Hx^{73/155}.$$

Case II $\quad x^{1/155} \ll D \ll N^{1/6}$.

Similar to Lemma 5.2, we can show that $S_{F,\delta}(H,D,M)$ can be written as a sum of type I sums of the form

$$S_{\mathrm{I}}(H,D,M_1,M_2) := \sum_{h\sim H} c_h \sum_{\substack{d\sim D, m_1\sim M_1, m_2\sim M_2 \\ d^2m_1m_2\asymp N}} g(d)a_{m_1}e\left(\frac{hx}{d^2m_1m_2+\delta}\right)$$

with $M_1 \ll (N/D^2)^{1/3}$ and $a_{m_1} \ll m_1^\varepsilon$, and type II sums of the form

$$S_{\mathrm{II}}(H,D,M_1,M_2) := \sum_{h\sim H} c_h \sum_{\substack{d\sim D, m_1\sim M_1, m_2\sim M_2 \\ d^2m_1m_2\asymp N}} g(d)a_{m_1}b_{m_2}e\left(\frac{hx}{d^2m_1m_2+\delta}\right)$$

with $(N/D^2)^{1/3} \ll M_1 \ll (N/D^2)^{1/2}$ and $a_{m_1} \ll m_1^\varepsilon,\ b_{m_2} \ll m_2^\varepsilon$. Since the proof is almost the same as that of Lemma 5.1, we omit the details.

For $S_{\mathrm{I}}(H,D,M_1,M_2)$, we estimate the sum over m_2 by Lemma 2.2 with the exponent pair $(2/7,4/7)$ and estimate the sums over other variables trivially. We get that

$$\begin{aligned} S_{\mathrm{I}}(H,D,M_1,M_2) &\ll \left(\frac{D^3M_1^2M_2^2}{x} + H(Hx)^{2/7}D^{3/7}M_1^{5/7}\right)N^{\varepsilon} \\ &\ll \left(\frac{N^2}{Dx} + \frac{H^{9/7}x^{2/7}N^{5/21}}{D^{1/21}}\right)N^{\varepsilon} \\ &\ll \left(\frac{N^2}{x} + H^{9/7}x^{2/7}N^{5/21}\right)N^{\varepsilon} \\ &\ll Hx^{1394/3255+\varepsilon} \\ &\ll Hx^{73/155} \end{aligned} \tag{6.5}$$

by noting that $M_1 \ll (N/D^2)^{1/3},\ D \gg 1,\ N \ll x^{82/155},\ H \ll MDx^{-73/155}$.

For $S_{\mathrm{II}}(H,D,M_1,M_2)$, we get by (4.19) of Lemma 4.5 that

$$\begin{aligned} S_{\mathrm{II}}(H,D,M_1,M_2)x^{-\varepsilon} &\ll \frac{H^{\frac12}N^{\frac56}}{D^{\frac76}} + \frac{Hx^{\frac14}N^{\frac38}}{D^{\frac12}} + \frac{Hx^{\frac16}N^{\frac{7}{12}}}{D^{\frac23}} + \frac{H^2x}{DN^2} \\ &\ll Hx^{73/155} \end{aligned} \tag{6.6}$$

by recalling that $x^{73/155} \ll N \ll x^{82/155},\ x^{1/155} \ll D \ll D^{1/6},\ H \ll DMx^{-73/155}$.

From (6.5) and (6.6) we see that (6.4) holds for $x^{1/155} \ll D \ll N^{1/6}$.

Case III $D \ll x^{1/155}$.

We have

$$\begin{aligned} S_{F,\delta}(H,D,M)x^{-\varepsilon} &\ll \sum_{d\sim D}\left|\sum_{h\sim H} c_h \sum_{\substack{m\sim M\\ d^2m\asymp N}} f(m)e\left(\frac{hx}{d^2m+\delta}\right)\right| \\ &\ll \sum_{d\sim D}\left|\sum_{h\sim H} c_h \sum_{m\sim N_d} f(m)e\left(\frac{hx_d}{m+\delta_d}\right)\right| \end{aligned} \tag{6.7}$$

where $x_d = x/d^2,\ N_d = N/d^2 \asymp M,\ \delta_d = \delta/d^2$. We shall show that the estimate

$$\sum_{h\sim H} c_h \sum_{m\sim N_d} f(m)e\left(\frac{hx_d}{m+\delta_d}\right) \ll Hx_d^{8/17+\varepsilon} \tag{6.8}$$

holds.

If $N_d \ll x_d^{8/17}$, then (6.8) is trivial. Now suppose $N_d \gg x_d^{8/17}$. It is easy to see that $H \ll N_d x_d^{-8/17}$ and $N_d \ll x_d^{9/17}$. So (6.8) follows from the estimate (5.5).

From (6.7) and (6.8) we get the estimate

$$S_{F,\delta}(H,D,M)x^{-\varepsilon} \ll H\sum_{d\sim D} x_d^{8/17} \ll x^{8/17}D^{1/17} \ll Hx^{73/155}$$

for $D \ll x^{1/155}$.

This completes the proof of Theorem 1.7.

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