Deformation of Rigid Conjugate Self-dual Galois Representations

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Abstract   In this article, we study deformations of conjugate self-dual Galois representations. The study is twofold. First, we prove an R=T type theorem for a conjugate self-dual Galois representation with coefficients in a finite field, satisfying a certain property called rigid. Second, we study the rigidity property for the family of residue Galois representations attached to a symmetric power of an elliptic curve, as well as to a regular algebraic conjugate self-dual cuspidal representation.

1 Introduction

Let F/F^+ be a CM extension of number fields with $\mathsf{c} \in \mathrm{Gal}(\overline{F}/F^+)$ a complex conjugation. In this article, we study deformations of conjugate self-dual Galois representations of F. The study is twofold. First, we prove an R=T type theorem (Theorem 3.38) for a conjugate self-dual Galois representation $\bar{r}$ of F with coefficients in a finite field, satisfying a certain property called *rigid* (Definition 3.36). It is worth mentioning that unlike many other references in the field, we neither assume that the characteristic of the coefficient field is relatively split in F/F^+ nor assume that $\bar{r}$ only ramifies at places that are split in F. Second, we study the rigidity property for the family of residue Galois representations attached to a symmetric power of an elliptic curve, as well as to a regular algebraic conjugate self-dual cuspidal representation Π. In particular, we show in Theorem 4.8 that if Π has a supercuspidal component, then the residue Galois representation is rigid whenever the characteristic is large enough.

The main purpose of this article is to make preparation for our work [21] in which we prove major cases toward the Beilinson–Bloch–Kato conjecture on the relation between Selmer groups and L-functions. To see how Galois deformation is used in the study of Selmer groups, we refer to that article.

The two main theorems are both technical to state. In order to give some flavor of what we can prove in this article, we now state a result (Theorem 1.1 below) that follows from the combination of the two main theorems and is relatively straightforward to formulate. Also, it seems to us that this kind of result is new in the literature.

More precisely, we consider the following data:

• an RACSDC (that is, regular algebraic conjugate self-dual cuspidal) representation Π of $\mathrm{GL}_N(\mathbb{A}_F)$ for some $N \geq 2$, with the archimedean weights $\xi = (\xi_{\tau,1} \leq \cdots \leq \xi_{\tau,N})_\tau \in (\mathbb{Z}^N)^{\mathrm{Hom}(F,\mathbb{C})}$ (Definition 2.8),

• a number field $E \subseteq \mathbb{C}$ such that one has a family of Galois representations

$$\{\rho_{\Pi,\lambda} \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_N(E_\lambda)\}_\lambda$$

indexed by primes of E satisfying $\rho_{\Pi,\lambda}^{\mathsf{c}} \simeq \rho_{\Pi,\lambda}^{\vee}(1-N)$ as in Proposition 2.9 and Definition 2.10 (see Remark 2.11),

• a finite set Σ^+ of nonarchimedean places of F^+ such that for every nonarchimedean place w of F not above Σ^+, Π_w is unramified and the underlying rational prime of w is unramified in F.

For every prime λ of E, we write ℓ_λ its underlying rational prime and O_λ the ring of integers of E_λ. Let Λ_{Π,Σ^+} be the set of primes λ of E such that either $\rho_{\Pi,\lambda}$ is not residually absolutely irreducible, or ℓ_λ underlies Σ^+, or $\ell_\lambda - N - 1$ is strictly smaller than the maximal distance of integers in ξ. For every prime λ of E not in Λ_{Π,Σ^+}, we can define

• a commutative O_λ-algebra $\mathsf{R}_{\mathscr{S}_\lambda}^{\mathrm{univ}}$ that classifies conjugate self-dual deformations of the residue representation of $\rho_{\Pi,\lambda}$ that are crystalline with regular Fontaine–Laffaille weights at places above ℓ_λ and unramified outside Σ^+ and places above ℓ_λ, and

• a maximal ideal $\mathfrak{m}_\lambda$ of $\mathbb{T}_N^{\ell_\lambda}$ with the residue field O_λ/λ determined by the Satake parameters of Π, where $\mathbb{T}_N^{\ell_\lambda}$ is the abstract spherical unitary Hecke algebra of rank N away from Σ^+ and places above ℓ_λ.

Consider pairs (V, K) in which

- V is a (nondegenerate) hermitian space over F (with respect to c) of rank N that is split at every nonarchimedean place of F^+ not in Σ^+, and

- $K = \prod_v K_v$ is a neat open compact subgroup of $U(V)(\mathbb{A}_{F^+}^\infty)$ such that K_v is the stabilizer of a self-dual lattice for $v \notin \Sigma^+$,

satisfying that there exists a cuspidal automorphic representation π of $U(V)(\mathbb{A}_{F^+})$ with nonzero K-invariants whose automorphic base change is Π. Every pair (V, K) as above gives a Shimura variety $\mathrm{Sh}(V, K)$, which is a quasi-projective smooth scheme over $\mathbb{C}$ of dimension $d(V)$. For $\lambda \notin \Lambda_{\Pi,\Sigma^+}$, the weights ξ give rise to an O_λ-linear local system $\mathscr{L}_{\xi,\lambda}$ on $\mathrm{Sh}(V, K)$; and we let T_λ be the image of $\mathbb{T}_N^{\ell_\lambda}$ in $\mathrm{End}_{\mathscr{O}}(\mathrm{H}_{\mathrm{\acute{e}t}}^{d(V)}(\mathrm{Sh}(V, K), \mathscr{L}_{\xi,\lambda}))$.

We refer to §3.6 for more details of the above constructions.

Theorem 1.1 *Let the setup be as above. Suppose that there exists a nonarchimedean place of F at which Π is supercuspidal. Then Λ_{Π,Σ^+} is a finite set. Moreover, there exists a finite set Λ'_{Π,Σ^+} of primes of E containing Λ_{Π,Σ^+}, such that for every $\lambda \notin \Lambda'_{\Pi,\Sigma^+}$ and every pair (V, K) as above with $d(V) \leq 1$, we have*

(1) There is a canonical isomorphism $\mathsf{R}_{\mathscr{L}_\lambda}^{\mathrm{univ}} \xrightarrow{\sim} \mathsf{T}_{\lambda,\mathfrak{m}_\lambda}$ of local complete intersection commutative O_λ-algebras.

(2) The $\mathsf{T}_{\lambda,\mathfrak{m}_\lambda}$-module $\mathrm{H}_{\mathrm{\acute{e}t}}^{d(V)}(\mathrm{Sh}(V, K), \mathscr{L}_{\xi,\lambda})_{\mathfrak{m}_\lambda}$ is finite and free.

The above theorem is a consequence of the two main results of this article, namely, Theorem 3.38 and Theorem 4.8. We have a similar consequence when $d(V)$ is general, but under an extra assumption on certain vanishing of localized cohomology off middle degree.

The article is organized as follows. In Section 2, we make some preparations for conjugate self-dual representations, unitary Hecke algebras, and automorphic representations. In Section 3, we study both local and global deformations of conjugate self-dual Galois representations. In Section 4, we study the rigidity property for symmetric powers of elliptic curves and automorphic Galois representations.

Notations and Conventions

- All rings are commutative and unital; and ring homomorphisms preserve units.

- Throughout the article, we fix an integer $N \geq 1$. Denote by M_N (resp. GL_N) the scheme over $\mathbb{Z}$ of N-by-N square matrices (resp. invertible square matrices).

- We fix a CM extension F/F^+ of number fields and an algebraic closure $\overline{F}$ of F, with $c \in \mathrm{Gal}(\overline{F}/F^+)$ a complex conjugation. Put $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ and $\Gamma_{F^+} := \mathrm{Gal}(\overline{F}/F^+)$. In this article, all hermitian spaces over F are with respect to the convolution c and are nondegenerate.

- Denote by Σ_∞ (resp. Σ_∞^+) the set of complex embeddings of F (resp. F^+). For $\tau \in \Sigma_\infty$, we denote by $\tau^c := \tau \circ c \in \Sigma_\infty$ its conjugation and $\underline{\tau} := \tau|_{F^+} \in \Sigma_\infty^+$ the underlying element.

- For every rational prime p, denote by Σ_p^+ the set of all p-adic places of F^+.

- Denote by Σ_{bad}^+ the union of Σ_p^+ for all p that ramifies in F.

- Denote by $\eta := \eta_{F/F^+} : \Gamma_{F^+} \to \{\pm 1\}$ the character associated to the extension F/F^+.

- For every prime ℓ, denote by $\epsilon_\ell : \Gamma_{F^+} \to \mathbb{Z}_\ell^\times$ the ℓ-adic cyclotomic character.

- For every place v of F^+, we

  - put $F_v := F \otimes_{F^+} F_v^+$; and define $\delta(v)$ to be 1 (resp. 2) if v splits (resp. does not split) in F;
  - fix an algebraic closure $\overline{F}_v^+$ of F_v^+ containing $\overline{F}$; and put $\Gamma_{F_v^+} := \mathrm{Gal}(\overline{F}_v^+/F_v^+)$ as a subgroup of Γ_{F^+};
  - for a homomorphism r from Γ_{F^+} to another group, denote by r_v the restriction of r to the subgroup $\Gamma_{F_v^+}$.

- For every nonarchimedean place w of F, we

  - identify the Galois group Γ_{F_w} with $\Gamma_{F_v^+} \cap \Gamma_F$ (resp. $\mathfrak{c}(\Gamma_{F_v^+} \cap \Gamma_F)\mathfrak{c}$), where v is the underlying place of F^+, if the embedding $F \hookrightarrow \overline{F}_v^+$ induces (resp. does not induce) the place w;
  - let $\mathrm{I}_{F_w} \subseteq \Gamma_{F_w}$ be the inertia subgroup;
  - denote by $\phi_w \in \Gamma_{F_w}$ an *arithmetic* Frobenius element.

2 Preparation

2.1 Extension of Essentially Conjugate Self-dual Representations

In this subsection, we collect some notion and facts on the extension of essentially conjugate self-dual representations.

Notation 2.1 We recall the group scheme $\mathscr{G}_N$ from [14, §1].[2)] Put

$$\mathscr{G}_N := (\mathrm{GL}_N \times \mathrm{GL}_1) \rtimes \{1, \mathfrak{c}\}$$

with $\mathfrak{c}^2 = 1$ and

$$\mathfrak{c}(g, \mu)\mathfrak{c} = (\mu\, {}^\mathrm{t}g^{-1}, \mu)$$

for $(g, \mu) \in \mathrm{GL}_N \times \mathrm{GL}_1$. In what follows, we will often regard GL_N as a subgroup of $\mathscr{G}_N$ via the embedding $g \mapsto (g, 1, 1)$. Denote by $\nu \colon \mathscr{G}_N \to \mathrm{GL}_1$ the homomorphism such that $\nu|_{\mathrm{GL}_N \times \mathrm{GL}_1}$ is the projection to the factor GL_1 and that $\nu(\mathfrak{c}) = -1$. We have the adjoint action ad of $\mathscr{G}_N$ on M_N, given by

$$\mathrm{ad}(g, \mu)(x) = gxg^{-1}, \quad \mathrm{ad}(\mathfrak{c})(x) = -\,{}^\mathrm{t}x$$

for $x \in \mathrm{M}_N$ and $(g, \mu) \in \mathrm{GL}_N \times \mathrm{GL}_1$.

Let $\tilde{\Gamma}$ be a topological group, and $\Gamma \subseteq \tilde{\Gamma}$ an open subgroup of index at most two.

Notation 2.2 Let R be a (topological) ring.

- For a (continuous) homomorphism $\rho \colon \Gamma \to \mathrm{GL}_N(R)$, we denote by $\rho^\vee \colon \Gamma \to \mathrm{GL}_N(R)$ the contragredient homomorphism, which is defined by the formula $\rho^\vee(x) = {}^\mathrm{t}\rho(x)^{-1}$ for every $x \in \Gamma$.

- For a (continuous) homomorphism $\rho \colon \Gamma \to \mathrm{GL}_N(R)$ and an element $\gamma \in \tilde{\Gamma}$ that normalizes Γ, we let $\rho^\gamma \colon \Gamma \to \mathrm{GL}_N(R)$ be the (continuous) homomorphism defined by $\rho^\gamma(x) = \rho(\gamma x \gamma^{-1})$ for every $x \in \Gamma$.

- For a (continuous) homomorphism

$$r \colon \tilde{\Gamma} \to \mathscr{G}_N(R)$$

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2) In fact, a better notion seems to be the C-group introduced in [7].

such that the image of $r|_\Gamma$ lies in $\mathrm{GL}_N(R) \times R^\times$, we denote

$$r^\natural \colon \Gamma \to \mathrm{GL}_N(R) \times R^\times \to \mathrm{GL}_N(R)$$

the composition of $r|_\Gamma$ with the projection to $\mathrm{GL}_N(R)$.

Lemma 2.3 *Suppose that $[\tilde{\Gamma} : \Gamma] = 2$. Let R be a (topological) ring and $\chi \colon \tilde{\Gamma} \to R^\times$ a (continuous) character. We have*

(1) If $r \colon \tilde{\Gamma} \to \mathscr{G}_N(R)$ is a (continuous) homomorphism satisfying $r^{-1}(\mathrm{GL}_N(R) \times R^\times) = \Gamma$ and $\nu \circ r = \chi$, then for every $\gamma \in \tilde{\Gamma} \setminus \Gamma$, we have

$$r^{\natural,\gamma} = B \circ \chi r^{\natural,\vee} \circ B^{-1},$$

where B is obtained from $r(\gamma) = (B, -\chi(\gamma), \mathfrak{c})$.

(2) Let $\rho \colon \Gamma \to \mathrm{GL}_N(R)$ be a (continuous) homomorphism, γ an element in $\tilde{\Gamma} \setminus \Gamma$, and $B \in \mathrm{GL}_N(R)$ such that $\rho^\gamma = B \circ \chi \rho^\vee \circ B^{-1}$ and $B\,^\mathrm{t}B^{-1} = \mu_B \chi(\gamma)^{-1} \rho(\gamma^2)$ for some $\mu_B \in \{\pm 1\}$. Then there exists a unique (continuous) homomorphism

$$r \colon \tilde{\Gamma} \to \mathscr{G}_N(R)$$

satisfying $r|_\Gamma = (\rho, \chi|_\Gamma, 1)$ and $r(\gamma) = (B, \mu_B \chi(\gamma), \mathfrak{c})$.

(3) Suppose in (2) that R is a field and ρ is absolutely irreducible. If ρ^γ and $\chi\rho^\vee$ are conjugate, then ρ induces a (continuous) homomorphism $r \colon \tilde{\Gamma} \to \mathscr{G}_N(R)$ satisfying $r|_\Gamma = (\rho, \chi)$, unique up to changing the $\mathrm{GL}_N(R)$-component of $r(\gamma)$ by a scalar in $R^\times$.

Proof Part (1) is a special case of [14, Lemma 2.1.1].

For (2), we check that

$$r(\gamma^2) = (B, \mu_B\chi(\gamma), \mathfrak{c}) \cdot (B, \mu_B\chi(\gamma), \mathfrak{c}) = (\mu_B\chi(\gamma)B\,^\mathrm{t}B^{-1}, \chi(\gamma^2), 1) = (\rho(\gamma^2), \chi(\gamma^2), 1).$$

Since $\tilde{\Gamma}$ is generated by Γ and γ, we obtain a unique (continuous) homomorphism $r \colon \tilde{\Gamma} \to \mathscr{G}_N(R)$ as in (2).

For (3), by Schur's lemma, the element B is unique up to scalar in $R^\times$, which implies the existence and also the uniqueness of μ_B. Thus, (3) follows immediately. $\qquad\square$

2.2 Unitary Satake Parameters and Unitary Hecke Algebras

In this subsection, we introduce the notion of unitary Satake parameters and unitary Hecke algebras.

Definition 2.4 (Abstract Satake parameter) *Let L be a ring. For a multi-subset $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_N\} \subseteq L$, we put*

$$P_{\boldsymbol{\alpha}}(T) \coloneqq \prod_{i=1}^{N}(T - \alpha_i) \in L[T].$$

Consider a nonarchimedean place v of F^+ not in Σ_{bad}^+.

(1) Suppose that v is inert in F. We define an (abstract) Satake parameter in L at v of rank N to be a multi-subset $\boldsymbol{\alpha} \subseteq L$ of cardinality N. We say that $\boldsymbol{\alpha}$ is unitary *if $P_{\boldsymbol{\alpha}}(T) = (-T)^N \cdot P_{\boldsymbol{\alpha}}(T^{-1})$.*

(2) Suppose that v splits in F. We define an (abstract) Satake parameter in L at v of rank N to be a pair $\boldsymbol{\alpha} \coloneqq (\boldsymbol{\alpha}_1; \boldsymbol{\alpha}_2)$ of multi-subsets $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \subseteq L$ of cardinality N, indexed by the two

places w_1, w_2 *of* F *above* v. *We say that* $\boldsymbol{\alpha}$ *is* unitary *if* $P_{\boldsymbol{\alpha}_1}(T) = c \cdot T^N \cdot P_{\boldsymbol{\alpha}_2}(T^{-1})$ *for some constant* $c \in L^{\times}$.

Let v be a nonarchimedean place of F^+ not in Σ_{bad}^+. Let $\Lambda_{N,v}$ be the unique up to isomorphism nondegenerate hermitian module over $O_{F_v} = O_F \otimes_{O_{F^+}} O_{F_v^+}$ of rank N, and $\mathrm{U}_{N,v}$ its unitary group over $O_{F_v^+}$. Under a suitable basis, the associated hermitian form of $\Lambda_{N,v}$ is given by the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Consider the *local spherical Hecke algebra*

$$\mathbb{T}_{N,v} \coloneqq \mathbb{Z}[\mathrm{U}_{N,v}(O_{F_v^+}) \backslash \mathrm{U}_{N,v}(F_v^+) / \mathrm{U}_{N,v}(O_{F_v^+})],$$

with the unit element $\mathbb{1}_{\mathrm{U}_{N,v}(O_{F_v^+})}$. Let $\mathrm{A}_{N,v}$ be the maximal split diagonal subtorus of $\mathrm{U}_{N,v}$, and $\mathrm{X}_*(\mathrm{A}_{N,v})$ be its cocharacter group. Then there is a well-known Satake transform

$$\mathbb{T}_{N,v} \to \mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})] \simeq \mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{X}_*(\mathrm{A}_{N,v})] \qquad (2.1)$$

as a homomorphism of algebras. Choose a uniformizer ϖ_v of F_v^+.

Construction 2.5 Let L be a ring over $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$. Let $\boldsymbol{\alpha}$ be a unitary Satake parameter in L at v of rank N. There are two cases.

(1) Suppose that v is inert in F. Then a set of representatives of $\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})$ can be taken as

$$\{(\varpi_v^{t_1}, \ldots, \varpi_v^{t_N}) \mid t_1, \ldots, t_N \in \mathbb{Z} \text{ satisfying } t_i + t_{N+1-i} = 0 \text{ for all } 1 \le i \le N\}.$$

Choose an ordering of $\boldsymbol{\alpha}$ as $(\alpha_1, \ldots, \alpha_N)$ satisfying $\alpha_i \alpha_{N+1-i} = 1$; we have a unique homomorphism

$$\mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})] \to L$$

of rings over $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$ sending the class of $(\varpi_v^{t_1}, \ldots, \varpi_v^{t_N})$ to $\prod_{i=1}^{\lfloor \frac{N}{2} \rfloor} \alpha_i^{t_i}$. Composing with the Satake transform (2.1), we obtain a ring homomorphism

$$\phi_{\boldsymbol{\alpha}} \colon \mathbb{T}_{N,v} \to L.$$

It is independent of the choices of the uniformizer ϖ_v and the ordering of $\boldsymbol{\alpha}$.

(2) Suppose that v splits in F into two places w_1 and w_2. Then a set of representatives of $\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})$ can be taken as

$$\left\{ \left(\mathrm{diag}(\varpi_v^{t_1}, \ldots, \varpi_v^{t_N}), \mathrm{diag}(\varpi_v^{-t_N}, \ldots, \varpi_v^{-t_1}) \right) \big| t_1, \ldots, t_N \in \mathbb{Z} \right\},$$

where the first diagonal matrix (resp. the second diagonal matrix) is regarded as an element in $\mathrm{A}_{N,v}(F_{w_1})$ (resp. $\mathrm{A}_{N,v}(F_{w_2})$). Choose orders in $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ as $(\alpha_{1,1}, \ldots, \alpha_{1,N})$ and $(\alpha_{2,1}, \ldots, \alpha_{2,N})$ satisfying $\alpha_{1,i} \alpha_{2,N+1-i} = 1$; we have a unique homomorphism

$$\mathbb{Z}[\|v\|^{\pm\delta(v)/2}][\mathrm{A}_{N,v}(F_v^+) / \mathrm{A}_{N,v}(O_{F_v^+})] \to L$$

of $\mathbb{Z}[\|v\|^{\pm\delta(v)/2}]$-rings sending the class of $(\mathrm{diag}(\varpi_v^{t_1}, \ldots, \varpi_v^{t_N}), \mathrm{diag}(\varpi_v^{-t_N}, \ldots, \varpi_v^{-t_1}))$ to $\prod_{i=1}^{N} \alpha_{1,i}^{t_i}$. Composing with the Satake transform (2.1), we obtain a ring homomorphism

$$\phi_{\boldsymbol{\alpha}} \colon \mathbb{T}_{N,v} \to L.$$

It is independent of the choices of the uniformizer ϖ_v, the order of the two places of F above v, and the orders in $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.

Definition 2.6 (Abstract unitary Hecke algebra) *For a finite set Σ^+ of nonarchimedean places of F^+ containing Σ_{bad}^+, we define the* abstract unitary Hecke algebra away from Σ^+ *to be the restricted tensor product*

$$\mathbb{T}_N^{\Sigma^+} \coloneqq \bigotimes_v{}' \mathbb{T}_{N,v}$$

over all $v \notin \Sigma_\infty^+ \cup \Sigma^+$ with respect to unit elements. It is a ring.

2.3 Automorphic Representations

In this subsection, we collect some facts concerning automorphic representations.

Notation 2.7 We denote by $\mathbb{Z}_{\leq}^N$ the subset of $\mathbb{Z}^N$ consisting of nondecreasing sequences. For a finite set T and an element $\xi = (\xi_\tau)_{\tau \in T} \in (\mathbb{Z}_{\leq}^N)^T$, put

$$a_\xi \coloneqq \min_{\tau \in T}\{\xi_{\tau,1}\}, \quad b_\xi \coloneqq \max_{\tau \in T}\{\xi_{\tau,N}\} + N - 1.$$

Let w be a nonarchimedean place of F. For every irreducible admissible (complex) representation Π of $\mathrm{GL}_N(F_w)$, every rational prime ℓ, and every isomorphism $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, we denote by $\mathrm{WD}(\iota_\ell\Pi)$ the (Frobenius semisimple) Weil–Deligne representation associated to $\iota_\ell\Pi$ via the local Langlands correspondence [18].

Definition 2.8 *We say that a (complex) representation Π of $\mathrm{GL}_N(\mathbb{A}_F)$ is RACSDC (that is, regular algebraic conjugate self-dual cuspidal) if*

(1) *Π is an irreducible cuspidal automorphic representation;*

(2) *$\Pi \circ \mathsf{c} \simeq \Pi^\vee$;*

(3) *for every archimedean place w of F, Π_w is regular algebraic (in the sense of [13, Definition 3.12]).*

If Π is RACSDC, then there exists a unique element $\xi_\Pi = (\xi_{\tau,1}, \ldots, \xi_{\tau,N})_\tau \in (\mathbb{Z}_{\leq}^N)^{\Sigma\infty}$, which we call the archimedean weights *of Π, satisfying $\xi_{\tau,i} = -\xi_{\tau^c, N+1-i}$ for every τ and i, such that Π_τ (as a representation of $\mathrm{GL}_N(\mathbb{C})$) is isomorphic to the (irreducible) principal series representation induced by the characters*

$$(\mathrm{arg}^{1-N+2\xi_{\tau,1}}, \mathrm{arg}^{3-N+2\xi_{\tau,2}}, \ldots, \mathrm{arg}^{N-3+2\xi_{\tau,N-1}}, \mathrm{arg}^{N-1+2\xi_{\tau,N}}),$$

where $\mathrm{arg} \colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the argument character *defined by the formula $\mathrm{arg}(z) \coloneqq z/\sqrt{z\overline{z}}$.*

Proposition 2.9 *Let Π be an RACSDC representation of $\mathrm{GL}_N(\mathbb{A}_F)$ with the archimedean weights $\xi = \xi_\Pi$.*

(1) *For every place w of F, Π_w is tempered.*

(2) *For every rational prime ℓ and every isomorphism $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, there is a semisimple continuous homomorphism*

$$\rho_{\Pi,\iota_\ell} \colon \Gamma_F \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell),$$

unique up to conjugation, satisfying that

(a) *for every nonarchimedean place w of F, the Frobenius semisimplification of the associated Weil–Deligne representation of $\rho_{\Pi,\iota_\ell}|_{\Gamma_{F_w}}$ is isomorphic to $\mathrm{WD}(\iota_\ell \Pi_w |\det|_w^{\frac{1-N}{2}})$;*

(b) *for every place w of F above ℓ, the representation $\rho_{\Pi,\iota_\ell}|_{\Gamma_{F_w}}$ is de Rham (crystalline if Π_w is unramified) with regular Hodge–Tate weights contained in the range $[a_\xi, b_\xi]$;*

(c) *$\rho^{\mathrm{c}}_{\Pi,\iota_\ell}$ and $\rho^\vee_{\Pi,\iota_\ell}(1-N)$ are conjugate.*

Proof   Part (1) is [9, Theorem 1.2]. For (2), the Galois representation ρ_{Π,ι_ℓ} is constructed in [12, Theorem 3.2.3]; the local-global compatibility (2a) is obtained in [9, Theorem 1.1] and [10, Theorem 1.1]; (2b) is obtained in [12, Theorem 3.2.3]; and (2c) follows from (2a) and the Chebotarev density theorem. $\square$

Definition 2.10   *Let Π be an RACSDC representation of $\mathrm{GL}_N(\mathbb{A}_F)$. We say that a subfield $E \subseteq \mathbb{C}$ is a* strong coefficient field *of Π if E is a number field; and for every prime λ of E, there exists a continuous homomorphism*

$$\rho_{\Pi,\lambda} \colon \Gamma_F \to \mathrm{GL}_N(E_\lambda),$$

necessarily unique up to conjugation, such that for every isomorphism $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ inducing the prime λ, $\rho_{\Pi,\lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ and ρ_{Π,ι_ℓ} are conjugate, where ρ_{Π,ι_ℓ} is the homomorphism from Proposition 2.9 (2).

Remark 2.11   By [12, Proposition 3.2.5], a strong coefficient field of Π exists when Π is RACSDC.

Let V be a hermitian space over F of rank N, and π an irreducible admissible representation of $\mathrm{U}(\mathrm{V})(\mathbb{A}_{F^+})$. An *automorphic base change* of π is defined to be an automorphic representation $\mathrm{BC}(\pi)$ of $\mathrm{GL}_N(\mathbb{A}_F)$ that is a finite isobaric sum of discrete automorphic representations such that $\mathrm{BC}(\pi)_v \simeq \mathrm{BC}(\pi_v)$ holds for all but finitely many nonarchimedean places v of F^+ such that π_v is unramified. By the strong multiplicity one property for GL_N [23], if $\mathrm{BC}(\pi)$ exists, then it is unique up to isomorphism. Moreover, for every nonarchimedean place v of F^+ that is nonsplit in F, we have a notion of local base change, which is defined by [24] when $N \leq 3$ and by [19, 22] for general N.

