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Finite Time Blow-up for Heat Flows of Self-induced Harmonic Maps

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Abstract Let M^n be an embedded closed submanifold of $\mathbb{R}^{k+1}$ or a smooth bounded domain in $\mathbb{R}^n$, where $n \geq 3$. We show that the local smooth solution to the heat flow of self-induced harmonic map will blow up at a finite time, provided that the initial map u_0 is in a suitable nontrivial homotopy class with energy small enough.

Keywords Heat flow of self-induced harmonic map, Landau–Lifshitz–Gilbert equation, Finite time blow-up.

MR(2010) Subject Classification 35B44, 35K40, 53C21, 58E20

1 Introduction

1.1 Background

In physics, the Landau–Lifshtiz (LL) equation, deduced in [19, 27], is a fundamental evolution equation for the ferromagnetic spin chain and was proposed on the phenomenological background in studying the dispersive theory of magnetization of ferromagnets. In fact, this equation describes the Hamiltonian dynamics corresponding to the micromagnetic energy, which is defined as follows.

We assume that a ferromagnetic material occupies a smooth bounded domain $\Omega \subset \mathbb{R}^3$. Let u, denoting magnetization vector, be a mapping from Ω into a unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The

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micromagnetic energy of map u is defined by

$$\mathcal{E}(u) := Q\int_\Omega \Phi(u)\,dx + \int_\Omega |\nabla u|^2\,dx + \int_\Omega |h_d(u)|^2\,dx.$$

Here ∇ denote the gradient operator and dx is the volume element of $\mathbb{R}^3$.

In the above functional, the first and second terms are the anisotropy energy with positive quality factor Q and exchange energy, respectively. $\Phi(u)$ is a real function on $\mathbb{S}^2$. The last term is the self-induced energy, and $h_d(u)$ is the demagnetizing field, which solves the following Maxwell equations

$$\begin{cases}\nabla\times h_d = 0,\\ \nabla\cdot(h_d + u\chi_\Omega) = 0\end{cases} \tag{1.1}$$

in $\mathbb{R}^3$, where χ_Ω is the characteristic function of Ω.

The Landau–Lifshitz–Gilbert (LLG) equation with dissipation can be written as

$$u_t = -\alpha u\times h - \beta u\times(u\times h),$$

where "$\times$" denotes the cross production in $\mathbb{R}^3$ and the local field h of $\mathcal{E}(u)$ can be derived as

$$h := -\frac{1}{2}\frac{\delta\mathcal{E}(u)}{\delta u} = \Delta u + h_d - \frac{Q}{2}\nabla_u\Phi.$$

Here $\beta \geq 0$ is the damping parameter and $\alpha\in\mathbb{R}$ such that $\alpha^2+\beta^2 = 1$. Mathematically speaking, the LLG equation is a hybrid of the heat flow and the Schrödinger flow for the energy $\mathcal{E}$.

In the following context, we restrict ourselves to the regime of soft and small ferromagnetic particle Ω_η with $\eta > 0$, where $\Omega_\eta = \{\eta x\,|\,x\in\Omega\}$ and $|\Omega|$ denotes the volume of Ω. "Soft" refers to the case when $Q = 0$, and "small" means that $|\Omega_\eta| \leq \eta^3|\Omega| \ll 1$. Then the micromagnetic energy becomes

$$\mathcal{E}(u) := \int_{\Omega_\eta} |\nabla u|^2\,dx + \int_{\Omega_\eta} |h_d(u)|^2\,dx.$$

Hence, by setting $u(x) = m(\eta x)$, we consider the rescaled micromagnetic energy

$$\mathcal{E}_\eta(u) := \eta^{-1}\mathcal{E}(m) = \int_\Omega |\nabla u|^2\,dx + \eta^2\int_\Omega |h_d(u)|^2\,dx,$$

where we set $h_d(u)(x) := h_d(m)(\eta x) : \Omega\to\mathbb{R}^3$, which also solves equation (1.1).

In fact, the non-local field $h_d(u) = -\nabla f$, where f is the solution of the Possion equation

$$\Delta f = \mathrm{div}(u\chi_\Omega),$$

and hence, which has a precise formula

$$f(x) = \int_\Omega \nabla_y G(x,y)u(y)dy,$$

where $G(x,y) = \frac{1}{4\pi|x-y|}$ is the fundamental function in $\mathbb{R}^3$. Therefore, $h_d(u)$ can be presented as

$$h_d(u)(x) = -\nabla_x\int_\Omega \nabla_y G(x,y)u(y)dy.$$

On the other hand, it is natural to extend the definition of h_d for $u:\Omega\to\mathbb{S}^{n-1}\hookrightarrow\mathbb{R}^n$ with $n\geq 3$ as

$$h_d(u)(x)=-\nabla_x\int_\Omega\nabla_y G(x,y)u(y)dy \tag{1.2}$$

in distribution sense, where Ω is a domain in $\mathbb{R}^n$ and $G(x,y)=\frac{c_n}{|x-y|^{n-2}}$ is the fundamental solution for the Laplace operator Δ on $\mathbb{R}^n$.

A smooth map $u:\Omega\to\mathbb{S}^{n-1}$ is called self-induced harmonic map if it is a critical point of $\mathcal{E}_\eta$ satisfying the following Euler–Lagrange equation

$$\Delta u=-|\nabla u|^2u-\eta^2(h_d(u)-\langle h_d(u),u\rangle u),$$

which explains the multiple magnetic phenomena in the static case.

The heat flow of self-induced harmonic map (i.e. LLG equation of the case $\beta=1$) with Neumann initial boundary condition satisfies the following equation

$$\begin{cases}\partial_t u=\Delta u+|\nabla u|^2u+\eta^2(h_d(u)-\langle h_d(u),u\rangle u), & (x,t)\in\Omega\times\mathbb{R}^+,\\ \left.\dfrac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0, & (x,t)\in\partial\Omega\times\mathbb{R}^+,\\ u(x,0)=u_0:\Omega\to\mathbb{S}^{n-1}, \end{cases} \tag{1.3}$$

where ν is the unit outer normal vector of $\partial\Omega$.

In general, one can also give the following extension of h_d for $u:M^n\to\mathbb{S}^k\hookrightarrow\mathbb{R}^{k+1}$ (see Section 2), where M^n is an embedded closed submanifold of $\mathbb{R}^{k+1}$ with $\dim(M)=n$. Therefore, from the viewpoint of mathematics it is of interest to consider the following heat flow of self-induced harmonic map u with initial smooth map $u_0:M\to\mathbb{S}^k$

$$\begin{cases}\partial_t u=\Delta u+|\nabla u|^2u+\eta^2(h_d(u)-\langle h_d(u),u\rangle u), & (x,t)\in M\times\mathbb{R}^+\\ u(x,0)=u_0:M\to\mathbb{S}^k. \end{cases} \tag{1.4}$$

1.2 Related works

Since LLG equation is an important topic in both mathematics and physics, there has been tremendous interest in developing the well-posedness of LLG equation and its related topics. Here, we list only a few of results that are closely related to our work in the present paper.

In 1964, Eells and Sampson [18] obtained the existence of local smooth solution to heat flow of harmonic map on closed Riemannian manifold. Moreover, the solution exists globally and converges to a harmonic map, provided that the target compact manifold has nonpositive sectional curvature. Later, Hamilton generalized this result to the case of Dirichlet boundary problem in [21]. Inspirted by the work of Sacks–Uhlenbeck [34], Chen and Struwe [36] proved that the global solution to heat flow of harmonic map exists, if the initial energy is small. A similar global existence of solution to LLG equation in $\mathbb{R}^2$ was obtained by Carbou [4].

In general settings, by using the L^p-spectral theory of the Laplace operator (see Section 7 of Chapter 1 in [37]), Taylor could give the existence of short-time regular solution in $C^1([0,T)\times\bar{M})\cap C^\infty((0,T)\times\bar{M})$ to the semi-linear parabolic equation in both Dirichlet and Neumann problems when M is a compact manifold with boundary, by providing the initial data $u_0\in C^\infty(\bar{M})$ (see Section 1 and Section 3 in Chapter 15 of [37], also refer to [28]). Since the equations

(1.3) and (1.4) are both strictly semi-linear parabolic equations, their locally regular solutions indeed exist.

Recently, the local existence of very regular solution to LLG equation ($\alpha \neq 0$) with Neumann boundary condition was addressed by applying the delicate Galerkin approximation method and adding compatibility initial-boundary condition in [5]. And then, inspired by the method used in [5], we obtained the locally very regular solution of LLG equation with spin-polarized transport in [7]. Later, we generalized our previous work [7] (or Carbou's result in [5]) to a Landau–Lifshitz–Gilbert flow with target being compact symplectic manifold, cf. [8]. For the most challenging case of $\beta = 0$, the local regular solution to Schrödinger flow defined on closed manifold was given by Ding and Wang in [16]. Very recently, we get the existence and uniqueness of local smooth solutions to LL equation (i.e. Schrödinger flow into sphere) with Neumann boundary condition in [9, 10].

On the other hand, in the past three decades, a great deal of mathematical effort in blow-up analysis has been devoted to studying the phenomenon for finite-time singularities of heat flow of harmonic maps. In 1989, Coron and Gildaglia [14] gave finite-time blow-up examples to heat flow of harmonic maps, for some certain symmetric initial data from $\mathbb{R}^n$ or $\mathbb{S}^n$ to $\mathbb{S}^n$ with $n \geq 3$. Another approach to show the occurrence of finite-time singularities was built up by Ding [15] for heat flow of harmonic map in dimension 3, by using a monotonicity formula of almost harmonic map and applying the delicate blow up analysis; and then by Chen and Ding in [11] for higher dimension $n \geq 3$, based on both the Struwe's parabolic energy monotonicity formula (see [12]) and Ding's method in [15]. Their results guarantee that such singular examples will occur if the initial data u_0 is in some nontrivial homotopy class with small energy. In fact, there exists topological conditions of spaces, which can insure the condition of initial map u_0, see [39]. When $n = 2$, such finite-time singular examples were shown by Chang, Ding and Ye in [6].

Later, the similar results of finite-time singularity were addressed by Grotowshi [20] for Yang–Mills heat flow with initial connection in a trivial $SO(n)$-bundle over $\mathbb{R}^n$ with $n \geq 5$. Under a similar setting as in [11], Naito in [32] gave finite-time blow-up solutions to Yang–Mills heat flow with initial data in a nontrivial principal bundle $SO(n)$ over $\mathbb{S}^n$ with $n \geq 5$. S.J. Ding and C.Y. Wang in [17] applied a similar approach as in [11] to show finite-times singular examples for Landau–Lifshitz equation in lower dimensions.

We would like to remark that the critical ingredient in [11] is the ε-regularity of heat flow of harmonic map (see [12]), which is based on both the parabolic monotonicity formula and the Bochner identity for the heat flow of harmonic map. Since in lack of both the parabolic monotonicity formula and the Bochner formula for Landau–Lifshitz equation, the authors in [17] used a similar treatment as in [29] and [30] to get a slice monotonicity formula in lower dimension and then obtained the ε-regular estimate of solutions by applying the dual of BMO and $\mathcal{H}_1$. This method has been also used by [30, 31] and [24] respectively to get partially regular results of harmonic maps and almost harmonic maps.

1.3 Main Results and Strategy

In this paper, we focus on the aspect of blow-up for the heat flow of self-induced harmonic map satisfying (1.3) or (1.4). Since the nonlocal potential $h_d(u)$ can be well estimated by u (see

Section 5 in Appendix), there holds a parabolic monotonicity formula by a similar argument as in [12]. Hence, we also use the dual of BMO and $\mathcal{H}_1$ to show the ε-regularity of self-induced harmonic map. By combining the parabolic monotonicity formula in Theorem 2.10 and the ε-regularity Theorem 2.6, we can get the following main results.

Theorem 1.1 *Let $n \geq 3$, and M^n be an embedded closed submanifold in $\mathbb{R}^{k+1}$ with induced metric. There exists constants ε, η_0 such that if*

$$0 < \eta \leq \eta_0,$$

and $u_0 \in C^\infty(M^n, \mathbb{S}^k)$ is in a nontrivial homotopic class $[u_0]$ with $E_{[u_0]} = 0$ (defined in (1.5)), which satisfies

$$\int_M |\nabla u_0|^2 dv \leq \varepsilon^2,$$

then the local smooth solution u of (1.4) with initial map u_0 blows up in finite time. Moreover, let T be the maximal existence time of u, we have

$$T \leq C(\varepsilon^2 + \eta_0^2)^{\frac{2}{n-2}},$$

where C is a positive constant depending only on the geometry of M and $\mathbb{S}^k$.

Remark 1.2 Let $[u_0] = \{f \in C(M,N) \,|\, f \text{ is homotopic to } u_0\}$. Then, there holds

$$E_{[u_0]} := \inf_{f \in [u_0] \cap W^{1,2}(M,N)} \int_M |\nabla f|^2 dv = 0 \tag{1.5}$$

if we provide one of the following three topological conditions:

(1) $\pi_1(M) = 0$ and $\pi_2(M) = 0$,

(2) $\pi_1(M) = 0$ and $\pi_2(N) = 0$,

(3) $\pi_1(N) = 0$ and $\pi_2(N) = 0$,

to see [39] for more details.

In particular, let $M = \mathbb{S}^n$ and $N = \mathbb{S}^n$ with $n \geq 3$. Let $u_0 : \mathbb{S}^n \to \mathbb{S}^n$ be a smooth map with $\deg(u_0) \neq 0$. Then u_0 is not homotopically trivial. Meanwhile, $E_{[u_0]} = 0$, since $\pi_1(\mathbb{S}^n) = 0$ and $\pi_2(\mathbb{S}^n) = 0$. Some other examples were given by Ding–Wang by using the Hopf map (to see [17]).

For a domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega \neq \emptyset$, we can get a similar result as in Theorem 1.1 for the Neumann boundary problem (1.3).

To proceed, we denote $[u_0]_\nu := \{v \in C^1(\bar{\Omega}, \mathbb{S}^{n-1}) \,|\, \text{there is a } C^0([0,1], C^1(\bar{\Omega})) \text{ map } \varphi : \bar{\Omega} \times [0,1] \to \mathbb{S}^{n-1} \text{ such that } \varphi(x,0) = u_0,\ \varphi(x,1) = v, \text{ and } \frac{\partial \varphi(\cdot,t)}{\partial \nu}|_{\partial\Omega} = 0\}$, and then define

$$E_{[u_0]_\nu} = \inf_{v \in \overline{[u_0]_\nu}} \int_\Omega |\nabla v|^2 dx,$$

where ν is the unit outer normal vector of $\partial\Omega$ and $\overline{[u_0]_\nu}$ is the completion of $[u_0]_\nu$ in $W^{1,2}$-norm. We say $[u_0]_\nu$ is a nontrivial class (denoted by $[u_0]_\nu \neq 0$), if constant maps are not in $[u_0]_\nu$.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with $n \geq 3$. There exists constants ε, η_0 such that if*

$$0 < \eta \leq \eta_0,$$

and $u_0 \in C^\infty(\bar{\Omega}, \mathbb{S}^{n-1})$ *is in a nontrivial class* $[u_0]_\nu$ *with* $E_{[u_0]_\nu} = 0$*, which satisfies*

$$\int_\Omega |\nabla u_0|^2 dv \le \varepsilon^2,$$

then the local smooth solution u *of* (1.3) *with initial map* u_0 *blows up in finite time. Moreover, letting* T *be the maximal existence time of* u*, we have*

$$T \le C(\varepsilon^2 + \eta_0^2)^{\frac{2}{n-2}},$$

where C *is a positive constant depending only on* n *and* $|\Omega|$*.*

Remark 1.4 Let $\Omega = \{x \in \mathbb{R}^n \,|\, 1 < |x| < 2\}$. For $n \ge 4$, by using the facts $\pi_1(\mathbb{S}^{n-1}) = 0$ and $\pi_2(\mathbb{S}^{n-1}) = 0$, we have $E_{[f]} = 0$, where $[f]$ is any nontrivial homotopy class of maps from $\mathbb{S}^{n-1}$ into itself. Since

$$\left\{ u(x) = g\left(\frac{x}{|x|}\right) \,\middle|\, g \in [f] \cap C^1(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}) \right\} \subset [u_0]_\nu$$

for some $u_0 = g_0(\frac{x}{|x|})$ with $g_0 \in [f] \cap C^1(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$, and

$$\int_\Omega |\nabla u|^2 dv = \omega_{n-1} \int_1^2 r^{n-3} dr \int_{\mathbb{S}^{n-1}} |\nabla g|^2 d\theta = \frac{\omega_{n-1}(2^{n-2}-1)}{n-2} \int_{\mathbb{S}^{n-1}} |\nabla g|^2 d\theta,$$

there holds

$$E_{[u_0]_\nu} = 0 \quad \text{and} \quad [u_0]_\nu \ne 0.$$

Now we give an illustration of the main ideas in our proof. We will first prove Theorem 1.1 in Section 3, and then Theorem 1.3 in Section 4. The proof of Theorem 1.1 consists of two steps. In the first step, we use the ε-regularity obtained in Theorem 2.6 to eliminate the case of not blowing up along heat flow of self-induced harmonic map. Then, in the second step, by using parabolic monotonicity formula in Theorem 2.10, we take a similar but more complicated argument as in [11] to give

$$T \le C(\varepsilon^2 + \eta^2)^{\frac{2}{n-2}}$$

for small ε and η, where T is the extreme existence time of solution and $\varepsilon^2 = \|\nabla u_0\|_{L^2(M)}$. Both of the two steps need fine estimates on h_d, which is given in Theorem 5.5 in Appendix 5.