Proposition 2.12   *Take an RACSDC representation Π of $\mathrm{GL}_N(\mathbb{A}_F)$ with $\xi_\Pi = (\xi_\tau)_\tau$ the archimedean weights. Let V be a hermitian space over F of rank N and $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{U}(\mathrm{V})(\mathbb{A}_{F^+})$ such that $\Pi \simeq \mathrm{BC}(\pi)$. Then*

(1) *For every nonarchimedean place v of F^+, $\mathrm{BC}(\pi_v) \simeq \Pi_v$.*

(2) *For every $\tau \in \Sigma_\infty$, $\pi_{\underline{\tau}}$ is a discrete series representation of Harish-Chandra parameter*

$$\left\{ \frac{1-N}{2} + \xi_{\tau,1}, \frac{3-N}{2} + \xi_{\tau,2}, \ldots, \frac{N-3}{2} + \xi_{\tau,N-1}, \frac{N-1}{2} + \xi_{\tau,N} \right\}$$

after we identity $\mathrm{U}(\mathrm{V})(F_{\underline{\tau}})$ as a subgroup of $\mathrm{GL}_N(\mathbb{C})$ via $\tau \colon F \otimes_{F^+, \underline{\tau}} \mathbb{R} \xrightarrow{\sim} \mathbb{C}$.

Proof   This follows from [19, Theorem 1.7.1] for generic packets. $\square$

Corollary 2.13   *Take an RACSDC representation Π of $\mathrm{GL}_N(\mathbb{A}_F)$. Let V be a hermitian space over F of rank N that is even, and $\pi = \bigotimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{U}(\mathrm{V})(\mathbb{A}_{F^+})$ such that $\Pi \simeq \mathrm{BC}(\pi)$. If v is a nonarchimedean place of F^+ that is inert in F (with*

w the unique place of F above it) such that V_v is not split and that π_v has nonzero invariants under a special maximal open compact subgroup of $U(V)(F_v^+)$, then the monodromy operator of $\mathrm{WD}(\iota_\ell \Pi_w)$ is conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \oplus 1_{N-2}$ for every rational prime ℓ and every isomorphism $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$.

Proof Write $N = 2r$ for a positive integer r. By Proposition 2.12, we know that Π_w is isomorphic to $\mathrm{BC}(\pi_v)$. Since Π_w is tempered by Proposition 2.9 (1), π_v is also tempered. Since π_v has nonzero invariants under a special maximal open compact subgroup of $U(V)(F_v^+)$, the cuspidal support of π_v is of the form $((F^\times)^{r-1} \times U(V_2)(F_v^+), \chi \boxtimes \mathbf{1})$, where V_2 is the anisotropic hermitian space of rank 2 over F_w and χ is a unitary unramified character of $(F^\times)^{r-1}$. In particular, the cuspidal support of Π_w is of the form

$$((F^\times)^{r-1} \times \mathrm{GL}_2(F_w) \times (F^\times)^{r-1}, \chi \boxtimes \mathrm{St}_2 \boxtimes \chi^{-1}),$$

where St_2 denotes the Steinberg representation of $\mathrm{GL}_2(F_w)$. The corollary follows immediately. $\square$

3 Deformation

In this section, we fix an *odd* rational prime ℓ and a subfield $E \subseteq \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$. We denote by $\mathscr{O}$ the ring of integers of E, by λ its maximal ideal, and by $k := \mathscr{O}/\lambda$ the residue field. Following [14], we denote by $\mathscr{C}_{\mathscr{O}}^f$ the category of Artinian local rings over $\mathscr{O}$ with residue field k, and by $\mathscr{C}_{\mathscr{O}}$ the category of Noetherian complete local rings over $\mathscr{O}$ that are inverse limits of objects of $\mathscr{C}_{\mathscr{O}}^f$. For an object R of $\mathscr{C}_{\mathscr{O}}$, we shall denote by $\mathfrak{m}_R$ its maximal ideal. For an $\mathscr{O}$-valued character, we will use the same notation for its induced R-valued character for every object R of $\mathscr{C}_{\mathscr{O}}$. Recall the character $\eta \colon \Gamma_{F^+} \to \{\pm 1\}$ associated to the extension F/F^+ and the ℓ-adic cyclotomic character $\epsilon_\ell \colon \Gamma_{F^+} \to \mathbb{Z}_\ell^\times (\subseteq \mathscr{O}^\times)$.

3.1 Deformation Problems

In this subsection, we introduce the notion of deformation problems. Let $\tilde{\Gamma}$ be a topological group, and $\Gamma \subseteq \tilde{\Gamma}$ an open subgroup of index at most two.

Notation 3.1 We consider a pair $(\bar{r}, \chi)$, where
- $\bar{r} \colon \tilde{\Gamma} \to \mathscr{G}_N(k)$ is a homomorphism,
- $\chi \colon \tilde{\Gamma} \to \mathscr{O}^\times$ a continuous homomorphism, known as the *similitude character*,

subject to the relation $\bar{r}^{-1}(\mathrm{GL}_N(k) \times k^\times) = \Gamma$ and $\nu \circ \bar{r} = \chi$.

The following definition slightly generalizes [14, Definition 2.2.1].

Definition 3.2 *A lifting of $\bar{r}$ to an object R of $\mathscr{C}_{\mathscr{O}}$ is a continuous homomorphism $r \colon \tilde{\Gamma} \to \mathscr{G}_N(R)$ satisfying $r \bmod \mathfrak{m}_R = \bar{r}$ and $\nu \circ r = \chi$. We say that two liftings are equivalent if they are conjugate by an element in $1 + \mathrm{M}_N(\mathfrak{m}_R) \subset \mathrm{GL}_N(R) \subset \mathscr{G}_N(R)$. By a deformation of $\bar{r}$, we mean an equivalence class of liftings of $\bar{r}$.*[3]

Now suppose that Γ is topologically finitely generated. Then there exists a universal lifting

$$r^{\mathrm{univ}} \colon \tilde{\Gamma} \to \mathscr{G}_N(\mathsf{R}_{\bar{r}}^{\mathrm{loc}})$$

---

3) Strictly speaking, a lifting or a deformation of $\bar{r}$ depends on the similitude character χ. But we choose to follow the terminology in [14] by not spelling the characters out, as the relevance on the similitude character is always clear from the context.

of $\bar{r}$ to an object $\mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ of $\mathscr{C}_{\mathscr{O}}$ such that, for every object R of $\mathscr{C}_{\mathscr{O}}$, the set of liftings of $\bar{r}$ to R is in natural bijection with $\mathrm{Hom}_{\mathscr{C}_{\mathscr{O}}}(\mathsf{R}_{\bar{r}}^{\mathrm{loc}}, R)$. Since Γ is topologically finitely generated, it is well-known that $\mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ is Noetherian; and there exist natural isomorphisms

$$\mathrm{Hom}_k(\mathfrak{m}_{\mathsf{R}_{\bar{r}}^{\mathrm{loc}}}/(\mathfrak{m}_{\mathsf{R}_{\bar{r}}^{\mathrm{loc}}}^2, \lambda), k) \simeq \mathrm{Hom}_{\mathscr{C}_{\mathscr{O}}}(\mathsf{R}_{\bar{r}}^{\mathrm{loc}}, k[\varepsilon]/(\varepsilon^2)) \simeq \mathrm{Z}^1(\tilde{\Gamma}, \mathrm{ad}\,\bar{r}),$$

where $\mathrm{Z}^1(\tilde{\Gamma}, \mathrm{ad}\,\bar{r})$ denotes the group of 1-cocycles of $\tilde{\Gamma}$ with values in the adjoint representation $(\mathrm{ad}\,\bar{r}, \mathrm{M}_N(k))$. Let $\widehat{\mathrm{GL}}_{N,\mathscr{O}}$ be the completion of the group scheme $\mathrm{GL}_{N,\mathscr{O}}$ along its unit section, which acts naturally on $\mathrm{Spf}\,\mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ by conjugation.

Definition 3.3 *A* local deformation problem *of $\bar{r}$ is a closed formal subscheme $\mathscr{D}$ of $\mathrm{Spf}\,\mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ that is stable under the action of $\widehat{\mathrm{GL}}_{N,\mathscr{O}}$.*

Definition 3.4 *For a local deformation problem $\mathscr{D}$ of $\bar{r}$, we define the* tangent space *of $\mathscr{D}$, denoted by $\mathrm{L}(\mathscr{D})$, to be the image of the subspace*

$$\mathrm{L}^1(\mathscr{D}) := \mathrm{Hom}_k(\mathfrak{m}_{\mathsf{R}_{\bar{r}}^{\mathrm{loc}}}/(\mathfrak{m}_{\mathsf{R}_{\bar{r}}^{\mathrm{loc}}}^2, \mathscr{I}, \lambda), k) \subseteq \mathrm{Z}^1(\tilde{\Gamma}, \mathrm{ad}\,\bar{r})$$

under the natural map $\mathrm{Z}^1(\tilde{\Gamma}, \mathrm{ad}\,\bar{r}) \to \mathrm{H}^1(\tilde{\Gamma}, \mathrm{ad}\,\bar{r})$, where $\mathscr{I} \subseteq \mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ is the closed ideal defining $\mathscr{D}$.

Note that we have the identity

$$\dim_k \mathrm{L}^1(\mathscr{D}) = N^2 + \dim_k \mathrm{L}(\mathscr{D}) - \dim_k \mathrm{H}^0(\tilde{\Gamma}, \mathrm{ad}\,\bar{r}). \tag{3.1}$$

Remark 3.5 Later, when we consider a nonarchimedean place v of F^+ and take $\tilde{\Gamma} = \Gamma_{F_v^+}$, the subgroup Γ we implicitly take is always $\Gamma_{F_v^+} \cap \Gamma_F$.

Now we apply Notation 3.1 and Definition 3.2 to the case where $\tilde{\Gamma} = \Gamma_{F^+}$ and $\Gamma = \Gamma_F$.

Definition 3.6 *A* global deformation problem *is a tuple $(\bar{r}, \chi, \mathrm{S}, \{\mathscr{D}_v\}_{v \in \mathrm{S}})$, where*

• $(\bar{r}, \chi)$ *is a pair as in Notation 3.1;*

• S *is a finite set of nonarchimedean places of F^+ containing all ℓ-adic places and those places v such that $\bar{r}_v$ is ramified;*

• $\mathscr{D}_v$ *is a local deformation problem of $\bar{r}_v$ (Remark 3.5) for each $v \in \mathrm{S}$.*

We take a global deformation problem $\mathscr{S} := (\bar{r}, \chi, \mathrm{S}, \{\mathscr{D}_v\}_{v \in \mathrm{S}})$. For $v \in \mathrm{S}$, we denote by $\mathscr{I}_v$ the closed ideal of $\mathsf{R}_{\bar{r}_v}^{\mathrm{loc}}$ defining $\mathscr{D}_v$. For a subset $\mathrm{T} \subseteq \mathrm{S}$, put

$$\mathsf{R}_{\mathscr{S},\mathrm{T}}^{\mathrm{loc}} := \widehat{\bigotimes_{v \in \mathrm{T}}} \mathsf{R}_{\bar{r}_v}^{\mathrm{loc}}/\mathscr{I}_v, \tag{3.2}$$

where the completed tensor product is taken over $\mathscr{O}$. Recall from [14, Definition 2.2.1] that a T-*framed lifting* of $\bar{r}$ to an object R of $\mathscr{C}_{\mathscr{O}}$ is a tuple $(r; \{\beta_v\}_{v \in \mathrm{T}})$, where r is a lifting of $\bar{r}$ to R (Definition 3.2), and $\beta_v \in 1 + \mathrm{M}_N(\mathfrak{m}_R)$ for $v \in \mathrm{T}$. Two T-framed liftings $(r; \{\beta_v\}_{v \in \mathrm{T}})$ and $(r'; \{\beta_v'\}_{v \in \mathrm{T}})$ of $\bar{r}$ to R are said to be equivalent, if there exists $x \in 1 + \mathrm{M}_N(\mathfrak{m}_R)$ such that $r' = x^{-1} \circ r \circ x$ and $\beta_v' = x^{-1}\beta_v$ for every $v \in \mathrm{T}$. A T-*framed deformation* of $\bar{r}$ is an equivalence class of T-framed liftings of $\bar{r}$. We say that a T-framed lifting $(r; \{\beta_v\}_{v \in \mathrm{T}})$ is *of type $\mathscr{S}$* if r_v belongs to $\mathscr{D}_v$ for every $v \in \mathrm{S}$, and is unramified for every $v \notin \mathrm{S}$. Note that being of type $\mathscr{S}$ is a property invariant under the conjugate action by $1 + \mathrm{M}_N(\mathfrak{m}_R)$. Thus it makes sense to speak of T-framed deformation of type $\mathscr{S}$. Let $\mathrm{Def}_{\mathscr{S}}^{\square_{\mathrm{T}}} : \mathscr{C}_{\mathscr{O}} \to \mathsf{Set}$ be the functor that sends an object R to the set of T-framed deformations of $\bar{r}$ to R of type $\mathscr{S}$.

Let $\Gamma_{F^+,\mathrm{S}}$ be the Galois group of the maximal sub-extension of $\overline{F}/F^+$ that is unramified outside S. Recall the cohomology group $\mathrm{H}^i_{\mathscr{S},\mathrm{T}}(\Gamma_{F^+,\mathrm{S}}, \mathrm{ad}\,\bar{r})$ for $i \geq 0$ introduced after [14, Definition 2.2.7]. By [14, Lemma 2.3.4], these are finite dimensional k-vector spaces, and satisfy $\mathrm{H}^i_{\mathscr{S},\mathrm{T}}(\Gamma_{F^+,\mathrm{S}}, \mathrm{ad}\,\bar{r}) = 0$ for $i > 3$.

Proposition 3.7 *Assume that $\bar{r}|_{\Gamma_F}$ is absolutely irreducible. Then for every subset $\mathrm{T} \subseteq \mathrm{S}$, the functor $\mathrm{Def}^{\Box_\mathrm{T}}_{\mathscr{S}}$ is represented by a Noetherian ring $\mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}$ in $\mathscr{C}_{\mathscr{O}}$. Put $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}} := \mathsf{R}^{\Box_\emptyset}_{\mathscr{S}}$.*

(1) There is a canonical isomorphism

$$\mathrm{Hom}_k(\mathfrak{m}_{\mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}}/(\mathfrak{m}^2_{\mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}}, \lambda, \mathfrak{m}_{\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}}), k) \simeq \mathrm{H}^1_{\mathscr{S},\mathrm{T}}(\Gamma_{F^+,\mathrm{S}}, \mathrm{ad}\,\bar{r}),$$

where we regard $\mathfrak{m}_{\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}}$ as its image under the tautological homomorphism $\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}} \to \mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}$. Moreover, if $\mathrm{H}^2_{\mathscr{S},\mathrm{T}}(\Gamma_{F^+,\mathrm{S}}, \mathrm{ad}\,\bar{r}) = 0$ and for $v \in \mathrm{S} \setminus \mathrm{T}$, $\mathscr{D}_v$ is formally smooth over $\mathscr{O}$, then $\mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}$ is a power series ring over $\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}$ in $\dim_k \mathrm{H}^1_{\mathscr{S},\mathrm{T}}(\Gamma_{F^+,\mathrm{S}}, \mathrm{ad}\,\bar{r})$ variables.

(2) The choice of a lifting $r^{\mathrm{univ}}_{\mathscr{S}} : \Gamma_{F^+} \to \mathscr{G}_N(\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}})$ in the universal deformation determines an extension of the tautological homomorphism $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}} \to \mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}$ to an isomorphism

$$\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}[[X_{v;i,j}]]_{v \in \mathrm{T}; 1 \leq i,j \leq N} \xrightarrow{\sim} \mathsf{R}^{\Box_\mathrm{T}}_{\mathscr{S}}$$

such that, for every $v \in \mathrm{T}$, the universal frame at v is given by $\beta_v = 1 + (X_{v;i,j})_{1 \leq i,j \leq N}$.

Proof These are exactly [14, Proposition 2.2.9 & Corollary 2.2.13] except that they consider only local deformation problems at split places (that is, they assume that all places in S are split in F). However, the same argument can be applied to the general case without change. $\square$

3.2 Fontaine–Laffaille Deformations

In this subsection, we study Fontaine–Laffaille deformations at ℓ-adic places. We take a place v of F^+ above ℓ; and let w be the place of F above v induced by the inclusion $F \subseteq \overline{F}^+_v$. We assume that ℓ is unramified in F, and denote by $\sigma \in \mathrm{Gal}(F_w/\mathbb{Q}_\ell)$ the absolute Frobenius element.

Assumption 3.8 The field E contains the image of every embedding of F_w into $\overline{\mathbb{Q}}_\ell$.

We first assume that E satisfies Assumption 3.8. Put $\Sigma_w := \mathrm{Hom}_{\mathbb{Z}_\ell}(O_{F_w}, \mathscr{O})$. Following [14], we use a covariant version of the Fontaine–Laffaille theory [15]. Let $\mathscr{MF}_{\mathscr{O},w}$ be the category of $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}$-modules M of finite length equipped with

• a decreasing filtration $\{\mathrm{Fil}^i M\}_{i \in \mathbb{Z}}$ by $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}$-submodules that are O_{F_w}-direct summands, satisfying $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{\ell-1} M = 0$, and

• a Frobenius structure, that is, $\sigma \otimes 1$-linear maps $\Phi^i : \mathrm{Fil}^i M \to M$ for $i \in \mathbb{Z}$, satisfying the relations $\Phi^i|_{\mathrm{Fil}^{i+1} M} = \ell\Phi^{i+1}$ and $\sum_{i \in \mathbb{Z}} \Phi^i \mathrm{Fil}^i M = M$.

Let $\mathscr{MF}_{k,w}$ be the full subcategory of $\mathscr{MF}_{\mathscr{O},w}$ of objects that are annihilated by λ. For every integer b satisfying $0 \leq b \leq \ell - 2$, let $\mathscr{MF}^{[0,b]}_{\mathscr{O},w}$ be the full subcategory of $\mathscr{MF}_{\mathscr{O},w}$ consisting of objects M satisfying $\mathrm{Fil}^{b+1} M = 0$. In particular, we have $\mathscr{MF}^{[0,\ell-2]}_{\mathscr{O},w} = \mathscr{MF}_{\mathscr{O},w}$ by definition.

For an object M of $\mathscr{MF}_{\mathscr{O},w}$, there is a canonical decomposition

$$M = \bigoplus_{\tau \in \Sigma_w} M_\tau,$$

where $M_\tau \coloneqq M \otimes_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \tau \otimes 1} \mathscr{O}$. Then we have $\operatorname{Fil}^i M = \bigoplus_{\tau \in \Sigma_w} \operatorname{Fil}^i M_\tau$ with $\operatorname{Fil}^i M_\tau = M_\tau \cap \operatorname{Fil}^i M$, and that Φ^i induces $\mathscr{O}$-linear maps

$$\Phi_\tau^i \colon \operatorname{Fil}^i M_\tau \to M_{\tau \circ \sigma^{-1}}.$$

We put

$$\operatorname{gr}^i M_\tau \coloneqq \operatorname{Fil}^i M_\tau / \operatorname{Fil}^{i+1} M_\tau,$$
$$\operatorname{gr}^\bullet M_\tau \coloneqq \bigoplus_i \operatorname{gr}^i M_\tau,$$
$$\operatorname{gr}^\bullet M \coloneqq \bigoplus_{\tau \in \Sigma_w} \operatorname{gr}^\bullet M_\tau.$$

We define the set of τ-*Fontaine–Laffaille weights* of M to be

$$\operatorname{HT}_\tau(M) \coloneqq \{i \in \mathbb{Z} \mid \operatorname{gr}^i M_\tau \neq 0\}.$$

We say that M has *regular Fontaine–Laffaille weights* if $\operatorname{gr}^i M_\tau$ is generated over $\mathscr{O}$ by at most one element for every $\tau \in \Sigma_w$ and every $i \in \mathbb{Z}$.

Let $\mathscr{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}$ be the category of $\mathscr{O}$-modules of finite length equipped with a continuous action of Γ_{F_w}. In [14, 2.4.1], the authors defined an exact fully faithful, covariant $\mathscr{O}$-linear functor

$$\mathbf{G}_w \colon \mathscr{MF}_{\mathscr{O},w} \to \mathscr{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}$$

whose essential image is closed under taking sub-objects and quotient objects. The length of an object M in $\mathscr{MF}_{\mathscr{O},w}$ as an $\mathscr{O}$-module equals $[F_w : \mathbb{Q}_\ell]$ times the length of $\mathbf{G}_w(M)$ as an $\mathscr{O}$-module. For two objects M_1, M_2 of $\mathscr{MF}_{\mathscr{O},w}$, we have a canonical isomorphism

$$\operatorname{Hom}_{\mathscr{MF}_{\mathscr{O},w}}(M_1, M_2) \xrightarrow{\sim} \mathrm{H}^0(F_w, \operatorname{Hom}_\mathscr{O}(\mathbf{G}_w(M_1), \mathbf{G}_w(M_2)))$$

and a canonical injective map

$$\operatorname{Ext}^1_{\mathscr{MF}_{\mathscr{O},w}}(M_1, M_2) \hookrightarrow \operatorname{Ext}^1_{\mathscr{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}}(\mathbf{G}_w(M_1), \mathbf{G}_w(M_2)),$$

where the target is canonically isomorphic to $\mathrm{H}^1(F_w, \operatorname{Hom}_k(\mathbf{G}_w(M_1), \mathbf{G}_w(M_2)))$ if M_1 and M_2 are both objects of $\mathscr{MF}_{k,w}$.

Example 3.9 For an integer b satisfying $0 \leq b \leq \ell - 2$ and an object R of $\mathscr{C}_\mathscr{O}^f$, we have an object $R\{b\}$ of $\mathscr{MF}_{\mathscr{O},w}$ defined as follows: the underlying $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}$-module is simply $(O_{F_w} \otimes_{\mathbb{Z}_\ell} R)e_b$, with the filtration given by

$$\operatorname{Fil}^i R\{b\} = \begin{cases} (O_{F_w} \otimes_{\mathbb{Z}_\ell} R)e_b & \text{if } i \leq b; \\ 0 & \text{if } i > b. \end{cases}$$

Finally, the Frobenius structure is determined by $\Phi^b(e_b) = e_b$. Then we have

$$\mathbf{G}_w(R\{b\}) \simeq R(-b)|_{\Gamma_{F_w}}$$

as $\mathscr{O}[\Gamma_{F_w}]$-modules.

Construction 3.10 We construct a functor $-^\sigma \colon \mathscr{MF}_{\mathscr{O},w} \to \mathscr{MF}_{\mathscr{O},w}$ as follows: for an object M of $\mathscr{MF}_{\mathscr{O},w}$, the underlying $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}$-module of M^σ is $O_{F_w} \otimes_{O_{F_w},\sigma} M$ with the induced filtration and Frobenius structure. Then we have $M_\tau^\sigma = M_{\tau \circ \sigma^{-1}}$ for every $\tau \in \Sigma_w$, and that

$\mathbf{G}_w(M^\sigma)$ is isomorphic to $\mathbf{G}_w(M)$ but with the action of Γ_{F_w} twisted by the absolute Frobenius of F_w: if we denote by ρ and ρ_σ the actions of Γ_{F_w} on $\mathbf{G}_w(M)$ and $\mathbf{G}_w(M^\sigma)$, respectively, then they satisfy

$$\rho_\sigma(g) = \rho(\tilde{\sigma}^{-1} g \tilde{\sigma}),$$

where $\tilde{\sigma} \in \mathrm{Gal}(\overline{F}_v^+/\mathbb{Q}_\ell)$ is a lift of the absolute Frobenius.

We now let $\mathbf{-}^{\mathsf{c}} \colon \mathscr{MF}_{\mathscr{O},w} \to \mathscr{MF}_{\mathscr{O},w}$ be the $[F_v^+ : \mathbb{Q}_\ell]$-th iteration of the functor $\mathbf{-}^\sigma$ constructed above.

For an object R of $\mathscr{C}_{\mathscr{O}}^f$ and an integer b with $0 \le b \le \ell - 2$, let $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$ be the full subcategory of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}$ consisting of objects M such that M is finite free over $O_{F_w} \otimes_{\mathbb{Z}_\ell} R$ and that the $O_{F_w} \otimes_{\mathbb{Z}_\ell} R$-submodule $\mathrm{Fil}^i M$ is a direct summand for every i. Let M be an object of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$. Then $\mathbf{G}_w(M)$ is finite free over R. Thus $\mathbf{G}_w$ induces a functor $\mathbf{G}_w \colon \mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R) \to R[\Gamma_{F_w}]^{\mathrm{f.r.}}$, where $R[\Gamma_{F_w}]^{\mathrm{f.r.}}$ denotes the category of finite free R-modules equipped with a continuous action by Γ_{F_w}. We have a functor

$$\mathbf{-}^\vee\{b\} \colon \mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)^{\mathrm{op}} \to \mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$$

defined as follows: for an object M of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$, $M^\vee\{b\}$ is the object of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$ such that:
- its underlying $O_{F_w} \otimes_{\mathbb{Z}_\ell} R$-module is $\mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} R}(M, O_{F_w} \otimes_{\mathbb{Z}_\ell} R)$,
- $\mathrm{Fil}^i M^\vee\{b\} = \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} R}(M/\mathrm{Fil}^{b+1-i} M, O_{F_w} \otimes_{\mathbb{Z}_\ell} R)$,
- for every $f \in \mathrm{Fil}^i M^\vee\{b\}$ and every $m \in \mathrm{Fil}^j M$, we have

$$\Phi^i(f)(\Phi^j(m)) = \begin{cases} \ell^{b-i-j} f(m)^\sigma & \text{if } i + j \le b; \\ 0 & \text{if } i + j > b. \end{cases}$$

It is clear that $M^\vee\{b\}$ is a well-defined object of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$ (see [14, p. 34]), and that $\mathbf{G}_w(M^\vee\{b\}) = \mathbf{G}_w(M)^\vee(-b)$.

Now suppose that we have an isomorphism of $R[\Gamma_{F_w}]$-modules $\mathbf{G}_w(M^{\mathsf{c}}) \simeq \mathbf{G}_w(M)^\vee(-b)$. Since the functor $\mathbf{G}_w$ is fully faithful, giving such an isomorphism is equivalent to giving an isomorphism $M^{\mathsf{c}} \simeq M^\vee\{b\}$ in $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$, or equivalently an $O_{F_w} \otimes_{\mathbb{Z}_\ell} R$-bilinear perfect pairing

$$\langle\,,\,\rangle \colon M^{\mathsf{c}} \times M \to O_{F_w} \otimes_{\mathbb{Z}_\ell} R, \tag{3.3}$$

such that the induced R-bilinear perfect pairings $\langle\,,\,\rangle_\tau \colon M_{\tau^{\mathsf{c}}} \times M_\tau \to R$ for $\tau \in \Sigma_w$ satisfy the following conditions:

(1) For every $i, j \in \mathbb{Z}$, every $x \in \mathrm{Fil}^i M_\tau$, and every $y \in \mathrm{Fil}^j M_\tau$, $\langle \Phi_{\tau^{\mathsf{c}}}^i x, \Phi_\tau^j y \rangle_{\tau \circ \sigma^{-1}}$ equals $\ell^{b-i-j} \langle x, y \rangle_\tau$ (resp. 0) if $i + j \le b$ (resp. $i + j > b$).

(2) For every $i \in \mathbb{Z}$, the annihilator of $\mathrm{Fil}^i M_\tau$ under $\langle\,,\,\rangle_\tau$ is $\mathrm{Fil}^{b+1-i} M_{\tau^{\mathsf{c}}}$; in particular, $\langle\,,\,\rangle_\tau$ induces an R-linear isomorphism $\mathrm{gr}^i M_\tau \simeq \mathrm{Hom}_R(\mathrm{gr}^{b-i} M_{\tau^{\mathsf{c}}}, R)$.