Theorem 1.3 is proved by a similar but more involved manner, since we need boundary regular estimates of u. For the case $\eta = 0$, that is, u is a heat flow of harmonic map, there is a standard treatment to deal with the boundary regularity, which is to extend u across the boundary by a transformation of reflection (see the definition in (6.4)). Let $\bar{u}$ denote this extension of u. It is not difficult to check that $\bar{u}$ is a strong solution of equation (1.3) with $\eta = 0$ on a bigger domain than Ω, when u satisfies the Neumann boundary condition. This transfers the regular estimates near the boundary to the interior ones. However, when $\eta > 0$, it seems difficult for us to get the regular estimates of h_d near the boundary by applying the above method, due to h_d being defined globally in (1.2), so as u. Fortunately, we can develop the global L^p-theory for heat equation with the Neumann boundary condition in a more general setting in Appendix 6 to overcome this difficulty. Since the ε-regularity in Theorem 2.6 and the parabolic monotonicity also hold near boundary in the case of Neumann problem (see Corallary 4.1 and Remark 4.3), after some modifications about the proof of Theorem 1.1, the desired result in Theorem 1.3 can be established.

The rest of our paper is organized as follows. In Section 2, we introduce the basic setting of self-induced harmonic maps and some critical preliminary lemmas, including the ε-regularity theorem of self-induced harmonic maps and the parabolic monotonicity formula for heat flow of self-induced harmonic maps. In Section 3, we give the proof of Theorem 1.1. The proof of theorem 1.3 will be built up in Section 4. Finally, the estimates of h_d and the global L^p-estimates of heat equations in more general settings are given in Appendix 5 and Appendix 6 respectively.

2 Preliminary

2.1 The Self-induced Dirichlet Energy Functional

Suppose that M^n $(n \geq 3)$ is an embedded closed submanifold of $\mathbb{R}^{k+1}$, which is equipped with the metric g induced by the Euclidean metric $\langle\, ,\, \rangle$ of $\mathbb{R}^{k+1}$. The tangent bundle of M, denoted by TM, is a subbundle of the trivial vector bundle $E = M \times \mathbb{R}^{k+1}$. Let $\pi : E \to TM$ be the bundle projection, and $\mathscr{F} = \Gamma(E)$ denote the space of smooth sections for vector bundle E, which is identified with the space of smooth map from M to $\mathbb{R}^{k+1}$, since E is trivial. The completeness of $\Gamma(E)$ under the $W^{s,2}$ Sobolev norm is denoted by $W^{s,2}(\mathscr{F})$ for any $s \in \mathbb{N}$. Denote

$$W^{1,2}(M, \mathbb{S}^k) := \{u \in W^{1,2}(\mathscr{F}) \,|\, u : M \to \mathbb{S}^k \hookrightarrow \mathbb{R}^{k+1}, \text{ for a.e. x} \in M\}.$$

For any $u \in W^{1,2}(\mathscr{F})$, the nonlocal potential is defined by

$$h_d(u)(x) = -\nabla_x f = -\nabla_x \int_M \langle \nabla_y G(x,y), \pi(u)(y) \rangle dv_y \tag{2.1}$$

in distribution sense, where ∇ denotes the Levi–Civita connection induced by metric g and $G(x,y)$ is a Green's function on M (refer to Lemma 5.1). Here and in the following context, without loss of generality and for simplicity, we denote $\pi(u)(x) = \pi_x(u(x))$. The function $f : M \to \mathbb{R}$ satisfies the following equation

$$\Delta f = \mathrm{div}(\pi(u)). \tag{2.2}$$

This implies

$$\mathrm{div}(h_d + \pi(u)) = 0. \tag{2.3}$$

Then we define the self-induced Dirichlet functional with nonlocal potential for a section $u \in W^{1,2}(M, \mathbb{S}^k)$ by

$$\mathcal{E}_\eta(u) := \int_M |\nabla u|^2 dv + \eta^2 \int_M |h_d(u)|^2 dv, \tag{2.4}$$

where η is a nonnegative scalar constant, and dv is the volume form.

2.2 Self-induced Harmonic Map

By definition, $h_d : L^2(M, \mathbb{R}^{k+1}) \to L^2(M, TM)$ is a linear operator. Then, the gradient $\frac{\delta \mathcal{E}}{\delta u}$ of this energy functional $\mathcal{E}_\eta$ is obtained from the following lemma.

Lemma 2.1 *For any u and w in $W^{1,2}(\mathscr{F})$, there holds*

$$\int_M \langle h_d(u), h_d(w) \rangle \, dv = -\int_M \langle h_d(u), w \rangle \, dv.$$

Proof Since $\Delta f(u) = \operatorname{div}(\pi(u))$ and $u \in W^{1,2}(M, \mathbb{S}^k)$, it implies $h_d(u) \in W^{1,2}(M, \mathbb{R}^{k+1})$ by the classical L^2-elliptic estimates.

Thus, by using Stokes formula, there holds

$$\begin{aligned}\int_M \langle h_d(u), h_d(w)\rangle\, dv &= -\int_M \langle \nabla f(u), h_d(w)\rangle\, dv = \int_M \operatorname{div}(h_d(w)) f(u) dv \\ &= -\int_M \operatorname{div}(\pi(w)) f dv = -\int_M \langle h_d(u), w\rangle\, dv.\end{aligned}$$

□

Proposition 2.2 *For any $u \in W^{1,2}(M, \mathbb{S}^k)$, the gradient of self-induced Dirichlet energy functional $\mathcal{E}_\eta$ at u is*

$$\frac{1}{2}\frac{\delta \mathcal{E}_\eta(u)}{\delta u} = \tau(u) - \eta^2(h_d(u) - \langle h_d(u), u\rangle u),$$

where $\tau(u) = -\Delta u - |\nabla u|^2 u$ is the tension field.

Proof Let $u_t = \frac{u+t\varphi}{|u+t\varphi|}$ for any $\varphi \in C^\infty(M, \mathbb{R}^{k+1})$ with t small enough. Then we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} E_\eta(u_t) &= 2\int_M \langle \nabla u, \nabla(\varphi - \langle \varphi, u\rangle u\rangle dv + 2\eta^2 \int_M \langle h_d(u), h_d(\varphi - \langle \varphi, u\rangle u)\rangle dv \\ &= 2\int_M \langle \nabla u, \nabla \varphi\rangle - \langle |\nabla u|^2 u, \varphi\rangle dv - 2\eta^2 \int_M \langle h_d(u), \varphi - \langle \varphi, u\rangle u\rangle dv.\end{aligned}$$

That is,

$$\frac{\delta E_\eta(u)}{\delta u} = 2\tau(u) - 2\eta^2(h_d(u) - \langle h_d(u), u\rangle u),$$

in the distribution sense. □

A smooth map $u: M \to \mathbb{S}^k \hookrightarrow \mathbb{R}^{k+1}$ is called a self-induced harmonic map if it satisfies the Euler–Lagrange equation

$$\tau(u) - \eta^2(h_d(u) - \langle h_d(u), u\rangle u) = 0. \tag{2.5}$$

Remark 2.3 The same results as in the above Lemma 2.1 and Proposition 2.2 also hold true in the case mentioned in Section 1 where the underlying space is a bounded domain $\Omega \subset \mathbb{R}^n$ (cf. [3]).

2.2.1 Epsilon Regularity of Self-induced Harmonic Map

In this subsection, the ε-regularity for self-induced harmonic map will be obtained, which is contributed to show Theorem 1.1 in Section 3. Let $u \in C^\infty(M, \mathbb{S}^k)$ be a self-induced harmonic map. Due to the fine regular estimates (see Theorem 5.5), the term involving h_d can be considered as a good disturbance of harmonic map. And hence, the techniques used to approach harmonic map can also be adopted to deal with the self-induced harmonic maps u. We will see that many properties of self-induced harmonic maps is analogous to harmonic maps, such as the monotonicity formula and the ε-regularity (also cf. [3]).

Let $\mathrm{inj}_M > 0$ be the injective radius of M, $B_r(p)$ denote the geodesic ball with radius r and center at $p \in M$, where $r \le \mathrm{inj}_M$. Choose $i_0(\le \mathrm{inj}_M)$ be a positive constant such that there holds

$$|g_{ij} - \delta_{ij}| \le Ar^2, \quad |\partial g_{ij}| \le Ar,$$

under a normal coordinates $(x^1, x^2, \ldots, x^n)$ on $B_{i_0}(p)$, where A is a constant depending only on the geometry of M.

Denote the scaling invariant energies by

$$I_r(u) := \frac{1}{r^{n-2}} \int_{B_r(p)} |\nabla u|^2 dv,$$

and

$$\bar{I}_r(u) := \frac{1}{r^{n-2}} \bigg(\int_{B_r(p)} |\nabla u|^2 dv + \eta^2 \int_{B_r(p)} |h_d(u)|^2 dv \bigg),$$

where $h_d(u) = -\nabla f \in \Gamma(TM)$.

Theorem 2.4 *There exists a constant C depending only on the geometry of M, such that if $u \in C^2(M, \mathbb{S}^k)$ is a self-induced harmonic map, then for any $p \in M$, there holds*

$$\mathrm{e}^{C\rho} \bar{I}_\rho(u) \le \mathrm{e}^{C\sigma} \bar{I}_\sigma(u)$$

for $0 < \rho \le \sigma < i_0$.

Proof Let φ_t be the one parameter transformation group induced by a smooth vector field $X \in \Gamma(TM)$, and $u_t = u \circ \varphi$. Then, there holds

$$\begin{aligned}\mathcal{E}_\eta(u_t, g) &= \int_M |\nabla u_t|_g^2 dv_g + \eta^2 \int_M |h_d(u_t)|^2 dv_g \\ &= \int_M |\nabla u|^2_{\varphi^*_{-t} g} dv_{\varphi^*_{-t} g} + \eta^2 \int_M |h_d(u)|^2_{\varphi^*_{-t} g} dv_{\varphi^*_{-t} g},\end{aligned}$$

because of $h_d(u_t) = (\varphi_t)_* h_d(u)$, where $(\varphi_t)_*$ is the push-in map induced by automorphism φ_t.

Since u is the critical point of $\mathcal{E}_\eta$, we obtain

$$\begin{aligned}0 = \frac{d}{dt}\bigg|_{t=0} \mathcal{E}_\eta(u_t, g) = \frac{d}{dt}\bigg|_{t=0} \mathcal{E}_\eta(u, g_{\varphi^*_{-t} g}) = -\frac{1}{2} \int_M (|\nabla u|^2 + \eta^2 |h_d(u)|^2) \mathrm{div} X dv_g \\ + \int_M g^{ij} (\langle \nabla u(\nabla_{e_i} X), \nabla u(e_j)\rangle + \langle h_d(u), \nabla_{e_i} X\rangle \langle h_d(u), e_j\rangle) dv_g,\end{aligned}$$

where $e_i = \frac{\partial}{\partial x^i}$ for $0 \le i \le n$.

Let $X = \chi(r) r \frac{\partial}{\partial r} = \chi(d(p,x)) x^i \frac{\partial}{\partial x^i}$, where

$$\chi(r) = \begin{cases} 1 & \text{if } r \le \rho; \\ \dfrac{\sigma - r}{\sigma - \rho} & \text{if } \rho < r \le \sigma; \\ 0 & \text{if } r > \sigma. \end{cases}$$

Thus, a simple calculation shows

$$\begin{aligned}&\mathrm{div} X = n\chi(r) + r\chi'(r) + O(r^2), \\ &g^{ij} \langle \nabla u(\nabla_{e_i} X), \nabla u(e_j)\rangle = |\nabla u|^2 (\chi(r) + O(r^2)) + \chi'(r) \bigg(r \bigg| \frac{\partial u}{\partial r} \bigg|^2 + O(r^3) |\nabla u|^2 \bigg),\end{aligned}$$

and

$$\begin{aligned}g^{ij} \langle h_d(u), (\nabla_{e_i} X)\rangle \langle h_d(u), e_j\rangle &= |h_d(u)|^2 (\chi(r) + O(r^2)) \\ &\quad + \chi'(r) \bigg(r \bigg| \bigg\langle h_d(u), \frac{\partial}{\partial r} \bigg\rangle \bigg|^2 + O(r^3) |h_d(u)|^2 \bigg).\end{aligned}$$

Therefore, we have

$$\begin{aligned}0=&(n-2)\int_{B_\sigma(p)}(\chi(r)+O(r^2)(|\nabla u|^2+\eta^2|h_d(u)|^2)dv_g\\&-\frac{1}{\sigma-\rho}\int_{B_\sigma(p)/B_\rho(p)}(r+O(r^3))(|\nabla u|^2+\eta^2|h_d(u)|^2)\\&-2r\left(\left|\frac{\partial u}{\partial r}\right|^2+\left|\left\langle h_d(u),\frac{\partial}{\partial r}\right\rangle\right|^2\right)dv_g.\end{aligned}$$

Letting $\rho\to\sigma$, it follows

$$\sigma\partial_\sigma\bar{E}_\sigma-(n-2)\bar{E}_\sigma(u)\geq -C\sigma^2\bar{E}_\sigma(u),$$

Thus, there holds

$$\bar{I}'_\sigma(u)=\sigma^{1-n}(\sigma\partial_t\bar{E}_\sigma-(n-2)\bar{E}_\sigma(u))\geq -C\bar{I}_\sigma(u),$$

which gives

$$\mathrm{e}^{C\rho}\bar{I}_\rho(u)\leq \mathrm{e}^{C\sigma}\bar{I}_\sigma(u),$$

for any $0<\rho\leq\sigma<i_0$. □

It is not difficult to show the above monotonicity formula also holds for $u\in W^{2,2}(M,\mathbb{S}^k)$. This formula combining with the dual of BMO and $\mathcal{H}_1$ implies the following ε-regularity theorem by using a similar method applied to harmonic map, see [17, 29, 30]. To this end, we firstly obtain an improved Poincaré inequality, which can be presented as the following lemma.

Lemma 2.5 *There exists a positive constant C_0 such that for any $\theta\in(0,1)$, there exists $\varepsilon_0(\theta)$ such that for any solution $u\in W^{2,2}(B_r(x_0),\mathbb{S}^k)$ to (2.5) and $0<r\leq i_0$, if u satisfies*

$$\hat{I}_r(u):=\frac{1}{r^{n-2}}\int_{B_r}|\nabla u|^2dv+\frac{\eta^2}{r^{n-2}}\int_{B_r}|h_d(u)|dv\leq\varepsilon_0^2(\theta),$$

there holds

$$\frac{1}{(\theta r)^n}\int_{B_{\theta r}}|u-u_{r\theta}|^2dv\leq C_0\theta^2\hat{I}_r(u),$$

where $u_{r\theta}$ is the mean value of u on $B_{r\theta}$.

Proof In general, we assume that $r=1$ by scaling. On the contrary, suppose that for any $C>0$, there exist a $\theta\in(0,1)$ and a sequence $\{u_j\}\subset W^{2,2}(B_1,\mathbb{S}^k)$ solving (2.5), which satisfy

$$\Delta u_j=-|\nabla u_j|^2u_j-\eta^2(h_d(u_j)-\langle h_d(u_j),u_j\rangle u_j),$$

such that

$$\hat{I}_1(u_j)=\varepsilon_j^2\to 0,$$

as $j\to\infty$. But,

$$\int_{B_\theta}|u_j-(u_j)_\theta|^2dv>C\theta^{n+2}\varepsilon_j^2,$$

where $(u_j)_\theta$ is the mean value of u_j on B_θ.

Letting $v_j(x)=\frac{1}{\varepsilon_j}(u_j-(u_j)_\theta)$, it follows

$$\int_{B_1}|\nabla v_j|^2dv\leq 1.$$

The Poincaré inequality (to see Lemma 6.14 in [28]) implies that $\{v_j\}$ is a bounded sequence in $W^{1,2}(B_1, \mathbb{R}^{k+1})$. Hence, there exists a $v \in W^{1,2}(B_1, \mathbb{R}^{k+1})$, which satisfies

$$v_j \rightharpoonup v \text{ weakly in } W^{1,2}(B_1, \mathbb{R}^{k+1}),$$
$$v_j \to v \text{ strongly in } L^2(B_1, \mathbb{R}^{k+1}).$$

On the other hand, for any cut-off function $\varphi \in C_0^\infty(B_1, \mathbb{R}^{k+1})$, it follows

$$\begin{aligned}\int_{B_1} \langle \nabla v_j, \nabla \varphi \rangle \, dv &= \frac{1}{\varepsilon_j} \int_{B_1} |\nabla u_j|^2 \langle u_j, \varphi \rangle \, dv + \frac{\eta^2}{\varepsilon_j} \int_{B_1} \langle (h_d(u_j) - \langle h_d(u_j), u_j \rangle u_j), \varphi \rangle \, dv \\ &\leq \frac{|\varphi|_{L^\infty}}{\varepsilon_j} \left(\int_{B_1} |\nabla u_j|^2 dv + |h_d(u_j)| dv \right) \to 0,\end{aligned}$$

as $j \to \infty$.

Thus, v is a harmonic function in B_1. Lemma 3.3.12 in [24] implies

$$\int_{B_\theta} |v|^2 dv \leq C_0 \theta^2 \int_{B_\theta} |\nabla v|^2 dv \leq C_0 \theta^{n+2}$$

for some C_0, which is a contradiction.

If u is a solution to Equation (2.5) on $B_r(x_0)$, we set $v(x) = u(x_0 + rx)$, which is defined on $B_1(0)$ endowed with the metric $(\bar{g}_{ij}) = (g(x_0 + rx)_{ij})$. Then v satisfies the below equation

$$\Delta v(x) = |\nabla v|^2(x) + \eta^2 r^2 (h_d(rx) - \langle h_d(rx), v(x) \rangle \, v(x)).$$

By the same argument as that in the above we can obtain the desired results. □

By using Lemma 2.5, the following ε-regularity theorem is obtained.

Theorem 2.6 *There exists a constant ε such that if $u \in W^{2,2}(M, \mathbb{S}^k)$ is a solution of* (2.5) *and satisfies, for $0 < r_0 \leq i_0$ and $p \in M$,*

$$\bar{I}_{r_0}(u) \leq \varepsilon^2,$$

then, u is C^α-continuous on $B_{\frac{r_0}{2}(p)}$ for any $\alpha \in (0,1)$. Moreover, there holds

$$[u]_{C^\alpha(B_{\frac{r_0}{2}})} \leq C \left(r_0^{2-n-2\alpha} \int_{B_{r_0}} |\nabla u|^2 + \eta^2 \right)^{\frac{1}{2}},$$

where constant C depends only on α and the geometry of M and $\mathbb{S}^k$.

Proof Since it is a local result, we may assume B_{r_0} is a Euclidean ball with the Euclidean metric g_E. Moreover, one can modify without difficulties the following argument to show the result is also true in general case.

Since $\bar{I}_{r_0}(u) \leq \varepsilon^2$, the monotonicity formula implies

$$\bar{I}_r(u)(y) = r^{2-n} \int_{B_r(y)} (|\nabla u|^2 + |h_d(u)|^2) dv \leq \mathrm{e}^{Cr_0} 2^{n-2} \varepsilon^2,$$

for any $y \in B_{\frac{r_0}{2}}(p)$ and $r \leq \frac{r_0}{2}$. Next, our proof is divided into 3 steps.

Step 1 We claim that for any $\delta > 0$, there exist C_δ and ε_1 such that if

$$I_r(u) \leq \varepsilon_1^2,$$

there holds

$$\left(\frac{1}{4}r\right)^{2-n}\int_{B_{\frac{1}{4}r}}|\nabla u|^2dv\le\frac{\delta}{r^{n-2}}\int_{B_r}|\nabla u|^2dv+C\delta\eta^4r^a+\frac{C_\delta}{\delta}r^{-n}\int_{B_r}|u-u_r|^2dv,$$

where $a\in(0,4)$.

Without loss of generality, we assume $r=1$. Let ξ be a cut-off function with support in B_1 and $\xi=1$ in $B_{\frac{1}{4}}$. Then, a simple calculation shows

$$\begin{aligned}\int_{B_1}\xi|\nabla u|^2dv&=-\int_{B_1}\langle\operatorname{div}(\xi\nabla u),u-\bar{u}\rangle\,dv\\&=-\int_{B_1}\langle\nabla\xi\cdot\nabla u,u-u_1\rangle\,dv-\int_{B_1}\xi\,\langle\Delta u,u-u_1\rangle\,dv\\&=-\int_{B_1}\langle\nabla\xi\cdot\nabla u,u-u_1\rangle\,dv+\int_{B_1}\xi\left\langle|\nabla u|^2u-\lambda,u-\bar{u}\right\rangle dv\\&\quad+\int_{B_1}\xi\,\langle\lambda,u-u_1\rangle\,dv+\eta^2\int_{B_1}\xi\,\langle h_d(u)-\langle h_d(u),u\rangle\,u,u-u_1\rangle\,dv\\&=I+II+III+IV,\end{aligned}$$

where $u_1=\frac{1}{|B_1|}\int_{B_1}u\,dv$ and

$$\lambda=\frac{\int_{\mathbb{R}^n}|\nabla u|^2\xi u\,dv}{\int_{\mathbb{R}^n}\xi\,dv}.$$

Since $|u|=1$, then

$$|\nabla u|^2u=\sum_i\nabla u^j\cdot(u^i\nabla u^j-u^j\nabla u^i)e_i,$$

and

$$\operatorname{div}(u^i\nabla u^j-u^j\nabla u^i)=\{u,\omega\}_{ij}=u^i\omega^j-u^j\omega^i.$$

where $\{e_i\}$ is the standard frame of $\mathbb{R}^{k+1}$, and

$$\omega=\tau(u)=\eta^2(h_d(u)-\langle h_d(u),u\rangle u)\in L^2(B_1,\mathbb{R}^{k+1}).$$

Therefore, Lemma 2.6 in [17] implies $|\nabla u|^2u\in\mathcal{H}^1_{\rm loc}(B_1)$. And moreover there holds

$$\begin{aligned}\|\xi(|\nabla u|^2u-\lambda)\|_{\mathcal{H}^1(\mathbb{R}^n)}&\le C\||\nabla u|^2u\|_{\mathcal{H}^1_{\rm loc}}(B_1)\\&\le C(\|\nabla u\|^2_{L^2(B_1)}+\eta^4\|h_d(u)\|^2_{L^2(B_1)}).\end{aligned}$$

Hence, it follows

$$\begin{aligned}II&\le C(\|\nabla u\|^2_{L^2(B_1)}+\eta^4\|h_d(u)\|^2_{L^2(B_1)})\|u-\bar{u}\|_{\mathrm{BMO}(B_1)}\\&\le C(\|\nabla u\|^2_{L^2(B_1)}+\eta^4\|h_d(u)\|^2_{L^2(B_1)})\|\nabla u\|_{L^2(B_1)}.\end{aligned}$$

Since I and III can be easily estimated by using Hölder inequality, we obtain

$$\begin{aligned}\int_{B_{1/4}}|\nabla u|^2dv\le&\left(C\|\nabla u\|_{L^2(B_1)}+\frac{2}{3}\delta\right)(\|\nabla u\|^2_{L^2(B_1)}+\eta^4\|h_d(u)\|^2_{L^2(B_1)})\\&+\frac{C_\delta}{\delta}\int_{B_1}|u-u_1|^2dv.\end{aligned}\tag{2.6}$$

In general case, let $u \in W^{2,2}(B_r, \mathbb{S}^k)$ be a solution of (2.5). Let $v(y) = u(ry)$ on B_1 with Euclidean metric g_E, then v satisfies the following equation:

$$\Delta v(y) + |\nabla v|^2 v(y) = -\eta^2 r^2 (h_d(u)(ry) - \langle h_d(u)(ry), v(y)\rangle\, v(y)).$$

thus, taking almost the same arguments as in the above leads to

$$\int_{B_{1/4}} |\nabla v|^2 dv \le \left(C\|\nabla v\|_{L^2(B_1)} + \frac{2}{3}\delta\right)(\|\nabla v\|^2_{L^2(B_1)} + \eta^4 r^4 \|h_d(u)(r.)\|^2_{L^2(B_1)}) + \frac{C_\delta}{\delta}\int_{B_1} |v - v_1|^2 dv.$$

Scaling back to B_r and choosing $\varepsilon_1 \le \frac{\delta}{3C}$, we have

$$\begin{aligned}\left(\frac{1}{4r}\right)^{2-n} \int_{B_{1/4r}} |\nabla u|^2 dv &\le \delta(r^{2-n}\|\nabla u\|^2_{L^2(B_r)} + \eta^4 r^{4-n}\|h_d(u)\|^2_{L^2(B_r)}) \\ &\quad + \frac{C_\delta}{\delta} r^{-n} \int_{B_r} |u - u_r|^2 dv \\ &\le \delta r^{2-n}\|\nabla u\|^2_{L^2(B_r)} + C\delta\eta^4 r^a + \frac{C_\delta}{\delta} r^{-n}\int_{B_r} |u-u_r|^2 dv,\end{aligned}$$

for any $a \in (0, 4)$. Here we have used the fact

$$\begin{aligned} r^{4-n}\|h_d(u)\|^2_{L^2(B_r)} &\le Cr^{4-\frac{n}{p}}\left(\int_{B_r} |h_d(u)|^{2p} dv\right)^{\frac{1}{p}} \\ &\le Cr^{4-\frac{n}{p}} \le Cr^a \end{aligned}$$

if $p > \frac{n}{4}$.

Step 2 By combining the result in Step 1 with Lemma 2.5, we can show the required decay estimate by choosing suitable δ, θ, ε_0 and ε_1.

The following iteration argument is similar with that in [29]. For any $\alpha \in (0, 1)$, let $\delta = (\frac{1}{4})^4$, $\theta = \theta(\alpha) \le \min\{(\frac{\delta^2}{2C_0 C_\delta})^{\frac{1}{2-2\alpha}}, (\frac{7}{15})^{1/2}\}$, where C_0 and C_δ comes from Lemma 2.5 and the claim in step 1, respectively. Let $l \in \mathbb{N}$ such that $\theta(\alpha) = \frac{1}{4^l}$. The monotonicity formula 2.4 implies that if $\bar{I}_r(u) \le \varepsilon^2$ is small enough, then

$$I_\rho(u) \le C\bar{I}_r(u) \le C\varepsilon^2 \le \varepsilon_1^2,$$

for $0 < \rho \le r$. Hence, by choosing $\hat{I}_r(u) \le \min\{\varepsilon_0^2(4\theta), \dots, \varepsilon_0^2(4^i\theta)\}$, it follows

$$I_{4^i\theta r}(u) \le \delta I_{4^{i+1}\theta r}(u) + C\eta^4\delta(4^{i+1}\theta)^a r^a + \frac{C_0 C_\delta}{\delta}(4^{i+1}\theta)^2 \hat{I}_r(u),$$

for $0 \le i \le l-1$.

By induction on i, there holds

$$I_{\theta r}(u) \le \delta^l I_{4^l\theta r}(u) + C\eta^4 \frac{1}{1-\delta 4^a}\theta^a r^a + \frac{C_0 C_\delta}{1-16\delta}\left(\frac{\theta}{\delta}\right)^2 \hat{I}_r(u),$$

where we choose $a < 3$. Thus,

$$I_{\theta r}(u) \le \theta^{2\alpha}\left(\frac{8}{15} + \theta^{4-2\alpha}\right) I_r(u) + C\eta^4 r^a + C\eta^2 r^{2-n}\int_{B_r} |h_d(u)| dv,$$

where we choose $a > 2\alpha$.

Since $r^{2-n}\|h_d(u)\|_{L^1(B_r)} \le Cr^{2-\frac{n}{p}}(\int_{B_r} |h_d(u)|^p dv)^{\frac{1}{p}} \le Cr^{2-\frac{n}{p}}$ if $p > \frac{n}{2}$. It implies

$$I_{\theta r}(u) \le \theta^{2\alpha} I_r(u) + C\eta^2 r^\beta,$$

for any $\beta \in (0,2)$.

Therefore, Lemma 3.4 in Chapter 3 of [23] implies

$$I_\rho(u) \le C_\alpha\left(\left(\frac{\rho}{r}\right)^{2\alpha} I_r(u) + C\eta^2\rho^\beta\right),$$

for any $0 < \rho \le r$ and $\beta < 2\alpha$.

Step 3 The Hölder continuity of u.

Step 2 follows that, for any $y \in B_{\frac{r_0}{2}}(p)$ and any $\rho \le r \le \frac{r_0}{2}$, there holds

$$\rho^{-n}\int_{B_\rho(y)} |u - u_{B_\rho(y)}|^2 \le C_{\alpha,\beta}\rho^{2\alpha'}\left(\frac{1}{r^{2\alpha'}} I_r(u) + \eta^2\right),$$

where $u_{B_\rho(y)}$ is the mean value of u on $B_\rho(y)$ and $2\alpha' = \beta$. Therefore, the Campanato space theory implies

$$[u]_{C^{\alpha'}(B_{\frac{r_0}{2}})} \le C\left(r_0^{2-n-2\alpha'}\int_{B_{r_0}} |\nabla u|^2 + \eta^2\right)^{\frac{1}{2}}.$$

□

Theorem 2.6 implies the following gap result.

Corollary 2.7 *There exists positive constants* ε_0 *and* η_0 *such that if* u *is a* $W^{2,2}(M, \mathbb{S}^k)$ *solution to* (2.5), $\|\nabla u\|^2_{L^2} \le \varepsilon_0^2$ *and* $\eta \le \eta_0$, *then* u *is a null-homotopic map.*

Proof We choose a finite covering $\{B_{r/2}(x_i)\}_{i=1}^N$ of M, such that there holds

$$\bar{I}_r(u)_i = r^{2-n}\int_{B_r(x_i)} |\nabla u|^2 + \eta^2|h_d(u)|^2 dv \le r^{2-n}(\varepsilon_0^2 + C\eta^2 r^l) \le \varepsilon^2,$$

for any $0 < i \le N \le Cr^{-n}$, where $l \in (0,n)$ and $r = \frac{i_0}{2}$.

Thus, the ε-regularity theorem 2.6 implies

$$\begin{aligned}\mathrm{osc}_M u &\le C\sum_{i=1}^N r^\alpha [u]_{C^\alpha}(B_{\frac{r}{2}}(x_i)) \le Cr^{\alpha-n}(r^{1-\frac{n}{2}-\alpha}\varepsilon_0 + \eta)\\ &\le C(i_0)(\varepsilon_0 + \eta_0).\end{aligned}$$

Then by choosing ε_0 and η_0 small enough, it follows that $u(M)$ is contained in a geodesic convex ball B of $\mathbb{S}^k$. Thus, u is null homotopic since B is contractible. □

2.3 Parabolic Monotonicity Formula of the Heat Flow of Self-induced Harmonic Map

In this section, a parabolic version of monotonicity formula will be given by using a similar argument with that for heat flow of harmonic maps by Struwe [12] and Hamilton [22].

A smooth map $u(x,t) : M \times [0,T) \to \mathbb{S}^k$ is called the heat flow of self-induced harmonic map if it solves the negative gradient flow of the self-induced energy $\mathcal{E}_\eta(u)$ defined in (2.4). Namely, u is a smooth solution of the following equation with initial smooth map $u_0 : M \to \mathbb{S}^k$:

$$\begin{cases}\partial_t u = -\dfrac{1}{2}\dfrac{\delta\mathcal{E}_\eta(u)}{\delta u} = \Delta u + |\nabla u|^2 u + \eta^2(h_d(u) - \langle h_d(u), u\rangle u),\\ u(x,0) = u_0.\end{cases} \tag{2.7}$$

where T is the maximal existence time.

Remark 2.8 Since the nonlocal field $h_d(u)$ is well estimated by u in Theorem 5.5 and Theorem 5.7, by using almost the same arguments as that in Section 1 and Section 3 in Chapter 15 of [37] (also refer to [28]), we can show the existence of local regular solution to (2.7) or (1.3).

The following energy identity for the heat flow of self-induced harmonic map, which will be used in the coming context, is obtained directly.