From now on, E will not necessarily be subject to Assumption 3.8.

Definition 3.11 *Let R be an object of $\mathscr{C}_{\mathscr{O}}$, and $\rho \colon \Gamma_{F_w} \to \mathrm{GL}_N(R)$ a continuous representation.*

(1) Let a, b be integers satisfying $0 \le b - a \le \ell - 2$. For E satisfying Assumption 3.8, we say that ρ is crystalline with (regular) Fontaine–Laffaille weights in $[a, b]$ *if, for every quotient R' of R in $\mathscr{C}_{\mathscr{O}}^f$, $\rho(a) \otimes_R R'$ lies in the essential image of the functor $\mathbf{G}_w \colon \mathscr{MF}_{\mathscr{O},w}^{[0,b-a]}(R') \to \mathscr{O}[\Gamma_{F_w}]^{\mathrm{f.l.}}$ (and that $\mathbf{G}_w^{-1}(\rho(a) \otimes_R R')$ has regular Fontaine–Laffaille weights).*

(2) *For E in general, we say that ρ is* regular Fontaine–Laffaille crystalline *if there exists a finite unramified extension $E_\dagger$ of E contained in $\overline{\mathbb{Q}}_\ell$ that satisfies Assumption 3.8 with the ring of integer $\mathscr{O}_\dagger$, such that $\rho \otimes_{\mathscr{O}} \mathscr{O}_\dagger$ is crystalline with regular Fontaine–Laffaille weights in $[a, b]$ in the sense of (1) for some integers a, b satisfying $0 \le b - a \le \ell - 2$.*

Now we consider a pair $(\bar{r}, \chi)$ from Notation 3.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F = \Gamma_{F_w}$.

Definition 3.12   *Suppose that $\bar{r}^\natural$ is regular Fontaine–Laffaille crystalline. We define $\mathscr{D}^{\mathrm{FL}}$ to be the local deformation problem of $\bar{r}$ that classifies the liftings $r \colon \Gamma_{F_v^+} \to \mathscr{G}_N(R)$ of $\bar{r}$ to objects R of $\mathscr{C}_{\mathscr{O}}$ such that $r^\natural$ is regular Fontaine–Laffaille crystalline.*

Lemma 3.13   *Suppose that $\bar{r}^\natural$ is regular Fontaine–Laffaille crystalline and that $\chi = \eta_v^\mu \epsilon_{\ell,v}^c$ for some $c \in \mathbb{Z}$ and $\mu \in \mathbb{Z}/2\mathbb{Z}$. Then*

$$\dim_k \mathrm{L}(\mathscr{D}^{\mathrm{FL}}) - \dim_k \mathrm{H}^0(F_v^+, \mathrm{ad}\,\bar{r}) = [F_v^+ : \mathbb{Q}_\ell] \cdot \frac{N(N-1)}{2}.$$

Proof   For this lemma, we may assume that E satisfies Assumption 3.8. After replacing $\bar{r}$ by $\bar{r}(a)$ for some integer a, we may assume that $\bar{r}^\natural$ is crystalline with regular Fontaine–Laffaille weights in $[0, b]$ with $0 \le b \le \ell - 2$. In this case, $\mathscr{D}^{\mathrm{FL}}$ simply classifies liftings r such that $r^\natural$ is crystalline with Fontaine–Laffaille weights in $[0, b]$.

Suppose first that v is split in F. Then we have $F_w = F_v^+$, and that a lifting r in $\mathscr{D}^{\mathrm{FL}}(R)$ of $\bar{r}$ is of the form $r = (\rho, \epsilon_{\ell,v}^c) \colon \Gamma_{F_w} \to \mathrm{GL}_N(R) \times R^\times$ such that for every Artinian quotient R' of R, $\rho \otimes_R R'$ lies in the essential image of the functor $\mathbf{G}_w$. Then the lemma is exactly [14, Corollary 2.4.3].

Suppose now that v is inert in F. Then we must have $c = -b$. Denote by $\Gamma_{w/v}$ the Galois group of the quadratic extension F_w/F_v^+. Then the restriction map induces an isomorphism

$$\mathrm{H}^1(F_v^+, \mathrm{ad}\,\bar{r}) \xrightarrow{\sim} \mathrm{H}^1(F_w, \mathrm{ad}\,\bar{r})^{\Gamma_{w/v}}.$$

Put $M := \mathbf{G}_w^{-1}(\bar{r}^\natural)$. Then the deformations of $\bar{r}$ to $k[\varepsilon]/(\varepsilon^2)$ that lie in the essential image of $\mathbf{G}_w$ are classified by $\mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)$, which is canonically a $\Gamma_{w/v}$-stable subspace of $\mathrm{H}^1(F_w, \mathrm{ad}\,\bar{r})$. Therefore, we have

$$\mathrm{L}(\mathscr{D}^{\mathrm{FL}}) = \mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M) \cap \mathrm{H}^1(F_w, \mathrm{ad}\,\bar{r})^{\Gamma_{w/v}} = \mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)^{\Gamma_{w/v}}.$$

In fact, the induced action of $\Gamma_{w/v}$ on $\mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)$ can be described as follows. Recall the functor $\mathbf{-}^{\mathsf{c}}$ in Construction 3.10. Then $\mathbf{G}_w(M^{\mathsf{c}})$ is isomorphic to $\bar{r}^{\natural,\mathsf{c}}|_{\Gamma_{F_w}}$. Since $\bar{r}^{\natural,\mathsf{c}}$ and $\bar{r}^{\natural,\vee}(-b)$ are conjugate, we have $M^{\mathsf{c}} \simeq M^\vee\{b\}$. We fix such an isomorphism, hence obtain a pairing $\langle\ ,\ \rangle$ (3.3) with $R = k$. Then for an element $[P] \in \mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)$ represented by an extension $0 \to M \to P \to M \to 0$, the image of $[P]$ under the action of the (unique) non-trivial element in $\Gamma_{w/v}$ is obtained by applying the functor $(\mathbf{-}^{\mathsf{c}})^\vee\{b\}$ to $0 \to M \to P \to M \to 0$.

To compute $\mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)^{\Gamma_{w/v}}$, we recall first the following long exact sequence in [14, Lemma 2.4.2]:

$$0 \to \mathrm{End}_{\mathscr{MF}_{k,w}}(M) \to \mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)$$
$$\xrightarrow{\alpha} \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M)$$
$$\xrightarrow{\beta} \mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M) \to 0, \tag{3.4}$$

where

- $\bullet$ $\mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)$ denotes the $O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}$-submodule of $\mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)$ of endomorphisms that preserve the filtration;
- $\bullet$ the map α takes an element $f \in \mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)$ to $(f \Phi^i - \Phi^i f)_{i \in \mathbb{Z}}$; and
- $\bullet$ the map β is defined as follows: if $\varphi = (\varphi^i)_{i \in \mathbb{Z}}$ is a $\sigma \otimes 1$-linear map from $\mathrm{gr}^\bullet M$ to M, then $\beta(\varphi)$ is given by the extension class of $E = M \oplus M$ with the filtration $\mathrm{Fil}^i E = \mathrm{Fil}^i M \oplus \mathrm{Fil}^i M$ and the Frobenius structure

$$\Phi_E^i := \begin{pmatrix} \Phi^i & \varphi^i \\ 0 & \Phi^i \end{pmatrix}.$$

To prove the lemma, we need to derive an analogous long exact sequence similar to (3.4) but with the last term $\mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)^{\Gamma_{w/v}}$. For the first term, note that we have a canonical isomorphism $\mathrm{End}_{\mathscr{MF}_{k,w}}(M) \simeq \mathrm{H}^0(F_w, \mathrm{ad}\,\bar{r})$, which contains $\mathrm{H}^0(F_v^+, \mathrm{ad}\,\bar{r})$ as a submodule. For the second term, let $\mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)^+$ be the submodule of $\mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)$ consisting of elements $f = (f_\tau)_{\tau \in \Sigma_w}$ such that $-f_{\tau^c}$ is the adjoint of f_τ under the pairing $\langle\ ,\ \rangle_\tau$ for every $\tau \in \Sigma_w$. For the third term, let $\mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M)^+$ denote by the submodule of $\mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M)$ consisting of $\varphi = (\varphi^i)_{i \in \mathbb{Z}}$ such that

$$\langle \Phi_{\tau^c}^i(x), \varphi_\tau^{b-i}(y) \rangle_\tau + \langle \varphi_{\tau^c}^i(x), \Phi_\tau^{b-i}(y) \rangle_\tau = 0 \tag{3.5}$$

is satisfied for every $x \in \mathrm{gr}^i M_{\tau^c}$ and $y \in \mathrm{gr}^{b-i} M_\tau$.

Then (3.4) induces an exact sequence

$$0 \to \mathrm{H}^0(F_v^+, \mathrm{ad}\,\bar{r}) \to \mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)^+$$
$$\xrightarrow{\alpha} \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M)^+$$
$$\xrightarrow{\beta} \mathrm{Ext}^1_{\mathscr{MF}_{k,w}}(M, M)^{\Gamma_{w/v}} \to 0$$

of k-vector spaces. We now compute the dimension of the middle two terms. From the description of $\mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)^+$, it is clear that f_{τ^c} is determined by f_τ for every $\tau \in \Sigma_w$. On the other hand, for each fixed τ, all possible choices of f_τ form a k-vector space of dimension $\frac{N(N+1)}{2}$. Thus, we have

$$\dim_k \mathrm{Fil}^0 \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}}(M, M)^+ = [F_v^+ : \mathbb{Q}_\ell] \cdot \frac{N(N+1)}{2}.$$

For $\mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M)^+$, we first note that the map

$$\bigoplus_i \Phi_\tau^i \colon \mathrm{gr}^\bullet M_\tau \to M_{\tau \circ \sigma^{-1}}$$

is an isomorphism for every $\tau \in \Sigma_w$. It follows from (3.5) that $\varphi_{\tau^c} := \bigoplus_i \varphi_{\tau^c}^i$ is determined by $\varphi_\tau := \bigoplus_i \varphi_\tau^i$. On the other hand, for each fixed τ, all the possible choices of $\varphi_\tau \colon \mathrm{gr}^\bullet M_\tau \to M_{\tau \circ \sigma^{-1}}$ form a k-vector space of dimension N^2. Thus, we have

$$\dim_k \mathrm{Hom}_{O_{F_w} \otimes_{\mathbb{Z}_\ell} \mathscr{O}, \sigma \otimes 1}(\mathrm{gr}^\bullet M, M)^+ = [F_v^+ : \mathbb{Q}_\ell] \cdot N^2.$$

The lemma follows immediately. $\qquad\square$

Proposition 3.14 *Suppose that $\bar{r}^\natural$ is regular Fontaine–Laffaille crystalline and that $\chi = \eta_v^\mu \epsilon_{\ell,v}^c$ for some $c \in \mathbb{Z}$ and $\mu \in \mathbb{Z}/2\mathbb{Z}$. Then the local deformation problem $\mathscr{D}^{\mathrm{FL}}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension $N^2 + [F_v^+ : \mathbb{Q}_\ell] \cdot \frac{N(N-1)}{2}$.*

Proof By Lemma 3.13, it suffices to show that $\mathscr{D}^{\mathrm{FL}}$ is formally smooth over $\mathscr{O}$. For this, we may again assume that E satisfies Assumption 3.8. Moreover, we may assume that $\bar{r}^{\natural}$ is crystalline with regular Fontaine–Laffaille weights in $[0, b]$ with $0 \leq b \leq \ell - 2$ as in the proof of Lemma 3.13. In this case, $\mathscr{D}^{\mathrm{FL}}$ simply classifies liftings r such that $r^{\natural}$ is crystalline with Fontaine–Laffaille weights in $[0, b]$.

When v is split in F/F^{+}, the proposition has been proved in [14, Lemma 2.4.1].

Now we suppose that v is inert in F. Then we must have $c = -b$. Fix a subset $\Sigma_w^{+} \subset \Sigma_w$ such that $\Sigma_w = \Sigma_w^{+} \coprod \Sigma_w^{+,\mathsf{c}}$. Let R be an object of $\mathscr{C}_{\mathscr{O}}^{f}$ and $I \subset R$ an ideal satisfying $\mathfrak{m}_R I = (0)$. Let r be a lifting of $\bar{r}$ to R/I, and put $M := \mathbf{G}_w^{-1}(r^{\natural})$, which is an object of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R/I)$.

Recall the functor $-^{\mathsf{c}}$ in Construction 3.10. Then $\mathbf{G}_w(M^{\mathsf{c}})$ is isomorphic to $r^{\natural,\mathsf{c}}|_{\Gamma_{F_w}}$. Since $r^{\natural,\mathsf{c}}$ and $r^{\natural,\vee}(-b)$ are conjugate, we have $M^{\mathsf{c}} \simeq M^{\vee}\{b\}$. We fix such an isomorphism, hence obtain a pairing $\langle\ ,\ \rangle$ (3.3) with $R = R/I$. Let $m_{\tau,1} < \cdots < m_{\tau,N}$ be the (regular) τ-Hodge–Tate weights of M for every $\tau \in \Sigma_w$. Then there exists a basis $e_{\tau,1}, \ldots, e_{\tau,N}$ of M_{τ} over R/I satisfying $\mathrm{Fil}^{m_{\tau,N+1-i}} M_{\tau} = \bigoplus_{j=1}^{i} (R/I) e_{\tau,j}$ for every $1 \leq i \leq N$. By duality, we have $m_{\tau^{\mathsf{c}},i} + m_{\tau,N+1-i} = b$. Then we may choose the basis $(e_{\tau,i})$ such that $\langle e_{\tau^{\mathsf{c}},i}, e_{\tau,j} \rangle_{\tau} = \delta_{i,N+1-j}$ for every $\tau \in \Sigma_w^{+}$ and every $1 \leq i, j \leq N$.

We now define an object $\widetilde{M} = \bigoplus_{\tau \in \Sigma_w} \widetilde{M}_{\tau}$ of $\mathscr{MF}_{\mathscr{O},w}^{[0,b]}(R)$ that reduces to M, together with a perfect pairing $\widetilde{M}^{\mathsf{c}} \times \widetilde{M} \to O_{F_w} \otimes_{\mathbb{Z}_{\ell}} R$ as in (3.3) that reduces to the pairing $\langle\ ,\ \rangle$, as follows. As an R-module, we take $\widetilde{M}_{\tau} = R^{\oplus N}$ with the basis $(\widetilde{e}_{\tau,i})$ that lifts the basis $(\widetilde{e}_{\tau,i})$ of M_{τ}. We lift $\langle\ ,\ \rangle_{\tau}$ to an R-bilinear perfect pairing $\widetilde{M}_{\tau^{\mathsf{c}}} \times \widetilde{M}_{\tau} \to R$ such that $\langle \widetilde{e}_{\tau^{\mathsf{c}},i}, \widetilde{e}_{\tau,j} \rangle_{\tau} = \delta_{i,N+1-j}$ still holds for every $\tau \in \Sigma_w^{+}$ and every $1 \leq i, j \leq N$. For the filtration, we put $\mathrm{Fil}^m \widetilde{M}_{\tau} := \bigoplus_{j=1}^{i} R\widetilde{e}_{\tau,j}$ for m satisfying $m_{\tau,N-i} < m \leq m_{\tau,N+1-i}$. Then $\widetilde{M} \otimes_R R/I$ is isomorphic to M as filtered $O_{F_w} \otimes R/I$-modules; and the condition (2) in Construction 3.10 holds for $\widetilde{M}$ as well. For the Frobenius structure on $\widetilde{M}$, we first define maps $\widetilde{\Phi}_{\tau}^{m_{\tau,i}} \colon \mathrm{Fil}^{m_{\tau,i}} \widetilde{M}_{\tau} \to \widetilde{M}_{\tau \circ \sigma^{-1}}$ for $\tau \in \Sigma_w^{+}$ by the recursive induction on i. For $i = N$, we take $\widetilde{\Phi}_{\tau}^{m_{\tau,N}}$ to be an arbitrary lift of $\Phi_{\tau}^{m_{\tau,N}} \colon \mathrm{Fil}^{m_{\tau,N}} M_{\tau} \to M_{\tau \circ \sigma^{-1}}$ for $\tau \in \Sigma_w^{+}$. For $i \leq N - 1$, we take $\widetilde{\Phi}_{\tau}^{m_{\tau,i}}$ to be a lift of $\Phi_{\tau}^{m_{\tau,i}} \colon \mathrm{Fil}^{m_{\tau,i}} M_{\tau} \to M_{\tau \circ \sigma^{-1}}$ that restricts to $\ell^{m_{\tau,i} - m_{\tau,i+1}} \widetilde{\Phi}_{\tau}^{m_{\tau,i+1}}$ on $\mathrm{Fil}^{m_{\tau,i+1}} \widetilde{M}_{\tau}$. By Nakayama's lemma, we have

$$\widetilde{M}_{\tau \circ \sigma^{-1}} = \sum_{i} \widetilde{\Phi}_{\tau}^{m_{\tau,i}}(\mathrm{Fil}^{m_{\tau,i}} \widetilde{M}_{\tau})$$

for every $\tau \in \Sigma_w^{+}$. Finally, we define $\widetilde{\Phi}_{\tau^{\mathsf{c}}}^{i} \colon \widetilde{M}_{\tau^{\mathsf{c}}} \to \widetilde{M}_{\tau^{\mathsf{c}} \circ \sigma^{-1}}$ for $\tau \in \Sigma_w^{+}$ to be the unique R-linear map satisfying the condition (1) in Construction 3.10 for $\widetilde{M}$. This finishes the construction of $\widetilde{M}$ together with an isomorphism $\widetilde{M}^{\mathsf{c}} \simeq \widetilde{M}^{\vee}\{b\}$, which give rise to a lifting $\widetilde{r}$ of $\bar{r}$ to R that reduces to r by Lemma 2.3. Thus, $\mathscr{D}^{\mathrm{FL}}$ is formally smooth over $\mathscr{O}$.

The proposition is proved. $\square$

At the end, we remark that in the self-dual (not conjugate self-dual) case, the Fontaine–Laffaille deformations have been studied in [4].

3.3   Representations of the Tame Group

In this subsection, we study conjugate self-dual representations of the tame group, and define the notion of minimally ramified deformations of such representations.

Definition 3.15 *Let $q \geq 1$ be a positive integer coprime to ℓ. We define the q-tame group, denoted by T_q, to be the semidirect product topological group $t^{\mathbb{Z}_\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$ where ϕ_q maps t to t^q, that is, $\phi_q t \phi_q^{-1} = t^q$. For every integer $b \geq 1$, we identify T_{q^b} as a subgroup of T_q topologically generated by t and $\phi_{q^b} = \phi_q^b$.*

We consider a reductive group G over $\mathscr{O}$, together with a surjective homomorphism $\nu \colon G \to H$ over $\mathscr{O}$, where H is an algebraic group over $\mathscr{O}$ of multiplicative type. Consider a pair $(\bar{\varrho}, \mu)$ in which $\bar{\varrho} \colon \mathrm{T}_q \to G(k)$ and $\mu \colon \mathrm{T}_q \to H(\mathscr{O})$ are continuous homomorphisms satisfying $\nu \circ \bar{\varrho} = \bar{\mu}$ and $\mu(t) = 1$. Similar to the case in §3.1, let $\mathsf{R}_{\bar{\varrho}}^{\mathrm{loc}}$ be the object in $\mathscr{C}_{\mathscr{O}}$ that parameterizes liftings ϱ of $\bar{\varrho}$ satisfying $\nu \circ \varrho = \mu$.[4] The following proposition generalizes the tame case of [26, Theorem 2.5].

Proposition 3.16 *The ring $\mathsf{R}_{\bar{\varrho}}^{\mathrm{loc}}$ is a local complete intersection, flat and of pure relative dimension d over $\mathscr{O}$, where d is the relative dimension of the kernel of ν over $\mathscr{O}$.*

Proof We follow the same line as in the proof of [26, Theorem 2.5]. Let G_0 and G_1 be the fibers at 1 and $\mu(\phi_q)$ of the homomorphism ν, respectively. Define the subscheme $\mathscr{M}(G, q)$ of $G_0 \times_{\mathrm{Spec}\,\mathscr{O}} G_1$ such that for every object R of $\mathscr{C}_{\mathscr{O}}$, $\mathscr{M}(G, q)(R)$ consists of pairs $(\boldsymbol{A}, \boldsymbol{B}) \in G_0(R) \times G_1(R)$ satisfying

$$\boldsymbol{B} \boldsymbol{A} \boldsymbol{B}^{-1} = \boldsymbol{A}^q. \tag{3.6}$$

It suffices to show that $\mathscr{M}(G, q)$ is a local complete intersection, flat and of pure relative dimension d over $\mathscr{O}$, since $\mathsf{R}_{\bar{\varrho}}^{\mathrm{loc}}$ is the completion of $\mathscr{M}(G, q)$ at the k-point $(\bar{\varrho}(t), \bar{\varrho}(\phi_q))$.

First, we show that every geometric fiber of $\mathscr{M}(G, q) \to \mathrm{Spec}\,\mathscr{O}$ is of pure dimension d. Consider the natural projection

$$p \colon \mathscr{M}(G, q) \to G_0$$

to the first factor. Take a geometric point $\mathrm{Spec}\,K \to \mathrm{Spec}\,\mathscr{O}$. For a point $\boldsymbol{A}_0 \in G_0(K)$ in the image of $p(K)$, let $Z(\boldsymbol{A}_0)$ be the centralizer of $\boldsymbol{A}_0$ in $G_{0,K}$ as a closed subscheme of $G_{0,K}$, and $C(\boldsymbol{A}_0)$ the conjugacy class of $\boldsymbol{A}_0$, which is a locally closed subscheme of $G_{0,K}$ isomorphic to $G_{0,K}/Z(\boldsymbol{A}_0)$. Then $C(\boldsymbol{A}_0)$ lies in the image of $p(K)$. For every point $\boldsymbol{A} \in C(\boldsymbol{A}_0)(K)$, the fiber $p(K)^{-1}(\boldsymbol{A})$ is a torsor under the group $Z(\boldsymbol{A})$, which is conjugate to $Z(\boldsymbol{A}_0)$. Thus, $p(K)^{-1}(C(\boldsymbol{A}_0))$ is irreducible of dimension

$$\dim p(K)^{-1}(C(\boldsymbol{A}_0)) = \dim C(\boldsymbol{A}_0) + \dim Z(\boldsymbol{A}_0)$$
$$= \dim G_{0,K}$$
$$= d.$$

To continue, we choose an embedding $e \colon G_K \to \mathrm{GL}_{m,K}$ of algebraic groups over K for some integer $m \geq 1$. By (3.6), the image of $e(K) \circ p(K)$ consists only of matrices whose generalized eigenvalues are $(q^{m!} - 1)$-th roots of unity, hence finitely many conjugacy classes in $\mathrm{GL}_m(K)$. We claim that the image of $p(K)$ consists of finitely many conjugacy classes in $G_0(K)$ as well, which implies that $\mathscr{M}(G, q)_K$ is of pure dimension d. In fact, we have the following commutative

---

[4] Here, once again we omit the similitude character μ in the ring $\mathsf{R}_{\bar{\varrho}}^{\mathrm{loc}}$, in order to be consistent with the previous convention.

diagram

$$
\begin{array}{ccc}
G_0(K)/\!\!/ G_0(K) & \longrightarrow & \mathrm{GL}_m(K)/\!\!/ \mathrm{GL}_m(K) \\
\downarrow & & \downarrow \\
(G_{0,K}/\!\!/ G_{0,K})(K) & \longrightarrow & (\mathrm{GL}_{m,K}/\!\!/ \mathrm{GL}_{m,K})(K)
\end{array}
$$

of sets, in which the bottom map is finite since the morphism $G_{0,K}/\!\!/ G_{0,K} \to \mathrm{GL}_{m,K}/\!\!/ \mathrm{GL}_{m,K}$ is; and the left vertical map is also finite due to the finiteness of unipotent conjugacy classes of a reductive group [27, Théorème 4.1]; it follows that the upper horizontal map is finite as well.

The above discussion shows that the morphism $\mathscr{M}(G,q) \to \operatorname{Spec}\mathscr{O}$ is of pure relative dimension d. Now we take a closed point $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}})$ of $\mathscr{M}(G,q)$, which induces a homomorphism

$$
\mathscr{O}_{\mathscr{M}(G,q),(\bar{\boldsymbol{A}},\bar{\boldsymbol{B}})} \to \mathscr{O}_{G_0,\bar{\boldsymbol{A}}} \widehat{\otimes}_{\mathscr{O}} \mathscr{O}_{G_1,\bar{\boldsymbol{B}}}
$$

of corresponding complete local rings. As both G_0 and G_1 are smooth over $\mathscr{O}$ of pure relative dimension d, both $\mathscr{O}_{G_0,\bar{\boldsymbol{A}}}$ and $\mathscr{O}_{G_1,\bar{\boldsymbol{B}}}$ are power series rings over $\mathscr{O}$ in d variables. The relation (3.6), or equivalently, the relation $\boldsymbol{A} = \boldsymbol{B}^{-1}\boldsymbol{A}^q\boldsymbol{B}$, is defined by d equations in $\mathscr{O}_{G_0,\bar{\boldsymbol{A}}} \widehat{\otimes}_{\mathscr{O}} \mathscr{O}_{G_1,\bar{\boldsymbol{B}}}$. In other words, $\mathscr{M}(G,q)$ is a local complete intersection, hence Cohen–Macaulay. Therefore, $\mathscr{M}(G,q)$ is flat over $\mathscr{O}$. The proposition is proved. $\square$

Take an integer $n \geq 1$. Now we apply the above discussion to the homomorphism $\nu\colon \mathscr{G}_n \to \mathrm{GL}_1$ in Notation 2.1. Consider a pair $(\bar{\varrho}, \mu)$ from Notation 3.1 with $\tilde{\Gamma} = \mathrm{T}_q$ and $\Gamma = \mathrm{T}_{q^2}$, such that $\mu(t) = 1$. In particular, $\bar{\varrho}\colon \mathrm{T}_q \to \mathscr{G}_n(k)$ is a homomorphism and $\mu\colon \mathrm{T}_q \to \mathscr{O}^\times$ is a (continuous) similitude character. Write

$$
\bar{\varrho}(t) = \bar{\boldsymbol{A}} = (\bar{A}, 1, 1), \quad \bar{\varrho}(\phi_q) = \bar{\boldsymbol{B}} = (\bar{B}, -\mu(\phi_q), \mathfrak{c}) \tag{3.7}
$$

for $\bar{A}, \bar{B} \in \mathrm{GL}_n(k)$. For a lifting ϱ of $\bar{\varrho}$ to an object R of $\mathscr{C}_{\mathscr{O}}$, we write $\varrho(t) = \boldsymbol{A} = (A, 1, 1)$ and $\varrho(\phi_q) = \boldsymbol{B} = (B, -\mu(\phi_q), \mathfrak{c})$. Then the pair (A, B) reduce to $(\bar{A}, \bar{B})$, and satisfy the relation

$$
B\,{}^{t}A^{-1}B^{-1} = A^q. \tag{3.8}
$$

Corollary 3.17 *The ring* $\mathrm{R}_{\bar{\varrho}}^{\mathrm{loc}}$ *is a local complete intersection, flat and of pure relative dimension* n^2 *over* $\mathscr{O}$.

Proof   This follows immediately from Proposition 3.16 since the kernel of $\nu\colon \mathscr{G}_n \to \mathrm{GL}_1$ is of dimension n^2. $\square$

From now till the end of this subsection, we assume $\ell \geq n$. For every integer $m \geq 1$, we denote by J_m the standard upper triangular nilpotent Jordan block

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
 & 0 & 1 & \cdots & 0 \\
 & & \ddots & \ddots & \vdots \\
 & & & 0 & 1 \\
 & & & & 0
\end{pmatrix}
$$

of size m.