Lemma 2.9 *Suppose u is a smooth solution of (2.7) (or (1.3)), then for any $0 < t < T$, there holds*

$$\mathcal{E}_\eta(u_t) + \int_0^t \int_M \left| \frac{\delta \mathcal{E}_\eta(u)}{\delta u} \right|^2 (x,t) dv dt = \mathcal{E}_\eta(u_0),$$

where $u_t(x) = u(x,t)$.

Now, we are in the position to give the parabolic type monotonicity formula for the heat flow of self-induced harmonic map. Let $B_\rho(p) \subset M$ be a geodesic ball with $\rho \le i_0$, $(x^1, x^2, \ldots, x^n)$ be a normal coordinate on $B_\rho(p) = \exp_p^{-1} B_\rho(p)$ with $x(p) = 0$. Define a weighted energy in the following form

$$\phi(r,u,t) = \phi_{p,t_0}(r,u,t) := \frac{1}{2} r^2 \int_{B_\rho(0)} |\nabla u|^2(x,t) \Gamma(x,0;t,t_0) \varphi^2(x) \sqrt{\det(g)} dx.$$

Here

$$\Gamma(x,0;t,t_0) = [4\pi(t_0 - t)]^{-n/2} \exp - \frac{|x|^2}{4(t_0 - t)}$$

is the backward heat kernel on $\mathbb{R}^n$ for $t < t_0 < T$, and $\varphi(x)$ is a cut-off function with support in B_ρ, $\varphi = 1$ on $B_{\frac{\rho}{2}}(0)$.

Let $t = t_0 - r^2$, where $0 < r < \min\{\sqrt{t_0}, \rho\}$. Then the weighted energy can be rewritten as

$$\phi(r) = \frac{1}{2} C_n r^{2-n} \int_{\mathbb{R}^n} |\nabla u|^2(x, t_0 - r^2) \exp\left(- \frac{|x|^2}{4r^2} \right) \varphi^2(x) \sqrt{\det(g)} dx. \tag{2.8}$$

Theorem 2.10 *There exists a constant C (depending only on the geometry of M and $\mathbb{S}^k$) such that if u is a smooth solution to the heat flow (2.7), then for any $0 < s \le r < \min\{\sqrt{t_0}, \rho\}$ with $t_0 < T$ and $\rho \le i_0$, the following properties hold.*

(1) *For $n = 3$, there holds*

$$\phi(s) \le \mathrm{e}^{C(r-s)} \phi(r) + C \mathcal{E}_\eta(u_0)(r-s).$$

(2) *For $n > 3$, there holds*

$$\phi(s) \le \mathrm{e}^{C(r-s)} \phi(r) + C \mathcal{E}_\eta(u_0)(r-s) + C\eta^4 (r^{l+1} - s^{l+1}),$$

where $l \in (1,3)$ is a constant depending only on n.

Proof Without loss of generality, we assume $C_n = 1$. Since Equation (2.7) is invariant under translation $(x,t) \to (x, t - t_0)$, we may shift $(0, t_0)$ to $(0,0)$. Let $u_r(x,t) = u(rx, r^2 t)$. It follows

$$\phi(r) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_r|^2 (x,-1) \phi_r^2(x) \exp\left(- \frac{|x|^2}{4} \right) \sqrt{\det(g_r(x))} dx,$$

where $\varphi_r(x) = \varphi(rx)$, $g_r(x) = g(rx) = g_{ij}(rx) dx^i \otimes dx^j$. Thus, we have

$$\frac{d\phi}{dr}(r_0) = \frac{1}{r_0} \frac{d\phi(rr_0)}{dr} \bigg|_{r=1}$$

$$
\begin{aligned}
&= \frac{1}{2} r_0^{1-n} \frac{d}{dr}\Big|_{r=1} \int_{\mathbb{R}^n} |\nabla u_r|^2_{g_r}(x, -r_0^2)\varphi_r^2(x) \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g_r)}dx \\
&= r_0^{1-n} \int_{\mathbb{R}^n} \Big\langle \nabla u, \nabla \frac{u_r}{dr}\Big|_{r=1}\Big\rangle \varphi^2(x) \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx \\
&\quad + r_0^{1-n} \int_{\mathbb{R}^n} \frac{dg_r^{ij}}{dr}\Big|_{r=1} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \varphi^2(x) \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx \\
&\quad + r_0^{1-n} \int_{\mathbb{R}^n} |\nabla u|^2 \varphi(x) \frac{d\varphi_r}{dr}\Big|_{r=1} \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx \\
&\quad + r_0^{1-n} \int_{\mathbb{R}^n} |\nabla u|^2 \varphi^2(x) \exp\Big(-\frac{|x|^2}{4r_0^2}\Big) \frac{d}{dr}\Big|_{r=1}\sqrt{\det(g_r)}dx \\
&= I + II + III + IV,
\end{aligned}
$$

Next, we estimate the above four terms step by steps.

Step 1 The estimates of I.

$$
\begin{aligned}
I &= r_0^{1-n} \int_{\mathbb{R}^n} \Big\langle \nabla^*\Big(\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\nabla u\Big), \nabla u\cdot x - 2r_0^2\partial_t u\Big\rangle \sqrt{\det(g)}dx \\
&= -r_0^{1-n} \int_{\mathbb{R}^n} \Big\langle -\tau(u) - \frac{x}{2r_0^2}\cdot_g \nabla u, \nabla u\cdot x - 2r_0^2\partial_t u\Big\rangle \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\
&\quad - 2r_0^{1-n} \int_{\mathbb{R}^n} \langle \nabla\varphi(x)\cdot_g \nabla u, \nabla u\cdot x - 2r_0^2\partial_t u\rangle \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi(x)\sqrt{\det(g)}dx \\
&= I_1 + I_2,
\end{aligned}
$$

where $\tau(u) = -\Delta u - |\nabla u|^2 u$.

A simple calculation shows

$$
\begin{aligned}
I_1 &= 2\eta^2 r_0^{3-n} \int_{\mathbb{R}^n} \Big\langle -\tau(u) - g^{ij}\frac{x_i}{2r_0^2}\frac{\partial u}{\partial x_j}, h_d(u) - \langle h_d(u), u\rangle u\Big\rangle \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\
&\quad + 2r_0^{3-n} \int_{\mathbb{R}^n} \Big| -\tau(u) - g^{ij}\frac{x_i}{2r_0^2}\frac{\partial u}{\partial x_j}\Big|^2 \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\
&\quad - r_0^{1-n} \int_{\mathbb{R}^n} \Big\langle -\tau(u) - g^{ij}\frac{x_i}{2r_0^2}\frac{\partial u}{\partial x_j}, (I-g)^{ij}x^i\frac{\partial u}{\partial x^j}\Big\rangle \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\
&\geq -8\eta^4 r_0^{3-n} \int_{\mathbb{R}^n} |h_d(u)|^2 \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\
&\quad + r_0^{3-n} \int_{\mathbb{R}^n} \Big| -\tau(u) - \frac{x}{2r_0^2}\cdot\nabla u\Big|^2 \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\
&\quad - Cr_0^{2-n} \int_{\mathbb{R}^n} \frac{|I-g|^2|x|^2}{r_0^3}|\nabla u|^2 \exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx.
\end{aligned}
$$

The last term of I_1 can be estimated as follows

$$
\text{The last term of } I_1 \leq C\phi(r_0) + C\int_{B_\rho(0)} |\nabla u|^2\sqrt{\det(g)}dx,
$$

since we have the fact in [12]:

$$
\frac{|x|^m}{r_0^s} r_0^{-n} \exp\Big(-\frac{|x|^2}{4r_0^2}\Big) \leq Cr_0^{-n}\exp\Big(-\frac{|x|^2}{4r_0^2}\Big) + C, \tag{2.9}
$$

for $m > s \geq 0$ and $\max\{|x|, r_0\} \leq \rho$.

Next, we obtain the estimate of I_2 as follows

$$\begin{aligned} I_2 \geq & -Cr_0^{3-n}\int_{\mathbb{R}^n}|\nabla u|^2|\nabla\varphi|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx \\ & -\frac{1}{2}r_0^{3-n}\int_{\mathbb{R}^n}\Big|-\tau(u)-\frac{x}{2r_0^2}\cdot_g\nabla u\Big|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\ & -C8\eta^4r_0^{3-n}\int_{\mathbb{R}^n}|h_d(u)|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\ & -Cr_0^{2-n}\int_{\mathbb{R}^n}\frac{|I-g|^2|x|^2}{r_0^3}|\nabla u|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx. \end{aligned}$$

Thus, by combining the estimates of I_1 and I_2, there holds

$$\begin{aligned} I \geq & -C\eta^4r_0^{3-n}\int_{\mathbb{R}^n}|h_d(u)|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx - C\phi(r_0) \\ & -Cr_0^{3-n}\int_{\mathbb{R}^n}|\nabla u|^2|\nabla\varphi|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx - C\int_{B_\rho(0)}|\nabla u|^2\sqrt{\det(g)}dx \\ \geq & -C\phi(r_0) - C(\rho)\int_{B_\rho}|\nabla u|^2\sqrt{\det(g)}dx - C\eta^4|M|^{1-\frac{1}{q}}, \end{aligned}$$

where we have used the fact

$$\begin{aligned} r_0^{3-n}&\int_{\mathbb{R}^n}|h_d(u)|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \\ &\leq Cr_0^{3-n}\|h_d(u)\|^2_{L^{2q'}(B_\rho(p))}\Big\|\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\Big\|_{L^q(\mathbb{R}^n)} \\ &\leq Cr_0^{3-n+\frac{n}{q}}\|h_d(u)\|^2_{L^{2q'}(B_\rho(p))} \leq Cr_0^{3-n+\frac{n}{q}}, \end{aligned}$$

for $1 < q \leq \frac{n}{n-3}$ with $n > 3$ and $\frac{1}{q}+\frac{1}{q'} = 1$, where we used Lemma 5.4.

In the case of $n = 3$, there holds

$$r_0^{3-n}\int_{\mathbb{R}^n}|h_d(u)|^2\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\varphi^2(x)\sqrt{\det(g)}dx \leq C\int_{B_\rho}|h_d(u)|^2\sqrt{\det(g)}dx.$$

Step 2 The estimates of II.

$$II \geq -C\phi(r_0) - C\int_{B_\rho(p)}|\nabla u|^2\sqrt{\det(g)}dx,$$

where we have used the inequality (2.9) and the fact

$$\Big|\frac{\partial g_r^{ij}}{\partial r}\Big|_{r=1} \leq C|x|^2.$$

Step 3 The estimates of III.

$$\begin{aligned} III &\geq -r_0^{1-n}\int_{B_{\rho(p)}/B_{\frac{\rho}{2}(p)}}|\nabla u|^2|\nabla\varphi||x|\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx \\ &\geq -C(\rho)\int_{B_\rho(p)}|\nabla u|^2\sqrt{\det(g)}dx. \end{aligned}$$

Step 4 The estimates of IV.

$$
\begin{aligned}
IV &\geq -r_0^{1-n}\int_{\mathbb{R}^n}|\nabla u|^2\varphi^2(x)\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)g^{ij}|(\partial_r g_r)_{ij}|_{r=1}|\sqrt{\det(g)}dx\\
&\geq -Cr_0^{2-n}\int_{\mathbb{R}^n}|\nabla u|^2\varphi^2(x)\exp\Big(-\frac{|x|^2}{4r_0^2}\Big)\sqrt{\det(g)}dx - C\int_{B_\rho(p)}|\nabla u|^2\sqrt{\det(g)}dx\\
&\geq -C\phi(r_0) - C\int_{B_\rho(p)}|\nabla u|^2\sqrt{\det(g)}dx,
\end{aligned}
$$

where the inequality (2.9) has been used.

Therefore, there holds

$$\phi'(r) > -C\phi(r) - C\mathcal{E}_\eta(u_0), \quad \text{for } n=3;$$

and

$$\phi'(r) > -C\phi(r) - C\mathcal{E}_\eta(u_0) - C\eta^4 r^l, \quad \text{for } n>3,$$

where $l\in(1,3)$ is a positive constant depending only on n. Therefore, the above differential inequalities imply the desire result in this theorem. □

3 The Proof of Theorem 1.1

Ding in [15] has ever shown a similar result with Theorem 1.1 for the heat flows of harmonic maps with Dirichlet boundary condition. But Ding used a key lemma (Lemma 3.2 in [15]) which doesn't work for the heat flows of self-induced harmonic maps, since the self-induced harmonic map is not defined locally. In this section, we intend to prove Theorem 1.1. To this end, we need to establish a similar result with Lemma 3.2 in [15] since we can not use the maximum principle in the present situation.

Lemma 3.1 *Let u be a smooth solution to (2.7). Assume that* $\max\{1, \frac{16}{i_0^2}\} < e(t_0)$ *satisfying*

$$e(t)\leq e(t_0) \quad \textit{for } 0<t_0-\frac{2}{e(t_0)}<t<t_0.$$

Then there exists a constant $C>0$ (depending only on the geometry of M and $\mathbb{S}^k$) such that if $\delta>0$ and $C\delta<\frac{1}{4}$, we have

$$e(t)\leq(1-2C\delta)^{-1}e(t_0) \quad \textit{for } t_0<t\leq t_0+\frac{\delta}{e(t_0)},$$

where $e(t)=\sup_{x\in M}|\nabla u|^2(x,t)$.

Proof Let $\bar{t}>t_0$ be a time such that

$$e(t)\leq e(\bar{t}) \quad \text{for } t_0<t<\bar{t}.$$

Assume that $e(u)(\bar{x},\bar{t})=e(\bar{t})$ for some $(\bar{x},\bar{t})\in M\times(0,T)$. Let

$$v(x,t)=u\bigg(\bar{x}+\frac{x}{\sqrt{e(\bar{t})}},\bar{t}+\frac{t}{e(\bar{t})}\bigg).$$

The assumption implies $v(x,t)$ is well-defined on $B_4(0)\times[-2,0]$ endowed with metric $\bar{g}(x)=(\bar{g}_{ij}(x))=(g(\bar{x}+\frac{x}{\sqrt{e(\bar{t})}})_{ij})$, under which $e(v)=|\nabla v|^2_{\bar{g}}\leq 1$. Moreover, it satisfies the following equation:

$$\frac{\partial v}{\partial t}-\Delta v=|\nabla v|^2 v+\eta^2(\tilde{h}_d(v)-\langle\tilde{h}_d(v),v\rangle v),$$

where

$$\tilde{h}_d(v)(x) = e(\bar{t})^{-1} h_d(u)\left(\bar{x} + \frac{x}{\sqrt{e(\bar{t})}}, \bar{t} + \frac{t}{e(\bar{t})}\right).$$

Let $\tilde{f}(x,t) = f(u)(\bar{x} + \frac{x}{\sqrt{e(\bar{t})}}, \bar{t} + \frac{t}{e(\bar{t})})$. Then $\tilde{h}_d(v) = \nabla_{\bar{g}} \tilde{f}$ and

$$\Delta_{\bar{g}} \tilde{f} = \frac{1}{e(\bar{t})} \mathrm{div}_g(\pi(v)) = \sum_{i,j} \bar{g}^{ij} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle - \frac{1}{e(\bar{t})} \langle H(x), v(x) \rangle,$$

where H is the mean curvature field of M. The fact $\lambda = e^{-1/2}(\bar{t}) < 1$, since $e(t_0) > 1$, gives the high order estimate $|\partial^k \bar{g}_{ij}| \le |\partial^k g_{ij}|$. And hence, the standard locally L^p estimate with bootstrap technique implies there exists a constant $C > 0$ (depending only on the geometry of M and $\mathbb{S}^k$) such that we have

$$\sup_{-1 \le t \le 0} \|v\|_{C^{3,\alpha}(B_2)} \le C, \tag{3.1}$$

where $\alpha \in (0,1)$ (also see Corollaries 6.6 and 6.8).

A simple calculation shows

$$\left| \frac{\partial e(v)}{\partial t} \right| \le |\Delta |\nabla v|^2| + C(1 + |\nabla^2 v|^2) + C(|\nabla v|)\eta^2 (|\tilde{h}_d(v)| + |\nabla \tilde{h}_d(v)|).$$

Thus,

$$e(v)(0,0) \le e(v)(0,t) + C|t| \quad \text{for } t \in [-1, 0].$$

By rescaling back to the original coordinates, we have

$$e(\bar{t}) = e(u)(\bar{x}, \bar{t}) \le e(t) + C(\bar{t} - t)e^2(\bar{t}) \quad \text{for } t \in \left[\bar{t} - \frac{1}{e(\bar{t})}, \bar{t}\right].$$

Let $\bar{t}$ be the least time in $(t_0, t_0 + \frac{\delta}{e(t_0)})$ such that

$$e(\bar{t}) = \frac{e(t_0)}{1 - 2C\delta}.$$

If such $\bar{t}$ does not exist, we are done. Therefore, we have

$$\bar{t} - \frac{1}{e(\bar{t})} \le t_0 + \frac{\delta}{e(t_0)} - \frac{1 - 2C\delta}{e(t_0)} \le t_0 - \frac{1 - (2C+1)\delta}{e(t_0)} \le t_0,$$

where we have chosen $(2C+1)\delta < 1$. Thus,

$$e(\bar{t}) \le e(t_0) + C(\bar{t} - t_0)e^2(\bar{t}) \le \frac{e(t_0)}{1 - 2C\delta}\left(1 + \frac{C\delta(4C\delta - 1)}{1 - 2C\delta}\right) < \frac{e(t_0)}{1 - 2C\delta},$$

if $C\delta < \frac{1}{4}$, which is a contradiction.

Therefore, we have

$$e(t) \le (1 - 2C\delta)^{-1} e(t_0),$$

for $t \in (t_0, t_0 + \frac{\delta}{e(t_0)})$. □

Remark 3.2 In the case of Neumann boundary, namely the case that u is a local smooth solution to Equation (1.3), Lemma 3.1 also holds. Its proof is totally similar with that in the above argument, the only thing we should check is the bound of $\sup_{-1 \le t \le 0} |v|_{C^{3,\alpha}(B_2 \cap \tilde{\Omega})}$, where

$$v(x,t) = u\left(\bar{x} + \frac{x}{\sqrt{e(\bar{t})}}, \bar{t} + \frac{t}{e(\bar{t})}\right)$$

is defined on $\tilde{\Omega} = \{(x,t) \in \mathbb{R}^n \times [-2,0] \,|\, \bar{x} + \frac{x}{\sqrt{e(\bar{t})}} \in \bar{\Omega}\}$ for fixed $(\bar{x},\bar{t}) \in \bar{\Omega} \times [0,T)$ with $e(\bar{t}) > 1$. In fact, v satisfies the following equation:

$$\frac{\partial v}{\partial t} - \Delta v = |\nabla v|^2 v + \eta^2(\tilde{h}_d(v) - \langle \tilde{h}_d(v), v\rangle v)$$

with Neumann boundary condition $\frac{\partial v}{\partial \nu}|_{\partial\tilde{\Omega}}(.,t) = 0$, where

$$\tilde{h}_d(v) = \frac{1}{e(\bar{t})} h_d(u)\biggl(\bar{x} + \frac{x}{\sqrt{e(\bar{t})}}, \bar{t} + \frac{t}{e(\bar{t})}\biggr).$$

Therefore, the global L^p-estimates in Appendix 6 imply

$$\sup_{-1\le t\le 0} |v|_{C^{3,\alpha}(B_2\cap\tilde{\Omega})} \le C,$$

for some constant C depending only on n.

In the last part of this section, we use the results developed in Sections 2.2.1 and 2.3 to give the proof of Theorem 1.1.

Proof of Theorem 1.1 The proof is divided into two steps as follows.

Step 1 The solution u blows up at T, that is $\sup_{t\in[0,T)} e(t) = \infty$.

On the contrary, we assume there is a bound

$$\sup_{0\le t<T} e(t) \le C \tag{3.2}$$

for some constant C. In this case, we claim that $T = \infty$. We prove the claim by contradiction. Assume that $T < \infty$, by applying the regular estimates in Remark 6.9, the bound (3.2) tells us there holds

$$|u|_{C^m(M\times[\delta,T))} \le C(m,\delta,C,|M|),$$

for any $m > 0$ and some small positive δ. And hence, it is not difficult to show T is an extensional time, which is a contradiction.

The energy identity in Lemma 2.9 shows

$$\int_0^\infty \left\|\frac{\delta\mathcal{E}_\eta(u)}{\delta u}\right\|^2_{L^2(M)} dt \le \mathcal{E}_\eta(u_0),$$

then there exists a sequence $\{t_i\}$ such that $\|\frac{\delta\mathcal{E}_\eta(u)}{\delta u}\|^2_{L^2(M)}(t_i) \to 0$ as $t_i \to \infty$. On the other hand, the Schauder estimates with a bootstrap technique imply

$$\|u_{t_i}\|_{C^{2,\alpha}(M)} \le C.$$

And hence, without loss of generality, we conclude that the sequence $u_{t_i}(x) \to u_\infty(x)$ in $C^{2,\alpha}$ norms when $t_i \to \infty$, which implies

$$\frac{\delta\mathcal{E}_\eta(u_\infty)}{\delta u_\infty} = 0.$$

The energy inequality follows

$$\mathcal{E}_\eta(u_\infty) \le \mathcal{E}_\eta(u_0) \le \varepsilon^2.$$

Thus, if ε and η_0 are small enough, Corollary 2.7 implies u_∞ is null homotopic map, which is a contradiction.

Step 2 The estimate for upper bound of T.

We claim that, if

$$\sup_{t\in[0,T)} e(t)=\infty, \tag{3.3}$$

where T is the maximal existence time of solution u, then there is a constant $C>0$ depending only on the geometry of M and $\mathbb{S}^k$ such that

$$\min\{i_0^2,T\}\le\begin{cases}C\mathcal{E}_\eta(u_0)^{\frac{2}{n-2}} & \text{for } n=3,\\ C(\mathcal{E}_\eta(u_0)+\eta^4)^{\frac{2}{n-2}} & \text{for } n>3.\end{cases}$$

Next, we show the claim. By (3.3), there exists a sequence of $t_i\to T$ such that

$$e(x_i,t_i)=e(t_i)\to\infty \quad \text{and} \quad e(t)\le e(t_i) \quad \text{for } t\in[0,t_i],$$

where $x_i\in M$. Letting $\lambda_i=\frac{1}{\sqrt{e(t_i)}}$, Lemma 3.1 implies

$$e(t)\le\frac{1}{1-2C\delta}\lambda_i^{-2},$$

where $t\in[0,t_i+\lambda_i^2\delta]\subset[0,T)$ and $\frac{4}{\lambda_i}\le i_0$.

Set $v_i(x,t)=u(x_i+\lambda_i x,t_i+\lambda_i^2 t)$ for $(x,t)\in B_{\lambda_i^{-1}\rho}(0)\times[-\lambda_i^{-2}t_i,\delta]$ endowed with the metric $g_i(x)=g(x_i+\lambda_i x)$, for some small $\rho<i_0$. Then, v_i satisfies the following equation

$$\partial_t v_i-\Delta v_i=|\nabla v_i|^2 v_i+\eta^2(\tilde{h}(v_i)-\langle\tilde{h}(v_i),v_i\rangle v_i), \tag{3.4}$$

on $B_{\lambda_i^{-1}\rho}(0)\times[-\lambda_i^{-2}t_i,\delta]$, where $\tilde{h}_d(v_i)(x)=\lambda_i^2 h_d(u)(x_i+\lambda_i x,t_i+\lambda_i^2 t)$.

Let $P_l=B_{l^2}(0)\times[-\frac{l^2}{2},\frac{l^2}{2}]$ be a parabolic cylinder for some $l>0$, and

$$d=\sup_{P_l}(|\tilde{h}_d(v_i)|+|\nabla\tilde{h}_d(v_i)|).$$

Since $e_i=|\nabla v_i|^2\le\frac{1}{1-2C\delta}\le 2$ and d depends only on δ and M, the Bochner formula shows

$$\partial_t e_i-\Delta_{g_i}e_i\le Ce_i+C\eta^2 d,$$

where the constant C is independent of i, due to $g_i\to g_E$ (the Euclidean metric) when $\lambda_i\to 0$ and the uniformly bound of e_i. Let $h_i=\exp(-Ct)e_i$. Then

$$\partial_t h_i-\Delta_i h_i\le\eta^2 d.$$

The Nash–Moser iteration (one can refer Theorem 6.17 in [28]) shows

$$1=h_i(0,0)\le C\left\{\left(\frac{1}{l^2|B_l|}\int_{P_l}h_i^2(x,t)d\nu_i dt\right)^{\frac{1}{2}}+\eta^2 dl^2\right\}.$$

By choosing $C\eta^2 dl^2\le\frac{1}{2}$, i.e. $l^2\le\frac{1}{2Cd\eta^2}$, we have

$$1\le\frac{4C\exp(Cl^2/2)}{l^2|B_l|}\int_{P_l}|\nabla v_i|^2 d\nu_i dt.$$

Let $R^2=\lambda_i^2\delta-\lambda_i^2 t=\lambda_i^2 s^2$. For $t\in(-\frac{l^2}{2},\frac{l^2}{2})$, by choosing $l^2\le\delta$ we can see that there holds

$$\frac{\delta}{2}<s^2=\delta-t<\frac{3\delta}{2}.$$

Let $t_0 = t_i + \lambda_i^2\delta$. By choosing $l^2 \le \frac{\delta}{2}$ we have

$$\begin{aligned} l^{2-n}\int_{B_l}|\nabla v_i|^2 d\nu_i &= (\lambda_i l)^{2-n}\int_{B_{\lambda_i l}(x_i)}|\nabla u|^2(\cdot, t_i+\lambda_i^2 t)d\nu \\ &\le \left(\frac{l}{s}\right)^{2-n} R^{2-n}\int_{B_{R\frac{l}{s}}(x_i)}|\nabla u|^2(\cdot, t_0-R^2)d\nu \\ &\le CR^{2-n}\int_{B_R(x_i)}|\nabla u|^2(\cdot, t_0-R^2)d\nu \\ &\le CR^{2-n}\int_{B_R(x_i)}|\nabla u|^2(\cdot, t_0-R^2)\exp\left(-\frac{|x|^2}{4R^2}\right)\varphi^2(x)\sqrt{\det(g)}dx \\ &\le CR^{2-n}\int_{\mathbb{R}^n}|\nabla u|^2(\cdot, t_0-R^2)\exp\left(-\frac{|x|^2}{4R^2}\right)\varphi^2(x)\sqrt{\det(g)}dx \\ &= C\phi(R), \end{aligned}$$

where we have used the facts: $\exp(\frac{|x|^2}{4R^2})$ is bounded in $B_R(x_i)$ and $\varphi(x)=1$ on $B_R(x_i)$, when $i\to\infty$.

Therefore, for small $l^2 \le \min\{\frac{\delta}{2}, \frac{1}{2Cd\eta^2}\}$, the monotonicity formulas in Theorem 2.10

$$\begin{aligned} 1 &\le \frac{4C\exp(Cl^2/2)}{l^2|B_l|}\int_{P_l}|\nabla v_i|^2 d\nu_i dt \le \frac{C}{l^4}\sup_{t\in[-\frac{l^2}{2},\frac{l^2}{2}]}\left(l^{2-n}\int_{B_l}|\nabla v_i|^2 d\nu_i\right) \\ &\le C(\delta)\times\begin{cases} R_0^{2-n}\mathcal{E}_\eta(u_0) & \text{for } n=3, \\ R_0^{2-n}\mathcal{E}_\eta(u_0)+\eta^4 R_0^{l+1} & \text{for } n>3. \end{cases} \end{aligned}$$

Here $R_0 \le \min\{i_0, \sqrt{t_0}\}$ for i large enough. And hence, letting $t_0 \to T$, the claim follows.

On the other hand, if $\sqrt{T} > i_0$, then $0 < i_0 \le C(\varepsilon^2+\eta_0^2)^{\frac{2}{n-2}}$ for ε and η_0 both small enough, which is a contradiction. Hence, $T \le i_0^2$, and the result follows. □

4 Estimates for the Boundary Case and the Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3 by modifying the previous argument on Theorem 1.1. To proceed, we first establish an ε-regularity result for self-induced harmonic map and a parabolic monotonicity inequality for the heat flow of self-induced harmonic map with Neumann boundary conditions. We only give the sketches of the proofs for these two results, since the arguments go almost the same as that in Sections 2.2.1 and 2.3.

4.1 ε-regularity and Gap Theorem for Boundary Case

Let $u:\Omega\to\mathbb{S}^{n-1}$ be a $W^{2,2}$ map on a smooth domain $\Omega\subset\mathbb{R}^n$ equipped with Euclidean metric g_E, which solves the following equation

$$\begin{cases} \tau(u) = \eta^2(h_d(u) - \langle h_d(u), u\rangle u); \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

Let $\bar{u}$ be the extension of u by transformation of reflection associated to $\partial\Omega$ ((6.4) and (6.5) in Section 6). One can check that $\bar{u}\in W^{2,2}(\hat{\Omega}, \mathbb{S}^n)$, which satisfies the below equation

$$\tau_{\bar{g}}(\bar{u}) = \eta^2(\bar{h}_d(\bar{u}) - \langle \bar{h}_d(\bar{u}), \bar{u}\rangle\,\bar{u})$$

on $\hat{\Omega}$, where $\bar{g}$ and $\bar{h}_d(\bar{u})$ are the extensions of g_E and $h_d(u)$ under this reflection respectively. Here $\Omega \subset\subset \hat{\Omega}$.

On the other hand, the estimates of $h_d(u)$ in Theorem 5.7 implies

$$\left\|\bar{h}_d(\bar{u})\right\|_{L^p(\hat{\Omega})} \leq C \left\|\bar{u}\right\|_{L^p(\hat{\Omega})},$$

for any $p \in (1, \infty)$.

Therefore, the monotonicity formula for almost harmonic maps (to see Proposition 2.1 in [15] or [31]) implies that there exists positive constants C such that

$$I_\rho(\bar{u}) \leq C I_\sigma(\bar{u}) + C\eta^2\sigma^\gamma, \tag{4.2}$$

for any $B_\sigma(p) \subset \hat{\Omega}$, $\gamma \in (0, 4)$ and $0 < \rho \leq \sigma$.

By using this monotonicity formula, by almost the same arguments as in Theorem 2.6 we get a similar regularity result as follows in Neumann case.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domian with $n \geq 3$. There exist positive constants ε and η_0 such that if $u \in W^{2,2}(\Omega, \mathbb{S}^{n-1})$ is a solution of (1.4) and there hold*

$$0 < \eta \leq \eta_0 \quad \text{and} \quad I_{r_0}(u) \leq \varepsilon^2$$

for $r_0 > 0$ and $p \in \bar{\Omega}$, then u is C^α-continuous on $B_{\frac{r_0}{2}(p)} \cap \Omega$ for any $\alpha \in (0, 1)$. Moreover, there holds

$$[u]_{C^\alpha(B_{\frac{r_0}{2}} \cap \Omega)} \leq C\left(r_0^{2-n-2\alpha} \int_{B_{r_0} \cap \Omega} |\nabla u|^2 + \eta_0^2\right)^{\frac{1}{2}},$$

where constant C is dependent only on n and α.

As a direct corollary, we have

Corollary 4.2 *There exist positive constants ε_0 and η_0 such that if u is a $W^{2,2}(\Omega, \mathbb{S}^{n-1})$ solution to (4.1), $\|\nabla u\|_{L^2}^2 \leq \varepsilon_0^2$ and $\eta \leq \eta_0$, then u is a null-homotopic map.*

4.2 Parabolic Monotonicity Inequality for the Boundary Case

Let u be a local smooth solution to

$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u + \eta^2(h_d(u) - \langle h_d(u), u\rangle u), & (x,t) \in \Omega \times [0, T), \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, & (x,t) \in \partial\Omega \times [0, T), \\ u(x, 0) = u_0 : \Omega \to \mathbb{S}^{n-1}, \end{cases} \tag{4.3}$$

where T is the maximal existence time. By definition of $h_d(u)$, Lemma 2.1 also holds for map $u \in W^{1,2}(\Omega, \mathbb{R}^n)$. This implies that the local smooth solution u to Equation (4.3) satisfies the following energy identity

$$\mathcal{E}_\eta(u_t) + \int_0^t \int_M \left|\frac{\delta \mathcal{E}_\eta(u)}{\delta u}\right|^2 (x,t) dv dt = \mathcal{E}_\eta(u_0),$$

for any $0 < t < T$ and we denote $u_t(x) = u(x, t)$.