Denote by $\mathcal{N}_n$ (resp. $\mathcal{U}_n$) the closed subscheme of M_n (resp. GL_n) defined by the equation $X^n = 0$ (resp. $(A-1)^n = 0$). For every object R of $\mathscr{C}_{\mathscr{O}}$, we have the *truncated exponential map*

$\exp\colon \mathcal{N}_n(R) \to \mathcal{U}_n(R)$ defined by the formula

$$\exp X = 1 + X + \cdots + \frac{X^{n-1}}{(n-1)!},$$

which is a bijection. Its inverse is given by the *truncated logarithm map* $\log\colon \mathcal{U}_n(R) \to \mathcal{N}_n(R)$ defined by the formula

$$\log A = \sum_{i=1}^{n-1} (-1)^{i-1} \frac{(A-1)^i}{i}.$$

Let $\mathfrak{P}_n$ be the set of partitions of n. By the classification of nilpotent orbits in GL_n, for $K = k, E$, we have canonical surjective maps $\pi\colon \mathcal{N}_n(K) \to \mathfrak{P}_n$ such that the fibers of π are exactly the orbits in $\mathcal{N}_n(K)$ under the conjugate action of $\mathrm{GL}_n(K)$.

By the continuity of $\bar{\varrho}$, we know that $\bar{A}$ in (3.7) is unipotent, which implies $\bar{A} \in \mathcal{U}_n(k)$. Put $\bar{X} := \log \bar{A} \in \mathcal{N}_n(k)$. Following [5, Definition 3.9], we define the functor $\mathrm{Nil}_{\bar{X}}\colon \mathscr{C}_{\mathscr{O}} \to \mathsf{Set}$ that sends an object R of $\mathscr{C}_{\mathscr{O}}$ to the set of elements $X \in \mathcal{N}_n(R)$ that reduce to $\bar{X}$ and are of the form $C X_0 C^{-1}$, where X_0 is an element in $\mathcal{N}_n(\mathscr{O})$ satisfying $\pi(X_0) = \pi(\bar{X})$ and $C \in \mathrm{GL}_n(R)$, where we regard X_0 as an element in $\mathcal{N}_n(E)$ in the notation $\pi(X_0)$.

Definition 3.18 *We say that a lifting ϱ of $\bar{\varrho}$ to an object R of $\mathscr{C}_{\mathscr{O}}$ is* minimally ramified *if there exists an element $X \in \mathrm{Nil}_{\bar{X}}(R)$ such that $\varrho^{\natural}(t) = \exp X$.*

Let $\mathscr{D}_{\bar{\varrho}}^{\min}$ be the local deformation problem of $\bar{\varrho}$ (Definition 3.3) that classifies minimally ramified liftings of $\bar{\varrho}$.

Proposition 3.19 *The local deformation problem $\mathscr{D}_{\bar{\varrho}}^{\min}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension n^2.*

Proof We follow the approach of [5, Proposition 5.6], where a similar result for symplectic or orthogonal representations was proved.

Consider the morphism $\alpha\colon \mathscr{D}_{\bar{\varrho}}^{\min} \to \mathrm{Nil}_{\bar{X}}$ that sends a lifting ϱ to $\log A$ if $\varrho^{\natural}(t) = A$. In the definition of $\mathrm{Nil}_{\bar{X}}$, we may fix the nilpotent element $X_0 \in \mathcal{N}_n(\mathscr{O})$. Moreover, up to conjugation in $\mathrm{GL}_n(\mathscr{O})$, we may assume

$$X_0 = \begin{pmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_r} \end{pmatrix},$$

where $n = n_1 + \cdots + n_r$. Let $Z_n(X_0)$ be the centralizer of X_0 in $\mathrm{GL}_{n,\mathscr{O}}$, which is a closed subscheme of $\mathrm{GL}_{n,\mathscr{O}}$. By [5, Remark 4.18], $Z_n(X_0)$ is smooth over $\mathscr{O}$. By [5, Lemma 3.11], $\mathrm{Nil}_{\bar{X}}$ is represented by a formal power series ring over $\mathscr{O}$ in $n^2 - \dim_{\mathscr{O}} Z_n(X_0)$ variables, where $\dim_{\mathscr{O}} Z_n(X_0)$ denotes the relative dimension of $Z_n(X_0)$ over $\mathscr{O}$. Thus, it suffices to show that α is represented by a formal scheme formally smooth of pure relative dimension $\dim_{\mathscr{O}} Z_n(X_0)$ over $\mathrm{Nil}_{\bar{X}}$.

Take a lifting ϱ of $\bar{\varrho}$ to an object R of $\mathscr{C}_{\mathscr{O}}$. Then $\varrho(\phi_q)$ has the form $(B, -\mu(\phi_q), \mathfrak{c})$ with $B \in \mathrm{GL}_n(R)$ that reduces to $\bar{B}$ and satisfies

$$B\,{}^{\mathrm{t}}X B^{-1} = -qX \tag{3.9}$$

by (3.8). For each given $X \in \mathrm{Nil}_{\bar{X}}$, if there exists $B \in \mathrm{GL}_n(R)$ that reduces to $\bar{B}$ and satisfies (3.9), then the set of all elements B form a torsor under the group

$$\widehat{Z}_n(X)(R) := \{g \in 1 + \mathrm{M}_n(\mathfrak{m}_R) \mid gXg^{-1} = X\},$$

which is isomorphic to the group of R-valued points of the formal completion of the group scheme $Z_n(X_0)$ along the unit section. Thus, to finish the proof, it suffices to show that the equation (3.9) admits at least one solution for B that reduces to $\bar{B}$.

Assume first $X = X_0$ in $\mathcal{N}_n(R)$. Then

$$B_0 := \begin{pmatrix} A_{n_1} & & \\ & \ddots & \\ & & A_{n_r} \end{pmatrix}$$

is a solution to (3.9), where

$$A_{n_i} := \begin{pmatrix} & & & (-q)^{n_i-1} \\ & & \cdot^{\cdot^{\cdot}} & \\ & -q & & \\ 1 & & & \end{pmatrix}.$$

In the general case, we write $X = CX_0C^{-1}$ for some $C \in \mathrm{GL}_n(R)$. Then $B := CB_0 \,{}^t\!C$ satisfies the equation (3.9). Up to multiplying C by an element in $Z_n(X_0)(R)$ from the right, we can make $B \in \mathrm{GL}_n(R)$ to reduce to $\bar{B}$. This finishes the proof of the proposition. $\square$

Recall from Definition 3.4 that $\mathrm{L}(\mathscr{D}_{\bar{\varrho}}^{\min}) \subseteq \mathrm{H}^1(\mathrm{T}_q, \mathrm{ad}\,\bar{\varrho})$ is the tangent space of the local deformation problem $\mathscr{D}_{\bar{\varrho}}^{\min}$.

Corollary 3.20 *We have* $\dim_k \mathrm{L}(\mathscr{D}_{\bar{\varrho}}^{\min}) = \dim_k \mathrm{H}^0(\mathrm{T}_q, \mathrm{ad}\,\bar{\varrho})$.

Proof Suppose that $\mathscr{D}_{\bar{\varrho}}^{\min} = \mathrm{Spf}\,\mathsf{R}_{\bar{\varrho}}^{\min}$. By (3.1), we have

$$\dim_k \mathfrak{m}_{\mathsf{R}_{\bar{\varrho}}^{\min}}/(\lambda, \mathfrak{m}_{\mathsf{R}_{\bar{\varrho}}^{\min}}^2) = \dim_k \mathrm{L}(\mathscr{D}_{\bar{\varrho}}^{\min}) + n^2 - \dim_k \mathrm{H}^0(\mathrm{T}_q, \mathrm{ad}\,\bar{\varrho}).$$

From this, the corollary follows immediately from Proposition 3.19. $\square$

To end this subsection, we record the following lemma concerning decomposition of representations of the q-tame group, in which part (1) will be used later and part (2) is only for complement.

Lemma 3.21 *Let* $(\bar{\rho}, \bar{M})$ *be an unramified representation of* $\mathrm{T}_q = t^{\mathbb{Z}_\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$ *over* k *of dimension* N. *Suppose that* $\bar{M}$ *admits a decomposition*

$$\bar{M} = \bar{M}_1 \oplus \cdots \oplus \bar{M}_s$$

stable under the action of $\bar{\rho}(\phi_q)$ *such that the characteristic polynomials of* $\bar{\rho}(\phi_q)$ *on* M_i *are mutually coprime for* $1 \le i \le s$. *Let* (ρ, M) *be a lifting of* $(\bar{\rho}, \bar{M})$ *to an object* R *of* $\mathscr{C}_{\mathscr{O}}$. *Then*

(1) *There is a unique decomposition*

$$M = M_1 \oplus \cdots \oplus M_s$$

of free R-*modules, such that* M_i *is stable under the action of* $\rho(\phi_q)$ *and it is a lifting of* $\bar{M}_i$ *as a* ϕ_q-*module.*

(2) *Write $\rho(t) = (\rho(t)_{i,j})$ with $\rho(t)_{i,j} \in \mathrm{Hom}_R(M_j, M_i)$. Suppose that q is not an eigenvalue for the canonical action of ϕ_q on $\mathrm{Hom}_k(\bar{M}_j, \bar{M}_i)$ for all $i \neq j$. Then we have $\rho(t)_{i,j} = 0$ for all $i \neq j$; in other words, the decomposition in (1) is stable under the whole group T_q.*

Proof   Part (1) is elementary, which we leave to the readers as an exercise.

For (2), we choose a basis of M over R adapted to the decomposition of M in (1). We identify $\rho(t)$ and $\rho(\phi_q)$ with their matrices under this basis. We have $\rho(\phi_q)_{i,j} = 0$ for $i \neq j$ since each M_i is stable under $\rho(\phi_q)$. Let $J \subset R$ be the ideal generated by the coefficients of $\rho(t)_{i,j}$ for $i \neq j$. We have to show that $J = 0$. By Nakayama's lemma, it suffices to show that $J = \mathfrak{m}_R J$. As

$$\rho(t)^q = (1 + (\rho(t) - 1))^q = 1 + q(\rho(t) - 1) + \sum_{a \geq 2} \binom{q}{a}(\rho(t) - 1)^a,$$

and $\rho(t) \equiv 1 \mod \mathfrak{m}_R$, we have

$$(\rho(t)^q)_{i,j} \equiv q\rho(t)_{i,j} \mod \mathfrak{m}_R J$$

for $i \neq j$. The relation $\phi_q t = t^q \phi_q$ implies that

$$\rho(\phi_q)_{i,i}\rho(t)_{i,j} = (\rho(t)^q)_{i,j}\rho(\phi_q)_{j,j} \equiv q\rho(t)_{i,j}\rho(\phi_q)_{j,j} \mod \mathfrak{m}_R J.$$

For every $1 \leq i \leq s$, denote by $P_i \in R[X]$ the characteristic polynomial of the square matrix $\rho(\phi_q)_{i,i}$. Then it follows that

$$\rho(t)_{i,j} P_i(q\rho(\phi_q)_{j,j}) \equiv P_i(\rho(\phi_q)_{i,i})\rho(t)_{i,j} \equiv 0 \mod \mathfrak{m}_R J$$

for $i \neq j$. By assumption, if $\bar{\alpha}$ is an eigenvalue of $\bar{\rho}(\phi_q)_{i,i}$, then $q^{-1}\bar{\alpha}$ is not an eigenvalue of $\bar{\rho}(\phi_q)_{j,j}$. It follows that $P_i(q\rho(\phi_q)_{j,j})$ is invertible, hence $\rho(t)_{i,j} \equiv 0 \mod \mathfrak{m}_R J$.

The lemma is proved. □

3.4   Minimally Ramified Deformations

In this subsection, we define and study the minimally ramified deformations at places coprime to the odd prime ℓ. Thus, we take a nonarchimedean place v of F^+ that is not above ℓ.

Let $\mathrm{I}_{F_v^+} \subseteq \Gamma_{F_v^+}$ be the inertia subgroup, and $\mathrm{P}_{F_v^+}$ the maximal closed subgroup of $\mathrm{I}_{F_v^+}$ of pro-order coprime to ℓ. Put $\mathrm{T}_v := \Gamma_{F_v^+}/\mathrm{P}_{F_v^+}$. Similarly, for every place w of F above v, we have Γ_{F_w}, I_{F_w}, P_{F_w}, and T_w. Finally, put $\mathrm{T}_w^+ := \Gamma_{F_v^+}/\mathrm{P}_{F_w}$.

Remark 3.22   The group T_v is a $\|v\|$-tame group (Definition 3.15).

- When w is split over v, we have $\mathrm{P}_{F_w} = \mathrm{P}_{F_v^+}$, $\mathrm{T}_w^+ = \mathrm{T}_v$, and $\mathrm{T}_w = \mathrm{T}_w^+ = \mathrm{T}_v$.
- When w is unramified over v, we have $\mathrm{P}_{F_w} = \mathrm{P}_{F_v^+}$, $\mathrm{T}_w^+ = \mathrm{T}_v$, and that the subgroup T_w of T_v is a $\|v\|^2$-tame group.
- When w is ramified over v, we have $\mathrm{P}_{F_v^+}/\mathrm{P}_{F_w} \simeq \mathbb{Z}/2\mathbb{Z}$, that the natural map $\mathrm{T}_w \to \mathrm{T}_v$ is an isomorphism, and a canonically split short exact sequence

$$1 \to \mathrm{P}_{F_v^+}/\mathrm{P}_{F_w} \to \mathrm{T}_w^+ \to \mathrm{T}_v \to 1.$$

We first recall some facts about extensions of representations of P_{F_w} from [14]. For an irreducible representation τ of P_{F_w} with coefficients in k, we put

$$\Gamma_\tau := \{\sigma \in \Gamma_{F_w} \mid \tau^\sigma \simeq \tau\},$$

where we recall that τ^σ denotes the representation given by $\tau^\sigma(g) = \tau(\sigma g\sigma^{-1})$ for $g \in \mathrm{P}_{F_w}$. Let T_τ be the image of Γ_τ in $\mathrm{T}_w = \Gamma_{F_w}/\mathrm{P}_{F_w}$. As P_{F_w} is normal in $\Gamma_{F_v^+}$, we may similarly define

$$\Gamma_\tau^+ := \{\sigma \in \Gamma_{F_v^+} \mid \tau^\sigma \simeq \tau\},$$

and denote by T_τ^+ its image in T_w^+.

Lemma 3.23 *We have the following properties for τ:*

(1) *the dimension of τ is coprime to ℓ; and τ has a unique deformation to a representation $\tilde\tau$ of P_{F_w} over $\mathscr{O}$;*

(2) *$\tilde\tau$ in (1) admits a unique extension to a representation of $\Gamma_\tau \cap \mathrm{I}_{F_w}$ over $\mathscr{O}$ whose determinant has order coprime to ℓ;*

(3) *there exists an extension of $\tilde\tau$ in (2) to a representation of Γ_τ over $\mathscr{O}$.*

Proof　This is [14, Lemma 2.4.11]. $\qquad\square$

Now we consider a pair $(\bar{r}, \chi)$ from Notation 3.1 with $\tilde\Gamma = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F = \Gamma_{F_w}$. Our first goal is to define the notion of *minimally ramified liftings* of $\bar{r}$ (Definition 3.29). Recall from Notation 2.2 that we have the induced homomorphism $\bar{r}^\natural : \Gamma_{F_w} \to \mathrm{GL}_N(k)$, which satisfies that $\bar{r}^\natural|_{\mathrm{P}_{F_w}}$ is semisimple.

When v is split in F, minimally ramified deformations have already been defined and studied in [14, 2.4.4]. Thus, we now assume that v is nonsplit in F, hence w is the unique prime of F above v.

Assumption 3.24　The residue field k of E contains a subfield $k^\flat$ of degree two such that every irreducible summand of $\bar{r}^\natural|_{\mathrm{P}_{F_w}} \otimes_k \overline{k}$ is defined over $k^\flat$.

We first assume that E satisfies Assumption 3.24. For an irreducible representation τ of P_{F_w} with coefficients in k, we put

$$M_\tau(\bar{r}) := \mathrm{Hom}_{k[\mathrm{P}_{F_w}]}(\tau, \bar{r}^\natural).$$

Then $\tau \otimes_k M_\tau(\bar{r})$ is canonically the τ-isotypic component of $\bar{r}^\natural|_{\mathrm{P}_{F_w}}$. As τ extends to a representation of Γ_τ, the k-vector space $M_\tau(\bar{r})$ is equipped with a natural action by T_τ; and $\tau \otimes_k M_\tau(\bar{r})$ is equipped with a natural action by Γ_τ.

We denote by $\mathfrak{T} = \mathfrak{T}(\bar{r})$ the set of isomorphism classes of irreducible representations τ of P_{F_w} such that $M_\tau(\bar{r}) \neq 0$. Then Γ_{F_w} acts on $\mathfrak{T}$ by conjugation, whose orbits we denote by $\mathfrak{T}/\Gamma_{F_w}$. For $\tau \in \mathfrak{T}$, we write $[\tau]$ for its orbit in $\mathfrak{T}/\Gamma_{F_w}$.

Choose an element $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$. By Lemma 2.3, the homomorphism $\bar{r}$ is determined by an element $\bar\Psi \in \mathrm{GL}_N(k)$ satisfying

$$\bar{r}^{\natural,\gamma} = \bar\Psi \circ \chi\bar{r}^{\natural,\vee} \circ \bar\Psi^{-1}, \quad \bar\Psi\,{}^t\bar\Psi^{-1} = -\chi(\gamma)^{-1}\bar{r}^\natural(\gamma^2).$$

In what follows, we will adopt the following simplified notation: for a representation τ of a subgroup of $\Gamma_{F_v^+}$, we write τ^* for $\chi\tau^\vee$. Due to the existence of $\bar\Psi$, we know that if $\tau \in \mathfrak{T}$, then $\tau^{\gamma,*} \in \mathfrak{T}$ as well. As $\gamma^2 \in \Gamma_{F_w}$, the assignment $\tau \mapsto \tau^{\gamma,*}$ induces an involution on the set $\mathfrak{T}/\Gamma_{F_w}$, which does not depend on the choice of γ.

Construction 3.25　We now would like to construct a Γ_{F_w}-stable partition $\mathfrak{T} = \mathfrak{T}_1 \sqcup \mathfrak{T}_2 \sqcup \mathfrak{T}_3$. For each subset $\mathfrak{T}_i$, we will specify, for every $\tau \in \mathfrak{T}_i$, an extension of $\tilde\tau$ in Lemma 3.23 (2) to a representation of Γ_τ with coefficients in $\mathscr{O}$ in a compatible way, specified below.

We start from the following observation. Suppose that $[\tau] = [\tau^{\gamma,*}]$ in $\mathfrak{T}/\Gamma_{F_w}$. Then there exists an element $h \in \Gamma_{F_w}$, unique up to left multiplication by an element in Γ_τ, such that $\tau^{\gamma,*} \simeq \tau^{h^{-1}}$, or equivalently, $\tau^{h\gamma} \simeq \tau^*$. Then we have $(h\gamma)^2 \in \Gamma_\tau$ but $h\gamma \notin \Gamma_\tau$. Denote by $\tilde{\Gamma}_\tau$ the subgroup of $\Gamma_{F_v^+}$ generated by Γ_τ and $h\gamma$, which contains Γ_τ as a subgroup of index two. Let $\tilde{T}_\tau$ be the image of $\tilde{\Gamma}_\tau$ in T_w^+, which contains T_τ as a subgroup of index two.

(1) We define $\mathfrak{T}_1$ to be the subset of $\mathfrak{T}$ consisting of τ such that $[\tau] \neq [\tau^{\gamma,*}]$. We choose a subset $\mathfrak{T}_1^\heartsuit \subseteq \mathfrak{T}_1$ such that $\{\tau, \tau^{\gamma,*} \mid \tau \in \mathfrak{T}_1^\heartsuit\}$ is a set of representatives for the Γ_τ-action on $\mathfrak{T}_1$. For each element $\tau \in \mathfrak{T}_1^\heartsuit$, we choose an extension of $\tilde{\tau}$ in Lemma 3.23 (2) to a representation of Γ_τ with coefficients in $\mathscr{O}$, which we still denote by $\tilde{\tau}$. For a general element $\tau \in \mathfrak{T}_1$, there are two cases. If $\tau \simeq \tau_1^h$ for (unique) $\tau_1 \in \mathfrak{T}_1^\heartsuit$ and some $h \in \Gamma_{F_w}$, then we choose $\tilde{\tau}$ to be $\tilde{\tau}_1^h$, as the extension to $\Gamma_\tau = h^{-1}\Gamma_{\tau_1}h$. If $\tau \simeq (\tau_1^h)^{\gamma,*}$ for (unique) $\tau_1 \in \mathfrak{T}_1^\heartsuit$ and some $h \in \Gamma_{F_w}$, then we choose $\tilde{\tau}$ to be $(\tilde{\tau}_1^h)^{\gamma,*}$, as the extension to $\Gamma_\tau = \gamma^{-1}h^{-1}\Gamma_{\tau_1}h\gamma$.

(2) We define $\mathfrak{T}_2$ to be the subset of $\mathfrak{T}$ consisting of τ such that $[\tau] = [\tau^{\gamma,*}]$, and that the images of Γ_τ and $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/I_{F_v^+}$ are different. We choose a subset $\mathfrak{T}_2^\heartsuit \subseteq \mathfrak{T}_2$ of representatives for the Γ_τ-action on $\mathfrak{T}_2$. For each element $\tau \in \mathfrak{T}_2^\heartsuit$, we choose an extension $\tilde{\tau}$ from Lemma 3.27 (1) below to a representation of Γ_τ with coefficients in $\mathscr{O}$. For $\tau \in \mathfrak{T}_2$ in general, we have $\tau \simeq \tau_2^h$ for (unique) $\tau_2 \in \mathfrak{T}_2^\heartsuit$ and some $h \in \Gamma_{F_w}$; and we choose $\tilde{\tau}$ to be $\tilde{\tau}_2^h$, as the extension to $\Gamma_\tau = h^{-1}\Gamma_{\tau_2}h$.

(3) We define $\mathfrak{T}_3$ to be the subset of $\mathfrak{T}$ consisting of τ such that $[\tau] = [\tau^{\gamma,*}]$, and that the images of Γ_τ and $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/I_{F_v^+}$ are the same. We choose a subset $\mathfrak{T}_3^\heartsuit \subseteq \mathfrak{T}_3$ of representatives for the Γ_τ-action on $\mathfrak{T}_3$. For each element $\tau \in \mathfrak{T}_3^\heartsuit$, we choose an extension $\tilde{\tau}$ from Lemma 3.27 (2) below to a representation of Γ_τ with coefficients in $\mathscr{O}$. For $\tau \in \mathfrak{T}_3$ in general, we have $\tau \simeq \tau_3^h$ for (unique) $\tau_3 \in \mathfrak{T}_3^\heartsuit$ and some $h \in \Gamma_{F_w}$; and we choose $\tilde{\tau}$ to be $\tilde{\tau}_3^h$, as the extension to $\Gamma_\tau = h^{-1}\Gamma_{\tau_3}h$.

In addition, we put $\mathfrak{T}^\heartsuit := \mathfrak{T}_1^\heartsuit \sqcup \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit$.

Remark 3.26 The partition $\mathfrak{T} = \mathfrak{T}_1 \sqcup \mathfrak{T}_2 \sqcup \mathfrak{T}_3$ does not depend on the choice of γ. Moreover, if $\mathfrak{T}_3$ is nonempty, then w is ramified over v.

Lemma 3.27 *Let $\tau \in \mathfrak{T}$ be an element of dimension d.*

(1) *If $\tau \in \mathfrak{T}_2$, then the representation $\tilde{\tau}$ in Lemma 3.23 (2) extends to a representation of Γ_τ with coefficients in $\mathscr{O}$ such that $\tilde{\tau}^{\gamma'} \simeq \tilde{\tau}^*$ still holds for every $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$.*

(2) *If $\tau \in \mathfrak{T}_3$, then the representation $\tilde{\tau}$ in Lemma 3.23 (2) extends to a representation of Γ_τ with coefficients in $\mathscr{O}$ such that $\tilde{\tau}^{\gamma'} \simeq \tilde{\tau}^*$ still holds for every $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$.*

Proof We fix a splitting $\Gamma_{F_v^+} \simeq P_{F_v^+} \rtimes T_v$ and an isomorphism $T_v \simeq T_q = t^{\mathbb{Z}_\ell} \rtimes \phi_q^{\hat{\mathbb{Z}}}$ with the q-tame group (Definition 3.15) where $q = \|v\|$. Then we have the induced splitting $\Gamma_\tau \simeq P_{F_w} \rtimes T_\tau$, where $T_\tau = t_\tau^{\mathbb{Z}_\ell} \rtimes \phi_\tau^{\hat{\mathbb{Z}}}$ is a subgroup of T_q, with $t_\tau = t^{\ell^a}$ and $\phi_\tau = \phi_q^b$ for unique integers $a \geq 0$ and $b > 0$. To extend $\tilde{\tau}$ in Lemma 3.23 (2) to a representation of Γ_τ, it suffices to specify $\tilde{\tau}(\phi_\tau)$.

For (1), there are two cases.

First, we suppose that w is unramified over v. Then b is even; and $\tilde{T}_\tau$ is the image of $\tilde{\Gamma}_\tau$ in T_v. Then $\tilde{T}_\tau$ is generated by T_τ and an element $\gamma' \in T_v$ of the form $(\tilde{t}_\tau, \phi_q^{b/2})$ such that $\gamma'^2 = (\tilde{t}_\tau^{q^{b/2}+1}, \phi_q^b)$ lies in Γ_τ. As $[\tau] = [\tau^{\gamma,*}]$, we have $\tau^{\gamma'} \simeq \tau^*$. We choose a basis of τ, hence regard τ as a homomorphism $\tau: P_{F_w} \to GL_d(k)$. By Lemma 3.23 (1)–(2), we have a continuous homomorphism $\tilde{\tau}: \Gamma_\tau \cap I_{F_w} \to GL_d(\mathscr{O})$ such that $\tilde{\tau}|_{P_{F_w}}$ is a lifting of τ, unique

up to conjugation in $1 + \mathrm{M}_d(\lambda)$. In particular, there is an element $B \in \mathrm{GL}_d(\mathscr{O})$, unique up to scalar in $\mathscr{O}^\times$, such that $\tilde{\tau}^{\gamma'}(g) = B\tilde{\tau}^*(g)B^{-1}$ for every $g \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. Since $\phi_\tau = \tilde{t}_\tau^{-q^{b/2}-1}\gamma'^2$, we have

$$
\begin{aligned}
\tilde{\tau}(\phi_\tau g \phi_\tau^{-1}) &= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1}\gamma'^2 g \gamma'^{-2}\tilde{t}_\tau^{q^{b/2}+1}) \\
&= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})\tilde{\tau}(\gamma'^2 g \gamma'^{-2})\tilde{\tau}(\tilde{t}_\tau^{q^{b/2}+1}) \\
&= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})B\tilde{\tau}^*(\gamma'^{-1}g\gamma')B^{-1}\tilde{\tau}(\tilde{t}_\tau^{q^{b/2}+1}) \\
&= \tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})(B\,{}^{\mathrm{t}}B^{-1})\tilde{\tau}(g)({}^{\mathrm{t}}BB^{-1})\tilde{\tau}(\tilde{t}_\tau^{q^{b/2}+1})
\end{aligned}
$$

for every $g \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. We put $\tilde{\tau}(\phi_\tau) := -\chi(\phi_q^{b/2})\tilde{\tau}(\tilde{t}_\tau^{-q^{b/2}-1})(B\,{}^{\mathrm{t}}B^{-1})$. Then we obtain the desired extension as in (1).

Second, we suppose that w is ramified over v. By the definition of $\mathfrak{T}_2$, the image of $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/\mathrm{I}_{F_v^+}$ contains $\phi_\tau^{\hat{\mathbb{Z}}}$ as a subgroup of index two. Thus, there exists an element $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$ such that $\gamma^2 = h\phi_\tau$ for some $h \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. The remaining argument is same to the above case.

For (2), by the definition of $\mathfrak{T}_3$, the image of $\tilde{\Gamma}_\tau$ in $\Gamma_{F_v^+}/\mathrm{I}_{F_v^+}$ coincides with $\phi_\tau^{\hat{\mathbb{Z}}}$. In particular, we can find an element $\gamma' \in \tilde{\Gamma}_\tau \setminus \Gamma_\tau$ contained in $\mathrm{I}_{F_v^+} \setminus \mathrm{I}_{F_w}$. By Lemma 3.23 (1)–(2), we have a continuous homomorphism $\tilde{\tau}: \Gamma_\tau \cap \mathrm{I}_{F_w} \to \mathrm{GL}_d(\mathscr{O})$ such that $\tilde{\tau}|_{\mathrm{P}_{F_w}}$ is a lifting of τ, unique up to conjugation in $1 + \mathrm{M}_d(\lambda)$. As we have $\tau^{\gamma'} \simeq \tau^*$ and $\tau^{\phi_\tau} \simeq \tau$, there are elements $A, B \in \mathrm{GL}_d(\mathscr{O})$ such that

$$
\tilde{\tau}^{\gamma'}(g) = A\tilde{\tau}^*(g)A^{-1}, \tag{3.10}
$$

$$
\tilde{\tau}^{\phi_\tau}(g) = B\tilde{\tau}(g)B^{-1}, \tag{3.11}
$$

for every $g \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. It follows from (3.10) that the desired element $\tilde{\tau}(\phi_\tau) \in \mathrm{GL}_d(\mathscr{O})$ has to satisfy the equation

$$
\chi(\phi_\tau)A\,{}^{\mathrm{t}}\tilde{\tau}(\phi_\tau)^{-1}A^{-1} = \tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}) = \tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})\tilde{\tau}(\phi_\tau), \tag{3.12}
$$

where we note that $\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1} \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. However, by (3.11), we have

$$
\tilde{\tau}^{\gamma'\phi_\tau\gamma'^{-1}}(g) = (\tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})B)\tilde{\tau}(g)(\tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})B)^{-1}
$$

for every $g \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. On the other hand, by (3.10) and (3.11), we have

$$
\tilde{\tau}^{\gamma'\phi_\tau\gamma'^{-1}}(g) = (A\,{}^{\mathrm{t}}B^{-1}A^{-1})\tilde{\tau}(g)(A\,{}^{\mathrm{t}}B^{-1}A^{-1})^{-1}
$$

for every $g \in \Gamma_\tau \cap \mathrm{I}_{F_w}$. Since τ is absolutely irreducible, it follows that there exists $\beta \in \mathscr{O}^\times$ such that

$$
A\,{}^{\mathrm{t}}B^{-1}A^{-1} = \beta \cdot \tilde{\tau}(\gamma'\phi_\tau\gamma'^{-1}\phi_\tau^{-1})B.
$$

Take an element $\alpha \in \mathscr{O}^\times$ such that $\alpha^2 = \beta\chi(\phi_\tau)$, which is possible by Assumption 3.24. Then it is clear that $\tilde{\tau}(\phi_\tau) = \alpha B \in \mathrm{GL}_d(\mathscr{O})$ is a solution to (3.12).

The lemma is proved. $\qquad\square$

Using Construction 3.25, we now discuss the structure of liftings of $\bar{r}$. Let $r: \Gamma_{F_v^+} \to \mathscr{G}_N(R)$ be a lifting of $\bar{r}$ to an object R of $\mathscr{C}_\mathscr{O}$. By Lemma 2.3, to give such a lifting r is equivalent to giving an element $\Psi \in \mathrm{GL}_N(R)$ that reduces to $\bar{\Psi}$ and satisfies

$$
r^{\natural,\gamma} = \Psi \circ \chi r^{\natural,\vee} \circ \Psi^{-1}, \quad \Psi\,{}^{\mathrm{t}}\Psi^{-1} = -\chi(\gamma)^{-1}r^\natural(\gamma^2).
$$

For every $\tau \in \mathfrak{T}$, put

$$M_\tau(r) := \mathrm{Hom}_{R[\mathrm{P}_{F_w}]}(\tilde{\tau} \otimes_{\mathscr{O}} R, r^{\natural}),$$

which is a finite free R-module equipped with the induced continuous action by T_τ. Denote by $m_\tau \geq 1$ the rank of $M_\tau(r)$. Let $\underline{\tau} \in \mathfrak{T}$ be the unique element such that $\tau^\gamma \simeq \underline{\tau}^*$. Choose an isomorphism $\iota_\tau \colon \tau^\gamma \xrightarrow{\sim} \underline{\tau}^*$, which, by Construction 3.25, lifts to an isomorphism $\iota_{\tilde{\tau}} \colon \tilde{\tau}^\gamma \xrightarrow{\sim} \underline{\tilde{\tau}}^*$ of representations of $\Gamma_{\underline{\tau}}$. Then we have successive isomorphisms

$$
\begin{aligned}
M_\tau(r)^\gamma &\xrightarrow{\sim} \mathrm{Hom}_{R[\mathrm{P}_{F_w}]}(\tilde{\tau}^\gamma \otimes_{\mathscr{O}} R, r^{\natural,\gamma}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{R[\mathrm{P}_{F_w}]}(\underline{\tilde{\tau}}^* \otimes_{\mathscr{O}} R, r^{\natural,*}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{R[\mathrm{P}_{F_w}]}(r^{\natural}, \underline{\tilde{\tau}} \otimes_{\mathscr{O}} R),
\end{aligned}
$$

where the second isomorphism is induced by $\iota_{\tilde{\tau}}$ and Ψ. As $\underline{\tau}$ is absolutely irreducible, we obtain a perfect R-bilinear pairing

$$M_\tau(r)^\gamma \times M_{\underline{\tau}}(r) \to \mathrm{End}_{R[\mathrm{P}_{F_w}]}(\underline{\tilde{\tau}} \otimes_{\mathscr{O}} R) = R,$$

which induces an isomorphism

$$\theta_{\tilde{\tau},r} \colon M_\tau(r)^\gamma \xrightarrow{\sim} M_{\underline{\tau}}(r)^{\vee} := \mathrm{Hom}_R(M_{\underline{\tau}}(r), R)$$

of $R[\mathrm{T}_{\underline{\tau}}]$-modules. In particular, we have

$$
\begin{aligned}
r^{\natural} \simeq \Bigg(&\bigoplus_{\tau \in \mathfrak{T}_1^{\heartsuit}} (\mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau(r)) \oplus \mathrm{Ind}_{\Gamma_{\tau}^\gamma}^{\Gamma_{F_w}}(\tilde{\tau}^{\gamma,*} \otimes_{\mathscr{O}} M_\tau(r)^{\gamma,\vee})) \Bigg) \\
&\oplus \Bigg(\bigoplus_{\tau \in \mathfrak{T}_2^{\heartsuit} \sqcup \mathfrak{T}_3^{\heartsuit}} \mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau(r)) \Bigg) \qquad (3.13)
\end{aligned}
$$

as representations of Γ_{F_w}.

Now for every τ, we fix an isomorphism $b_\tau \colon M_\tau(\bar{r}) \xrightarrow{\sim} k^{\oplus m_\tau}$ of k-vector spaces, and let $\mathrm{T}_\tau \to \mathrm{GL}_{m_\tau}(k)$ be the induced homomorphism. There are two cases.

(a) Suppose that $\tau \in \mathfrak{T}_1$. Then $M_{\underline{\tau}}(r)$ is determined by $M_\tau(r)$. If we choose an isomorphism $M_\tau(r) \simeq R^{\oplus m_\tau}$ of R-modules that reduces to b_τ, then we obtain a continuous homomorphism

$$\varrho_\tau \colon \mathrm{T}_\tau \to \mathrm{GL}_{m_\tau}(R)$$

that reduces to $\mathrm{T}_\tau \to \mathrm{GL}_{m_\tau}(k)$.

(b) Suppose that $\tau \in \mathfrak{T}_2 \sqcup \mathfrak{T}_3$. Let h be element from Construction 3.25. Then $\theta_{\tilde{\tau},r}$ induces an isomorphism $M_\tau(r)^{h\gamma} \xrightarrow{\sim} M_\tau(\tau)^{\vee}$ of $R[\mathrm{T}_\tau]$-modules. Applying Lemma 2.3 (3) to $\mathrm{T}_\tau \to \mathrm{GL}_{m_\tau}(k)$, we obtain a homomorphism

$$\bar{\varrho}_\tau \colon \tilde{\mathrm{T}}_\tau \to \mathscr{G}_{m_\tau}(k)$$

satisfying $\bar{\varrho}_\tau^{-1}(\mathrm{GL}_{m_\tau}(k) \times k^\times) = \mathrm{T}_\tau$ and $\nu \circ \bar{\varrho}_\tau = \eta_v^{\mu_\tau}$ for some $\mu_\tau \in \mathbb{Z}/2\mathbb{Z}$ determined by $\tilde{\tau}$.[5] In general, if we choose an isomorphism $M_\tau(r) \simeq R^{\oplus m_\tau}$ of R-modules that reduces to b_τ, then we obtain a continuous homomorphism

$$\varrho_\tau \colon \tilde{\mathrm{T}}_\tau \to \mathscr{G}_{m_\tau}(R)$$

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[5] In fact, when $\tau \in \mathfrak{T}_2$, one can always modify $\tilde{\tau}$ to make $\mu_\tau = 0$; but when $\tau \in \mathfrak{T}_3$, μ_τ is determined by τ.

that reduces to $\bar\varrho_\tau$ and satisfies $\nu \circ \rho_\tau = \eta_v^{\mu_\tau}$.

The following proposition is the counterpart of [14, Corollary 2.4.13] when v is nonsplit in F.

Proposition 3.28 *Suppose that E satisfies Assumption 3.24. We keep the choices of $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$, those in Construction 3.25, ι_τ, and b_τ. For every object R of $\mathscr{C}_{\mathscr{O}}$, the assignment*

$$r \mapsto (\varrho_\tau)_{\tau \in \mathfrak{T}^\heartsuit}$$

establishes a bijection between deformations of $\bar r$ to R and equivalence classes of tuples $(\varrho_\tau)_{\tau \in \mathfrak{T}^\heartsuit}$ where

(a) *for $\tau \in \mathfrak{T}_1^\heartsuit$, $\varrho_\tau \colon \mathrm{T}_\tau \to \mathrm{GL}_{m_\tau}(R)$ is a continuous homomorphism that reduces to $\bar\varrho_\tau$;*

(b) *for $\tau \in \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit$, $\varrho_\tau \colon \tilde{\mathrm{T}}_\tau \to \mathscr{G}_{m_\tau}(R)$ is a continuous homomorphism that reduces to $\bar\varrho_\tau$ and satisfies $\nu \circ \rho_\tau = \eta_v^{\mu_\tau}$.*

Here, two tuples $(\varrho_\tau)_{\tau \in \mathfrak{T}^\heartsuit}$ and $(\varrho_\tau')_{\tau \in \mathfrak{T}^\heartsuit}$ are said to be equivalent if ϱ_τ and ϱ_τ' are conjugate by elements in $1 + \mathrm{M}_{m_\tau}(\mathfrak{m}_R)$ for every $\tau \in \mathfrak{T}^\heartsuit$.

Proof We now attach to every tuple $(\varrho_\tau)_{\tau \in \mathfrak{T}^\heartsuit}$ as in the statement a lifting r explicitly. Denote by M_τ the $R[\mathrm{T}_\tau]$-module corresponding to ϱ_τ. Consider

$$M := \left(\bigoplus_{\tau \in \mathfrak{T}_1^\heartsuit} (\mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde\tau \otimes_{\mathscr{O}} M_\tau) \oplus \mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee})) \right)$$
$$\oplus \left(\bigoplus_{\tau \in \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit} \mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde\tau \otimes_{\mathscr{O}} M_\tau) \right),$$

which is a free R-module of rank N, equipped with a continuous action by Γ_{F_w}. Moreover, we have $M \otimes_R R/\mathfrak{m}_R \simeq \bar r^\natural$ as representations of Γ_{F_w} by (3.13). Thus, we may fix an isomorphism $M \simeq R^{\oplus N}$ such that the induced continuous homomorphism $\rho = \rho_M \colon \Gamma_{F_w} \to \mathrm{GL}_N(R)$ reduces to $\bar r^\natural$. Thus, by Lemma 2.3, to construct the desired lifting r from ρ, it amounts to finding an element $\Psi \in \mathrm{GL}_N(R)$ satisfying

$$\rho^\gamma = \Psi \circ \chi \rho^\vee \circ \Psi^{-1}, \quad \Psi\,{}^t\Psi^{-1} = -\chi(\gamma)^{-1}\rho(\gamma^2). \tag{3.14}$$

We will construct Ψ as a direct sum of Ψ_τ for $\tau \in \mathfrak{T}^\heartsuit$.

For $\tau \in \mathfrak{T}_1^\heartsuit$, we note that $\tilde\tau(\gamma^{-2}) \otimes \varrho_\tau(\gamma^{-2})$ induces an isomorphism

$$\mathrm{Ind}^{\gamma^{-2}} \colon \mathrm{Ind}_{\Gamma_{\tau^{\gamma^2}}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma^2} \otimes_{\mathscr{O}} M_\tau^{\gamma^2}) \simeq (\mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^{\gamma,*} \xrightarrow{\sim} \mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde\tau \otimes_{\mathscr{O}} M_\tau). \tag{3.15}$$

Thus, we obtain an isomorphism

$$(\mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde\tau \otimes_{\mathscr{O}} M_\tau) \oplus \mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^{\gamma,*}$$
$$\xrightarrow{\sim} \mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde\tau \otimes_{\mathscr{O}} M_\tau) \oplus \mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}) \tag{3.16}$$

as the composition of the canonical isomorphism

$$(\mathrm{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde\tau \otimes_{\mathscr{O}} M_\tau) \oplus \mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^{\gamma,*}$$
$$\xrightarrow{\sim} \mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}) \oplus (\mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^{\gamma,*},$$

and the isomorphism

$$\mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}) \oplus (\mathrm{Ind}_{\Gamma_{\tau^\gamma}}^{\Gamma_{F_w}}(\tilde\tau^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^{\gamma,*}$$

$$\xrightarrow{\sim} \operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau) \oplus \operatorname{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}}(\tilde{\tau}^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee})$$

given by the matrix

$$\begin{pmatrix} 0 & -\chi(\gamma)\operatorname{Ind}^{\gamma^{-2}} \\ 1 & 0 \end{pmatrix}$$

in which $\operatorname{Ind}^{\gamma^{-2}}$ is the isomorphism (3.15). We now let Ψ_τ be the matrix representing the isomorphism

$$(\operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau) \oplus \operatorname{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}}(\tilde{\tau}^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^*$$
$$\xrightarrow{\sim} (\operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau) \oplus \operatorname{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}}(\tilde{\tau}^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}))^\gamma$$

induced from (3.16) by duality.

For $\tau \in \mathfrak{T}_2^\heartsuit \sqcup \mathfrak{T}_3^\heartsuit$, let h be the element in Construction 3.25. Put $\gamma' \coloneqq h\gamma$, which is an element in $\tilde{\Gamma}_\tau \backslash \Gamma_\tau$. The homomorphism $\varrho_\tau \colon \tilde{\mathrm{T}}_\tau \to \mathscr{G}_{m_\tau}(R)$ induces an isomorphism $M_\tau^{\gamma'} \xrightarrow{\sim} M_\tau^\vee$ by Lemma 2.3 (1), which induces an isomorphism $M_\tau^{\gamma,\vee} \xrightarrow{\sim} M_\tau^{h^{-1}}$. On the other hand, by Lemma 3.27, we have an isomorphism $\tilde{\tau}^{\gamma,*} \simeq \tilde{\tau}^{h^{-1}}$. Thus, we obtain an isomorphism

$$(\operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau))^{\gamma,*} \xrightarrow{\sim} \operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau) \tag{3.17}$$

as the composition of the canonical isomorphism

$$(\operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau))^{\gamma,*} \xrightarrow{\sim} \operatorname{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}}(\tilde{\tau}^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}),$$

the isomorphism

$$\operatorname{Ind}_{\Gamma_{\tau\gamma}}^{\Gamma_{F_w}}(\tilde{\tau}^{\gamma,*} \otimes_{\mathscr{O}} M_\tau^{\gamma,\vee}) \xrightarrow{\sim} \operatorname{Ind}_{\Gamma_{\tau h^{-1}}}^{\Gamma_{F_w}}(\tilde{\tau}^{h^{-1}} \otimes_{\mathscr{O}} M_\tau^{h^{-1}})$$

specified above, and the isomorphism

$$\operatorname{Ind}_{\Gamma_{\tau h^{-1}}}^{\Gamma_{F_w}}(\tilde{\tau}^{h^{-1}} \otimes_{\mathscr{O}} M_\tau^{h^{-1}}) \xrightarrow{\sim} \operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau)$$

given by the action of h^{-1}. We now let Ψ_τ be the matrix representing the isomorphism

$$(\operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau))^* \xrightarrow{\sim} (\operatorname{Ind}_{\Gamma_\tau}^{\Gamma_{F_w}}(\tilde{\tau} \otimes_{\mathscr{O}} M_\tau))^\gamma$$

induced from (3.17) by duality.

Finally, we put $\Psi \coloneqq \bigoplus_{\tau \in \mathfrak{T}^\heartsuit} \Psi_\tau$. Then (3.14) follows by construction. In other words, we have assigned a lifting r from the tuple $(\varrho_\tau)_{\tau \in \mathfrak{T}^\heartsuit}$. It is straightforward to check that such assignment is inverse to the assignment in the proposition. The proposition follows. $\qquad\square$

From now on, E will not necessarily be subject to Assumption 3.24. Using Proposition 3.28, we can define minimally ramified liftings of $\bar{r}$ for E in general.

Definition 3.29 *Suppose that $l \geq N$. Choose a finite unramified extension $E_\dagger$ of E contained in $\overline{\mathbb{Q}}_\ell$ that satisfies Assumption 3.24, with the ring of integers $\mathscr{O}_\dagger$ and the residue field $k_\dagger$. We also keep the choices of $\gamma \in \Gamma_{F_v^+} \backslash \Gamma_{F_w}$, those in Construction 3.25, ι_τ, and b_τ, as in Proposition 3.28 (with respect to E).*

We say that a lifting r of $\bar{r}$ to some object R of $\mathscr{C}_{\mathscr{O}}$ is minimally ramified *if in the tuple $(\varrho_\tau)_{\tau \in \mathfrak{T}^\heartsuit}$ corresponding to the lifting $r \otimes_{\mathscr{O}} \mathscr{O}_\dagger$ from Proposition 3.28, every homomorphism ϱ_τ is a minimally ramified lifting of $\bar{\varrho}_\tau$ in the following sense.*

(1) *For $\tau \in \mathfrak{T}_1^{\heartsuit}$, minimally ramified liftings of $\bar{\varrho}_\tau$ is defined in Definition* 3.18 (*which is equivalent to* [14, Definition 2.4.14]).

(2) *For $\tau \in \mathfrak{T}_2^{\heartsuit}$, note that $\tilde{\mathrm{T}}_\tau$ is isomorphic to the q_τ-tame group for some power q_τ of $\|v\|$ under which the subgroup T_τ is the q_τ^2-tame group. Thus, we may define minimally ramified liftings of $\bar{\varrho}_\tau$ using Definition* 3.18 (*with respect to the similitude character $\eta_v^{\mu_\tau}$, which is trivial on T_τ*);

(3) *For $\tau \in \mathfrak{T}_3^{\heartsuit}$, note that $\tilde{\mathrm{T}}_\tau \simeq \mathrm{T}_\tau \times \mathbb{Z}/2\mathbb{Z}$. Then, by Lemma* 2.3, *we may regard the homomorphism ϱ_τ as a continuous homomorphism $\varrho_\tau \colon \mathrm{T}_\tau \to G(R)$, where G is a symplectic* (*resp. orthogonal*) *group of rank m_τ if μ_τ is 0 (*resp.* 1). *Thus, we may define minimally ramified liftings of $\bar{\varrho}_\tau$ using* [5, Definition 5.4].

Remark 3.30   It is straightforward to check that Definition 3.29 do not depend on the choices of $E_\dagger$, $\gamma \in \Gamma_{F_v^+} \setminus \Gamma_{F_w}$, those in Construction 3.25, ι_τ, and b_τ.

Now we allow v to be a nonarchimedean place of F^+ that is not above ℓ, but not necessarily nonsplit in F. Again, we consider a pair $(\bar{r}, \chi)$ from Notation 3.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F$.

Corollary 3.31   *Let w be a place of F above v and suppose that $\ell \nmid \prod_{i=1}^N (q_w^i - 1)$. Let $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ be an isomorphism, and R an object of $\mathscr{C}_{\mathscr{O}}$ contained in $\overline{\mathbb{Q}}_\ell$. Let r_1 and r_2 be two R-valued liftings of $\bar{r}$. If Π_1 and Π_2 are two irreducible admissible representations of $\mathrm{GL}_N(F_w)$ such that $r_i^\natural \otimes_R \overline{\mathbb{Q}}_\ell$ is the induced representation of $\mathrm{WD}(\iota_\ell \Pi_i)$ for $i = 1, 2$, then Π_1 and Π_2 are in the same Bernstein component.*

Proof   Choose a finite unramified extension $E_\dagger$ of E contained in $\overline{\mathbb{Q}}_\ell$ satisfying Assumption 3.24, with the ring of integers $\mathscr{O}_\dagger$ and the residue field $k_\dagger$. Let $R_\dagger$ be the subring of $\overline{\mathbb{Q}}_\ell$ generated by R and $\mathscr{O}_\dagger$, which is an object of $\mathscr{C}_{\mathscr{O}_\dagger}$. Then both $r_1 \otimes_R R_\dagger$ and $r_2 \otimes_R R_\dagger$ are liftings of $\bar{r} \otimes_k k_\dagger$. By Proposition 3.28 (resp. Lemma 3.23) when v is nonsplit (resp. split) in F and the condition that $\ell \nmid \prod_{i=1}^N (q_w^i - 1)$, $\mathrm{WD}(\iota_\ell \Pi_1)|_{\mathrm{I}_{F_w}}$ and $\mathrm{WD}(\iota_\ell \Pi_2)|_{\mathrm{I}_{F_w}}$ are conjugate. Then it is well-known that Π_1 and Π_2 are in the same Bernstein component (see, for example, [29, Lemma 3.2] for a proof). $\square$

Definition 3.32   *When $\ell \geq N$, we define $\mathscr{D}^{\min}$ to be the local deformation problem of $\bar{r}$ that classifies minimally ramified liftings in the sense of Definition* 3.29 (*resp.* [14, Definition 2.4.14]) *when v is nonsplit* (*resp. split*) *in F.*

Proposition 3.33   *We have*

(1) *The ring $\mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ is a reduced local complete intersection, flat and of pure relative dimension N^2 over $\mathscr{O}$.*

(2) *Every irreducible component of $\mathrm{Spf}\, \mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ is a local deformation problem* (*Definition* 3.3).

(3) *When $\ell \geq N$, $\mathscr{D}^{\min}$ is an irreducible component of $\mathrm{Spf}\, \mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ and is formally smooth over $\mathrm{Spf}\, \mathscr{O}$ of pure relative dimension N^2.*

Proof   For this proposition, we may assume that E satisfies Assumption 3.24.

For (1), when v splits in F, this is [26, Theorem 2.5]. Thus, we may assume that v is nonsplit in F. By Proposition 3.28, $\mathsf{R}_{\bar{r}}^{\mathrm{loc}}$ is a power series ring over

$$\widehat{\bigotimes_{\tau \in \mathfrak{T}^{\heartsuit}}} \mathsf{R}_{\bar{\varrho}_\tau}^{\mathrm{loc}}.$$

We now claim that for every $\tau \in \mathfrak{T}^\heartsuit$, $\mathsf{R}^{\mathrm{loc}}_{\bar\varrho_\tau}$ a local complete intersection, flat and equidimensional. Indeed, for $\tau \in \mathfrak{T}^\heartsuit_1$, this is [26, Theorem 2.5]; for $\tau \in \mathfrak{T}^\heartsuit_2$, this is Corollary 3.17; for $\tau \in \mathfrak{T}^\heartsuit_3$, this is Proposition 3.16 for G a symplectic or orthogonal group with the trivial similitude character. On the other hand, by [3, Theorem 3.3.2] or [6, Theorem 1], we know that $\mathsf{R}^{\mathrm{loc}}_{\bar r}[1/\ell]$ is reduced and of pure dimension $\dim \mathscr{G}_N = N^2$. Thus, $\mathsf{R}^{\mathrm{loc}}_{\bar r}$ is a local complete intersection, flat and of pure relative dimension N^2 over $\mathscr{O}$. Since $\mathsf{R}^{\mathrm{loc}}_{\bar r}$ is generically reduced and Cohen–Macaulay, it is reduced. (1) is proved.

For (2), take an irreducible component $\mathscr{D}$ of $\mathrm{Spf}\,\mathsf{R}^{\mathrm{loc}}_{\bar r}$, and let $\mathscr{L}_N$ be the formal completion of $\mathrm{GL}_{N,\mathscr{O}}$ along the unit section. Then the conjugation action induces a homomorphism $\mathscr{L}_N \times_{\mathrm{Spf}\,\mathscr{O}} \mathscr{D} \to \mathrm{Spf}\,\mathsf{R}^{\mathrm{loc}}_{\bar r}$ whose image contains $\mathscr{D}$. Since $\mathscr{L}_N$ is irreducible, the image is irreducible, hence has to be $\mathscr{D}$. In other words, $\mathscr{D}$ is a local deformation problem.

For (3), since $\mathscr{D}^{\mathrm{min}}$ is Zariski closed in $\mathrm{Spf}\,\mathsf{R}^{\mathrm{loc}}_{\bar r}$ from its definition, it suffices to show that $\mathscr{D}^{\mathrm{min}}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension N^2. When v splits in F, this is [14, Corollary 2.4.21]. Thus, we may assume that v is nonsplit in F. For $\tau \in \mathfrak{T}^\heartsuit$, let $\mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau}$ be the local deformation problem of $\bar\varrho_\tau$ classifying minimally ramified liftings of $\bar\varrho_\tau$ in various cases in Definition 3.29. By Proposition 3.28 and Definition 3.29, $\mathscr{D}^{\mathrm{min}}$ is formally smooth over

$$\prod_{\tau \in \mathfrak{T}^\heartsuit} \mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau}.$$

We claim that for every $\tau \in \mathfrak{T}^\heartsuit$, $\mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$. Indeed, for $\tau \in \mathfrak{T}^\heartsuit_1$, this is [14, Lemma 2.4.19]; for $\tau \in \mathfrak{T}^\heartsuit_2$, this is Proposition 3.19; for $\tau \in \mathfrak{T}^\heartsuit_3$, this is a part of [5, Theorem 1.1]. Thus, $\mathscr{D}^{\mathrm{min}}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$.