Next, we show a similar parabolic type monotonicity formula as that obtained in Theorem 2.10 for the boundary case. For any $x_0 \in \partial\Omega$, let $\{x\}$ be a chart near x_0 such that $\frac{\partial}{\partial x_n} = \nu$ and the domain $B_\rho(x_0) = \{x \in \bar{\Omega} \,|\, |x - x_0| \leq \rho\}$ is corresponding to the half ball $B_\rho^+ = \{x \in$

$\mathbb{R}^n \,|\, |x| \leq \rho,\ x_n \geq 0\}$. We can define a similar weighted energy $\phi^+(r)$ on half space $\mathbb{R}^n_+$ as follows,

$$\phi^+(r) = \frac{1}{2}C_n r^{2-n}\int_{\mathbb{R}^n_+}|\nabla u|^2(x, t_0 - r^2)\exp\left(-\frac{|x|^2}{4r^2}\varphi^2(x)\right)\sqrt{\det(g)}dx,$$

where $0 < r < \min\{\sqrt{t_0}, \rho\}$, and φ is a cutoff function with support in B^+_ρ. Then we have

Theorem 4.3 *There exists a constant C (depending only on the geometry of Ω) such that if u is a smooth solution of* (4.3)*, then for any $0 < s \leq r < \min\{\sqrt{t_0}, \rho\}$ with $t_0 < T$ and $\rho \leq r_0$, the following properties hold.*

(1) *For $n = 3$, $\phi^+(s) \leq \mathrm{e}^{C(r-s)}\phi^+(r) + C\mathcal{E}_\eta(u_0)(r - s)$;*

(2) *For $n > 3$, $\phi^+(s) \leq \mathrm{e}^{C(r-s)}\phi^+(r) + C\mathcal{E}_\eta(u_0)(r - s) + C\eta^4(r^{l+1} - s^{l+1})$, where $l \in (0, 3)$ is a constant.*

The proof of Theorem 4.3 is a similar argument with that in Theorem 2.10. The only thing we need to emphasize is that the condition $\frac{\partial u}{\partial \nu} = 0$ on boundary $\partial\Omega$ guarantees the boundary term is vanished when integration by parts in Step 1 of Theorem 2.10.

4.3 Proof of Main Result

With the above ε-regularity Theorem 4.1 and the parabolic monotonicity given in Theorem 4.3 at hand, we then prove Theorem 1.3 by applying almost same arguments as in the proof of Theorem 1.1.

Proof of Theorem 1.3 The proof is divided into two steps as follows.

Step 1 The solution u blows up at T, namely, $\sup_{t\in[0,T)} e(t) = \infty$.

On the contrary, by taking a similar argument as in Step 1 of Theorem 1.1 and using the global estimates in Corollary 6.8, we can show there exists a limiting map $u_\infty \in C^{2,\alpha}(\bar{\Omega}, \mathbb{S}^{n-1})$ which satisfies the below equation

$$\begin{cases} \Delta u_\infty = |\nabla u_\infty|^2 u_\infty + \eta^2(h_d(u) - \langle h_d(u), u_\infty\rangle\, u_\infty), \\ \left.\dfrac{\partial u_\infty}{\partial \nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{4.4}$$

with $E_\eta(u_\infty) \leq E_\eta(u_0)$.

Therefore, if ε and η_0 are small enough, Theorem 4.1 and Corollary 4.2 imply that the $u_\infty(\bar{\Omega})$ is in a contractible geodesic ball B of $\mathbb{S}^n$. Suppose that $\phi : B \to D_1$ is a smooth diffeomorphism, where D_1 is the unit disk in $\mathbb{R}^{n-1}$, then there exists a smooth map

$$F(x, t) := \phi^{-1} \circ (t\phi \circ u_\infty(x) + (1 - t)\phi(p_0))$$

satisfying $F(x, 0) = p_0$, $F(x, 1) = u_\infty$ and

$$\left.\frac{\partial F}{\partial \nu}\right|_{\partial\Omega} = td\phi^{-1}(t\phi \circ u_\infty(x) + (1 - t)\phi(p_0)) \circ d\phi(u_\infty(x)) \circ \left.\frac{\partial u_\infty}{\partial \nu}\right|_{\partial\Omega} = 0.$$

Since $u_\infty \in [u_0]_\nu$, this implies that constant map is in $[u_0]_\nu$, which is a contradiction with $[u_0]_\nu \neq 0$.

Step 2 The estimate for upper bound of T.

Since Lemma 3.1 also holds for the boundary case, one can refer to Remark 3.2 for the details, by a similar argument as that in Step 2 of Theorem 1.1, we can also show that there exists a sequence of maps $\{v_i(x,t) = u(x_i + \lambda_i x, t_i + \lambda_i^2 t)\}$ satisfying

$$\partial_t v_i = \Delta v_i + |\nabla v_i|^2 v_i + \eta^2(\tilde{h}_d(v_i) - \langle \tilde{h}_d(v_i), v_i\rangle v_i), \tag{4.5}$$

on $\Omega_i \times [-\lambda_i^{-2} t_i, \delta]$, where $\Omega_i = \{x \in \mathbb{R}^n \mid x_i + \lambda_i x \in \Omega\}$ and $\tilde{h}_d(v_i)(x) = \lambda_i^2 h_d(u)(x_i + \lambda_i x, t_i + \lambda_i^2 t)$ is a vector-valued function from $\tilde{\Omega}$ to $\mathbb{R}^n$. In fact, $\lambda_i \to 0$ when $i \to \infty$, and there holds

$$\sup_{\bar{\Omega}_i} |\nabla v_i|^2 \le \frac{1}{1 - 2C\delta} \le 2 \quad \text{and} \quad |\nabla v_i|(0,0) = 1.$$

Thus, without loss of generality, we assume $(0,0) \in \partial\Omega_i$, the global estimates in Corollary 6.8 imply that there exists a constant $C > 0$ depending only on δ such that we have a bound

$$\sup_{-l^2 \le t \le l^2} \sum_{0 \le 2s + r < 4} |\partial_x^r \partial_t^s v_i|_{C^\alpha(B_l^+(0))} \le C,$$

for some small $l < \delta$. Therefore, by an almost same blow-up argument as in [11], we can see that there holds the following result, replacing the one obtained by Nash–Morse iteration in Theorem 1.1,

$$1 = |\nabla v_i|(0,0) \le \frac{C}{l^{n+2}} \int_{P_l^+} |\nabla v_i|^2 dx,$$

where $P_l^+ = B_l^+(0) \times [-l^2, l^2]$. Again by applying the parabolic monotonicity formula in Theorem 4.3, we take a similar argument as in Theorem 1.1 to obtain the desire result. □

5 Appendix: Estimates of Potential h_d

In this section, our goal is to get regular estimates of h_d. In the sense of distribution, h_d is defined by

$$h_d(u) = \nabla \int_M \langle \nabla_y G(x,y), \pi(u)(y) \rangle \, dv_y,$$

which is a singular integral involved the Green's function $G(x,y)$ with u.

Since the main difference between the perturbed harmonic map and harmonic map is the nonlocal potential, it is essential to obtain some regularity estimates of h_d when u is regular. To proceed, we need to recall some basic facts about the Green function $G(x,y)$ and the Calderón–Zygmund singular integral theory, which can be found in [1] and [2, 35] respectively.

Lemma 5.1 *Let (M^n, g) be a compact Riemannian manifold without boundary. There exists a positive Green function $G(x,y) \in C^\infty(M \times M \setminus \mathrm{diag}(M \times M), \mathbb{R})$ such that the following properties hold.*

(1) *$G(x,y) = G(y,x)$ and $G(x,y) > 0$ for any $x \neq y$.*

(2) *$\Delta_x G(x,y) = -\delta_x(y)$ for fixed $y \in M$, where δ is the Delta function.*

(3) *For any fixed y, there holds*

$$G(x,y) = c_n (d(x,y))^{2-n}(1 + o(1)),$$

where $d(x,y)$ is the distance function on M.

(4) *There exists constant C such that for $0 \le i \le 3$, we have*

$$|\nabla^i G(x,y)| \le \frac{C}{(d(x,y))^{n-2+i}}.$$

Next, we give a brief introduction to the Calderón–Zygmund decomposition theory for metric spaces satisfying volume doubling condition, which will be used to give the L^p-estimate of $h_d(u)$ for $u \in L^p(M, \mathbb{R}^{k+1})$, $1 < p < \infty$. More details can be found in [2].

Let (M, g) be a complete Riemannian manifold. We call that M satisfies the volume doubling condition if there exists a constant C such that for any geodesic ball $B_r(p)$ and $B_{2r}(p)$, there holds

$$\mu(B_{2r}) \le C\mu(B_r). \tag{5.1}$$

Since a compact manifold (M, g) naturally meets Condition 5.1, the following lemma is obtained by a similar argument as in Lemma 7.3.5 in [2]. There Auscher discussed the Calderón–Zygmund decomposition theorem by using an overlapping-balls-covering technique (see Theorem 2.3.4 in [2]).

Lemma 5.2 *Suppose (M^n, g) is a compact Riemannian manifold without boundary. Let $f \in L^1(M)$ and $\delta > 0$. Then there exist a $C(n)$ and a decomposition of $f = g + b$ almost everywhere on M, such that the following arguments hold.*

(1) *$g \in L^\infty(M)$ with $\|g\|_{L^\infty} \le C(n)\delta$,*

(2) *$b = \sum_i b_i$ with support being in B_i and $\int_M b_i dv = 0$, where $\{B_i\}$ are geodesic balls,*

(3) *$\int_{B_i} |b_i| dv \le C(n)\delta \operatorname{vol}(B_i)$,*

(4) *$\{B_i\}$ has the bounded overlapping property, that is, $\sum_i \chi_{B_i} \le C(n)$,*

(5) *$\sum_i \operatorname{vol}(B_i) \le \frac{C(n)}{\delta}\|f\|_{L^1(M)}$.*

By using Lemma 5.2, one can obtain the following Calderón–Zygmund (C-Z) singular integral theorem, whose proof can be found in [2, Corollary 7.3.8] and [35, Theorem 1 in p. 29].

Lemma 5.3 *Suppose (M^n, g) is a compact Riemannian manifold without boundary. Let $T : L^2(M) \to L^2(M)$ be a bounded linear operator given by $T(f)(x) = K * f(x)$, where $K(x, y) = K(y, x)$ satisfies the following Hörmander conditions:*

$$|K(x, y)| \le \frac{C}{d(x, y)^n}, \qquad |\nabla K(x, y)| \le \frac{C}{d(x, y)^{n+1}}, \tag{5.2}$$

where $d(x, y)$ is the distance function on M, and $n \ge 3$. Then, there exists $C(n, p)$ for $p \in (1, \infty)$ such that for any $u \in L^p(M)$, it holds

$$\|T(u)\|_{L^p} \le C(n, p)\|u\|_{L^p}.$$

Now, we are in the position to show the estimates of h_d.

Theorem 5.4 *Let $p \in (1, \infty)$. There exists a constant C, such that for any $u \in L^p(M, \mathbb{R}^{k+1})$, the potential $h_d(u) \in L^p(M, \mathbb{R}^{k+1})$, which satisfies*

$$\|h_d(u)\|_{L^p} \le C\|u\|_{L^p}.$$

That is, $h_d : L^p(M, \mathbb{R}^{k+1}) \to L^p(M, \mathbb{R}^{k+1})$ is a bounded linear operator.

Proof Without loss of generality, we assume $u \in C^\infty(M, \mathbb{R}^{k+1})$.

Step 1 We claim that $h_d : L^2(M, \mathbb{R}^{k+1}) \to L^2(M, \mathbb{R}^{k+1})$ is bounded.

Let f solve $\Delta f = \operatorname{div}(\pi(u))$ satisfying (2.1) . Then f is smooth by the classical elliptic theory. By choosing test function $v = f$, then we have

$$\int_M |\nabla f|^2 dv = \int_M \langle \nabla f, \pi(u)\rangle dv.$$

The Hölder inequality implies

$$\int_M |h_d(u)|^2 dv \le \int_M |\pi(u)|^2 dv \le \int_M |u|^2 dv.$$

Step 2 We claim $h_d(u) = K * \pi(u) + h_0\pi(u)$, where K is a C-Z singular integral operator and h_0 is a bounded constant operator.

A simple calculation shows

$$\begin{aligned} f(x) = & \int_M G(x,y)\mathrm{div}(\pi(u))(y)dv_y = \lim_{\varepsilon\to 0}\int_{M\setminus B_\varepsilon(x)} G(x,y)\mathrm{div}(\pi(u))(y)dv_y \\ = & -\lim_{\varepsilon\to 0}\int_{M\setminus B_\varepsilon(x)} \langle \nabla_y G(x,y), \pi(u)(y)\rangle\, dv_y + \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} G(x,y)\pi(u)(y)\cdot \nu dv_y, \\ = & -\int_M \langle \nabla_y G(x,y), \pi(u)(y)\rangle\, dv_y. \end{aligned}$$

Here we have used the fact

$$\left|\int_{\partial B_\varepsilon(x)} G(x,y)\pi(u)(y)\cdot\nu dv_y\right| \le \frac{C\varepsilon}{|\partial B_\varepsilon(x)|}\int_{\partial B_\varepsilon(x)} |\pi(u)|(y)dv_y \to 0$$

as $\varepsilon \to 0$, by using the estimate of Green's function and the Bishop's volume comparison theorem.

For any $\varphi \in C^\infty(M, TM)$, then

$$\begin{aligned} \langle \nabla f, \varphi\rangle := & -\int_M f\mathrm{div}(\varphi)dv = \int_M\int_M \langle \nabla_y G(x,y), \pi(u)(y)\rangle\, dv_y \mathrm{div}(\varphi)(x)dv_x \\ = & \int_M \left\langle \int_M \nabla_y G(x,y)\mathrm{div}(\varphi)(x)dv_x, \pi(u)(y)\right\rangle dv_y \\ = & -\int_M \left\langle \int_M \langle\nabla_x\nabla_y G(x,y), \varphi(x)\rangle dv_x, \pi(u)(y)\right\rangle dv_y \\ & -\int_M \left\langle \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(y)} \nabla_y G(x,y)(\varphi(x)\cdot\nu)dv_x, \pi(u)(y)\right\rangle dv_y, \end{aligned}$$

where the first equality is in the sense of distribution.

On the other hand, there holds

$$\begin{aligned} & \left|\int_{\partial B_\varepsilon(y)} \nabla_y G(x,y)\varphi(x)\cdot\nu dv_x - \int_{\partial B_\varepsilon(y)} \nabla_y G(x,y)\varphi(y)\cdot\nu dv_x\right| \\ & \le C\varepsilon^{1-n}\int_{\partial B_\varepsilon(y)} |\varphi(x)-\varphi(y)|dv_x \to 0, \end{aligned}$$

as $\varepsilon \to 0$, since φ is continuous. Hence, the last term in the above formula is as follows,

$$\lim_{\varepsilon\to 0}\int_M \left\langle \int_{\partial B_\varepsilon(y)} \nabla_y G(x,y)\otimes \nu dv_x, \pi(u)\otimes\varphi(y)\right\rangle d\nu_y,$$

which defines a bounded operator h_0, since

$$\sup_{0<\varepsilon<i_0}\int_{\partial B_\varepsilon(y)} |\nabla_y G(x,y)|dv_x \le C$$

by using the estimates of the Green's function in Lemma 5.1. Hence, it follows that

$$h_d(u) = K * \pi(u) + h_0\pi(u) = T(u) + h_0\pi(u).$$

where $K(x,y)=\nabla_x\nabla_y G(x,y)$ and $K(x,y)=K(y,x)$. The operator T satisfies the Hörmander condition (5.2) by the estimates of Green's function G in Lemma 5.1. Therefore, Theorem 5.3 implies h_d is bounded from $L^p(M,\mathbb{R}^{k+1})$ to itself for any $p\in(1,\infty)$. □

Theorem 5.5 *Let $p\in(1,\infty)$. There exists a constant C, such that for any $u\in W^{1,p}(M,\mathbb{R}^{k+1})$, the potential $h_d(u)\in W^{1,p}(M,TM)$, which satisfies*

$$\|h_d(u)\|_{W^{1,p}}\le C\|u\|_{W^{1,p}}.$$

That is, $h_d: W^{1,p}(M,\mathbb{R}^{k+1)}\to W^{1,p}(M,TM)$ is a bounded linear operator.

Proof Since $u\in W^{1,p}$, we have $\Delta f=\mathrm{div}(\pi(u))\in L^p$. By applying Theorem 2.3 in [38], there holds

$$\|f\|_{W^{2,p}}\le C(\|\Delta f\|_{L^p}+\|f\|_{W^{1,p}}).$$

The precise form

$$\Delta f=\mathrm{div}(\pi(u))=\mathrm{div}(u)-\langle H,u\rangle$$

gives

$$\|\Delta f\|_{L^p}\le C(\|u\|_{W^{1,p}}+\|H\|_{L^\infty}\|u\|_{L^p}).$$

Here $H(x)$ is the mean curvature of M.