It remains to compute the dimension. By (3.1), it suffices to show that

$$\dim_k \mathrm{L}(\mathscr{D}^{\mathrm{min}}) = \dim_k \mathrm{H}^0(F^+_v, \mathrm{ad}\,\bar r). \tag{3.18}$$

For every $\tau \in \mathfrak{T}^\heartsuit$, let $\mathrm{L}(\mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau})$ be the tangent space of the deformation problem $\mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau}$, which is a subspace of $\mathrm{H}^1(\mathrm{T}_\tau, \mathrm{ad}\,\bar\varrho)$ (resp. $\mathrm{H}^1(\tilde{\mathrm{T}}_\tau, \mathrm{ad}\,\bar\varrho)$) if $\tau \in \mathfrak{T}^\heartsuit_1$ (resp. $\tau \in \mathfrak{T}^\heartsuit_2 \sqcup \mathfrak{T}^\heartsuit_3$). By Proposition 3.28, we have

$$\dim_k \mathrm{L}(\mathscr{D}^{\mathrm{min}}) = \sum_{\tau \in \mathfrak{T}^\heartsuit} \dim_k \mathrm{L}(\mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau}). \tag{3.19}$$

We claim that

$$\dim_k \mathrm{L}(\mathscr{D}^{\mathrm{min}}_{\bar\varrho_\tau}) = \begin{cases} \dim_k \mathrm{H}^0(\mathrm{T}_\tau, \mathrm{ad}\,\bar\varrho_\tau) & \text{if } \tau \in \mathfrak{T}^\heartsuit_1; \\ \dim_k \mathrm{H}^0(\tilde{\mathrm{T}}_\tau, \mathrm{ad}\,\bar\varrho_\tau) & \text{if } \tau \in \mathfrak{T}^\heartsuit_2 \sqcup \mathfrak{T}^\heartsuit_3. \end{cases} \tag{3.20}$$

Indeed, for $\tau \in \mathfrak{T}^\heartsuit_1$, this is [14, Corollary 2.4.20]; for $\tau \in \mathfrak{T}^\heartsuit_2$, this is Corollary 3.20; for $\tau \in \mathfrak{T}^\heartsuit_3$, this is a part of [5, Theorem 1.1] as $\dim_k \mathrm{H}^0(\tilde{\mathrm{T}}_\tau, \mathrm{ad}\,\bar\varrho_\tau) = \dim_k \mathrm{H}^0(\mathrm{T}_\tau, \mathrm{ad}^0\,\bar\varrho_\tau)$. By (3.13) (for $\bar r$) and the fact that $\tau \otimes_k M_\tau(\bar r)$ is canonically the τ-isotypic component of $\bar r^\natural|_{\mathrm{P}_{F_w}}$, we have

$$\mathrm{H}^0(F^+_v, \mathrm{ad}\,\bar r) \simeq \left(\bigoplus_{\tau \in \mathfrak{T}^\heartsuit_1} \mathrm{H}^0(\mathrm{T}_\tau, \mathrm{ad}\,\bar\varrho_\tau) \right) \oplus \left(\bigoplus_{\tau \in \mathfrak{T}^\heartsuit_2 \sqcup \mathfrak{T}^\heartsuit_3} \mathrm{H}^0(\tilde{\mathrm{T}}_\tau, \mathrm{ad}\,\bar\varrho_\tau) \right).$$

Together with (3.19) and (3.20), we obtain (3.18).

The proposition is proved. $\qquad\square$

3.5 Level-raising Deformations

In this subsection, we discuss level-raising deformations. Assume $\ell \geq N \geq 2$. We take a nonarchimedean place v of F^+ that is inert in F and not above ℓ. Let w be the unique place of F above v. Recall that we have $\mathrm{T}_v = \Gamma_{F_v^+}/\mathrm{P}_{F_v^+}$ and $\mathrm{T}_w = \Gamma_{F_w}/\mathrm{P}_{F_w}$. Then T_v is isomorphic to the q-tame group and the subgroup T_w is the q^2-tame group (Definition 3.15), where $q = \|v\|$.

We consider a pair $(\bar{r}, \chi)$ from Notation 3.1 with $\tilde{\Gamma} = \Gamma_{F_v^+}$ and $\Gamma = \Gamma_{F_v^+} \cap \Gamma_F = \Gamma_{F_w}$, such that $\bar{r}$ is unramified and $\chi = \eta_v^\mu \epsilon_{\ell,v}^{1-N}$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$. Then by Lemma 3.23 (1), every lifting r of $\bar{r}$ to an object R of $\mathscr{C}_\mathscr{O}$ factors through T_v. In particular, we may apply the discussion in §3.3 to the pair $(\bar{r}, \chi)$.

Now assume $\ell \nmid (q^2 - 1)$ and that the generalized eigenvalues of $\bar{r}^\natural(\phi_w)$ in $\overline{\mathbb{F}}_\ell$ contain the pair $\{q^{-N}, q^{-N+2}\}$ exactly once. By Lemma 3.21 (1), for every lifting r of $\bar{r}$ to an object R of $\mathscr{C}_\mathscr{O}$, we have a canonical decomposition

$$R^{\oplus N} = M_0 \oplus M_1 \tag{3.21}$$

of free R-modules stable under the action of $r^\natural(\phi_w)$, such that if we write $P_0(T)$ for the characteristic polynomial of $r^\natural(\phi_w)$ on M_0, then $P_0(T) \equiv (T - q^{-N})(T - q^{-N+2}) \bmod \mathfrak{m}_R$.

Definition 3.34 *Let $(\bar{r}, \chi)$ be as above. We define $\mathscr{D}^{\mathrm{mix}}$ to be the local deformation problem of $\bar{r}$ (Definition 3.3) that classifies liftings r to an object R of $\mathscr{C}_\mathscr{O}$ such that in the decomposition (3.21), $r^\natural(\mathrm{I}_{F_w})$ preserves M_0 and acts trivially on M_1.*[6]

We define

(1) $\mathscr{D}^{\mathrm{unr}}$ to be the local deformation problem contained in $\mathscr{D}^{\mathrm{mix}}$ such that the action of $r^\natural(\mathrm{I}_{F_w})$ on M_0 is also trivial;

(2) $\mathscr{D}^{\mathrm{ram}}$ to be the local deformation problem contained in $\mathscr{D}^{\mathrm{mix}}$ such that the equality $P_0(T) = (T - q^{-N})(T - q^{-N+2})$ holds in $R[T]$.

It is clear that $\mathscr{D}^{\mathrm{unr}}$ coincides with $\mathscr{D}^{\mathrm{min}}$ from Definition 3.32.

Proposition 3.35 *Suppose that $\ell \nmid (q^2 - 1)$ and that the generalized eigenvalues of $\bar{r}^\natural(\phi_w)$ in $\overline{\mathbb{F}}_\ell$ contain the pair $\{q^{-N}, q^{-N+2}\}$ exactly once. Then the formal scheme $\mathscr{D}^{\mathrm{mix}}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}[[x_0, x_1]]/(x_0 x_1)$ of pure relative dimension $N^2 - 1$ such that the irreducible components defined by $x_0 = 0$ and $x_1 = 0$ are $\mathscr{D}^{\mathrm{unr}}$ and $\mathscr{D}^{\mathrm{ram}}$, respectively. In particular, $\mathscr{D}^{\mathrm{ram}}$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension N^2.*

Proof We fix an isomorphism $\mathrm{T}_v \simeq \mathrm{T}_q = t^{\mathbb{Z}_\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$ such that $\phi_w = \phi_q^2$. We write $k^{\oplus N} = \bar{M}_0 \oplus \bar{M}_1$ such that $\bar{r}^\natural(\phi_q^2)$ has eigenvalues q^{-N} and q^{-N+2} on $\bar{M}_0$. Without loss of generality, we may assume that $\bar{M}_0$ is spanned by the first two factors and $\bar{M}_1$ is spanned by the last $N-2$ factors. Thus, we obtain two unramified homomorphisms $\bar{r}_0 \colon \mathrm{T}_q \to \mathscr{G}_2(k)$ and $\bar{r}_1 \colon \mathrm{T}_q \to \mathscr{G}_{N-2}(k)$. Let $\mathscr{D}_0$ be the local deformation problem of $\bar{r}_0$ classifying liftings of $\bar{r}_0$. Let $\mathscr{D}_1$ be the local deformation problem of $\bar{r}_1$ classifying unramified liftings.

Suppose that $N \geq 3$. We say that lifting r of $\bar{r}$ to an object R of $\mathscr{C}_\mathscr{O}$ is *standard* if

$$r^\natural(t) = \begin{pmatrix} A_0 & 0 \\ 0 & 1_{N-2} \end{pmatrix}, \quad r(\phi_q) = \left(\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}, (-1)^{\mu+1} q^{1-N}, \mathfrak{c} \right)$$

---

6) Note that since $\ell \nmid (q^2 - 1)$, the characteristic polynomial of $r^\natural(t)$ on M_0 is automatically $(T - 1)^2$ for every $t \in \mathrm{I}_{F_w}$.

for some $A_0, B_0 \in \mathrm{GL}_2(R)$ and $B_1 \in \mathrm{GL}_{N-2}(R)$. Let $\mathscr{D}_{0,1}^{\mathrm{mix}} \subseteq \mathscr{D}^{\mathrm{mix}}$ be the locus of standard liftings. Then we have a natural isomorphism

$$\mathscr{D}_{0,1}^{\mathrm{mix}} \simeq \mathscr{D}_0 \times_{\mathrm{Spf}\,\mathscr{O}} \mathscr{D}_1$$

of formal schemes over $\mathrm{Spf}\,\mathscr{O}$.

For $n \geq 1$, denote by $\mathscr{L}_n$ the formal completion of $\mathrm{GL}_{n,\mathscr{O}}$ along the unit section. Then $\mathscr{L}_N$ acts on $\mathscr{D}^{\mathrm{mix}}$ by conjugation. We claim that $\mathscr{D}_{0,1}^{\mathrm{mix}}$ generates $\mathscr{D}^{\mathrm{mix}}$ under the action of $\mathscr{L}_N$. For this, it suffices to show that for every lifting r of $\bar{r}$ to an object R of $\mathscr{C}_{\mathscr{O}}$, the maps

$$B \colon M_0 \to R^{\oplus N} \to M_1, \quad B \colon M_1 \to R^{\oplus N} \to M_0$$

induced by B from Lemma 2.3 (1) for $\gamma = \phi_q$, are both zero. Since the two maps intertwine the actions $r^\natural$ and $r^{\natural,\vee} \otimes \epsilon_\ell^{1-N}$ of T_{q^2}, it suffices to show that the generalized eigenvalues of $r_0^{\natural,\vee} \otimes \epsilon_\ell^{1-N}(\phi_q^2)$ and the generalized eigenvalues of $r_1^\natural(\phi_q^2)$ are disjoint. However, this follows from the condition that the generalized eigenvalues of $\bar{r}^\natural(\phi_w)$ in $\overline{\mathbb{F}}_\ell$ contain the pair $\{q^{-N}, q^{-N+2}\}$ exactly once.

The above claim induces a canonical isomorphism

$$\mathscr{D}_{0,1}^{\mathrm{mix}} \times_{\mathrm{Spf}\,\mathscr{O}} (\mathscr{L}_2 \times_{\mathrm{Spf}\,\mathscr{O}} \mathscr{L}_{N-2} \backslash \mathscr{L}_N) \xrightarrow{\sim} \mathscr{D}^{\mathrm{mix}}.$$

By Proposition 3.19, $\mathscr{D}_1$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension $(N-2)^2$. Now since $\mathscr{L}_2 \times_{\mathrm{Spf}\,\mathscr{O}} \mathscr{L}_{N-2} \backslash \mathscr{L}_N$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension $N^2 - (N-2)^2 - 4$, it suffices to prove the proposition for $N = 2$.

Now we assume $N = 2$. After changing a basis, we may assume

$$\bar{r}(\phi_q) = (\bar{B}, (-1)^{\mu+1}q^{1-N}, \mathfrak{c}), \quad \bar{B} = \begin{pmatrix} 0 & (-1)^{\mu+1} \\ q & 0 \end{pmatrix}.$$

Then we have

$$\bar{r}^\natural(\phi_q^2) = (-1)^{\mu+1}q^{1-N}\bar{B}\,{}^\mathrm{t}\bar{B}^{-1} = \begin{pmatrix} q^{-N} & 0 \\ 0 & q^{-N+2} \end{pmatrix}.$$

For every object R of $\mathscr{C}_{\mathscr{O}}$, the set $\mathscr{D}^{\mathrm{mix}}(R)$ is bijective to the set of pairs (B, X) where $B \in \mathrm{GL}_2(R)$ and $X \in \mathrm{M}_2(\mathfrak{m}_R)$ satisfying $B \equiv \bar{B} \bmod \mathfrak{m}_R$, that the characteristic polynomial of X is T^2, and the relation

$$B\,{}^\mathrm{t}XB^{-1} = -qX. \tag{3.22}$$

Indeed, the bijection is given by $r(\phi_q) = (B, (-1)^{\mu+1}q^{1-N}, \mathfrak{c})$ and $r^\natural(t) = 1_2 + X$. We let $\mathscr{D}_0^{\mathrm{mix}}$ be the subscheme of $\mathscr{D}^{\mathrm{mix}}$ defined by the condition that $r^\natural(\phi_q^2) = (-1)^{\mu+1}q^{1-N}B\,{}^\mathrm{t}B^{-1}$ is a diagonal matrix. Take a lifting $r \in \mathscr{D}_0^{\mathrm{mix}}(R)$ corresponding to the pair (B, X); we must have

$$B = \begin{pmatrix} 0 & (-1)^{\mu+1}(1+x) \\ q(1+y) & 0 \end{pmatrix},$$

$$r^\natural(\phi_q^2) = \begin{pmatrix} q^{-N}(1+x)(1+y)^{-1} & 0 \\ 0 & q^{-N+2}(1+y)(1+x)^{-1} \end{pmatrix}$$

for some $x, y \in \mathfrak{m}_R$. Then by (3.22), $X = \begin{pmatrix} 0 & 0 \\ x_0 & 0 \end{pmatrix}$ for some $x_0 \in \mathfrak{m}_R$ satisfying $(x - y)x_0 = 0$. Put $x_1 := x - y$. Then we obtain an isomorphism

$$\mathscr{D}_0^{\mathrm{mix}} \simeq \mathrm{Spf}\, \mathscr{O}[[x_0, x_1, y]]/(x_0 x_1)$$

such that

- $x_0 = 0$ if and only if r is unramified;
- $x_1 = 0$ if and only if $P_0(T) = (T - q^{-N})(T - q^{-N+2})$, where P_0 is the characteristic polynomial of $r^\natural(\phi_w) = r^\natural(\phi_q^2)$.

Finally, note that $\mathscr{L}_2$ acts on $\mathscr{D}^{\mathrm{mix}}$ by conjugation, which induces a canonical isomorphism

$$\mathscr{D}_0^{\mathrm{mix}} \times_{\mathrm{Spf}\,\mathscr{O}} (\mathscr{L}_1 \times_{\mathrm{Spf}\,\mathscr{O}} \mathscr{L}_1 \backslash \mathscr{L}_2) \xrightarrow{\sim} \mathscr{D}^{\mathrm{mix}}.$$

The proposition (for $N = 2$) follows as $\mathscr{L}_1 \times_{\mathrm{Spf}\,\mathscr{O}} \mathscr{L}_1 \backslash \mathscr{L}_2$ is formally smooth over $\mathrm{Spf}\,\mathscr{O}$ of pure relative dimension 2. The entire proposition is now proved. □

3.6 An Almost Minimal R=T Theorem

In this subsection, we prove a version of the R=T theorem for a global Galois representation. Assume $N \geq 2$.

We take an element $\xi = (\xi_\tau)_\tau \in (\mathbb{Z}_{\leq}^N)^{\Sigma_\infty}$ satisfying $\xi_{\tau,i} = -\xi_{\tau^c, N+1-i}$ for every τ and i. Assume $\ell \geq (b_\xi - a_\xi) + 2$ (Notation 2.7) and that ℓ is unramified in F. Fix an isomorphism $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$ and assume that the complex algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_N$ determined by ξ can be defined over $\iota_\ell^{-1}E$.

We consider a pair $(\bar{r}, \chi)$ from Notation 3.1 with $\tilde{\Gamma} = \Gamma_{F^+}$ and $\Gamma = \Gamma_F$, in which $\chi = \eta^\mu \epsilon_\ell^{1-N}$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$. We take two finite sets $\Sigma_{\min}^+$ and Σ_{lr}^+ of nonarchimedean places of F^+ such that

- $\Sigma_{\min}^+$, Σ_{lr}^+, and Σ_ℓ^+ are mutually disjoint;
- $\Sigma_{\min}^+$ contains Σ_{bad}^+;
- every place $v \in \Sigma_{\mathrm{lr}}^+$ is inert in F and satisfies $\ell \nmid (\|v\|^2 - 1)$.

Definition 3.36 *We say that $\bar{r}$ is* rigid *for $(\Sigma_{\min}^+, \Sigma_{\mathrm{lr}}^+)$ if the following are satisfied:*

(1) For v in $\Sigma_{\min}^+$, every lifting of $\bar{r}_v$ is minimally ramified (Definition 3.29).

(2) For v in Σ_{lr}^+, the generalized eigenvalues of $\bar{r}_v^\natural(\phi_w)$ in $\overline{\mathbb{F}}_\ell$ contain the pair $\{\|v\|^{-N},$ $\|v\|^{-N+2}\}$ exactly once, where w is the unique place of F above v.

(3) For v in Σ_ℓ^+, $\bar{r}_v^\natural$ is regular Fontaine–Laffaille crystalline (Definition 3.11).

(4) For a nonarchimedean place v of F^+ not in $\Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+ \cup \Sigma_\ell^+$, the homomorphism $\bar{r}_v$ is unramified.

Suppose now that $\bar{r}$ is rigid for $(\Sigma_{\min}^+, \Sigma_{\mathrm{lr}}^+)$. Consider a global deformation problem (Definition 3.6)

$$\mathscr{S} := (\bar{r}, \eta^\mu \epsilon_\ell^{1-N}, \Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+ \cup \Sigma_\ell^+, \{\mathscr{D}_v\}_{v \in \Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+ \cup \Sigma_\ell^+})$$

where

- for $v \in \Sigma_{\min}^+$, $\mathscr{D}_v$ is the local deformation problem classifying all liftings of $\bar{r}_v$;
- for $v \in \Sigma_{\mathrm{lr}}^+$, $\mathscr{D}_v$ is the local deformation problem $\mathscr{D}^{\mathrm{ram}}$ of $\bar{r}_v$ from Definition 3.34;
- for $v \in \Sigma_\ell^+$, $\mathscr{D}_v$ is the local deformation problem $\mathscr{D}^{\mathrm{FL}}$ of $\bar{r}_v$ from Definition 3.12.

Then we have the *global universal deformation ring* $\mathrm{R}_{\mathscr{S}}^{\mathrm{univ}}$ from Proposition 3.7.

Remark 3.37 It is possible that $\bar{r}$ is rigid for two pairs $(\Sigma_{\min}^+, \Sigma_{\mathrm{lr}}^+)$ and $(\Sigma_{\min}^{+\prime}, \Sigma_{\mathrm{lr}}^{+\prime})$. Then $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$ and $\mathsf{R}_{\mathscr{S}'}^{\mathrm{univ}}$ are different in general, where $\mathscr{S}'$ denotes the corresponding global deformation problem for $(\Sigma_{\min}^{+\prime}, \Sigma_{\mathrm{lr}}^{+\prime})$.

Now we state an R=T theorem. Let V be a hermitian space over F of rank N such that V_v is not split for $v \in \Sigma_{\mathrm{lr}}^+$. Let $(p_{\underline{\tau}}, q_{\underline{\tau}})_{\underline{\tau} \in \Sigma_\infty^+}$ be the signature of V, and put $d(V) := \sum_{\underline{\tau} \in \Sigma_\infty^+} p_{\underline{\tau}} q_{\underline{\tau}}$.

Take a self-dual $\prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+} O_{F_v}$-lattice Λ in $V \otimes_F \mathbb{A}_F^{\Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+}$ and a neat open compact subgroup K of $\mathrm{U}(V)(\mathbb{A}_{F^+}^\infty)$ of the form

$$
\mathrm{K} = \prod_{v \in \Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+} \mathrm{K}_v \times \prod_{v \notin \Sigma_\infty^+ \cup \Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+} \mathrm{U}(\Lambda)(O_{F_v^+})
$$

in which K_v is special maximal for $v \in \Sigma_{\mathrm{lr}}^+$. We have the system of (complex) Shimura varieties $\{\mathrm{Sh}(V, \mathrm{K}') \mid \mathrm{K}' \subseteq \mathrm{K}\}$ associated to $\mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(V)$ indexed by open compact subgroups $\mathrm{K}' \subseteq \mathrm{K}$, which are quasi-projective smooth complex schemes of dimension $d(V)$.[7] The element ξ gives rise to a continuous homomorphism

$$
\prod_{v \in \Sigma_\ell^+} \mathrm{U}(\Lambda)(O_{F_v^+}) \to \mathrm{GL}_{\mathscr{O}}(L_\xi)
$$

where L_ξ is a finite free $\mathscr{O}$-module, hence induces an $\mathscr{O}$-linear (étale) local system $\mathscr{L}_\xi$ on $\mathrm{Sh}(V, \mathrm{K}')$ for every $\mathrm{K}' \subseteq \mathrm{K}$, compatible under restriction.

Let Σ^+ be a finite set of nonarchimedean places of F^+ containing $\Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+$. In particular, we have the abstract unitary Hecke algebra $\mathbb{T}_N^{\Sigma^+}$ (Definition 2.6). Let $\phi \colon \mathbb{T}_N^{\Sigma^+} \to k$ be the homomorphism such that

- for every nonarchimedean place v of F^+ not in Σ^+ that induces one place w of F, we have $\phi|_{\mathbb{T}_{N,v}} = \phi_{\boldsymbol{\alpha}}$ (Construction 2.5) where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ is the unitary abstract Hecke parameter at v (Definition 2.4) satisfying that $\{\alpha_1 \|v\|^{N-1}, \ldots, \alpha_N \|v\|^{N-1}\}$ are the generalized eigenvalues of $\bar{r}_v^\natural(\phi_w^{-1})$ in $\overline{\mathbb{F}}_\ell$;

- for every nonarchimedean place v of F^+ not in Σ^+ that splits into two places w_1 and w_2 of F, we have $\phi|_{\mathbb{T}_{N,v}} = \phi_{\boldsymbol{\alpha}}$ (Construction 2.5) where $\boldsymbol{\alpha} = ((\alpha_{1,1}, \ldots, \alpha_{1,N}); (\alpha_{2,1}, \ldots, \alpha_{2,N}))$ is the unitary abstract Hecke parameter at v (Definition 2.4) satisfying that for $i = 1, 2$, $\{\alpha_{i,1} \|v\|^{\frac{N-1}{2}}, \ldots, \alpha_{i,N} \|v\|^{\frac{N-1}{2}}\}$ are the generalized eigenvalues of $\bar{r}_v^\natural(\phi_{w_i}^{-1})$ in $\overline{\mathbb{F}}_\ell$.

We write $\mathfrak{m}$ for the kernel of ϕ.

Theorem 3.38 *Suppose that* $\Sigma_{\mathrm{lr}}^+ = \emptyset$ *if N is odd. Under the above setup, we assume*

  (D0) *(already assumed)* ℓ *is odd,* $\ell \geq (b_\xi - a_\xi) + 2$, *and* ℓ *is unramified in* F;

  (D1) $\ell \geq 2(N + 1)$;

  (D2) $\bar{r}^\natural|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ *is absolutely irreducible*;

  (D3) $\bar{r}$ *is rigid for* $(\Sigma_{\min}^+, \Sigma_{\mathrm{lr}}^+)$ *(Definition 3.36)*;

  (D4) *for every finite set* $\Sigma^{+\prime}$ *of nonarchimedean places of F^+ containing* Σ^+, *and every*

---

7) Strictly, we need to choose a CM type Φ of F to define the Deligne homomorphism for the Shimura varieties. More precisely, the Deligne homomorphism $\mathrm{h} \colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m \to (\mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(V)) \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\underline{\tau} \in \Sigma_\infty^+} \mathrm{U}(V_{\underline{\tau}})$ is the one such that for $z \in (\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m)(\mathbb{R}) = \mathbb{C}^\times$, the $\underline{\tau}$-component of $\mathrm{h}(z)$ equals $1_{p_{\underline{\tau}}} \oplus (\overline{z}/z) 1_{q_{\underline{\tau}}}$ after we identify $\mathrm{U}(V)(F_{\underline{\tau}_\infty}^+)$ with a subgroup of $\mathrm{GL}_N(\mathbb{C})$ via a complex basis of $V \otimes_{F, \tau} \mathbb{C}$ under which the hermitian form is given by $1_{p_{\underline{\tau}}} \oplus -1_{q_{\underline{\tau}}}$, where τ is the unique place of F above $\underline{\tau}$ in Φ.

open compact subgroup $K' \subseteq K$ *satisfying* $K'_v = K_v$ *for* $v \notin \Sigma^{+\prime}$, *we have*

$$\mathrm{H}^d_{\text{ét}}(\mathrm{Sh}(V, K'), \mathscr{L}_\xi \otimes_{\mathscr{O}} k)_{\mathbb{T}_N^{\Sigma^{+\prime}} \cap \mathfrak{m}} = 0$$

when $d \neq d(V)$.[8)]

Let T *be the image of* $\mathbb{T}_N^{\Sigma^+}$ *in* $\mathrm{End}_{\mathscr{O}}(\mathrm{H}^{d(V)}_{\text{ét}}(\mathrm{Sh}(V, K), \mathscr{L}_\xi))$. *If* $\mathsf{T}_\mathfrak{m} \neq 0$, *then*

(1) *There is a canonical isomorphism* $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}} \xrightarrow{\sim} \mathsf{T}_\mathfrak{m}$ *of local complete intersection rings over* $\mathscr{O}$.

(2) *The* $\mathsf{T}_\mathfrak{m}$-*module* $\mathrm{H}^{d(V)}_{\text{ét}}(\mathrm{Sh}(V, K), \mathscr{L}_\xi)_\mathfrak{m}$ *is finite and free.*

(3) *We have* $\mu \equiv N \mod 2$.

The rest of this subsection is devoted to the proof of the theorem. We will use the Taylor–Wiles patching argument following [14] and [28]. Put $\mathrm{S} := \Sigma^+_{\min} \cup \Sigma^+_{\mathrm{lr}} \cup \Sigma^+_\ell$. To prove the theorem, we may replace E by a finite unramified extension in $\overline{\mathbb{Q}}_\ell$. Thus, we may assume that k contains all eigenvalues of matrices in $\bar{r}^\natural(\Gamma_F)$.

Remark 3.39 By (D0), we know that F is not contained in $F^+(\zeta_\ell)$. Thus, by [28, Theorem A.9], (D1) and (D2) imply that $\bar{r}(\mathrm{Gal}(\overline{F}/F^+(\zeta_\ell)))$ is adequate in the sense of [28, Definition 2.3].

Recall that a prime v of F^+ is called a *Taylor–Wiles prime* for the global deformation problem $\mathscr{S}$ if

- $v \notin \mathrm{S}$; v splits in F; and $\|v\| \equiv 1 \mod \ell$;
- $\bar{r}_v$ is unramified;
- $\bar{r}^\natural_v(\phi_w)$ is not a scalar and admits an eigenvalue $\bar{\alpha}_v \in k$, called *special eigenvalue*, such that $\bar{r}^\natural_v(\phi_w)$ acts semisimply on the generalized eigenspace for $\bar{\alpha}_v$, where w is the place of F above v induced by the inclusion $F \subseteq \overline{F}^+_v$.

A *Taylor–Wiles system* is a tuple $(\mathrm{Q}, \{\bar{\alpha}_v\}_{v \in \mathrm{Q}})$ where Q is a (possibly empty) finite set of Taylor–Wiles primes, and $\bar{\alpha}_v$ is a special eigenvalue for every $v \in \mathrm{Q}$. For such a system, we write $r^\natural_v = r^\bullet_v \oplus r^\circ_v$ for every $v \in \mathrm{Q}$, where $r^\bullet_v$ (resp. r°_v) is the generalized eigenspace for $\bar{\alpha}_v$ (resp. for generalized eigenvalues other than α_v). Then we have another global deformation problem (see [28, Definition 4.1])

$$\mathscr{S}(\mathrm{Q}) := (\bar{r}, \eta^\mu \epsilon_\ell^{1-N}, \mathrm{S} \cup \mathrm{Q}, \{\mathscr{D}_v\}_{v \in \mathrm{S} \cup \mathrm{Q}})$$

where $\mathscr{D}_v$ is the same as in $\mathscr{S}$ for $v \in \mathrm{S}$; and for $v \in \mathrm{Q}$, $\mathscr{D}_v$ is the local deformation problem of $\bar{r}_v$ that classifies liftings r_v such that $r^\natural_v$ is of the form $r^\bullet_v \oplus r^\circ_v$ in which $r^\bullet_v$ is a lifting of $\bar{r}^\bullet_v$ on which I_{F_w} acts by scalars, and r°_v is an unramified lifting of $\bar{r}^\circ_v$.

For each $v \in \mathrm{Q}$, we

- put $d_v := \dim_k \bar{r}^\bullet_v$;
- let $\mathrm{P}_{d_v} \subseteq \mathrm{GL}_N$ be the standard upper-triangular parabolic subgroup corresponding to the partition $(N - d_v, d_v)$;
- let κ_v be the residue field of F^+_v, and Δ_v the maximal quotient of $\kappa_v^\times$ of ℓ-power order;
- fix an isomorphism $\mathrm{K}_v \simeq \mathrm{GL}_N(O_{F^+_v})$ and denote by $\mathrm{K}_{v,0} \subseteq \mathrm{K}_v$ the parahoric subgroup corresponding to P_{d_v};

---

8) This is automatic when $d(V) = 0$, and follows from (D2) when $d(V) = 1$.

- let $K_{v,1}$ be the kernel of the canonical map

$$K_{v,0} \to P_{d_v}(\kappa_v) \to GL_{d_v}(\kappa_v) \xrightarrow{\det} \kappa_v^\times \to \Delta_v.$$

We then

- put $\Delta_Q := \prod_{v \in Q} \Delta_v$; and let $\mathfrak{a}_Q$ be the augmentation ideal of $\mathscr{O}[\Delta_Q]$;
- put $\mathfrak{m}_Q := \mathfrak{m} \cap \mathbb{T}_N^{\Sigma^+ \cup Q}$;
- put

$$K_i(Q) = \prod_{v \notin Q} K_v \times \prod_{v \in Q} K_{v,i}$$

for $i = 0, 1$, which are open compact subgroups of K.

In particular, $K_1(Q)$ is a normal subgroup of $K_0(Q)$; and we have a canonical isomorphism

$$K_0(Q)/K_1(Q) \xrightarrow{\sim} \Delta_Q. \tag{3.23}$$

Note that when $Q = \emptyset$, $K_0(Q) = K_1(Q) = K$.

For $i = 0, 1$, we put

$$H_{K_i(Q)} := \mathrm{Hom}_{\mathscr{O}}(H_{\text{ét}}^{d(V)}(\mathrm{Sh}(V, K_i(Q)), \mathscr{L}_\xi), \mathscr{O}).$$

By (3.23), $H_{K_1(Q)}$ is canonically a module over $\mathscr{O}[\Delta_Q]$.

Lemma 3.40 *Let the situation be as in Theorem* 3.38. *The $\mathscr{O}[\Delta_Q]$-module $H_{K_1(Q),\mathfrak{m}_Q}$ is finite and free. Moreover, the canonical map*

$$H_{K_1(Q),\mathfrak{m}_Q}/\mathfrak{a}_Q \to H_{K_0(Q),\mathfrak{m}_Q}$$

is an isomorphism.

Proof   For a smooth complex scheme X and an abelian group A, we denote by X^{an} the underlying complex manifold and $C_\bullet(X^{\text{an}}, A)$ the complex of singular chains for X^{an} with coefficients in A.

For $i = 0, 1$ and every positive integer m, we denote by $K_i^m(Q)$ the kernel of the composite map

$$K_i(Q) \to K \to \mathfrak{u}_m := \prod_{v \in \Sigma_\ell^+} U(\Lambda)(O_{F_v^+}/\ell^m).$$

Then $K_i^m(Q)$ acts trivially on $L_\xi \otimes_{\mathscr{O}} \mathscr{O}/\ell^m$, hence $\mathscr{L}_\xi \otimes_{\mathscr{O}} \mathscr{O}/\ell^m$ becomes constant on $\mathrm{Sh}(V, K_i^m(Q))$. By (D4), we have

$$H_{\text{ét}}^{d(V)}(\mathrm{Sh}(V, K_i(Q)), \mathscr{L}_\xi)_{\mathfrak{m}_Q} = \varprojlim_m H_{\text{ét}}^{d(V)}(\mathrm{Sh}(V, K_i^m(Q)), L_\xi \otimes_{\mathscr{O}} \mathscr{O}/\ell^m)_{\mathfrak{m}_Q}^{\mathfrak{u}_m}$$

and is $\mathscr{O}$-torsion free. By Artin's comparison theorem between the singular cohomology and the étale cohomology, the dual complex

$$\mathrm{Hom}_{\mathscr{O}}(C_\bullet(\mathrm{Sh}(V, K_i^m(Q))^{\text{an}}, L_\xi \otimes_{\mathscr{O}} \mathscr{O}/\ell^m), \mathscr{O}/\ell^m)$$

calculates $H_{\text{ét}}^\bullet(\mathrm{Sh}(V, K_i^m(Q)), L_\xi \otimes_{\mathscr{O}} \mathscr{O}/\ell^m)$. By (D4), we have

$$H_d(\mathrm{Sh}(V, K_i^m(Q))^{\text{an}}, L_\xi \otimes_{\mathscr{O}} \mathscr{O}/\ell^m)_{\mathfrak{m}_Q} = 0$$

for $d \neq d(V)$.

On the other hand, $K_i^m(Q)$ is neat, which implies that $t^{-1}U(V)(F^+)t \cap K_i^m(Q)$ has no torsion elements for every $t \in U(V)(\mathbb{A}_{F^+}^\infty)$. By (the same proof of) [20, Lemma 6.9], for every $m \geq 1$, $C_\bullet(Sh(V, K_1^m(Q))^{an}, L_\xi \otimes_\mathscr{O} \mathscr{O}/\ell^m)$ is a perfect complex of free $(\mathscr{O}/\ell^m)[\Delta_Q]$-modules; and there is a canonical isomorphism

$$C_\bullet(Sh(V, K_1^m(Q))^{an}, L_\xi \otimes_\mathscr{O} \mathscr{O}/\ell^m) \otimes_{\mathscr{O}[\Delta_Q]} \mathscr{O}[\Delta_Q]/\mathfrak{a}_Q \simeq C_\bullet(Sh(V, K_0^m(Q))^{an}, L_\xi \otimes_\mathscr{O} \mathscr{O}/\ell^m)$$

of complexes of $(\mathscr{O}/\ell^m)[\mathbb{T}_N^{\Sigma^+ \cup Q}]$-modules. It follows easily that the canonical map

$$\left(H_{K_1(Q),\mathfrak{m}_Q} \otimes_\mathscr{O} \mathscr{O}/\ell^m\right)/\mathfrak{a}_Q \to H_{K_0(Q),\mathfrak{m}_Q} \otimes_\mathscr{O} \mathscr{O}/\ell^m$$

is an isomorphism, after localizing at $\mathfrak{m}_Q$ and taking u_m-invariants.

Then the lemma follows by passing to the limit for m. $\qquad\square$

We now discuss the existence of Taylor–Wiles systems. For each $v \in S$, we have the tangent space $L(\mathscr{D}_v) \subseteq H^1(F_v^+, ad\, \bar{r})$ from Definition 3.4. Let $L(\mathscr{D}_v)^\perp \subseteq H^1(F_v^+, ad\, \bar{r}(1))$ be the annihilator of $L(\mathscr{D}_v)$ under the local Tate duality induced by the perfect pairing $ad\, \bar{r} \times ad\, \bar{r}(1) \to k(1)$ sending (x, y) to $tr(xy)$. Recall that $\Gamma_{F^+,S}$ is the Galois group of the maximal sub-extension of $\overline{F}/F^+$ that is unramified outside S. For every subset $T \subseteq S$, we define $H^1_{\mathscr{L}^\perp,T}(\Gamma_{F^+,S}, ad\, \bar{r}(1))$ to be the kernel of the natural map

$$H^1(\Gamma_{F^+,S}, ad\, \bar{r}(1)) \to \bigoplus_{v \in S\backslash T} H^1(F_v^+, ad\, \bar{r}(1))/L(\mathscr{D}_v)^\perp.$$

Recall the rings $R^{loc}_{\mathscr{S},T}$ (3.2) and $R^{\square_T}_{\mathscr{S}(Q)}$ from Proposition 3.7. Moreover, $R^{\square_T}_{\mathscr{S}(Q)}$ is naturally an algebra over $R^{loc}_{\mathscr{S},T}$.

Lemma 3.41 *Let the situation be as in Theorem* 3.38. *Let* T *be a subset of* S. *For every integer* $b \geq \dim_k H^1_{\mathscr{L}^\perp,T}(\Gamma_{F^+,S}, ad\, \bar{r}(1))$ *and every integer* $n \geq 1$, *there is a Taylor–Wiles system* $(Q_n, \{\bar{\alpha}_v\}_{v \in Q_n})$ *satisfying*

(1) $|Q_n| = b$;

(2) $\|v\| \equiv 1 \bmod \ell^n$;

(3) $R^{\square_T}_{\mathscr{S}(Q_n)}$ *can be topologically generated over* $R^{loc}_{\mathscr{S},T}$ *by*

$$g_{b,T} := b - \sum_{v \in T \cap \Sigma_\ell^+} [F_v^+ : \mathbb{Q}_\ell] \frac{N(N-1)}{2} - N[F^+ : \mathbb{Q}] \frac{1 + (-1)^{\mu+1-N}}{2}$$

elements.

Proof By (3.1), Proposition 3.14 for $v \in \Sigma_\ell^+$, Proposition 3.33 for $v \in \Sigma_{min}^+$ (which is applicable by (D3)), and Proposition 3.35 for $v \in \Sigma_{lr}^+$, we have for every $v \in S$ that

$$\dim_k L(\mathscr{D}_v) - \dim_k H^0(F_v^+, ad\, \bar{r}(1)) = \begin{cases} [F_v^+ : \mathbb{Q}_\ell] \frac{N(N-1)}{2} & \text{if } v \mid \ell; \\ 0 & \text{if } v \nmid \ell. \end{cases}$$

Then the lemma follows from [28, Proposition 4.4] in view of Remark 3.39.[9] $\qquad\square$

Proof of Theorem 3.38 By definition, we have

$$H_{K,\mathfrak{m}} = Hom_\mathscr{O}(H^{d(V)}_{\text{ét}}(Sh(V, K), \mathscr{L}_\xi)_\mathfrak{m}, \mathscr{O}),$$

---

9) Strictly speaking, the set S in [28, Proposition 4.4] consists of only places split in F. But the same argument works in our case as well.

under which $\mathsf{T}_{\mathfrak{m}}$ is identified with the image of $\mathbb{T}_N^{\Sigma^+}$ in $\mathrm{End}_{\mathscr{O}}(H_{\mathrm{K},\mathfrak{m}})$ since $\mathrm{H}_{\mathrm{\acute{e}t}}^{d(\mathrm{V})}(\mathrm{Sh}(\mathrm{V},\mathrm{K}),\mathscr{L}_\xi)_{\mathfrak{m}}$ is $\mathscr{O}$-torsion free by (D4).

First, we need to construct a canonical homomorphism $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}} \to \mathsf{T}_{\mathfrak{m}}$. It is well-known that $\mathsf{T}_{\mathfrak{m}}[1/\ell]$ is a reduced ring finite over E. As $H_{\mathrm{K},\mathfrak{m}}$ is a finite free $\mathscr{O}$-module, $\mathsf{T}_{\mathfrak{m}}$ is a reduced ring finite flat over $\mathscr{O}$. Via ι_ℓ, every (closed) point x of $\mathrm{Spec}\,\mathsf{T}_{\mathfrak{m}}[1/\ell]$ gives rise to an RACSDC representation Π_x of $\mathrm{GL}_N(\mathbb{A}_F)$ (Definition 2.8), satisfying that

(a) the associated Galois representation ρ_{Π_x,ι_ℓ} from Proposition 2.9 (2) is residually isomorphic to $\bar{r}^\natural \otimes_k \overline{\mathbb{F}}_\ell$ (hence residually absolutely irreducible by (D2));

(b) $\Pi = \mathrm{BC}(\pi)$ for a cuspidal automorphic representation π of $\mathrm{U}(\mathrm{V})(\mathbb{A}_{F^+})$ satisfying that $(\pi^\infty)^{\mathrm{K}}$ appears (nontrivially) in $\mathrm{H}_{\mathrm{\acute{e}t}}^{d(\mathrm{V})}(\mathrm{Sh}(\mathrm{V},\mathrm{K}),\mathscr{L}_\xi \otimes_{\mathscr{O},\iota_\ell^{-1}} \mathbb{C})$;

(c) the archimedean weights of Π equals ξ, which follows from (b) and Proposition 2.12. The representation ρ_{Π_x,ι_ℓ} induces a continuous homomorphism

$$\rho_x \colon \Gamma_F \to \mathrm{GL}_N(\mathsf{T}_x),$$

where T_x denotes the localization of $\mathsf{T}_{\mathfrak{m}}[1/\ell]$ at x. By a theorem of Carayol [11, Théorème 2], the product homomorphism

$$\prod_{x \in \mathrm{Spec}\,\mathsf{T}_{\mathfrak{m}}[1/\ell]} \rho_x \colon \Gamma_F \to \mathrm{GL}_N\left(\prod_x \mathsf{T}_x\right)$$

is conjugate to some continuous homomorphism $\rho_{\mathfrak{m}} \colon \Gamma_F \to \mathrm{GL}_N(\mathsf{T}_{\mathfrak{m}})$ that is a lifting of $\bar{r}^\natural$. Moreover, by Proposition 2.9 (2)(c) and Lemma 2.3, we obtain a continuous homomorphism

$$r_{\mathfrak{m}} \colon \Gamma_{F^+} \to \mathscr{G}_N(\mathsf{T}_{\mathfrak{m}})$$

satisfying $r_{\mathfrak{m}}^\natural = \rho_{\mathfrak{m}}$ that is a lifting of $\bar{r}$. We claim that $r_{\mathfrak{m}}$ satisfies the global deformation problem $\mathscr{S}$. Indeed, by (b) and Proposition 2.12, $\Pi_{x,w}$ is unramified for nonarchimedean places w of F not above $\Sigma_{\min}^+ \cup \Sigma_{\mathrm{lr}}^+$. Thus, by (c) and Proposition 2.9 (2)(b), $\bar{r}_{\mathfrak{m},v}$ belongs to $\mathscr{D}_v^{\mathrm{FL}}$ for $v \in \Sigma_\ell^+$; and by Proposition 2.9 (2)(a), $\bar{r}_{\mathfrak{m},v}$ is unramified for $v \notin \mathrm{S}$. By (b), Corollary 2.13, and Proposition 2.9 (2)(a), $\bar{r}_{\mathfrak{m},v}$ belongs to $\mathscr{D}_v^{\mathrm{ram}}$ for $v \in \Sigma_{\mathrm{lr}}^+$.[10]

Therefore, by the universal property of $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$, we obtain a canonical homomorphism

$$\varphi \colon \mathsf{R}_{\mathscr{S}}^{\mathrm{univ}} \to \mathsf{T}_{\mathfrak{m}} \tag{3.24}$$

of rings over $\mathscr{O}$. Moreover, it is clear that our homomorphism $r_{\mathfrak{m}}$ satisfies [14, Proposition 3.4.4 (2)–(3)] as well, which implies that φ is surjective. Thus, it remains to show that φ is injective.

We follow the strategy for [28, Theorem 6.8]. Fix a subset $\mathrm{T} \subseteq \mathrm{S}$. We take an integer $n \geq 1$, and a Taylor–Wiles system $(\mathrm{Q}_n, \{\bar{\alpha}_v\}_{v \in \mathrm{Q}_n})$ from Lemma 3.41. For each $v \in \mathrm{Q}_n$, we

- let $\mathrm{Art}_v \colon F_v^\times \to \Gamma_{F_v^+}^{\mathrm{ab}}$ be the local Artin map;
- let $\varpi_v \in F_v^+$ be the uniformizer such that $\mathrm{Art}_v(\varpi_v)$ coincides with the image of ϕ_v^{-1} in $\Gamma_{F_v^+}^{\mathrm{ab}}$;
- let pr_{ϖ_v} be the commuting projection defined in [28, Propositions 5.9 & 5.12];

---

[10] This is not correct if N is odd, which is the only reason that we suppose that $\Sigma_{\mathrm{lr}}^+ = \emptyset$ if N is odd in the statement of the theorem.

• for every $\alpha \in O_{F_v^+}^\times$, let $\mathtt{V}_v^\alpha \in \mathbb{Z}[\mathrm{K}_{v,1} \backslash \mathrm{K}_{v,0} / \mathrm{K}_{v,1}]$ be the characteristic function of the double coset

$$\mathrm{K}_{v,1} \begin{pmatrix} 1_{N-1} & 0 \\ 0 & \alpha \end{pmatrix} \mathrm{K}_{v,1}.$$

For $i = 0, 1$, we put

$$M_{i,\mathrm{Q}_n} := \left(\prod_{v \in \mathrm{Q}_n} \mathrm{pr}_{\varpi_v} \right) H_{\mathrm{K}_i(\mathrm{Q}_n), \mathfrak{m}_{\mathrm{Q}_n}},$$

and let $\mathsf{T}_{i,\mathrm{Q}_n}$ be the image of $\mathbb{T}_N^{\Sigma^+ \cup \mathrm{Q}_n}$ in $\mathrm{End}_{\mathscr{O}}(M_{i,\mathrm{Q}_n})$. We also put

$$M := H_{\mathrm{K},\mathfrak{m}}.$$

Then the canonical map $M \to H_{\mathrm{K},\mathfrak{m}_{\mathrm{Q}_n}}$ is an isomorphism, hence we obtain canonical surjective homomorphisms

$$\mathsf{T}_{1,\mathrm{Q}_n} \twoheadrightarrow \mathsf{T}_{0,\mathrm{Q}_n} \twoheadrightarrow \mathsf{T}_{\mathfrak{m}}$$

of rings over $\mathscr{O}$. Similar to $\mathsf{T}_{\mathfrak{m}}$, we obtain a continuous homomorphism

$$r_{i,\mathrm{Q}_n} : \Gamma_{F^+} \to \mathscr{G}_N(\mathsf{T}_{i,\mathrm{Q}_n}),$$

which is a lifting of $\bar{r}$, for $i = 0, 1$. We have the following two claims:

(1) For every $v \in \mathrm{Q}_n$, there is a continuous character $\mathtt{v}_v \colon O_{F_v^+}^\times \to \mathsf{T}_{1,\mathrm{Q}_n}^\times$ that factors through Δ_v such that

• for every $\alpha \in O_{F_v^+}^\times$, the actions of $\mathtt{V}_v^\alpha$ and $\mathtt{v}_v(\alpha)$ on M_{1,Q_n} coincide;

• $r_{1,\mathrm{Q}_n,v}^\natural$ has a (unique) decomposition $r_{1,\mathrm{Q}_n,v}^\bullet \oplus r_{1,\mathrm{Q}_n,v}^\circ$ such that $r_{1,\mathrm{Q}_n,v}^\bullet$ is a lifting of $\bar{r}_v^\bullet$ on which $\mathrm{I}_{F_v^+}$ acts via the character $\mathtt{v}_v \circ \mathrm{Art}_v^{-1}$, and $r_{1,\mathrm{Q}_n,v}^\circ$ is an unramified lifting of $\bar{r}_v^\circ$.

(2) The composite map

$$M = H_{\mathrm{K},\mathfrak{m}_{\mathrm{Q}_n}} \to H_{\mathrm{K}_0(\mathrm{Q}_n),\mathfrak{m}_{\mathrm{Q}_n}} \xrightarrow{\prod_{v \in \mathrm{Q}_n} \mathrm{pr}_{\varpi_v}} M_{0,\mathrm{Q}_n}$$

is an isomorphism. In particular, the canonical homomorphism $\mathsf{T}_{0,\mathrm{Q}_n} \to \mathsf{T}_{\mathfrak{m}}$ is an isomorphism; and r_{0,Q_n} and $r_{\mathfrak{m}}$ are equivalent liftings of $\bar{r}$.

Indeed, these claims follow easily from [28, Propositions 5.9 & 5.12].

It follows from (1) that r_{1,Q_n} satisfies the global deformation problem $\mathscr{S}(\mathrm{Q}_n)$, which induces a canonical surjective homomorphism

$$\varphi_n \colon \mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}} \twoheadrightarrow \mathsf{T}_{1,\mathrm{Q}_n}$$

of rings over $\mathscr{O}$. We regard $\mathsf{T}_{1,\mathrm{Q}_n}$ as an $\mathscr{O}[\Delta_{\mathrm{Q}_n}]$-module by (1). Now we claim that $\mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}}$ is naturally an $\mathscr{O}[\Delta_{\mathrm{Q}_n}]$-module as well, and that φ_n is $\mathscr{O}[\Delta_{\mathrm{Q}_n}]$-linear. Indeed, take a universal lifting $r_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}}$ for $\bar{r}$ over $\mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}}$. Then for each $v \in \mathrm{Q}_n$, there is a unique character $\mathtt{v}_v^{\mathrm{univ}} \colon \Delta_v \to (\mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}})^\times$ such that $\mathrm{I}_{F_v^+}$ acts on $r_{\mathscr{S}(\mathrm{Q}_n),v}^{\mathrm{univ},\bullet}$ via the character

$$\mathrm{I}_{F_v^+} \xrightarrow{\mathrm{Art}_v^{-1}} O_{F_v^+}^\times \to \kappa_v^\times \to \Delta_v \xrightarrow{\mathtt{v}_v^{\mathrm{univ}}} (\mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}})^\times.$$

Then $\mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}}$ becomes a ring over $\mathscr{O}[\Delta_{\mathrm{Q}_n}]$ via the character $\prod_{v \in \mathrm{Q}_n} \mathtt{v}_v^{\mathrm{univ}} \colon \Delta_{\mathrm{Q}_n} \to (\mathsf{R}_{\mathscr{S}(\mathrm{Q}_n)}^{\mathrm{univ}})^\times$. Moreover, φ_n is a homomorphism of rings over $\mathscr{O}[\Delta_{\mathrm{Q}_n}]$ by the local-global compatibility. By

(2) and Lemma 3.40, we obtain a canonical commutative diagram

$$
\begin{array}{ccc}
\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathrm{Q}_n)}/\mathfrak{a}_{\mathrm{Q}_n} & \xrightarrow{\ \sim\ } & \mathsf{R}^{\mathrm{univ}}_{\mathscr{S}} \\
{\scriptstyle \varphi_n/\mathfrak{a}_{\mathrm{Q}_n}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\
\mathsf{T}_{1,\mathrm{Q}_n}/\mathfrak{a}_{\mathrm{Q}_n} & \xrightarrow{\ \sim\ } \mathsf{T}_{0,\mathrm{Q}_n} \xrightarrow{\ \sim\ } & \mathsf{T}_{\mathfrak{m}}
\end{array}
$$

of rings over $\mathscr{O}$, where all horizontal arrows are isomorphisms.

Choose universal liftings

$$
r^{\mathrm{univ}}_{\mathscr{S}} \colon \Gamma_{F^+} \to \mathscr{G}_N(\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}), \quad r^{\mathrm{univ}}_{\mathscr{S}(\mathrm{Q}_n)} \colon \Gamma_{F^+} \to \mathscr{G}_N(\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathrm{Q}_n)})
$$

for $\bar{r}$ over $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}$ and $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathrm{Q}_n)}$, respectively, such that $r^{\mathrm{univ}}_{\mathscr{S}} = r^{\mathrm{univ}}_{\mathscr{S}(\mathrm{Q}_n)} \bmod \mathfrak{a}_{\mathrm{Q}_n}$. By Proposition 3.7 (2), we obtain isomorphisms

$$
\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}[[X_{v;i,j}]]_{v\in\mathrm{T};1\le i,j\le N} \xrightarrow{\ \sim\ } \mathsf{R}^{\square_\mathrm{T}}_{\mathscr{S}}, \quad \mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathrm{Q}_n)}[[X_{v;i,j}]]_{v\in\mathrm{T};1\le i,j\le N} \xrightarrow{\ \sim\ } \mathsf{R}^{\square_\mathrm{T}}_{\mathscr{S}(\mathrm{Q}_n)}
$$

of rings over $\mathscr{O}$. In particular, we have a surjective homomorphism $\mathsf{R}^{\square_\mathrm{T}}_{\mathscr{S}} \to \mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}$, which makes $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}$ an algebra over $\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}$.

We put

$$
S_\infty := \mathscr{O}[[X_{v;i,j}]]_{v\in\mathrm{T};1\le i,j\le N}[[Y_1,\dots,Y_b]];
$$

and let $\mathfrak{a}_\infty \subseteq S_\infty$ be the augmentation ideal. Put

$$
R_\infty := \mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}[[Z_1,\dots,Z_{g_{b,\mathrm{T}}}]]
$$

where $g_{b,\mathrm{T}}$ is the number appearing in Lemma 3.41. Applying the usual patching lemma (see the proof of [1, Theorem 3.6.1], or [28, Lemma 6.10]), we have the following:

• There exists a homomorphism $S_\infty \to R_\infty$ of rings over $\mathscr{O}$ such that we have a surjective homomorphism $R_\infty/\mathfrak{a}_\infty R_\infty \to \mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}$ of rings over $\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}$.

• There exist an R_∞-module M_∞ and an isomorphism $M_\infty/\mathfrak{a}_\infty M_\infty \simeq M$ of $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}}$-modules.

• As an S_∞-module, M_∞ is finite and free.

In particular, we have

$$
\operatorname{depth}_{R_\infty}(M_\infty) \ge \dim S_\infty = 1 + |\mathrm{T}|N^2 + b.
$$

On the other hand, by Proposition 3.14 for $v \in \mathrm{T} \cap \Sigma^+_\ell$, Proposition 3.33 for $v \in \mathrm{T} \cap \Sigma^+_{\min}$ (which is applicable by (D3)), and Proposition 3.35 for $v \in \mathrm{T} \cap \Sigma^+_{\mathrm{lr}}$, we know that $\mathsf{R}^{\mathrm{loc}}_{\mathscr{S},\mathrm{T}}$ is a formal power series ring over $\mathscr{O}$ in

$$
|\mathrm{T}|N^2 + \sum_{v\in\mathrm{T}\cap\Sigma^+_\ell} [F^+_v : \mathbb{Q}_\ell]\frac{N(N-1)}{2}
$$

variables. It follows that R_∞ is a regular local ring of dimension

$$
1 + |\mathrm{T}|N^2 + \sum_{v\in\mathrm{T}\cap\Sigma^+_\ell} [F^+_v : \mathbb{Q}_\ell]\frac{N(N-1)}{2} + g_{b,\mathrm{T}}
$$

$$
= 1 + |\mathrm{T}|N^2 + b - N[F^+ : \mathbb{Q}]\frac{1 + (-1)^{\mu+1-N}}{2}.
$$

As $\dim R_\infty \ge \operatorname{depth}_{R_\infty}(M_\infty)$, we obtain Theorem 3.38 (3). By the Auslander–Buchsbaum theorem, M_∞ is a finite free R_∞-module.