On the other hand, we have $f(x)=-\int_M\langle\nabla_y G(x,y),\pi(u)(y)\rangle dv_y$. The Young inequality implies

$$\|f\|_{L^p}\le\|\nabla_y G(x,\cdot)\|_{L^1}\|u\|_{L^p}\le C\|u\|_{L^p}.$$

To combine with the result in Theorem 5.4, there holds

$$\|h_d(u)\|_{W^{1,p}}\le C\|u\|_{W^{1,p}}.$$ □

Remark 5.6 By using Theorem 2.3 in [38] again, we can also obtain higher regular estimates for h_d:

$$\|h_d(u)\|_{W^{k,p}}\le\|f\|_{W^{k+1,p}}\le C_k(\|u\|_{W^{k,p}}+\|H\#u\|_{W^{k-1,p}})\le C_k\|u\|_{W^{k,p}},$$

for $p\in(1,\infty)$ and $k>1$.

If $M=\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, similar estimates of h_d have been established by Carbou and others as the following theorem (cf. [4, 5, 26, 33]).

Theorem 5.7 *Let $1<p<\infty$. There for any $k\in\mathbb{N}$, if $u\in W^{k,p}(\Omega)$, the potential h_d belongs to $W^{k,p}(\Omega)$, and there exists a constant C_k such that*

$$\|h_d(u)\|_{W^{k,p}(\Omega)}\le C_k\|u\|_{W^{k,p}(\Omega)}.$$

6 Appendix: Global Estimates of Heat Equation with Neumann Boundary Condition

Let (M,g) be a compact Riemannian manifold with boundary ∂M (or $\partial M=\emptyset$), and $u:\bar M\times[0,T_0]\to\mathbb{R}$ be a smooth solution to the following heat equation

$$\begin{cases}\partial_t u-\Delta u=f(x,u), & (x,t)\in M\times(0,T_0],\\ \dfrac{\partial u}{\partial\nu}=0, & (x,t)\in\partial M\times[0,T_0],\\ u(x,0)=u_0, & x\in M.\end{cases}\tag{6.1}$$

with u_0 being a smooth initial data, where $f(x,u)$ is the nonhomogeneous term involving x and u, ν is the outer normal vector field. If $\partial M=\emptyset$, let u be a smooth solution to the following problem

$$\begin{cases} \partial_t u-\Delta u=f(x,u), & (x,t)\in M\times(0,T_0],\\ u(x,0)=u_0, & x\in M. \end{cases} \tag{6.2}$$

Let $r_0>0$ be the injective radius of M, $\{x_1,\dots,x_n\}$ be normal coordinates on geodesic ball $B_r(y)\subset M$ with $r\le r_0$. We denote $Q_{r,\delta}=B_r\times(\delta,T_0]$ for $0<\delta\le T_0$, and set

$$W^{2,1}_{2,0}(Q_{r,\delta})=\{u\in W^{2,1}_2(Q_{r,\delta})\,|\,u=0 \text{ on the parabolic boundary}\}.$$

To get regular estimates of the solution u to (6.1) or (6.2), we need to use the following local L^p-estimates for parabolic equations (cf. [13, 28]).

Lemma 6.1 *Let $1<p<\infty$. Suppose that $u\in W^{2,1}_{2,0}(Q_{r,\delta})\cap L^p(Q_{r,\delta})$ is a solution to $\partial_t u-\Delta u=f$ with $f\in L^p(Q_{r,\delta})$. Then there exists a constant C depending only on $\omega(r)$, p and n, such that there holds a bound:*

$$\|u\|_{W^{2,1}_p(Q_{r,\delta})}\le C(\|u\|_{L^p(Q_{r,\delta})}+\|f\|_{L^p(Q_{r,\delta})}),$$

where $\omega(r)=\sup_{B_r}\sum_{ij}|g_{ij}-\delta_{ij}|+\sup_{B_r}\sum_{ij}|\partial g_{ij}|$.

Remark 6.2 If B_r is equipped with the Euclidean metric g_E, the above constant C is dependent only of p and n.

With Lemma 6.1 at hand, we can show the global estimates for the regular solution u to (6.2) by taking an inductive argument and using a process of patching together the local estimates for u in the case $\partial M=\emptyset$. On the other hand, when $\partial M\neq\emptyset$, the analysis is more complicated, since we need to deal with the regular estimates for the solution u to (6.1) on the boundary, where u has no equation. By using the classical treatment for Neumann boundary problem, we will extend the solution across boundary by reflection, which transfers the boundary estimates into the locally interior ones. Therefore, global estimates of solution u for (6.1) can be obtained by a suitable combination of extending the solution and employing an argument of patching local estimates.

6.1 Regular Estimates of Solutions to Equation (6.2)

In this subsection, we will establish the L^p-theory of heat equation on a closed Riemannian manifold. Let (M,g) be a closed Riemannian manifold, and u be a smooth solution of (6.2) with smooth initial data u_0. To state the result precisely, we denote

$$\|u\|_{\tilde W^{k,p}(M\times[t_1,t_2])}=\sum_{i=0}^{k}\left\|\nabla^i u\right\|_{L^p(M\times[t_1,t_2])},$$

and

$$\|u\|_{\hat W^{k+2,p}(M\times[t_1,t_2])}=\|u\|_{\tilde W^{k+2,p}(M\times[t_1,t_2])}+\|\partial_t u\|_{\tilde W^{k,p}(M\times[t_1,t_2])},$$

for $k\in\mathbb{N}$ and $[t_1,t_2]\subset(0,T_0]$.

Theorem 6.3 *Let $2\le p<\infty$, and u be a smooth solution of (6.2). Then for any $0<\delta<T_0$ and $k\in\mathbb{N}$, there exists a positive constant $C_{\delta,k}$ depending only on k, δ and n such that there*

holds

$$\|u\|_{\hat{W}^{k+2,p}(M\times[\delta,T_0])} \le C_{\delta,k}(\|f\|_{\tilde{W}^{k,p}(M\times[\delta/2,T])} + \|u\|_{\tilde{W}^{k+1,p}(M\times[\delta/2,T_0])}).$$

Proof Since (M,g) is closed, there exists a uniformly local finite covering $\{B_r(x_i)\}$ for M such that $B_{r/2}(x_i)\cap B_{r/2}(x_j)=\emptyset$ for $i\neq j$ (cf. Lemma 1.6 in [25]). Let $\{\eta_i\}$ be a partition of unity corresponding to this covering. Then $u=\sum_i u_i$ where $u_i=\eta_i u$ has compact support in $B_r(x_i)$. Moreover, there exists a constant C depending on the geometry of M such that

$$\sum_i |\nabla\eta_i|(x) + \sum_i |\nabla^2\eta_i|(x) \le C, \quad I(x)=\mathrm{Card}\{i\,|\,x\in B_r(x_i)\}\le C, \tag{6.3}$$

for any $x\in M$. Then we divide our proof into two steps.

Step 1 We show the result when $k=0$.

Since u_i has support in $B_r(x_i)$, which satisfies the following equation

$$\partial_t u_i - \Delta u_i = 2\langle\nabla u,\nabla\eta_i\rangle + \Delta\eta_i u + \eta_i f = \bar{f}_i,$$

Theorem 6.1 implies that there exists a C_δ such that

$$\|u_i\|_{W_p^{2,1}(Q_{r,\delta})} \le C_\delta(\|u_i\|_{L^p(Q_{r,\delta/2})} + \|\bar{f}_i\|_{L^p(Q_{r,\delta/2})}),$$

where $Q_{r,\delta}=B_r\times[\delta,T]$. Next, we patch the above local results to get

$$\begin{aligned}\|u\|_{W_p^{2,1}(M\times[\delta,T_0])} &\le \sum_i \|u_i\|_{W_p^{2,1}(Q_{r,\delta}(x_i))}\\ &\le CC_\delta\sum_i(\|u\|_{\tilde{W}^{1,p}(Q_{r,\delta/2}(x_i))} + \|f\|_{L^p(Q_{r,\delta/2}(x_i))})\\ &\le C^2C_\delta(\|u\|_{\tilde{W}^{1,p}(M\times[\delta/2,T_0])} + \|f\|_{L^p(M\times[\delta/2,T_0])}).\end{aligned}$$

Step 2 Higher estimates for u by inducting on k.

We assume that the estimate in this theorem holds true for any $l\le k$. In the case of $k+1$, let X be any smooth vector field on M, then we can take a simple computation to show that $\nabla_X u$ satisfies the following equation locally

$$(\partial_t-\Delta)\nabla_X u = \nabla_X f + \sum_{\alpha=1}^n \Big[\frac{\partial}{\partial x^\alpha},X\Big]\nabla_{\frac{\partial}{\partial x^\alpha}}u + \nabla_{\frac{\partial}{\partial x^\alpha}}\Big(\Big[\frac{\partial}{\partial x^\alpha},X\Big]u\Big) = f_X,$$

where $[\cdot,\cdot]$ is the Lie bracket and $\{x^\alpha\}$ is a chart. Here,

$$f_X = \nabla_X f + \nabla^2 X\# u + \nabla X\#\nabla u,$$

where # denotes the linear combination.

Therefore, by using the assumption of induction, there holds

$$\begin{aligned}\|\nabla_X u\|_{\hat{W}^{k+2,p}(M\times[\delta,T_0])} &\le C_\delta(\|\nabla_X u\|_{\tilde{W}^{k+1,p}(M\times[\delta/2,T_0])} + \|f_X\|_{\tilde{W}(M\times[\delta/2,T_0])})\\ &\le C_{\delta,X}(\|\nabla_X f\|_{\tilde{W}^{k,p}(M\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{k+2,p}(M\times[\delta/2,T_0])}).\end{aligned}$$

For the covering $\{B_r(x_i)\}$ in Step 1 with $2r\le i_0$, there exists a constant C depending only on the geometry of M such that

$$I'(x) = \mathrm{card}\{i\,|\,x\in B_{2r}(x_i)\}\le C.$$

Let $X = \varphi\frac{\partial}{\partial x_i^\alpha}$ with $\varphi = 1$ on $B_r(x_i)$ and $\varphi = 0$ on $M \setminus B_{2r}(x_i)$, where $\{x_i^\alpha\}$ is the normal coordinates on $B_{2r}(x_i)$. Then there exists a constant $C_{\delta,k} > 0$ such that

$$\left\|\frac{\partial u}{\partial x_i^\alpha}\right\|_{\hat{W}^{k+2,p}(Q_{r,\delta}(x_i))} \le C_{\delta,k}\left(\left\|\frac{\partial f}{\partial x_i^\alpha}\right\|_{\tilde{W}^{k,p}(Q_{2r,\delta/2}(x_i))} + \|u\|_{\tilde{W}^{k+2,p}(Q_{2r,\delta/2}(x_i))}\right).$$

Since there exists a $C_l > 0$ such that

$$C_l^{-1}\|\nabla u\|_{W^{l,p}(B_r(x_i))} \le \sum_{\alpha=1}^{n}\left\|\frac{\partial u}{\partial x_i^\alpha}\right\|_{W^{l,p}(B_r(x_i))} \le C_l\|\nabla u\|_{W^{l,p}(B_r(x_i))}$$

for any $l \ge 0$. It follows that

$$\|u\|_{\hat{W}^{k+3,p}(Q_{r,\delta}(x_i))} \le C_{\delta,k}(\|f\|_{\tilde{W}^{k+1,p}(Q_{2r,\delta/2}(x_i))} + \|u\|_{\tilde{W}^{k+2,p}(Q_{2r,\delta}(x_i))}).$$

Therefore, by a similar patching argument as in Step 1, we have

$$\|u\|_{\hat{W}^{k+3,p}(M\times[\delta,T_0])} \le C_{\delta,k}(\|f\|_{\tilde{W}^{k+1,p}(M\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{k+2,p}(M\times[\delta/2,T_0])}).$$

Therefore, the proof is completed. □

6.2 Regular Estimates of Solutions to Equation (6.1)

In this subsection, we show that the regular solution u to the Neumann boundary problem (6.1) satisfies similar estimates as that in Theorem 6.3. For simplicity, we assume $(M, g) = (\Omega, g_E)$ is a bounded domain in $\mathbb{R}^n$ endowed with the Euclidean metric g_E, and the boundary $\partial\Omega$ is smooth. Let u be a smooth solution of Equation (6.1). Our theorem can be stated as follows.

Theorem 6.4 *Let u be a smooth solution to* (6.1) *on $\bar{\Omega}\times[0,T_0]$, Then for any $0<\delta<T_0$, $k\ge 0$ and $2 \le p < \infty$, there exists a $C_{\delta,k}$ such that we have*

$$\|u\|_{W_p^{2k+2,k+1}(\Omega\times[\delta,T_0])} \le C_{\delta,k}(\|f\|_{W_p^{2k,k}(\Omega\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{2k+1,p}(\Omega\times[\delta/2,T_0])}).$$

Before proving Theorem 6.4, we need to define the reflection associated to the boundary. Let $\Omega_\varepsilon = \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) \le \varepsilon\}$ with ε small enough. We define the reflection map by

$$R : \Omega_\varepsilon \longrightarrow \mathbb{R}^n, \quad x \longmapsto y = 2\pi(x) - x, \tag{6.4}$$

where $\pi : \Omega_\varepsilon \to \partial\Omega$ is the projection such that $\pi(x) = z$ with $|x - z| = \mathrm{dist}(x, \partial\Omega)$. In fact, π is well-defined and smooth if we choose ε small enough, so does R (cf. [17]). For simplicity, we denote

$$\hat{\Omega} = \bar{\Omega} \cup R(\Omega_\varepsilon).$$

Let $u : \Omega \to \mathbb{R}$ be a function. we define the extension of u and metric g_E by using the transformation of reflection associated to $\partial\Omega$ as follows,

$$\bar{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ u \circ R^{-1}(x), & x \in R(\Omega_\varepsilon). \end{cases} \tag{6.5}$$

And

$$\bar{g} = \begin{cases} g_E, & x \in \bar{\Omega}, \\ (R^{-1})^* g_E, & x \in R(\Omega_\varepsilon), \end{cases}$$

which is in $W^{1,\infty}(\tilde{\Omega})$.

Now we are in position to prove Theorem 6.4.

Proof Our proof is divided into three steps.

Step 1 We show the result when $k = 0$. Let η be a cut-off function such that $\eta|\partial_{\mathrm{Par}(\Omega\times(0,T_0])} = 0$, where $\partial_{\mathrm{Par}(\Omega\times(0,T_0])}$ is the parabolic boundary. $v = \eta u$ satisfies the below equation

$$\begin{cases} \partial_t v - \Delta v = -u(\partial_t - \Delta)\eta + 2\langle \nabla u, \nabla \eta\rangle + \eta f = \tilde{f}, \\ v|\partial_{\mathrm{Par}(\Omega\times(0,T_0])} = 0. \end{cases} \tag{6.6}$$

Then, Theorem 6.1 implies

$$\|u\|_{W_p^{2,1}(\Omega'\times[\delta,T_0])} \leq C_\delta(\|f\|_{L^p(\Omega\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{1,p}(\Omega\times[\delta/2,T_0])}).$$

where $\Omega' \times [\delta, T_0]$ is the level set of $\{\eta = 1\}$.

On the other hand, to get the global regular estimates of u, we need to extend u across the boundary $\partial\Omega$. Let x_0 be a point on $\partial\Omega$, U_0 be a neighborhood of x_0 in $\bar{\Omega}$. The Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ implies that $\bar{u}$ is a strong solution to equation

$$(\partial_t - \Delta_{\bar{g}})\bar{u} = \bar{f},$$

on $\Omega_0 = U_0 \cup R(U_0)$, where

$$\bar{f}(x) = \begin{cases} f(x), & x \in U_0, \\ f(R^{-1}(x)), & x \in R(U_0). \end{cases}$$

Hence, Theorem 6.1 again shows

$$\|u\|_{W_p^{2,1}(U_0'\times[\delta,T_0])} \leq C_\delta(\|f\|_{L^p(U_0\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{1,p}(U_0\times[\delta,T_0])}),$$

where $U_0' \subset\subset U_0$. Then by applying a similar patching process as in Theorem 6.3, the desire result can be obtained.

Step 2 Higher order estimates by inducting on k.

Assume that the result holds true for any $s \leq k$, we will show it also holds true for $s = k+1$. In order to apply a similar treatment as that used in Step 2 of Theorem 6.3, we classify the tangent vector fields on $\bar{\Omega}$ into two classes as follows.

(1) $A = \{X \in \chi(\Omega) \,|\, X|_{\partial\Omega} \in \chi(\partial\Omega) \text{ with } [X, \nu] = 0\}$, where ν is the outer normal vector of $\partial\Omega$. In this case, we mainly take $X = \frac{\partial}{\partial x_i}$ near the boundary under local coordinates $\{x_i\}$ with $\nu = \frac{\partial}{\partial x_n}$ or X has compact support in Ω.