Now we claim that the surjection $R_\infty/\mathfrak{a}_\infty R_\infty \to \mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$ is an isomorphism. Indeed, the R_∞-module $M = M_\infty/\mathfrak{a}_\infty M_\infty$ is finite free over $R_\infty/\mathfrak{a}_\infty R_\infty$; but the action of $R_\infty/\mathfrak{a}_\infty R_\infty$ factors through the surjective map $R_\infty/\mathfrak{a}_\infty R_\infty \to \mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$, implying that this surjection must be an isomorphism. From the claim, we know that M is a finite free $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$-module. In particular, the surjective homomorphism φ (3.24) is injective, hence an isomorphism. Theorem 3.38 (1)–(2) are proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4 Rigidity

4.1 Rigidity of Symmetric Powers of Elliptic Curves

In this subsection, we study rigidity of symmetric powers of elliptic curves.

Let A be an elliptic curve over F^+. For every rational prime ℓ, we fix an isomorphism $\mathrm{H}_{\mathrm{\acute{e}t}}^1(A_{\overline{F}}, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell^{\oplus 2}$, hence obtain a continuous homomorphism $\rho_{A,\ell} \colon \Gamma_{F^+} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$. Suppose that $N \geq 2$. We obtain a continuous homomorphism

$$r_{A,\ell} \colon \Gamma_{F^+} \to \mathscr{G}_N(\mathbb{Z}_\ell) = (\mathrm{GL}_N(\mathbb{Z}_\ell) \times \mathbb{Z}_\ell^\times) \rtimes \{1, \mathfrak{c}\}$$

by the formula $r_{A,\ell}(\gamma) = (\mathrm{Sym}^{N-1} \rho_{A,\ell}(\gamma), \eta_v^{N-1} \epsilon_{\ell,v}^{1-N}(\gamma), \mathfrak{c}(\gamma))$, where $\mathfrak{c}(\gamma) = \mathfrak{c}$ (resp. $\mathfrak{c}(\gamma) = 1$) if $\gamma \in \Gamma_{F^+} \setminus \Gamma_F$ (resp. $\gamma \in \Gamma_F$). Denote by $\bar{r}_{A,\ell}$ the composition of $r_{A,\ell}$ and the projection $\mathscr{G}_N(\mathbb{Z}_\ell) \to \mathscr{G}_N(\mathbb{F}_\ell)$.

Proposition 4.1 *Let v be a nonarchimedean place of F. For all but finitely many rational primes $\ell \geq N$, every lifting of $\bar{r}_{A,\ell,v}$ to an object R of $\mathscr{C}_{\mathbb{Z}_\ell}$ (with respect to the similitude character $\eta_v^N \epsilon_{\ell_v}^{1-N}$) is minimally ramified in the sense of Definition* 3.29.

Proof   For simplicity, we only prove the proposition for v nonsplit in F. The split case is similar and easier, which we leave to the readers. Thus, let w be the unique place of F above v. Fix a finite totally ramified extension F'_w of F_w in $\overline{F}_v^+$ such that $A' := A \otimes_{F^+} F'_w$ has either good or multiplicative reduction. Let T'_w be the image of the subgroup $\mathrm{Gal}(\overline{F}_v^+/F'_w)$ of Γ_{F_w} in $\mathrm{T}_w = \Gamma_{F_w}/\mathrm{P}_{F_w}$. We fix an isomorphism $\mathrm{T}_w \simeq \mathrm{T}_q = t^{\mathbb{Z}_\ell} \rtimes \phi_q^{\hat{\mathbb{Z}}}$ with the q-tame group, where $q = \|w\|$. We now assume $\ell \nmid [F'_w : F_w]$ and $\ell \nmid \prod_{i=1}^N (q^i - 1)$. Then $\mathrm{T}'_w = \mathrm{T}_w$. Let $\mathfrak{T} = \mathfrak{T}(\bar{r}_{A,\ell,v})$ be the set of isomorphism classes of absolutely irreducible representations of P_{F_w} appearing in $\bar{r}_{A,\ell,v}^\natural$ as before.

We first consider the case where A' has multiplicative reduction. Let u be the valuation of the j-invariant $j(A)$ in F_w, which is a negative integer. Assume further that ℓ is coprime to u. Then $\rho_{A',\ell}(t)$ is conjugate to either $1 + J_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $-(1 + J_2)$ in $\mathrm{GL}_2(\mathbb{Z}_\ell)$, which implies that $\mathrm{Sym}^{N-1} \rho_{A',\ell}(t)$ is conjugate to $1 + J_N$ or $(-1)^{N-1}(1 + J_N)$ in $\mathrm{GL}_N(\mathbb{Z}_\ell)$. It follows that $\mathfrak{T}$ is a singleton, say $\{\tau\}$; and every lifting ϱ_τ of $\bar{\varrho}_\tau$ is minimally ramified since $\ell \nmid \prod_{i=1}^N (q^i - 1)$. Thus, every lifting r of $\bar{r}_{A,\ell,v}$ is minimally ramified.

We then consider the case where A' has good reduction. Let $\alpha, \beta \in \overline{\mathbb{Q}}_\ell$ be the two eigenvalues of $\rho_{A',\ell}(\phi_q)$. Then α, β are Weil $q^{-1/2}$-numbers in $\overline{\mathbb{Q}}$, which depend only on A', not on ℓ. We further assume that ℓ satisfies that $\alpha, \beta \in (\overline{\mathbb{Z}}_{(\ell)})^\times$, and that the image of the set

$$\left\{ (\alpha/\beta)^{N-1}, (\alpha/\beta)^{N-3}, \dots, (\alpha/\beta)^{3-N}, (\alpha/\beta)^{1-N} \right\}$$

in $\overline{\mathbb{F}}_\ell^\times$ does not contain q. It follows that for every $\tau \in \mathfrak{T}$, every lifting ϱ_τ of $\bar{\varrho}_\tau$ is actually unramified by Lemma 3.21 (2) as $\ell \nmid \prod_{i=1}^N (q^i - 1)$, hence minimally ramified. Thus, every lifting r of $\bar{r}_{A,\ell,v}$ is minimally ramified.

Since in both cases, we only exclude finitely many rational primes ℓ, the proposition follows. $\qquad\square$

The proposition has the following immediate corollary.

Corollary 4.2 *Let Σ^+ be a finite set of nonarchimedean places of F^+ containing Σ^+_{bad} such that A has good reduction outside Σ^+. Then for all but finitely many rational primes ℓ, $\bar{r}_{A,\ell}$ is rigid for $(\Sigma^+, \emptyset)$ (Definition 3.36 with $\mathscr{O} = \mathbb{Z}_\ell$).*

Proof   We need to show that each of the four conditions in Definition 3.36 excludes only finitely many rational primes ℓ. By Proposition 4.1, Condition (1) excludes only finitely many ℓ. Condition (2) is empty. Condition (3) holds if ℓ satisfies $\ell \geq N+1$ and $\Sigma^+_\ell \cap \Sigma^+ = \emptyset$. Condition (4) is automatic. $\qquad\square$

4.2   Rigidity of Automorphic Galois Representations

In this subsection, we study rigidity of reduction of automorphic Galois representations.

Let Π be an RACSDC representation of $\mathrm{GL}_N(\mathbb{A}_F)$ (Definition 2.8) for $N \geq 2$, and denote by Σ^+_Π the smallest (finite) set of nonarchimedean places of F^+ containing Σ^+_{bad} such that Π_w is unramified for every nonarchimedean place w of F not above Σ^+_Π. Let $E \subseteq \mathbb{C}$ be a strong coefficient field of Π (Definition 2.10). Then for every prime λ of E, we have a continuous homomorphism $\rho_{\Pi,\lambda} \colon \Gamma_F \to \mathrm{GL}_N(E_\lambda)$.

When $\rho_{\Pi,\lambda}$ is residually absolutely irreducible, by Proposition 2.9 (2)(c) and Lemma 2.3 (3), we have an extension

$$\bar{\rho}_{\Pi,\lambda,+} \colon \Gamma_{F^+} \to \mathscr{G}_N(O_E/\lambda)$$

of the residue representation $\bar{\rho}_{\Pi,\lambda}$ of $\rho_{\Pi,\lambda}$, with the similitude character $\chi_\lambda := \eta^N \epsilon_\ell^{1-N}$.

Conjecture 4.3   *Let Π and E be as above. Fix a finite set Σ^+ of nonarchimedean place of F^+ containing Σ^+_Π. Then for all but finitely many primes λ of E, we have*

   (1) *$\rho_{\Pi,\lambda}$ is residually absolutely irreducible;*

   (2) *$\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ is absolutely irreducible, where ℓ is the underlying rational prime of λ;*

   (3) *$\bar{r}_{\Pi,\lambda} := \bar{\rho}_{\Pi,\lambda,+}$ is rigid for $(\Sigma^+, \emptyset)$ (Definition 3.36 with $\mathscr{O}$ the ring of integers of E_λ).*

Remark 4.4   When $N = 2$, Conjecture 4.3 is not hard to verify. Part (3) of Conjecture 4.3 was also studied in [17] under several simplifying restrictions.

Concerning Conjecture 4.3 (1)–(2), we have the following proposition.

Proposition 4.5   *Let Π and E be as above. Suppose that there exists a nonarchimedean place w of F such that Π_w is supercuspidal. Then*

   (1) *There exists a finite set Λ_1 of primes of E depending on Π_w only, such that for every $\lambda \notin \Lambda_1$, $\rho_{\Pi,\lambda}$ is residually absolutely irreducible.*

   (2) *There exists a finite set Λ_2 containing Λ_1 from (1) such that for every $\lambda \notin \Lambda_2$, the restriction $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ remains absolutely irreducible.*

The proof of part (2) was suggested to us by Toby Gee. Originally, our alternative argument needs to further assume for (2) that Π is a twist of the Steinberg representation at some nonarchimedean place of F not above Σ^+_{bad}.

Proof   Let W_{F_w} be the Weil group of F_w. Since Π_w is supercuspidal, we have the induced continuous representation $\rho_{\Pi_w} \colon \mathrm{W}_{F_w} \to \mathrm{GL}_N(\mathbb{C})$ via the local Langlands correspondence, which is irreducible. Fix an arithmetic Frobenius element ϕ_w in W_{F_w}, which determines a natural quotient map $\mathrm{W}_{F_w} \to \mathbb{Z}$ sending ϕ_w to 1. For every integer $b \geq 1$, let $\mathrm{W}_{F_w}^b$ be the inverse image of $b\mathbb{Z}$. Then there exist an absolutely irreducible representation τ of I_{F_w} and a character χ of $\mathrm{W}_{F_w}^b$, such that ρ_{Π_w} is isomorphic to $\mathrm{Ind}_{\mathrm{W}_{F_w}^b}^{\mathrm{W}_{F_w}} \tau \otimes \chi$, where b is the smallest positive integer satisfying $\tau^{\phi_w^b} \simeq \tau$. We may choose a finite extension E' of E inside $\mathbb{C}$, and a finite set Λ' of primes of E', such that both τ and χ are defined over $O_{E',(\Lambda')}$. In particular, the image of ρ_{Π_w} is contained in $\mathrm{GL}_N(O_{E',(\Lambda')})$, up to conjugation.

Now let Λ_1' be the smallest set of primes of E' containing Λ' such that every $\lambda' \notin \Lambda_1'$ satisfies

- w does not divide ℓ;
- the underlying rational prime does not divide $b|\mathrm{I}_{F_w}/\ker \rho_{\Pi_w}|$;
- $\bar{\tau}_{\lambda'} \coloneqq \tau \otimes_{O_{E',(\Lambda')}} \overline{O_{E'}/\lambda'}$ remains irreducible;
- b remains the smallest positive integer that satisfies $\bar{\tau}_{\lambda'}^{\phi_w^b} \simeq \bar{\tau}_{\lambda'}$.

Then Λ_1' is a finite set, satisfying that the composite map

$$\bar{\rho}_{\Pi_w,\lambda'} \colon \mathrm{W}_{F_w} \to \mathrm{GL}_N(O_{E',(\Lambda')}) \to \mathrm{GL}_N(\overline{O_{E'}/\lambda'})$$

is irreducible for $\lambda' \notin \Lambda_1'$. Let $\xi = \xi_\Pi$ be the archimedean weights of Π (Definition 2.8).

For (1), we let Λ_1 be the set of primes of E underlying Λ_1'; and then (1) follows by Proposition 2.9(2a).

For (2), let Λ_2 be the union of Λ_1 constructed in (1) above and all primes λ of F whose underlying rational prime ℓ satisfies either $\ell \leq N(b_\xi - a_\xi) + 1$ or $\Sigma_\ell^+ \cap \Sigma_\Pi^+ \neq \emptyset$. Take a prime $\lambda \notin \Lambda_2$. By (1), $\bar{\rho}_{\Pi,\lambda}$ is absolutely irreducible, whose coefficients we may assume to be just $\overline{O_E/\lambda}$. Since the degree of the extension $F(\zeta_\ell)/F$ is coprime to ℓ, the representation $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ is semisimple. We claim that $\bar{\rho}_{\Pi,\lambda}$ is an induction of an irreducible representation ρ' of $\mathrm{Gal}(\overline{F}/F')$ for some field extensions $F \subseteq F' \subseteq F(\zeta_\ell)$ such that $[F' : F]$ equals the number of irreducible summands of $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$. By [8, Lemma 4.3], it suffices to show that the irreducible summands of $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ are pairwise non-isomorphic. Since w is unramified in $F(\zeta_\ell)$, it suffices to check that the irreducible summands of $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{I}_{F_w}}$ are pairwise non-isomorphic, which is already known by our choice of Λ_1' above.

By our definition of Λ_2 and Proposition 2.9 (2)(b), $\bar{\rho}_{\Pi,\lambda}$ is crystalline with regular Fontaine–Laffaille weights in $[a_\xi, b_\xi]$ and $\ell > N(b_\xi - a_\xi) + 1 \geq (b_\xi - a_\xi) + 2$. Thus, we must have $F' = F$ by Lemma 4.7 below. Therefore, $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ remains absolutely irreducible, hence (2) follows.

The proposition is proved. $\square$

To finish the proof of Proposition 4.5 (2), we need two lemmas, both of which are suggested to us by Toby Gee. We start with some notation. For every finite extension L of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$, we put $\Gamma_L \coloneqq \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/L)$, denote by $\mathrm{I}_L \subseteq \Gamma_L$ the inertia subgroup and by $\mathrm{P}_L \subseteq \mathrm{I}_L$ the wild inertia subgroup, and put $\mathrm{T}_L \coloneqq \Gamma_L/\mathrm{P}_L$.

Lemma 4.6   *Let $\bar{\rho} \colon \Gamma_L \to \mathrm{GL}_N(\overline{\mathbb{F}}_\ell)$ be a continuous homomorphism such that $\bar{\rho}(\mathrm{P}_L) = \{1\}$. Then there exists a finite unramified extension L' of L inside $\overline{\mathbb{Q}}_\ell$ such that $\bar{\rho}|_{\Gamma_{L'}}$ is a direct sum of characters.*

Proof   Let $t \in I_L/P_L$ be a topological generator and $\phi \in T_L$ a lift of the arithmetic Frobenius. Then we have $T_L = t^{\prod_{p \neq \ell} \mathbb{Z}_p} \rtimes \phi^{\hat{\mathbb{Z}}}$ subject to the relation $\phi t \phi^{-1} = t^{\ell^a}$, where ℓ^a is the residue cardinality of L. We regard $\bar{\rho}$ as a representation of T_L. As I_L/P_L has pro-order prime to ℓ, the element $\bar{\rho}(t)$ is semisimple. Let $b \geq 1$ be an integer such that the eigenvalues of $\bar{\rho}(t)$ are contained in $\mathbb{F}_{\ell^{ab}}^{\times} \subseteq \overline{\mathbb{F}}_{\ell}^{\times}$. Then $\bar{\rho}(\phi^b)$ commutes with $\bar{\rho}(t)$. Let $c \geq 1$ be an integer such that $\bar{\rho}(\phi^{bc})$ is semisimple. Then the unique unramified extension of L inside $\overline{\mathbb{Q}}_{\ell}$ of degree bc satisfies the requirement of the lemma. $\qquad\square$

Lemma 4.7   *Consider a field tower $\mathbb{Q}_{\ell} \subseteq L \subseteq L' \subseteq \overline{\mathbb{Q}}_{\ell}$ in which $L/\mathbb{Q}_{\ell}$ is finite unramified and L'/L is finite Galois. Let $\bar{\psi} \colon \Gamma_{L'} \to \mathrm{GL}_m(\overline{\mathbb{F}}_{\ell})$ be a continuous homomorphism for some integer $m \geq 1$, and put $\bar{\rho} := \mathrm{Ind}_{\Gamma_{L'}}^{\Gamma_L} \bar{\psi}$. Suppose that*

(a) *$\bar{\rho}$ is crystalline with regular Fontaine–Laffaille weights in $[a, b]$ (Definition 3.11) for some integers a and b satisfying $0 \leq b - a \leq \ell - 2$; and*

(b) *$(b - a)[L' : L] < \ell - 1$.*

Then L' is unramified over L.

Proof   We may assume $a = 0$. We assume on the contrary that L'/L is ramified. Up to replacing L by the maximal subfield of L' that is unramified over L, we may assume that L'/L is totally ramified. Up to replacing $\bar{\psi}$ by its semisimplification, we may assume that $\bar{\psi}$ is semisimple. Since $|\Gamma_L/\Gamma_{L'}| = [L' : L]$ is coprime to ℓ, $\bar{\rho} = \mathrm{Ind}_{\Gamma_{L'}}^{\Gamma_L}(\bar{\psi})$ is also semisimple. As P_L is a pro-ℓ normal subgroup of Γ_L, we have $\bar{\rho}(P_L) = \{1\}$ by [25, Proposition 4]. By Lemma 4.6, there exists a finite unramified extension L''/L inside $\overline{\mathbb{Q}}_{\ell}$ such that $\bar{\rho}|_{\Gamma_{L''}}$ is a direct sum of characters. Up to replacing L by L'' and L' by $L'L''$, we may assume that $\bar{\rho}$ itself is a direct sum of characters of Γ_L, say $\chi_1, \ldots, \chi_N \colon \Gamma_L \to \overline{\mathbb{F}}_{\ell}^{\times}$. On the other hand, since $\bar{\rho} = \mathrm{Ind}_{\Gamma_{L'}}^{\Gamma_L}(\bar{\psi})$, there exist two distinct characters χ_i and χ_j such that $\chi_i \chi_j^{-1}$ is trivial on $I_{L'}$, which is the unique subgroup of I_L of index $[L' : L]$. However, Condition (a) implies that $\chi_i \chi_j^{-1}$ is a crystalline character of Fontaine–Laffaille weights contained in $[-b, b]$. By the Fontaine–Laffaille theory, we have

$$\chi_i \chi_j^{-1}|_{I_L} = \bigotimes_{\tau \colon L \hookrightarrow \overline{\mathbb{Q}}_{\ell}} \omega_{\tau}^{b_{\tau}},$$

where $\omega_{\tau} \colon I_L \to \overline{\mathbb{F}}_{\ell}^{\times}$ is the fundamental character of level 1 corresponding to τ, and b_{τ} is an integer in $[-b, b]$. Since the Fontaine–Laffaille weights of $\bar{\rho}$ are regular, we have $b_{\tau} \neq 0$ for all τ. Now condition (b) implies that $\chi_i \chi_j^{-1}$ can not be trivial on $I_{L'}$, which is a contradiction. The lemma is proved. $\qquad\square$

Concerning the entire Conjecture 4.3, we have the following theorem.

Theorem 4.8   *Let Π and E be as above. Suppose that there exists a nonarchimedean place of F at which Π is supercuspidal. Then Conjecture 4.3 holds for Π and E.*

Proof   Let $\xi = \xi_{\Pi}$ be the archimedean weights of Π (Definition 2.8). Let Λ_2 be the set in Proposition 4.5 (2). It suffices to prove (3) of Conjecture 4.3. We need to show that each of the four conditions in Definition 3.36 excludes only finitely many primes λ. Condition (2) is empty. Condition (3) holds if the underlying rational prime ℓ of λ satisfies $\ell \geq (b_{\xi} - a_{\xi}) + 2$ and $\Sigma_{\ell}^+ \cap \Sigma^+ = \emptyset$ by Proposition 2.9 (2)(b). Condition (4) is automatic.

It remains to consider Condition (1). Let λ be a prime of E not in Λ_2 whose underlying

rational prime ℓ satisfies

$$\Sigma_\ell^+ \cap \Sigma^+ = \emptyset, \quad \ell \geq 2(N+1), \quad \ell \geq (b_\xi - a_\xi) + 2, \quad \ell \geq q_w^N$$

for every place w of F above Σ^+. In particular, we have

(a) ℓ is unramified in F;

(b) Π_w is unramified for every place w of F above ℓ;

(c) $\bar{\rho}_{\Pi,\lambda}|_{\mathrm{Gal}(\overline{F}/F(\zeta_\ell))}$ is absolutely irreducible, which implies that $\bar{r}_{\Pi,\lambda}(\mathrm{Gal}(\overline{F}/F^+(\zeta_\ell)))$ is adequate by Remark 3.39;

(d) Proposition 3.14 holds for the local deformation problem $\mathscr{D}^{\mathrm{FL}}$ of $\bar{r}_{\Pi,\lambda,v}$ for every $v \in \Sigma_\ell^+$;

(e) Proposition 3.33 holds for $\bar{r}_{\Pi,\lambda,v}$ for every $v \in \Sigma^+$.

For a collection $\mathscr{D}_{\Sigma^+} = \{\mathscr{D}_v \,|\, v \in \Sigma^+\}$ in which $\mathscr{D}_v$ is an irreducible component of $\mathrm{Spf}\, \mathsf{R}^{\mathrm{loc}}_{\bar{r}_{\Pi,\lambda,v}}$ for $v \in \Sigma^+$, we define a global deformation problem (Definition 3.6)

$$\mathscr{S}(\mathscr{D}_{\Sigma^+}) \coloneqq (\bar{r}_{\Pi,\lambda}, \eta^N \epsilon_\ell^{1-N}, \Sigma^+ \cup \Sigma_\ell^+, \{\mathscr{D}_v\}_{v \in \Sigma^+ \cup \Sigma_\ell^+})$$

where for $v \in \Sigma^+$, $\mathscr{D}_v$ is the prescribed irreducible component (which is a local deformation problem by Proposition 3.33 (2)) in $\mathscr{D}_{\Sigma^+}$; and for $v \in \Sigma_\ell^+$, $\mathscr{D}_v$ is the local deformation problem $\mathscr{D}^{\mathrm{FL}}$ of $\bar{r}_{\Pi,\lambda,v}$ from Definition 3.12. Now by (a)–(e), and the same proof of [28, Theorem 10.1] (which assumes that $\Sigma^+ \cup \Sigma_\ell^+$ consists only of places split in F), we know that the global universal deformation ring $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathscr{D}_{\Sigma^+})}$ is a finite $\mathscr{O}$-module. Moreover, we have $\mu \equiv N \bmod 2$. By (d)–(e) and the same proof of [16, Lemma 5.1.3] (which assumes that $\Sigma^+ \cup \Sigma_\ell^+$ consists only of places split in F), we know that the Krull dimension of $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathscr{D}_{\Sigma^+})}$ is at least one. Thus, $\mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathscr{D}_{\Sigma^+})}[1/\ell]$ is nonzero. Fix an isomorphism $\iota_\ell \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. By choosing a $\overline{\mathbb{Q}}_\ell$-point of $\mathrm{Spec}\, \mathsf{R}^{\mathrm{univ}}_{\mathscr{S}(\mathscr{D}_{\Sigma^+})}[1/\ell]$, we obtain an RACSDC representation $\Pi(\mathscr{D}_{\Sigma^+})$ of $\mathrm{GL}_N(\mathbb{A}_F)$ satisfying

• $\Pi(\mathscr{D}_{\Sigma^+})$ is unramified outside Σ^+;

• for every place w of F above Σ^+, there is an open compact subgroup U_w of $\mathrm{GL}_N(F_w)$ depending only on Π_w, such that $\Pi(\mathscr{D}_{\Sigma^+})_w^{U_w} \neq \{0\}$;

• the archimedean weights of $\Pi(\mathscr{D}_{\Sigma^+})$ are contained in $[a_\xi, b_\xi - N + 1]$;

• $\rho_{\Pi(\mathscr{D}_{\Sigma^+}),\iota_\ell}$ and $\rho_{\Pi,\lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ are residually isomorphic.

In fact, the second property is a consequence of Proposition 2.9 (2)(a), Corollary 3.31 (which is applicable since $\ell \geq q_w^N$), and the fact that irreducible admissible representations lying on a given Bernstein component have a common level. Note that there are only finitely many RACSDC representations of $\mathrm{GL}_N(\mathbb{A}_F)$ up to isomorphism, satisfying the first three properties. By the strong multiplicity one property for GL_N [23], we know that for ℓ large enough, Π is the only RACSDC representation of $\mathrm{GL}_N(\mathbb{A}_F)$ up to isomorphism, satisfying all the four properties.

Now we claim that for two different collections $\mathscr{D}_{\Sigma^+}$ and $\mathscr{D}'_{\Sigma^+}$, the RACSDC representations $\Pi(\mathscr{D}_{\Sigma^+})$ and $\Pi(\mathscr{D}'_{\Sigma^+})$ are not isomorphic. Assuming this claim, then for ℓ large enough, we have only one collection, which is $\{\mathscr{D}_v^{\mathrm{min}} \,|\, v \in \Sigma^+\}$, that is, Definition 3.36 (1) is satisfied. The theorem is proved.

For the claim itself, we take a place $v \in \Sigma^+$. Then the local components of $\Pi(\mathscr{D}_{\Sigma^+})$ above v give rise to a continuous homomorphism $r \colon \Gamma_{F_v^+} \to \mathscr{G}_N(\overline{\mathbb{Q}}_\ell)$, which corresponds to a $\overline{\mathbb{Q}}_\ell$-point x_r in $\mathrm{Spec}\, \mathsf{R}^{\mathrm{loc}}_{\bar{r}_{\Pi,\lambda,v}}[1/\ell]$. Now the dimension of the tangent space of $\mathrm{Spec}\, \mathsf{R}^{\mathrm{loc}}_{\bar{r}_{\Pi,\lambda,v}}[1/\ell]$ at x_r is equal to

$$N^2 + \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^1(F_v^+, \mathrm{ad}\, r) - \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^0(F_v^+, \mathrm{ad}\, r)$$

$$= N^2 + \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^2(F_v^+, \mathrm{ad}\, r)$$
$$= N^2 + \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^0(F_v^+, (\mathrm{ad}\, r)(1))$$
$$\leq N^2 + \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^0(F_w, (\mathrm{ad}\, r^\natural)(1)),$$

where w is the place of F induced by the embedding $F \hookrightarrow \overline{F}_v^+$. However, since $\Pi(\mathscr{D}_{\Sigma^+})_w$ is generic, we have $\dim_{\overline{\mathbb{Q}}_\ell} \mathrm{H}^0(F_w, (\mathrm{ad}\, r^\natural)(1)) = 0$ by [2, Lemma 1.3.2 (1)]. Thus, by Proposition 3.33 (1), $\mathrm{Spec}\, \mathsf{R}^{\mathrm{loc}}_{\bar{r}_{\Pi,\lambda,v}}[1/\ell]$ is smooth at x_r, which implies that x_r can not lie on two irreducible components. The claim then follows. $\qquad\qquad\square$

Conflict of Interest   The authors declare no conflict of interest.

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