(2) $B = \{X \in \chi(\Omega) \,|\, X|_{\partial\Omega} \perp \chi(\partial\Omega)\}$. That is, B is the space of smooth vector field whose restriction on $\partial\Omega$ is orthogonal with the tangent space of $\partial\Omega$. In this case, we mainly take $X = \nu$ near the boundary.

Step 2.1 Estimates for the case $X \in A$.

Let φ_s be the solution of the following ordinary differential equation

$$\begin{cases} \dfrac{\partial \varphi_s}{\partial s} = X(\varphi_s), \\ \varphi_0 = \text{identity}. \end{cases} \tag{6.7}$$

Then, $\nabla_X u = \frac{\partial u\circ\varphi}{ds}|_{s=0}$ and $\frac{\partial \nabla_X u}{\partial \nu}|_{\partial\Omega} = \nabla_X(\frac{\partial u}{\partial \nu}|_{\partial\Omega}) + [X, \nu]u|_{\partial\Omega} = 0$.

Letting $X \in A$ and $Y \in A$, a simple calculation shows

$$\begin{cases}(\partial_t - \Delta)\nabla_X\nabla_Y u = \nabla_X\nabla_Y f + L = f_{XY},\\ \dfrac{\partial}{\partial\nu}\nabla_X\nabla_Y u|_{\partial\Omega} = \left(\nabla_X\nabla_Y\dfrac{\partial u}{\partial\nu} + [\nu, X]\nabla_Y u + \nabla_Y([\nu, X]u)\right)\Big|_{\partial\Omega} = 0,\end{cases} \tag{6.8}$$

where

$$L = \sum_{l+s+m=4,\, 1\le m<4} \nabla^l X \# \nabla^s Y \# \nabla^m u.$$

By applying the assumption of induction, we have

$$\|\nabla_X\nabla_Y u\|_{W_p^{2k+2,k+1}(\Omega\times[\delta,T_0])} \le C_{\delta,k}(\|f_{XY}\|_{W_p^{2k,k}(\Omega\times[\delta,T_0])} + \|\nabla_X\nabla_Y u\|_{\tilde{W}^{2k+1,p}(\Omega\times[\delta/2,T_0])}).$$

On the other hand, $\partial_t u$ satisfies the following evolved equation

$$\begin{cases}(\partial_t - \Delta)\partial_t u = \partial_t f,\\ \dfrac{\partial}{\partial\nu}(\partial_t u)|_{\partial\Omega} = 0.\end{cases} \tag{6.9}$$

Then, again the assumption of induction gives

$$\|\partial_t u\|_{W_p^{2k+2,k+1}(\Omega\times[\delta,T_0])} \le C_{\delta,k}(\|\partial_t f\|_{W_p^{2k,k}(\Omega\times[\delta/2,T_0])} + \|\partial_t u\|_{\tilde{W}^{2k+1,p}(\Omega\times[\delta/2,T_0])}).$$

Step 2.2 Estimates for the case of $X \in B$.

Let x_0 be a point on $\partial\Omega$. Without loss of generality, we assume $U_0 \subset \Omega_\varepsilon$ is a neighborhood of x_0 in $\bar{\Omega}$ such that $\partial U_0 \cap \partial\Omega \subset \mathbb{R}^{n-1}$. By choosing $X = \nu = \frac{\partial}{\partial x_n}$, we can show

$$\frac{\partial^2 u}{\partial x_n^2} = \Delta u - \sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2} = \partial_t u - f - \sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}.$$

The estimates in Step 2.1 implies

$$\begin{aligned}\left\|\frac{\partial^2 u}{\partial x_n^2}\right\|_{W_p^{2k+2,k+1}(U_0'\times[\delta,T_0])} &\le C_{\delta,k}(\|\partial_t u\|_{W_p^{2k+2,k+1}(U_0'\times[\delta,T_0])} + \|f\|_{W_p^{2k+2,k+1}(U_0'\times[\delta,T_0])})\\ &\quad + C_{\delta,k}\sum_{i=1}^{n-1}\left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_{W_p^{2k+2,k+1}(U_0'\times[\delta,T_0])}\\ &\le C_{\delta,k}(\|f\|_{W_p^{2k+2,k+1}(U_0\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{2k+3,p}(U_0\times[\delta/2,T_0])}),\end{aligned}$$

where U_0' has compact support in U_0.

Then by using a similar patching argument as in Step 2 of Theorem 6.3, the estimates in Step 2.1 and Step 2.2 imply

$$\|u\|_{W_p^{2k+4,k+2}(\Omega'\times[\delta,T_0])} \le C_{\delta,k}(\|f\|_{W_p^{2k+2,k+1}(\Omega\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{2k+3,p}(\Omega\times[\delta/2,T_0])})$$

and

$$\|u\|_{W_p^{2k+4,k+2}(\Omega_{2\varepsilon/3}\times[\delta,T_0])} \le C_{\delta,k}(\|f\|_{W_p^{2k+2,k+1}(\Omega_\varepsilon\times[\delta/2,T_0])} + \|u\|_{\tilde{W}^{2k+3,p}(\Omega_\varepsilon\times[\delta/2,T_0])}),$$

where $\Omega' = \Omega \setminus \Omega_{\varepsilon/2}$.

Therefore, the desired result follows from the above tw estimates. □

6.3 Applications to the Heat Flow of Self-induced Harmonic Maps

By using the L^p-estimates of heat equation we obtained in Theorem 6.3 and Theorem 6.4, we have the following estimates for heat flows of self-induced harmonic maps, which has been used in the preceding context.

Theorem 6.5 *Let u be a smooth solution of* (1.3). *We assume that* $\sup_{\bar{\Omega}\times[0,T_0]}|\nabla u|(x,t)\leq C$ *for some constant C. Then for any $2\leq p<\infty$ and $0<\delta<T_0$, there exists a constant $C(\delta,C,T_0,|\Omega|)$ such that we have*

$$\|u\|_{W^{4,2}_p(\Omega\times[2\delta,T_0])}\leq C(\delta,C,T_0,|\Omega|),$$

where $|\Omega|$ denotes the volume of Ω.

Proof Let

$$f(x,u)=|\nabla u|^2u+(h_d(u)-\langle h_d(u),u\rangle\, u).$$

Since $\sup_{\bar{\Omega}\times[0,T_0]}e(x,t)\leq C$, we can easily see that for any $p_0\in[2,\infty)$

$$\|f\|_{L^{p_0}(\Omega\times[0,T_0])}\leq C(p_0,|\Omega|).$$

On the other hand, the global L^p-estimate in Theorem 6.4 implies

$$\|u\|_{W^{2,1}_{p_0}(\Omega\times[\delta/2,T_0])}\leq C_\delta(\|f\|_{L^{p_0}}+\|u\|_{\tilde{W}^{1,p_0}(\Omega\times[0,T_0])})\leq C(p_0,T_0,|\Omega|).$$

The above estimate for u then implies that

$$\|f\|_{\tilde{W}^{1,p_0}(\Omega\times[\delta/2,T_0])}\leq C(p_0,T_0,|\Omega|),$$

since

$$\nabla f=\sum_{s+l+\gamma=2}\nabla^s u\#\nabla^l u\#\nabla^\gamma u+\sum_{s+l+\gamma=1}\nabla^s h_d(u)\#\nabla^l u\#\nabla^\gamma u.$$

To show the estimates of $\|u\|_{\tilde{W}^{3,p_0}}+\|\partial_t u\|_{\tilde{W}^{1,p_0}}$, we only need to estimate the bounds of $\frac{\partial^3 u}{(\partial x^n)^3}$ and $\frac{\partial}{\partial x^n}\partial_t u$, where $\{x^i\}$ is a chart at a neighborhood U_0 of any $x_0\in\partial\Omega$ in $\bar{\Omega}$ and $\frac{\partial}{\partial x^n}|_{\partial\Omega\cap U_0}=\nu$, since the other components of $\nabla^3 u$ and $\nabla\partial_t u$ can be estimated by an almost the same method as that in Step 2.1 of Theorem 6.4.

Without loss of generality, we assume $U_0\cap\partial\Omega\subset\mathbb{R}^{n-1}$. Then $\frac{\partial u}{\partial x^n}$ satisfies the following equation:

$$\begin{cases}(\partial_t-\Delta)\dfrac{\partial u}{\partial x^n}=\dfrac{\partial f}{\partial x^n},\\ \dfrac{\partial u}{\partial x^n}\Big|_{U_0\cap\partial\Omega}=0.\end{cases}\tag{6.10}$$

The global L^p-estimates in Dirichlet case (refer to [13, 28]) implies

$$\left\|\frac{\partial u}{\partial x^n}\right\|_{W^{2,1}_{p_1}(U_0'\times[\delta,T_0])}\leq C(\|f\|_{\tilde{W}^{1,p_0}}+\|u\|_{\tilde{W}^{1,p_0}})\leq C(p_0,T_0,|U_0|),$$

where $U_0'\subset\subset U_0$.

Then, by applying a covering argument, we can show

$$\|u\|_{\tilde{W}^{3,p_0}}+\|\partial_t u\|_{\tilde{W}^{1,p_0}}\leq C(p_0,T_0,|\Omega|).$$

Hence, this estimate again implies $f \in \tilde{W}^{2,p}$ with norm controlled by $C(p_0, T_0, |\Omega|)$, for any $2 \le p \le p_0 < \infty$, since

$$\nabla^2 f = \sum_{s+l+\gamma=3} \nabla^s u \# \nabla^l u \# \nabla^\gamma u + \sum_{s+l+\gamma=2} \nabla^s h_d(u) \# \nabla^l u \# \nabla^\gamma u.$$

To get a bound of $\|f\|_{W_p^{2,1}(\Omega\times[\delta,T_0])}$, it remains to show the estimate for $\partial_t f$. A simple calculation shows

$$\partial_t f = \nabla u \# \nabla \partial_t u \# u + |\nabla u|^2 \partial_t u + h_d(\partial_t u) + \partial_t u \# h_d(u) \# u.$$

Since $|\nabla u| + |h_d(u)| \le C$ and $\|h_d(\partial_t u)\|_{W^{1,p}(\Omega)} \le C \|\partial_t u\|_{W^{1,p}(\Omega)}$, there holds

$$\|\partial_t f\|_{L^p(\Omega\times[\delta,T_0])} \le \|\partial_t u\|_{W^{1,p}(\Omega\times[\delta/2,T_0])}.$$

Therefore, Theorem 6.4 implies

$$\|u\|_{W_p^{4,2}(\Omega\times[2\delta,T_0])} \le C(\|f\|_{W_p^{2,1}(\Omega\times[\delta,T_0])} + \|u\|_{\tilde{W}^{3,p}(\Omega\times[\delta,T_0])}) \le C,$$

where C is only dependent on p_2, $|\Omega|$, T_0 and $\sup_{\Omega\times[0,T_0]} |\nabla u|$. □

For the application of Theorem 6.5 to the blow-up analysis in Section 4, we consider the equation to some scaling of u (a smooth solution of (1.3)). That is, for $0 < \lambda \le 1$, let $v(x,t) = u(\lambda x, \lambda^2 t)$, where $(0,0) \in \bar{\Omega}$ and $\lambda^{-1} = \sup_{\bar{\Omega}\times[0,T_0]} |\nabla u|$. A simple calculation shows that v satisfies the following equation on $\Omega_\lambda = \{x \,|\, \lambda x \in \Omega\}$:

$$\begin{cases} \partial_t v - \Delta v = |\nabla v|^2 v + \eta^2 (h_d^\lambda(v) - \langle h_d^\lambda(v), v\rangle v), \\ \dfrac{\partial v}{\partial \nu}\Big|_{\partial\Omega_\lambda} = 0, \\ v(x,0) = u_0(\lambda x) : \Omega_\lambda \to \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}, \end{cases} \tag{6.11}$$

where $h_d^\lambda(v)(x,t) = \lambda^2 h_d(u)(\lambda x, \lambda^2 t)$ is a vector-valued function from Ω_λ to $\mathbb{R}^n$, and $\sup_{\Omega_\lambda\times[0,\lambda^{-2}T_0]} |\nabla v| \le 1$. Thus, the fine estimates of $h_d(u)$ imply

$$\left\|h_d^\lambda(v)\right\|_{W^{k,p}(\Omega_\lambda)} \le C\lambda^2 \|v\|_{W^{k,p}(\Omega_\lambda)}.$$

Given any $x_0 \in \bar{\Omega}_\lambda$, we assume $B_2^+(x_0) = \{x \in \bar{\Omega}_\lambda \,|\, |x - x_0| < 2\} \subset \bar{\Omega}_\lambda$. Theorem 6.5 implies the following result.

Corollary 6.6 *Let $p \in (n, \infty)$, and $[t_0, t_0 + 1] \subset [0, \lambda^{-1} T_0]$. Suppose that v is a smooth solution to Equation* (6.11)*. Then, there holds*

$$\|v\|_{W_p^{4,2}(B_2^+(x_0)\times[t_0+\delta,t_0+1])} \le C(\delta, p, |\Omega|),$$

for any $x_0 \in \bar{\Omega}_\lambda$ and some small $\delta > 0$.

Proof Let $p_0 > n$. Then, there holds

$$\|h_d^\lambda(v)\|_{W^{1,p_0}(\Omega_\lambda)} \le C\lambda^2 \|v\|_{W^{1,p_0}(\Omega_\lambda)} \le C\lambda^{2-\frac{n}{p_0}} |\Omega|^{\frac{1}{p_0}}.$$

It implies that for some constant C there holds true

$$\sup_{\Omega_\lambda\times[0,\lambda^{-2}T_0]} |h_d^\lambda(v)| \le C\lambda^{2-\frac{n}{p_0}}.$$

Without loss of generality, we assume $t_0 = 0$. A similar argument as in Theorem 6.5 shows that there exists $C(p, \delta, p_0)$ and $p_0 > n$ such that

$$\|v\|_{\tilde{W}^{3,p_0}(B_2^+(x_0)\times[\delta/2,1])} + \|\partial_t v\|_{\tilde{W}^{1,p_0}(B_2^+(x_0)\times[\delta/2,1])} \le C(\lambda, \delta, |\Omega|).$$

Therefore, Theorem 6.5 implies the result. □

By using the following embedding theorem in [13], we can get the point-wise estimates of solutions as follows.

Theorem 6.7 *Let $v \in W_p^{2l,l}(\Omega \times [0,T_0])$, where $\partial\Omega$ is smooth and $l \in \mathbb{N}$. Then, for $0 \leq r + 2s = \mu < 2l$, if $p > \frac{n+2}{2l-\mu}$ and $\frac{n+2}{p}$ is not an integer, there holds*

$$\partial_t^s \partial_x^r u \in C^\alpha(\bar{\Omega} \times [0,T_0]),$$

where $\alpha = 2l - \mu - (n+2)/p$. Moreover, we have

$$|\partial_t^s \partial_x^r u|_{C^\alpha(\bar{\Omega}\times[0,T_0])} \leq C \|u\|_{W_p^{2l.l}(\Omega\times[0,T_0])},$$

where C is only dependent of n, l, p, $\partial\Omega$, $\mathrm{diam}(\Omega)^{-1}$ and T_0^{-1}.

Corollary 6.8 *Let u be a smooth solution of Equation (1.3). We assume that $\sup_{\bar{\Omega}\times[0,T_0]} |\nabla u| \cdot (x,t) \leq C$ for some constant C. Then, there holds*

$$\sum_{0\leq 2s+r<4} |\partial_t^s \partial_x^r u|_{C^\alpha(\bar{\Omega}\times[\delta,T_0])} \leq C(\delta, C, T_0, |\Omega|),$$

for any $\alpha \in (0,1)$.

Remark 6.9 (1) Suppose that v is smooth solution to (6.11) with $\sup_{\Omega_\lambda\times[0,\lambda^{-2}T_0]} |\nabla v| \leq 1$, then there exists a constant C depending only on $|\Omega|$ and δ such that

$$\sum_{0\leq 2s+r<4} |\partial_t^s \partial_x^r v|_{C^\alpha(B_2^+(x_0)\times[t_0+\delta,t_0+1])} \leq C(\delta, |\Omega|),$$

for any $x_0 \in \bar{\Omega}_\lambda$ and $[t_0, t_0+1] \subset [0, \lambda^{-2}T_0]$.

(2) Let $k \geq 1$. Suppose u is a smooth solution of to (1.3) on $\bar{\Omega}\times[0,T)$ with $\sup_{\bar{\Omega}\times[0,T)} |\nabla u|(x, t) \leq C$. By considering the equation of $u_k = \partial_t^k u$ for any $k \in \mathbb{N}$, we can apply an argument of induction on k to show that for any $m > 0$, there holds

$$|u|_{C^m(\bar{\Omega}\times[\delta,T))} \leq C(m, \delta, C, |\Omega|).$$

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